## B-L model with $D_4 \times Z_4 \times Z_2$ symmetry for fermion mass hierarchies and mixings

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We construct a gauge B - L model with  $D_4 \times Z_4 \times Z_2$  symmetry that can explain the quark and lepton mass hierarchies and their mixings with the realistic CP phases via the type-I seesaw mechanism. Six quark mases, three quark mixing angles and CP phase in the quark sector can get the central values and Yukawa couplings in the quark sector are diluted a range of three orders of magnitude difference by the perturbation theory at the first order. For neutrino sector, the smallness of neutrino mass is achieved by the Type-I seesaw mechanism. Both inverted and normal neutrino mass hierarchies are in consistent with the experimental data. The prediction for the sum of neutrino masses for normal and inverted hierarchies, the effective neutrino masses and the Dirac CP phase are well consistent with all the recent limits.

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#### I. INTRODUCTION

The mass hierarchy problem is one of the most exciting issues in particle physics that require the extension of the Standard Model (SM). Some of the experimental data related to flavour problem including the origin of the quark mass hierarchy  $[1]m_u \ll m_c \ll m_t$  and  $m_d \ll m_s \ll m_b$ , the hierarchy of charged lepton mass  $m_e \ll m_\mu \ll m_\tau$  and the origin of the tiny of three quark mixing angles as well as the neutrino mass spectrum and mixings.

Because of mentioned issues, various SM extensions have been implemented such as symmetry extensions with scalars and/or fermion fields. The B - L model [2–8] is appreciated because the simplest way is to add three right-handed neutrinos for generating neutrino masses. Although this model solves many interesting problems such as dark matter [3], the muon anomalous magnetic moment [4, 8], leptogenesis [5, 6] and gravitational wave radiation [7], it cannot provide a satisfactory explanation for fermion masses and mixing observables. Non-Abelian discrete symmetries have seem to be the most powerful tool for reproducing the observed mass and mixing patterns of leptona

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and quarks (see, for example, Ref. [9]).  $D_4$  symmetry received much attention because it can provide a predictive depiction of the mentioned patterns [10–23], however, those previous works are essentially different from our current study for the following basic points:

- (1) Ref. [16] based on symmetries  $G_{BL} \times D_4 \times Z_4$  in which, for the quark sector, up to four  $SU(2)_L$  doublets and three singlets are introduced, and the obtained quark mixing matrix, whose "13", "23", "31" and "32" entries are zero, is not natural because in fact all the elements of the quark mixing matrix are non-zero [1].
- (2) Ref. [17] based on symmetries  $G_{SM} \times D_4 \times Z_2$  in which the realistic quark mixing pattern has not been considered and the quark mass hierarchy is not satisfied.
- (3) Ref. [18] based on symmetries  $G_{331} \times U(1)_{\mathcal{L}} \times D_4$  in which five  $SU(3)_L$  triplets are used, and the 1-2 mixing of the ordinary quarks is obtained if the  $D_4$  symmetry is violated with 1' symmetry instead of  $\underline{1}$  as usual.
- (4) Ref. [19] based on symmetries  $G_{331} \times U(1)_{\mathcal{L}} \times D_4$  in which the realistic quark mixing matrix is achieved satisfied, however, the quark mass hierarchy is not satisfied.
- (5) In Ref. [20], the obtained quark mixing matrix, whose "13", "23", "31" and "32" entries are zero, is not natural because in fact all the elements of the quark mixing matrix are non-zero [1], and the quark mass hierarchy is not satisfied.
- (6) In Ref. [21], the obtained quark mixing matrix, whose "13", "23", "31" and "32" entries are zero, is not natural because in fact all the elements of the quark mixing matrix are non-zero [1], and the quark mass hierarchy is not satisfied.
- (7) Ref. [22] based on symmetries  $G_{331} \times D_4 \times Z_4 \times Z_3^{(1)} \times Z_3^{(2)} \times Z_{16}$  in which two  $SU(3)_L$ triplets and six  $SU(3)_L$  singlets are used.
- (8) In Ref. [23], the quark mass hierarchy is a bit unnatural since the Yukawa couplings spread over the region from  $\mathcal{O}(10^{-3})$  to  $\mathcal{O}(1)$  (three orders of magnitude difference).

Hence, it would be desirable to propose another  $D_4$  flavor model which can overcome the mentioned limitations of previous studies, especially the quark mass hierarchy, the tiny of quark mixing angles, the neutrino mass spectrum and mixing pattern.

<sup>&</sup>lt;sup>1</sup>  $G_{BL} = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$  is the gauge symmetry of B - L model. <sup>2</sup>  $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$  is the SM gauge symmetry.

In this study, we propose another  $D_4$  model, which differs from those of Refs. [13, 16], by additionally introducing one doublet (H') put in  $\underline{1}'$  under  $D_4$  [13] and using one singlet instead of one doublet in the quark sector [16]. The properties under  $D_4$  of the right handed charged lepton  $(l_{1R})$  and of the right handed neutrino  $(\nu_{1R})$ , and the properties under  $Z_4$  of right-hand leptons  $l_{1R}$ ,  $l_{\alpha R}$ ,  $\nu_{1R}$ ,  $\nu_{\alpha R}$  and singlet scalars  $\chi$ ,  $\varphi$ ,  $\phi$  in our present work are completely different from those of Ref. [13, 16]. As a consequence, the charged-leptons, neutrinos and quarks mass hierarchies can be naturally achieved.

The rest of this work is layout as follows. We present the model description in section II. Sections III and IV are devoted to the quark and lepton masses and mixings, respectively. Section V is for the numerical analysis. We make some conclusions in Sec. VI.

#### II. THE MODEL

The total symmetry of the model is  $\Gamma = SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times D_4 \times Z_4 \times Z_2$  where lepton, quark and scalar fields, under  $D_4$  and  $Z_4$ , are essentially different from those of Refs. [13, 16]. Namely, in this study, the first families of the left handed quark, right handed up-and down quarks are assigned in  $\mathbf{1}_{+-}$ ; the two other families of quarks are assigned in  $\mathbf{2}$ . To explain the hierarchies of quark masses, one  $SU(2)_L$  doublet H' with B - L = 0 put in  $\mathbf{1}_{-+}$  under  $D_4$  together with three flavons  $\rho, \varphi$  and  $\phi$  with B - L = 0 respectively put in  $\mathbf{2}$  and  $\mathbf{1}_{+-}$  under  $D_4$  are additional introduced, i.e., the considered model contains two  $SU(2)_L$  doublets<sup>3</sup>. The particle and scalar contents of the model is shown in Table I.

Table I. Particle and scalar contents of the model ( $\alpha = 2, 3$ ).

Fields	$Q_{1L}$	$Q_{\alpha L}$	$u_{1R}$	$u_{\alpha R}$	$d_{1R}$	$d_{\alpha R}$	$\psi_{1L}$	$\psi_{\alpha L}$	$l_{1R}$	$l_{\alpha R}$	$\nu_{1R}$	$\nu_{\alpha R}$	H	H'	ρ	$\phi$	$\varphi$	$\chi$
$\mathrm{U}(1)_{B-L}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	-1	-1	-1	-1	-1	-1	0	0	0	0	0	2
$D_4$	$ 1_{+-} $	2	$1_{+-}$	2	$ 1_{+-} $	2	$ 1_{-+} $	2	$ 1_{-+} $	2	$ 1_{-+} $	2	$ 1_{+-} $	$ 1_{-+} $	2	$ 1_{+-} $	1	$1_{+-}$
$Z_4$	1	i	-1	-i	-1	-i	1	i	-1	-i	i	i	-1	-1	$\left -i\right $	1	-1	-1
$Z_2$	-	+	_	+	_	+	+	+	+	+	-	_	+	+	-	+	-	+

With the given particle content,  $\overline{Q}_{1L}u_{1R}$  transforms as  $(\mathbf{2}, \frac{1}{2}, 0, \mathbf{1}_{++}, -1)$  can couple to  $(\widetilde{H}\phi)_{\mathbf{1}_{++}}$ ;  $\overline{Q}_{\alpha L}u_{\alpha R} \sim (\mathbf{2}, \frac{1}{2}, 0, \mathbf{1}_{+-} + \mathbf{1}_{-+} + \mathbf{1}_{++} + \mathbf{1}_{--}, 1)$  can, respectively, couple to  $\widetilde{H}, \widetilde{H'}, (\widetilde{H}\phi)_{\mathbf{1}_{++}}$  and  $(\widetilde{H'}\phi)_{\mathbf{1}_{--}}; \overline{Q}_{1L}u_{\alpha R} \sim (\mathbf{2}, \frac{1}{2}, 0, \mathbf{2}, i)$  can couple to  $(\widetilde{H}\rho)_{\mathbf{2}}$  and  $(\widetilde{H'}\rho)_{\mathbf{2}}$ ; and  $\overline{Q}_{\alpha L}u_{1R} \sim (\mathbf{2}, \frac{1}{2}, 0, \mathbf{2}, -i)$ 

<sup>&</sup>lt;sup>3</sup> see, for instance [24, 25], for a review of the two-Higgs-doublet model (2HDM).

can couple to  $(\tilde{H}\rho^*)_2$  and  $(\tilde{H'}\rho^*)_2$  to form invariant terms that generate up-quark mass matrix. The situation is similar to the down quark sector. The Yukawa terms in the quark and lepton sectors are:

$$\begin{aligned} -\mathcal{L}_{Y}^{q} &= \frac{x_{1}^{u}}{\Lambda} (\overline{Q}_{1L} u_{1R})_{1++} (\widetilde{H}\phi)_{1++} + x_{2}^{u} (\overline{Q}_{\alpha L} u_{\alpha R})_{1+-} \widetilde{H} + x_{3}^{u} (\overline{Q}_{\alpha L} u_{\alpha R})_{1-+} \widetilde{H}' \\ &+ \frac{y_{1}^{u}}{\Lambda} (\overline{Q}_{\alpha L} u_{\alpha R})_{1++} (\widetilde{H}\phi)_{1++} + \frac{y_{2}^{u}}{\Lambda} (\overline{Q}_{\alpha L} u_{\alpha R})_{1--} (\widetilde{H'}\phi)_{1--} + \frac{z_{1}^{u}}{\Lambda} (\overline{Q}_{1L} u_{\alpha R})_{2} (\widetilde{H}\rho)_{2} \\ &+ \frac{z_{2}^{u}}{\Lambda} (\overline{Q}_{\alpha L} u_{1R})_{2} (\widetilde{H}\rho^{*})_{2} + \frac{z_{3}^{u}}{\Lambda} (\overline{Q}_{1L} u_{\alpha R})_{2} (\widetilde{H'}\rho)_{2} + \frac{z_{4}^{u}}{\Lambda} (\overline{Q}_{\alpha L} u_{1R})_{2} (\widetilde{H'}\rho^{*})_{2} \\ &+ \frac{x_{1}^{d}}{\Lambda} (\overline{Q}_{1L} d_{1R})_{1++} (H\phi)_{1++} + x_{2}^{d} (\overline{Q}_{\alpha L} d_{\alpha R})_{1+-} H + x_{3}^{d} (\overline{Q}_{\alpha L} d_{\alpha R})_{1-+} H' \\ &+ \frac{y_{1}^{d}}{\Lambda} (\overline{Q}_{\alpha L} d_{\alpha R})_{1++} (H\phi)_{1++} + \frac{y_{2}^{d}}{\Lambda} (\overline{Q}_{\alpha L} d_{\alpha R})_{1--} (H'\phi)_{1--} + \frac{z_{1}^{d}}{\Lambda} (\overline{Q}_{1L} d_{\alpha R})_{2} (H\rho^{*})_{2} + \frac{z_{3}^{d}}{\Lambda} (\overline{Q}_{1L} d_{\alpha R})_{2} (H'\rho)_{2} + \frac{z_{4}^{d}}{\Lambda} (\overline{Q}_{\alpha L} d_{1R})_{2} (H'\rho^{*})_{2} + \text{H.c}, \quad (1) \\ -\mathcal{L}_{lep}^{Y} &= \frac{h_{1}}{\Lambda} (\overline{\psi}_{1L} l_{1R})_{1++} (H\phi)_{1++} + h_{2} (\overline{\psi}_{\alpha L} l_{\alpha R})_{1--} (H'\phi)_{1--} \\ &+ \frac{x_{1}}{\Lambda} (\overline{\psi}_{1L} u_{\alpha R})_{2} (\widetilde{H}\rho^{*})_{2} + \frac{x_{2}}{\Lambda} (\overline{\psi}_{1L} u_{\alpha R})_{2} (\widetilde{H'}\rho^{*})_{2} \\ &+ \frac{x_{3}}{\Lambda} (\overline{\psi}_{\alpha L} l_{\alpha R})_{1++} (H\phi)_{1++} + \frac{h_{5}}{\Lambda} (\overline{\psi}_{\alpha L} l_{\alpha R})_{1--} (H'\phi)_{1--} \\ &+ \frac{x_{1}}{\Lambda} (\overline{\psi}_{1L} v_{\alpha R})_{2} (\widetilde{H}\rho^{*})_{2} + \frac{x_{2}}{\Lambda} (\overline{\psi}_{1L} v_{\alpha R})_{2} (\widetilde{H'}\rho^{*})_{2} \\ &+ \frac{x_{3}}{\Lambda} (\overline{\psi}_{\alpha L} v_{\alpha R})_{1-+} (\widetilde{H}\varphi)_{1++} + \frac{x_{4}}{\Lambda} (\overline{\psi}_{\alpha L} v_{\alpha R})_{1+-} (\widetilde{H'}\varphi)_{1+-} \\ &+ \frac{y_{1}}{2\Lambda} (\overline{\nu}_{1R}^{c} \nu_{1R})_{1++} (\phi\chi)_{1++} + y_{2} (\overline{\nu}_{\alpha R}^{c} \nu_{\alpha R})_{1+-} \chi + \frac{y_{3}}{2\Lambda} (\overline{\nu}_{\alpha R}^{c} \nu_{\alpha R})_{1++} (\phi\chi)_{1++} + y_{2} (\overline{\nu}_{\alpha R}^{c} \nu_{\alpha R})_{1+-} \chi + \frac{y_{3}}{2\Lambda} (\overline{\nu}_{\alpha R}^{c} \nu_{\alpha R})_{1++} (\phi\chi)_{1++} + H.c, \quad (2) \end{aligned}$$

where  $x_{1,2,3}^{u,d}$ ,  $y_{1,2}^{u,d}$  and  $z_{1,2,3,4}^{u,d}$  are the Yukawa-like couplings in the quark sector,  $h_{1,2,3,4,5}$ ;  $x_{1,2,3,4}$  and  $y_{1,2,3}$  are the Yukawa-like couplings in the lepton sector and  $\Lambda$  is the cut-off scale of the theory.

It is worthy to note that additional discrete symmetries  $D_4$ ,  $Z_4$  and  $Z_2$  play crucial roles in forbidding undesired terms to get the expected quark and lepton mass matrices which are listed in Table IV. For instance, in the absence of  $Z_2$ , there will be additional invariant terms  $(\overline{\psi}_{1L}l_{\alpha R})_2(H\rho)_2, (\overline{\psi}_{1L}l_{\alpha R})_2(H'\rho)_2, (\overline{\psi}_{\alpha L}l_{1R})_2(H\rho)_2$  and  $(\overline{\psi}_{\alpha L}l_{1R})_2(H'\rho)_2$  which contribute to the entries "12", "13", "21" and "31" of the charged lepton matrix. As a result, we cannot obtain the mass of charged leptons as expected since the charged lepton matrix cannot be diagonalized.

The vacuum expectation value (VEV) of the scalar fields get the form:

$$\langle H \rangle = (0 \quad v)^T, \quad \langle H' \rangle = (0 \quad v')^T, \quad \langle \varphi \rangle = v_{\varphi}, \quad \langle \phi \rangle = v_{\phi},$$
  
$$\langle \rho \rangle = (\langle \rho_1 \rangle, \quad \langle \rho_1 \rangle) \equiv (v_{\rho}, \quad v_{\rho}), \quad \langle \chi \rangle = v_{\chi}.$$
(3)

In fact, the electroweak symmetry breaking scale is of order about one hundred GeV,  $v^2 + v'^2 = (174 \text{ GeV})^2$ . Furthermore, in the 2HDM, the limits of the parameter  $t_{\beta} = \frac{v'}{v}$  are given by [26]  $t_{\beta} = \frac{v'}{v} \in [1.0.10.0]$  or [27]  $t_{\beta} = \frac{v'}{v} \in [1.0.3.0]$ . For the purpose of determining the scale of Yukawa

couplings, we consider the case of  $t_{\beta} = 1.424$ , i.e.,

$$v = 100 \,\text{GeV}, \ v' = 142.40 \,\text{GeV}.$$
 (4)

In addition, in order to satisfy the quark mass hierarchy, the VEV of singlets and the cut-off scale are assumed to be as follows

$$v_{\rho} = 5 \times 10^{11} \,\text{GeV}, \ v_{\phi} = 10^{11} \,\text{GeV}, \ \Lambda \simeq 10^{13} \,\text{GeV}.$$
 (5)

The models with more than one  $SU(2)_L$  scalar doublet as in this work, the Flavor Changing Neutral Current (FCNC) processes such as  $b \to s\gamma$  exist in the Higgs sector. However, they are suppressed by non-Abelian discrete symmetries [28, 29]. To make such process below the current experimental limits, some restrictions on the model parameters such as the Yukawa couplings and large masses for non SM scalars need to be imposed. The considered model contains many free parameters which allows us freedom to assume that the remaining scalars are sufficiently heavy to fulfil the current experimental limits. Furthermore, the first two lines of Eq. (2) imply that the off-diagonal Yukawa couplings in the charged-lepton sector are proportional to  $\frac{v_{\phi}}{\Lambda} \sim 10^{-2}$ . Therefore, the lepton flavor violation (LFV) processes, such as  $l_j \to l_i \gamma$ , are suppressed by the tiny factor  $\frac{v_{\phi}}{\Lambda} \frac{1}{m_H^2}$  associated with the mentioned small Yukawa couplings and the large mass scale of the heavy scalars  $m_H$  [30–33]. A detailed study of FCNC and LFV processes are beyond the scope of this work.

#### III. QUARK MASS AND MIXING

Using the Clebsch-Gordan coefficients of  $D_4$  symmetry [34], from Eq. (1), when the scalar fields get the VEVs as, Eq. (3), the up-and down-quark mass matrices take the following forms:

$$M_q = M_q^{(0)} + \delta M_q \quad (q = u, d), \tag{6}$$

where

$$M_q^{(0)} = \begin{pmatrix} a_{1q} & 0 & 0 \\ 0 & a_{2q} + a_{3q} & 0 \\ 0 & 0 & a_{2q} - a_{3q} \end{pmatrix}, \quad \delta M_q = \begin{pmatrix} 0 & c_{1q} + c_{3q} & c_{1q} - c_{3q} \\ c_{2q} + c_{4q} & 0 & b_{1q} + b_{2q} \\ c_{2q} - c_{4q} & b_{1q} - b_{2q} & 0 \end{pmatrix}, \quad (7)$$

with

$$a_{1q} = x_1^q v \frac{v_{\varphi}}{\Lambda}, \quad a_{2q} = x_2^q v, \quad a_{3q} = x_3^q v', \quad b_{1q} = y_1^q v \frac{v_{\phi}}{\Lambda}, \quad b_{2q} = y_2^q v' \frac{v_{\phi}}{\Lambda}, \\ c_{1q} = z_1^q v \frac{v_{\rho}}{\Lambda}, \quad c_{2q} = z_2^q v \frac{v_{\rho}}{\Lambda}, \quad c_{3q} = z_3^q v' \frac{v_{\rho}}{\Lambda}, \quad c_{4q} = z_4^q v' \frac{v_{\rho}}{\Lambda} \quad (q = u, d).$$
(8)

Expressions (6)-(8) show that, besides two doublets H and H', one singlet  $\varphi$  contributes to  $M_q^{(0)}$ while  $\delta M_q$  is due to the contribution of two singlets  $\rho$  and  $\phi$ . Without the contributions of  $\rho$ and  $\phi$ ,  $\delta M_q$  will be vanished and the quark mass matrices  $M_q$  in Eq. (6) reduce to the diagonal matrices  $M_q^{(0)}$ , i.e., the corresponding quark mixing matrix  $V_{CKM} = \mathbb{I}_{3\times 3}$  which was ruled out by the recent data. The realistic quark mixing angles are very small [1] which implies that the quark mixing matrix is very close to the identity matrix; thus, the second term  $\delta M_q$  in Eq.(7) can be considered as the perturbed parameter for generating the quark mixing pattern. As a consequence, the realistic quark mixing pattern can be achieved at the first order of perturbation theory. Indeed, at the first order of perturbed theory, the matrices  $\delta M_q$  contribute to the eigenvectors but they have no contribution to the eigenvalues of the quark mass matrices  $M_q$ . The quark masses are determined as

$$m_u = a_{1u}, \quad m_c = a_{2u} + a_{3u}, \quad m_t = a_{2u} - a_{3u},$$
  
$$m_d = a_{1d}, \quad m_s = a_{2d} + a_{3d}, \quad m_b = a_{2d} - a_{3d},$$
 (9)

and the corresponding perturbed quark mixing matrices are:

$$U_{L}^{u} = U_{R}^{u} = \begin{pmatrix} 1 & \frac{c_{1u} + c_{3u}}{m_{c} - m_{u}} & \frac{c_{1u} - c_{3u}}{m_{t} - m_{u}} \\ \frac{c_{4u} + c_{2u}}{m_{u} - m_{c}} & 1 & \frac{b_{2u} + b_{1u}}{m_{t} - m_{c}} \\ \frac{c_{4u} - c_{2u}}{m_{t} - m_{u}} & \frac{b_{2u} - b_{1u}}{m_{t} - m_{c}} & 1 \end{pmatrix}, \quad U_{L}^{d} = U_{R}^{d} = \begin{pmatrix} 1 & \frac{c_{1d} + c_{3d}}{m_{s} - m_{b}} & \frac{c_{1d} - c_{3d}}{m_{b} - m_{d}} \\ \frac{c_{4d} + c_{2d}}{m_{d} - m_{s}} & 1 & \frac{b_{2d} + b_{1d}}{m_{b} - m_{s}} \\ \frac{c_{4d} - c_{2d}}{m_{b} - m_{d}} & \frac{b_{2d} - b_{1d}}{m_{b} - m_{s}} & 1 \end{pmatrix}, \quad (10)$$

with  $b_{1,2q}$  and  $c_{1,2,3,4q}$  (q = u, d) are given in Eq. (8). For simplicity, we consider that case of  $y_{1q} = y_{2q} = y_q$  (q = u, d),  $z_{3d} = z_{1d} = z_d$ , i.e.,

$$b_{2d} = b_{1d} = b_d, \ b_{2u} = b_{1u} = b_u, \ c_{3d} = c_{1d}.$$
 (11)

The quark mixing matrix,  $V_{\text{CKM}} = V_L^u V_L^{d\dagger}$ , owns the following entries:

$$V_{\rm CKM}^{11} = 1 + \frac{2c_{1d}^*(c_{1u} + c_{3u})}{(m_u - m_c)(m_d - m_s)},$$

$$V_{\rm CKM}^{12} = \frac{2b_d^*(c_{1u} - c_{3u})}{(m_b - m_s)(m_t - m_u)} + \frac{c_{1u} + c_{3u}}{m_c - m_u} + \frac{c_{2d}^* + c_{4d}^*}{m_d - m_s},$$

$$V_{\rm CKM}^{13} = \frac{c_{1u} - c_{3u}}{m_t - m_u} + \frac{c_{4d}^* - c_{2d}^*}{m_b - m_d}, \quad V_{\rm CKM}^{21} = \frac{c_{2u} + c_{4u}}{m_u - m_c} + \frac{2c_{1d}^*}{m_s - m_d},$$

$$V_{\rm CKM}^{22} = 1 + \frac{4b_d^*b_u}{(m_b - m_s)(m_t - m_c)} + \frac{(c_{2u} + c_{4u})(c_{2d}^* + c_{4d}^*)}{(m_u - m_c)(m_d - m_s)},$$

$$V_{\rm CKM}^{23} = \frac{2b_u}{m_t - m_c} + \frac{(c_{2u} + c_{4u})(c_{2d}^* - c_{4d}^*)}{(m_b - m_d)(m_c - m_u)}, \quad V_{\rm CKM}^{31} = \frac{c_{4u} - c_{2u}}{m_t - m_u},$$

$$V_{\rm CKM}^{32} = \frac{2b_d^*}{m_b - m_s} + \frac{(c_{2u} - c_{4u})(c_{2d}^* + c_{4d}^*)}{(m_d - m_s)(m_u - m_t)}, \quad V_{\rm CKM}^{33} = 1 + \frac{(c_{2u} - c_{4u})(c_{2d}^* - c_{4d}^*)}{(m_b - m_d)(m_t - m_u)}.$$
(12)

Comparing the model results on the quark masses and quark mixing matrix in Eqs. (9) and (12) with their corresponding experimental constraints on  $V_{ij}^{exp}$  as shown in Tab. II (the second column), we get the explicit expressions of  $a_{1u,d}$ ,  $a_{2u,d}$ ,  $a_{3u,d}$ ,  $b_{u,d}$ ,  $c_{1u,d}$ ,  $c_{2u,d}$ ,  $c_{3u}$  and  $c_{4u,d}$  as functions of quark masses and quark mixing matrix elements as presented in Eqs. (B1) and (B2) of Appendix B.

Expressions (8), (11), (B1) and (B2) imply that the model parameters  $a_{1u,d}$ ,  $a_{2u,d}$ ,  $a_{3u,d}$ ,  $b_{u,d}$ ,  $c_{1u,d}$ ,  $c_{2u,d}$ ,  $c_{3u}$  and  $c_{4u,d}$  depend on the observed parameters in the quark sector, including quark masses  $m_u, m_c, m_t, m_d, m_s, m_b$  and quark mixing matrix elements  $V_{ij}^{\exp}$  (i, j = 1, 2, 3), that have been determined accurately [1]. At the best-fit points of mentioned parameters<sup>4</sup> given in Refs.[1], we obtain a prediction for the quark mixing matrix and the model's parameters in the quark sector as shown in Table II and Eq. (13), respectively.

Observable	Best-fit point [1]	The model prediction	Percent error (%)
$m_u[{ m MeV}]$	2.16	2.16	0
$m_c[\text{GeV}]$	1.27	1.27	0
$m_t [\text{GeV}]$	172.69	172.69	0
$m_d[{ m MeV}]$	4.67	4.67	0
$m_s[{ m MeV}]$	93.4	93.4	0
$m_b[\text{GeV}]$	4.18	4.18	0
$V_{\rm CKM}^{11}$	0.974352	0.974352	0
$V_{\rm CKM}^{12}$	0.224998	0.224998	0
$V_{\rm CKM}^{13}$	0.0015275 - 0.003359i	0.0015275 - 0.003359i	0
$V_{\rm CKM}^{21}$	-0.224865 - 0.000136871i	-0.224865 - 0.000136871i	0
$V_{\rm CKM}^{22}$	0.973492	0.973492	0
$V_{\rm CKM}^{23}$	0.0418197	0.0418197	0
$V_{ m CKM}^{31}$	0.00792247 - 0.00327i	0.00792247 - 0.00327i	0
$V_{\rm CKM}^{32}$	-0.0410911 - 0.000755113i	-0.0410911 - 0.000755113i	0
$V_{\rm CKM}^{33}$	0.999118	0.999118	0

Table II. The best-fit points for quark parameters taken from Ref.[1] and the model prediction.

<sup>&</sup>lt;sup>4</sup> The best-fit points in Table II correspond to the Wolfenstain parameters[1]:  $\lambda = 0.2250$ , A = 0.826,  $\bar{\rho} = 0.159$ and  $\bar{\eta} = 0.348$  which correspond to the mixing angles  $\sin \theta_{12}^q = 0.22500$ ,  $\sin \theta_{13}^q = 0.00369$ ,  $\sin \theta_{23}^q = 0.04182$  and  $\delta_{CP}^q = 1.444$ .

$$a_{1u} = 2.160 \times 10^{-3} \,\text{GeV}, \quad a_{2u} = 86.980 \,\text{GeV}, \quad a_{3u} = -85.710 \,\text{GeV},$$
  

$$b_u = (2.308 + 0.5413i) \,\text{GeV}, \quad c_{1u} = 8.414 + 3.028i \,\text{GeV},$$
  

$$c_{2u} = (-0.614 + 0.211i) \,\text{GeV}, \quad c_{3u} = (-8.269 - 3.170i) \,\text{GeV},$$
  

$$c_{4u} = (0.754 - 0.353i) \,\text{GeV}, \quad a_{1d} = 4.670 \times 10^{-3} \,\text{GeV}, \quad a_{2d} = 2.140 \,\text{GeV},$$
  

$$a_{3d} = -2.040 \,\text{GeV}, \quad b_d = (-8.658 + 0.262i)10^{-2} \,\text{GeV},$$
  

$$c_{1d} = (-5.080 + 4.973i)10^{-3} \,\text{GeV}, \quad c_{2d} = (0.193 - 0.077i) \,\text{GeV},$$
  

$$c_{4d} = (-0.204 + 0.087i) \,\text{GeV}.$$
(13)

The Jarlskog invariant in the quark sector,  $J_{CP}^q = \text{Im}[V_{us}V_{cb}V_{cs}^*V_{ub}^*]$ , is calculated from Eq. (12) with the model result in Table II (the third column) as  $J_{CP}^q = 3.08 \times 10^{-5}$ , which coincides with that of Ref. [1].

Next, comparing Eqs. (8) and (13) with the aid of Eqs. (4)-(5), ones obtain:

$$\begin{aligned} |x_{1u}| &= 2.16 \times 10^{-3}, \ |x_{2u}| &= 0.87, \ |x_{3u}| &= 0.60, \ |y_{1u}| &= 2.37, \\ |y_{2u}| &= 1.67, \ |z_{1u}| &= 1.79, \ |z_{2u}| &= 0.13, \ |z_{3u}| &= 1.24, \ |z_{4u}| &= 0.12, \\ |x_{1d}| &= 4.67 \times 10^{-3}, \ |x_{2d}| &= 2.14 \times 10^{-2}, \ |x_{3d}| &= 1.43 \times 10^{-2}, \\ |y_{1d}| &= 8.66 \times 10^{-2}, \ |y_{2d}| &= 6.08 \times 10^{-2}, \ |z_{1d}| &= 1.42 \times 10^{-3}, \\ |z_{2d}| &= 4.16 \times 10^{-2}, \ |z_{3d}| &= 10^{-2}, \ |z_{4d}| &= 3.11 \times 10^{-2}, \end{aligned}$$
(14)

which differ by about three orders of magnitude.

#### IV. LEPTON MASSES AND MIXINGS

Using the Clebsch-Gordan coefficients of  $D_4[34]$ , from Eq. (2), when the scalar fields get the VEVs, Eq. (3), we find charged leptons  $(M_l)$  and neutrino (Dirac and right-handed Majorana) mass matrices  $(M_D, M_R)$  as follows

$$M_{l} = \begin{pmatrix} a_{1} & 0 & 0 \\ 0 & a_{2} + a_{3} & a_{4} + a_{5} \\ 0 & a_{4} - a_{5} & a_{2} - a_{3} \end{pmatrix}, M_{D} = \begin{pmatrix} 0 & -a_{D} + b_{D} & a_{D} + b_{D} \\ 0 & c_{D} + d_{D} & 0 \\ 0 & 0 & -c_{D} + d_{D} \end{pmatrix}, M_{R} = \begin{pmatrix} a_{R} & 0 & 0 \\ 0 & b_{R} & c_{R} \\ 0 & c_{R} & b_{R} \end{pmatrix}, (15)$$

where

$$a_{1} = \left(\frac{v_{\phi}}{\Lambda}\right)vh_{1}, \ a_{2} = h_{2}v, \ a_{3} = h_{3}v', \ a_{4} = \left(\frac{v_{\phi}}{\Lambda}\right)vh_{4}, \ a_{5} = \left(\frac{v_{\phi}}{\Lambda}\right)v'h_{5}.$$
(16)  
$$a_{D} = \left(\frac{v_{\phi}}{\Lambda}\right)x_{1}v, \ h_{D} = \left(\frac{v_{\phi}}{\Lambda}\right)x_{2}v', \ c_{D} = \left(\frac{v_{\phi}}{\Lambda}\right)x_{2}v, \ d_{D} = \left(\frac{v_{\phi}}{\Lambda}\right)x_{2}v'$$

$$a_D = \left(\frac{1}{\Lambda}\right) x_1 v, \quad b_D = \left(\frac{1}{\Lambda}\right) x_2 v, \quad c_D = \left(\frac{1}{\Lambda}\right) x_3 v, \quad a_D = \left(\frac{1}{\Lambda}\right) x_4 v,$$
$$a_R = \frac{y_1}{\Lambda} v_{\chi} v_{\phi}, \quad b_R = y_2 v_{\chi}, \quad c_R = \frac{y_3}{\Lambda} v_{\chi} v_{\phi}.$$
(17)

• Charged-lepton sector: For simplicity, we consider the case of  $\arg h_3 = (\arg h_2 + \pi)$  and  $\arg h_5 = \arg h_4$ , i.e.,  $\arg a_3 = (\arg a_2 + \pi)$  and  $\arg a_5 = \arg a_4$ . Yukawa couplings  $h_i (i = 1 \div 5)$  are complex in general, therefore the matrix  $M_l$  is complex and its eigenvalues are complex. Let us first define a Hermitian matrix  $m_l^2 = M_l M_l^{\dagger}$ , given by

$$m_l^2 = M_l M_l^+ = \begin{pmatrix} A_0 & 0 & 0 \\ 0 & B_0 & \mathcal{D}_0 . e^{-i\theta} \\ 0 & \mathcal{D}_0 . e^{i\theta} & C_0 \end{pmatrix},$$
 (18)

where  $^{5}$ 

$$A_{0} = |a_{1}|^{2}, \quad B_{0} = (|a_{2}| - |a_{3}|)^{2} + (|a_{4}| + |a_{5}|)^{2}, \quad C_{0} = (|a_{2}| + |a_{3}|)^{2} + (|a_{4}| - |a_{5}|)^{2},$$
  

$$D_{0} = 2(|a_{2}||a_{4}| + |a_{3}||a_{5}|)c_{\alpha}, \quad G_{0} = -2(|a_{3}||a_{4}| + |a_{2}||a_{5}|)s_{\alpha}, \quad D_{0} = \sqrt{D_{0}^{2} + G_{0}^{2}}, \quad (19)$$

$$\theta = \arccos\left(\frac{D_0}{D_0}\right), \ \alpha = \arg a_2 - \arg a_4.$$
(20)

The matrix  $m_l^2$  in Eq. (18) is diagonalised by two mixing matrices  $V_{l(L,R)}$  with  $V_{lL}^+ m_l^2 V_{lR} = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$ , where

$$m_e^2 = A_0, \ m_{\mu,\tau}^2 = \frac{1}{2} \left( B_0 + C_0 \mp \sqrt{(B_0 - C_0)^2 + 4\mathcal{D}_0^2} \right), \tag{21}$$

$$V_{lL} = V_{lR} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\psi} & -s_{\psi} \cdot e^{-i\theta} \\ 0 & s_{\psi} \cdot e^{i\theta} & c_{\psi} \end{pmatrix},$$
(22)

where

$$s_{\psi} = \frac{1}{\sqrt{2}\sqrt{1 - \frac{B_0 - C_0}{B_0 - C_0 + \sqrt{(B_0 - C_0)^2 + 4\mathcal{D}_0^2}}}}.$$
(23)

Equations (19)-(21) and (23) yield the following relations:

$$|a_{1}| = m_{e}, |a_{2}| = \frac{|a_{4}|D_{0}s_{\alpha} + |a_{5}|c_{\alpha}G_{0}}{(|a_{4}|^{2} - |a_{5}|^{2})s_{2\alpha}}, |a_{3}| = \frac{|a_{4}|c_{\alpha}G_{0} + |a_{5}|D_{0}s_{\alpha}}{(|a_{5}|^{2} - |a_{4}|^{2})s_{2\alpha}}, |a_{4}| = \frac{a+b}{2}, |a_{5}| = \frac{a-b}{2},$$

$$(24)$$

<sup>5</sup> In this work, the following notations are used:  $s_{\psi} = \sin \psi$ ,  $c_{\psi} = \cos \psi$ ,  $s_{\theta} = \sin \theta$ ,  $c_{\theta} = \cos \theta$ ,  $t_{\alpha} = \tan \alpha$ ,  $t_{\theta} = \tan \theta$ ,  $s_{\delta} = \sin \delta_{CP}$ ,  $s_{ij} = \sin \theta_{ij}$ ,  $c_{ij} = \cos \theta_{ij}$  and  $t_{ij} = \tan \theta_{ij}$  (ij = 12, 13, 23).

where

$$a = \sqrt{\frac{\sqrt{(B_0 C_0 - x_0 + y_0)^2 - 4B_0 C_0 y_0} + B_0 C_0 - x_0 + y_0}{2C_0}},$$
  
$$b = \sqrt{\frac{\sqrt{(B_0 C_0 - x_0 + y_0)^2 - 4B_0 C_0 y_0} + B_0 C_0 + x_0 - y_0}{2B_0}},$$
(25)

$$x_0 = \frac{\left(c_{\alpha}G_0 + D_0 s_{\alpha}\right)^2}{s_{2\alpha}^2}, \quad y_0 = \frac{\left(c_{\alpha}G_0 - D_0 s_{\alpha}\right)^2}{s_{2\alpha}^2},$$
(26)

$$B_{0} = (m_{\mu}^{2} - m_{\tau}^{2}) s_{\psi}^{2} + m_{\tau}^{2}, \quad C_{0} = (m_{\mu}^{2} - m_{\tau}^{2}) s_{\psi}^{2} + m_{\mu}^{2},$$
  
$$D_{0} = (m_{\tau}^{2} - m_{\mu}^{2}) c_{\theta} s_{\psi} c_{\psi}, \quad G_{0} = (m_{\mu}^{2} - m_{\tau}^{2}) s_{\theta} s_{\psi} c_{\psi}.$$
 (27)

Expressions (16) and (24)-(27) imply that  $h_1$  depends on  $m_e$ ,  $\Lambda$ ,  $v_{\phi}$  and v;  $h_2$  depends on v,  $m_{\mu}$ ,  $m_{\tau}$ ,  $\psi$ ,  $\theta$ and  $\alpha$ ;  $h_3$  depends on v',  $m_{\mu}$ ,  $m_{\tau}$ ,  $\psi$ ,  $\theta$  and  $\alpha$ ; and  $h_4$  and  $h_5$  depend on v,  $\Lambda$ ,  $v_{\phi}$ ,  $m_{\mu}$ ,  $m_{\tau}$ ,  $\psi$ ,  $\theta$  and  $\alpha$ . As will see in Sec. V, with the observed charged leptons  $m_{e,\mu,\tau}$  [1] and the cut-off scale, the VEV scales of scalar fields in Eqs (4) and (5), there exist possible ranges of the model parameters such that the Yukawa couplings in the charged lepton sector,  $h_i$  ( $i = 1 \div 5$ ), differ by about two orders of magnitude, i.e., the charged lepton mass hierarchy is satisfied.

• Neutrino sector: The effective neutrino mass matrix arise from type-I seesaw mechanism  $M_{\nu} = -M_D M_R^{-1} M_D^T$ , obtained from Eq. (15), as follows:

$$M_{\nu} = \begin{pmatrix} A & -B_1 & -B_2 \\ -B_1 & C_1 & C_3 \\ -B_2 & C_3 & C_2 \end{pmatrix},$$
 (28)

where

$$A = \frac{2b_D^2}{b_R + c_R} + \frac{2a_D^2}{b_R - c_R}, \quad B_1 = \frac{(c_D + d_D) \left[ a_D (b_R + c_R) - b_D (b_R - c_R) \right]}{b_R^2 - c_R^2},$$
$$B_2 = \frac{(c_D - d_D) \left[ a_D (b_R + c_R) + b_D (b_R - c_R) \right]}{b_R^2 - c_R^2}, \quad C_1 = \frac{b_R (c_D + d_D)^2}{b_R^2 - c_R^2},$$
$$C_2 = \frac{b_R (c_D - d_D)^2}{b_R^2 - c_R^2}, \quad C_3 = \frac{c_R \left( c_D^2 - d_D^2 \right)}{b_R^2 - c_R^2}.$$
(29)

The mass matrix  $M_{\nu}$  in Eq.(28) owns three eigenvalues and the corresponding mixing matrix as

follows:

$$\lambda_{1} = 0, \quad \lambda_{2} = \frac{C_{2} - 2B_{2}n_{1} + An_{1}^{2} + n_{2}(2C_{3} - 2B_{1}n_{1} + C_{1}n_{2})}{n_{1}^{2} + n_{2}^{2} + 1},$$
  

$$\lambda_{3} = \frac{C_{2} - 2B_{2}t_{1} + At_{1}^{2} + t_{2}(2C_{3} - 2B_{1}t_{1} + C_{1}t_{2})}{t_{1}^{2} + t_{2}^{2} + 1},$$
(30)

$$\mathbf{R} = \begin{pmatrix} \frac{k_1}{\sqrt{1+k_1^2(1+k_2^2)}} & \frac{n_1}{\sqrt{n_1^2+n_2^2+1}} & \frac{t_1}{\sqrt{t_1^2+t_2^2+1}} \\ \frac{k_1k_2}{\sqrt{1+k_1^2(1+k_2^2)}} & \frac{n_2}{\sqrt{n_1^2+n_2^2+1}} & \frac{t_2}{\sqrt{t_1^2+t_2^2+1}} \\ \frac{1}{\sqrt{1+k_1^2(1+k_2^2)}} & \frac{1}{\sqrt{n_1^2+n_2^2+1}} & \frac{1}{\sqrt{t_1^2+t_2^2+1}} \end{pmatrix},$$
(31)

where new parameters  $k_{1,2}$ ,  $n_{1,2}$  and  $t_{1,2}$ , own explicit expressions in Appendix C, satisfy the following relations

$$k_1(n_1 + k_2n_2) + 1 = 0, \ k_1(t_1 + k_2t_2) + 1 = 0, \ n_1t_1 + n_2t_2 + 1 = 0,$$
 (32)

$$C_2 - B_2(k_1 + n_1) + C_3(k_1k_2 + n_2) + k_1 [An_1 + C_1k_2n_2 - B_1(k_2n_1 + n_2)] = 0,$$
(33)

$$C_2 - B_2(k_1 + t_1) + C_3(k_1k_2 + t_2) + k_1 [At_1 + C_1k_2t_2 - B_1(k_2t_1 + t_2)] = 0,$$
(34)

$$C_2 + C_3 n_2 + A n_1 t_1 - B_1 n_2 t_1 - B_2 (n_1 + t_1) + (C_3 - B_1 n_1 + C_1 n_2) t_2 = 0,$$
(35)

$$C_2 + k_1 \left[ 2C_3 k_2 - 2B_2 + k_1 (A - 2B_1 k_2 + C_1 k_2^2) \right] = 0.$$
(36)

Depending on the sign of  $\Delta m_{31}^2$ , the neutrino mass spectrum can be normal or inverted hierarchy [1]. In the considered model,  $0 = m_1 \equiv \lambda_1 < m_2 \equiv \lambda_2 < m_3 \equiv \lambda_3$  for NH and  $0 = m_3 \equiv \lambda_1 < m_1 \equiv \lambda_2 < m_2 \equiv \lambda_3$  for IH. Since the lightest neutrino mass is equal to zero, other neutrino masses and their sum are given by

$$\begin{cases} m_1 = 0, \quad m_2 = \sqrt{\Delta m_{21}^2}, \quad m_3 = \sqrt{\Delta m_{31}^2} \text{ for NH}, \\ m_1 = \sqrt{-\Delta m_{31}^2}, \quad m_2 = \sqrt{\Delta m_{21}^2 - \Delta m_{31}^2}, \quad m_3 = 0 \text{ for IH}. \end{cases}$$
(37)

$$\sum m_{\nu} = \begin{cases} \sqrt{\Delta m_{21}^2} + \sqrt{\Delta m_{31}^2} & \text{for NH,} \\ \sqrt{\Delta m_{21}^2 - \Delta m_{31}^2} + \sqrt{-\Delta m_{31}^2} & \text{for IH.} \end{cases}$$
(38)

The neutrino mass matrix  $M_{\nu}$  in Eq. (28) is diagonalized as

$$\mathbf{U}_{\nu}^{T}M_{\nu}\mathbf{U}_{\nu} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{2} & 0 \\ 0 & 0 & m_{3} \end{pmatrix}, \ \mathbf{U}_{\nu} = \begin{pmatrix} \frac{k_{1}}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} & \frac{n_{1}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{t_{1}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{k_{1}k_{2}}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} & \frac{n_{2}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{t_{2}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{1}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} & \frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{1}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{1}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} & \frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{1}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{n_{1}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{1}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} & \frac{1}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} \\ \frac{n_{2}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{1}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} & \frac{k_{1}k_{2}}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} \\ \frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{1}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} & \frac{1}{\sqrt{1+k_{1}^{2}(1+k_{2}^{2})}} \end{pmatrix} & \text{for IH,} \end{cases}$$

$$(39)$$

where  $\lambda_2, \lambda_3, k_{1,2}, n_{1,2}$  and  $t_{1,2}$  are given in Appendix C.

Expressions (30) and (32)-(36) yield:

$$\begin{cases} k_1 = \frac{n_1 t_1 + n_2^2 + 1}{t_1 (n_1^2 + n_2^2) + n_1}, & k_2 = \frac{n_2 (t_1 - n_1)}{n_1 t_1 + n_2^2 + 1}, & t_2 = -\frac{n_1 t_1 + 1}{n_2} & \text{for NH}, \\ n_2 = \frac{1 - k_1 n_1}{k_1 k_2}, & t_1 = \frac{k_1 (n_1 - k_1 k_2^2) - 1}{k_2 (k_1 + k_2^2) n_1 - 1}, & t_2 = \frac{k_2 (k_1 + n_1)}{-1 + k_1 (1 + k_2^2) n_1} & \text{for IH}, \end{cases}$$
(40)

$$A = -\frac{C_2 - B_2(k_1 + n_1) + C_1k_1k_2n_2 + C_3(k_1k_2 + n_2) - B_1k_1(k_2n_1 + n_2)}{k_1n_1}$$
(NH and IH), (41)

$$B_1 = \frac{C_3 + C_1 k_1 k_2}{k_1} + \frac{(C_2 - B_2 k_1 + C_3 k_1 k_2)(n_1 - t_1)}{(n_1 t_2 - n_2 t_1) k_1}$$
(NH and IH), (42)

$$B_2 = \frac{C_2}{k_1} + C_3 k_2 \quad \text{(NH and IH)}, \tag{43}$$

$$C_{1} = \frac{C_{3}(k_{1} - n_{1}) + \frac{(C_{2} - B_{2}n_{1} + C_{3}n_{2})(k_{1} - t_{1})}{t_{2} - k_{2}t_{1}} + \frac{(C_{2} - B_{2}k_{1} + C_{3}k_{1}k_{2})(n_{1} - t_{1})n_{1}}{n_{2}t_{1} - n_{1}t_{2}}}{k_{1}(k_{2}n_{1} - n_{2})}$$
 (NH and IH), (44)

$$C_{2} = \begin{cases} \frac{\sqrt{\Delta m_{21}^{2}}}{1+n_{1}^{2}+n_{2}^{2}} + \frac{\sqrt{\Delta m_{31}^{2}n_{2}^{2}}}{(1+n_{1}t_{1})^{2}+n_{2}^{2}(1+t_{1}^{2})} & \text{for NH,} \\ k_{1}^{2} \left( \frac{k_{2}^{2} \left(\sqrt{-\Delta m_{31}^{2}} - \sqrt{\Delta m_{21}^{2}-\Delta m_{31}^{2}}\right)}{1+2k_{1}n_{1}+k_{1}^{2} \left[n_{1}^{2}+k_{2}^{2}(1+n_{1}^{2})\right]} + \frac{(1+k_{2}^{2})\sqrt{\Delta m_{21}^{2}-\Delta m_{31}^{2}}}{1+k_{1}^{2}(1+k_{2}^{2})} \right) & \text{for IH,} \end{cases}$$

$$\left( \sqrt{\Delta m_{21}^{2}n_{2}} - \sqrt{\Delta m_{21}^{2}(1+n_{1}t_{1})n_{2}} - \epsilon \right)$$

$$(45)$$

$$C_{3} = \begin{cases} \frac{\sqrt{\Delta m_{21} n_{2}}}{1+n_{1}^{2}+n_{2}^{2}} - \frac{\sqrt{\Delta m_{31}(1+n_{1}(1)n_{2})}}{(1+n_{1}t_{1})^{2}+n_{2}^{2}(1+t_{1}^{2})} & \text{for NH,} \\ k_{1}k_{2} \left( \frac{\left(\sqrt{\Delta m_{21}^{2}-\Delta m_{31}^{2}}-\sqrt{-\Delta m_{31}^{2}}\right)(k_{1}n_{1}+1)}{1+2k_{1}n_{1}+k_{1}^{2}\left[n_{1}^{2}+k_{2}^{2}(1+n_{1}^{2})\right]} - \frac{\sqrt{\Delta m_{21}^{2}-\Delta m_{31}^{2}}}{1+k_{1}^{2}(1+k_{2}^{2})} \right) & \text{for IH.} \end{cases}$$
(46)

The corresponding leptonic mixing matrix is

$$\mathbf{U} = \mathbf{U}_{L}^{\dagger} \mathbf{U}_{\nu} = \begin{cases} \left( \begin{array}{ccc} \frac{k_{1}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} & \frac{n_{1}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{t_{1}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{c_{\psi}k_{1}k_{2}+e^{-i\theta}s_{\psi}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} & \frac{e^{-i\theta}\left(c_{\psi}e^{i\theta}n_{2}+s_{\psi}\right)}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{e^{-i\theta}\left(s_{\psi}+e^{i\theta}c_{\psi}t_{2}\right)}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{c_{\psi}-e^{i\theta}k_{1}k_{2}s_{\psi}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} & \frac{c_{\psi}-e^{i\theta}n_{2}s_{\psi}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{c_{\psi}-e^{i\theta}s_{\psi}t_{2}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} \\ \frac{n_{1}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{t_{1}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} & \frac{k_{1}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} \\ \frac{e^{-i\theta}\left(c_{\psi}e^{i\theta}n_{2}+s_{\psi}\right)}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{e^{-i\theta}\left(s_{\psi}+e^{i\theta}c_{\psi}t_{2}\right)}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}} & \frac{c_{\psi}-e^{i\theta}k_{1}k_{2}s_{\psi}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} \\ \frac{c_{\psi}-e^{i\theta}n_{2}s_{\psi}}}{\sqrt{n_{1}^{2}+n_{2}^{2}+1}} & \frac{c_{\psi}-e^{i\theta}s_{\psi}t_{2}}{\sqrt{t_{1}^{2}+t_{2}^{2}+1}}} & \frac{c_{\psi}-e^{i\theta}k_{1}k_{2}s_{\psi}}{\sqrt{\left(k_{2}^{2}+1\right)k_{1}^{2}+1}} \\ \end{array} \right)$$
for IH.

The lepton mixing matrix  $U_{PMNS}$ , in the standard parametrization, take the form:

$$U_{\rm MPNS} = \begin{pmatrix} c_{13}c_{12} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - e^{i\delta}c_{12}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - e^{i\delta}c_{12}c_{23}s_{13} & -c_{12}s_{23} - e^{i\delta}c_{23}s_{12}s_{13} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\eta_1} & 0 \\ 0 & 0 & e^{i\eta_2} \end{pmatrix}, \quad (48)$$

where  $s_{ij} = \sin \theta_{ij}$  and  $c_{ij} = \cos \theta_{ij}$  with  $\theta_{13}, \theta_{12}$  and  $\theta_{23}$  are the reactor, solar and atmospheric mixing angles, respectively;  $\delta_{CP}$  is the Dirac CP violation phase and  $\eta_{1,2}$  are the two Majorana CP violating phases. Comparing the entries "12" and "13" of two mixing matrices in (47) and (48) we get:

$$\eta_1 = 0, \ \eta_2 = \delta$$
 (both NH and IH). (49)

The lepton mixing angles, obtained from Eqs. (47) and (48), are:

$$s_{13}^{2} = |\mathbf{U}_{e3}|^{2} = \begin{cases} \frac{t_{1}^{2}}{t_{1}^{2} + t_{2}^{2} + 1} & \text{for NH,} \\ \frac{k_{1}^{2}}{1 + k_{1}^{2} \left(1 + k_{2}^{2}\right)} & \text{for IH,} \end{cases}$$
(50)

$$s_{12}^{2} = \frac{|\mathbf{U}_{e2}|^{2}}{1 - |\mathbf{U}_{e3}|^{2}} = \begin{cases} \frac{n_{1}^{2}(t_{1}^{2} + t_{2}^{2} + 1)}{(t_{2}^{2} + 1)(n_{1}^{2} + n_{2}^{2} + 1)} & \text{for NH,} \\ \frac{[1 + k_{1}^{2}(1 + k_{2}^{2})]t_{1}^{2}}{(1 + k_{1}^{2}k_{2}^{2})(1 + t_{1}^{2} + t_{2}^{2})} & \text{for IH,} \end{cases}$$
(51)

$$s_{23}^{2} = \frac{|\mathbf{U}_{\mu3}|^{2}}{1 - |\mathbf{U}_{e3}|^{2}} = \begin{cases} \frac{c_{\psi}^{2}t_{2}^{2} + s_{2\psi}c_{\theta}t_{2} + s_{\psi}^{2}}{t_{2}^{2} + 1} & \text{for NH,} \\ \frac{c_{\psi}^{2}k_{1}^{2}k_{2}^{2} + s_{\psi}^{2} + k_{1}k_{2}s_{2\psi}c_{\theta}}{1 + k_{1}^{2}k_{2}^{2}} & \text{for IH,} \end{cases}$$
(52)

The Jarlskog invariant in the active sector, determined from Eq. (47), takes the form [1, 35]

$$J_{CP}^{(l)} = \frac{n_1 t_1 (t_2 - n_2) s_{\psi} c_{\psi} s_{\theta}}{\left(n_1^2 + n_2^2 + 1\right) \left(1 + t_1^2 + t_2^2\right)}$$
(NH and IH). (53)

Comparing  $J_{CP}^{(l)}$  in Eq. (53) and that of the standard parametrization,  $J_{CP}^{(l)} = c_{12}c_{13}^2c_{23}s_{12}s_{13}s_{23}s_{\delta}$ , we obtain:

$$s_{\delta} = \frac{n_1 t_1 (t_2 - n_2) s_{\psi} c_{\psi} s_{\theta}}{\left(n_1^2 + n_2^2 + 1\right) \left(1 + t_1^2 + t_2^2\right) c_{12} c_{13}^2 c_{23} s_{12} s_{13} s_{23}}$$
(NH and IH). (54)

The effective neutrino masses [36], obtained from Eqs. (37), (39) and (47), possess the following forms:

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^{3} U_{ei}^{2} m_{i} \right| = \begin{cases} \frac{\sqrt{\Delta m_{21}^{2} n_{1}^{2}}}{1 + n_{1}^{2} + n_{2}^{2}} + \frac{\sqrt{\Delta m_{31}^{2} t_{1}^{2}}}{1 + t_{1}^{2} + t_{2}^{2}} & \text{for NH,} \\ \frac{\sqrt{-\Delta m_{31}^{2} n_{1}^{2}}}{1 + n_{1}^{2} + n_{2}^{2}} + \frac{\sqrt{\Delta m_{21}^{2} - \Delta m_{31}^{2} t_{1}^{2}}}{1 + t_{1}^{2} + t_{2}^{2}} & \text{for IH,} \end{cases}$$

$$(55)$$

$$m_{\beta} = \sqrt{\sum_{i=1}^{3} |U_{ei}|^2 m_i^2} = \begin{cases} \sqrt{\frac{\Delta m_{21} n_1}{1 + n_1^2 + n_2^2} + \frac{\Delta m_{31} n_1}{1 + t_1^2 + t_2^2}} & \text{for NH,} \\ \sqrt{\frac{(\Delta m_{21}^2 - \Delta m_{31}^2)t_1^2}{1 + t_1^2 + t_2^2} - \frac{\Delta m_{31}^2 n_1^2}{1 + n_1^2 + n_2^2}} & \text{for IH,} \end{cases}$$
(56)

From Eqs. (50)-(52), we can express  $n_{1,2}, t_1$  and  $s_{\delta}$  in terms of two constrained parameters  $c_{\theta}, s_{\psi}$ and five observable parameters  $\Delta m_{21}^2, \Delta m_{31}^2, s_{12}^2, s_{23}^2, s_{13}^2$  and as follows:

• For NH:

$$n_1 = \frac{s_{12}^2 c_{13}^4 t_1^2}{\sqrt{\left(c_{13}^2 t_1^2 - s_{13}^2\right) s_{12}^2 c_{12}^2 c_{13}^4 t_1^2 - s_{12}^2 s_{13}^2 c_{13}^2 t_1}}, \quad n_2 = \frac{(1 + n_1 t_1) s_{13}}{\sqrt{c_{13}^2 t_1^2 - s_{13}^2}}, \tag{57}$$

$$t_{1} = t_{13} \sqrt{\frac{s_{\psi}^{2}(s_{23}^{2} - c_{\psi}^{2}) + c_{\psi}^{2}(c_{23}^{2} + c_{2\theta}s_{\psi}^{2}) + 2\sqrt{c_{\theta}^{2}c_{\psi}^{2}s_{\psi}^{2}(s_{23}^{2}c_{23}^{2} - s_{\psi}^{2}c_{\psi}^{2}s_{\theta}^{2})}{(c_{\psi}^{2} - s_{23}^{2})^{2}}.$$
 (58)

• For IH:

$$k_{1} = -t_{13} \sqrt{\frac{s_{\psi}^{2}(s_{23}^{2} - c_{\psi}^{2}) + c_{\psi}^{2}(c_{23}^{2} + c_{2\theta}s_{\psi}^{2}) - 2\sqrt{c_{\theta}^{2}c_{\psi}^{2}s_{\psi}^{2}(s_{23}^{2}c_{23}^{2} - s_{\psi}^{2}c_{\psi}^{2}s_{\theta}^{2})}{\left(c_{\psi}^{2} - s_{23}^{2}\right)^{2}},$$
(59)

$$k_{2} = \frac{\sqrt{k_{1}^{2}c_{13}^{2} - s_{13}^{2}}}{k_{1}s_{13}}, \quad n_{1} = \frac{s_{12}c_{12}c_{13}^{2}\sqrt{k_{1}^{2}\left(k_{1}^{2}c_{13}^{2} - s_{13}^{2}\right)} - k_{1}c_{12}^{2}s_{13}^{2}c_{13}^{2}}{s_{13}^{4} + s_{12}^{2}c_{13}^{2}\left(s_{13}^{2} - k_{1}^{2}\right)}.$$
(60)

Expressions (40)-(46) and (54)-(60) show that the model parameters  $s_{\delta}, k_{1,2}, n_{1,2}$  and  $t_{1,2}$  depend on two constrained parameters  $c_{\theta}, s_{\psi}$  and three observable parameters  $s_{12}^2, s_{23}^2, s_{13}^2$  while  $A, B_{1,2}, C_{1,2,3}, \langle m_{ee} \rangle$  and  $m_{\beta}$  depend on two constrained parameters  $c_{\theta}, s_{\psi}$  and five observable parameters  $\Delta m_{21}^2, \Delta m_{31}^2, s_{12}^2, s_{23}^2, s_{13}^2$ .

#### V. NUMERICAL ANALYSIS

• For the charged lepton sector, using the values of  $\Lambda$ , the observed values of the charged lepton masses [1],  $m_e = 0.51099 \text{ MeV}, m_{\mu} = 105.65837 \text{ MeV}, m_{\tau} = 1776.86 \text{ MeV}$  and the VEV of scalar fields in Eqs. (4) and (5), with the help of Eqs. (16) and (24)-(27), we get  $|h_1| \simeq 10^{-2}$ , and  $h_{2,3,4,5}$ are still depend on three parameters  $\alpha$ ,  $\theta$  and  $\psi$ . In the case of  $s_{\alpha} = -0.95$  ( $\alpha = 288.2^{\circ}$ ), the Yukawa-like couplings  $h_{2,3,4,5}$  depend on two parameters  $\theta$  and  $\psi$  which are plotted in Figs. 1 and 2.



Figure 1.  $10^3 |h_2|$  (left panel) and  $10^3 |h_3|$  (right panel) versus  $c_\theta$  and  $s_\psi$  with  $c_\theta \in (0.29, 0.31)$  and  $s_\psi \in (0.25, 0.65)$ .



Figure 2.  $|h_4|$  (left panel) and  $|h_5|$  (right panel) versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$ .

Figures 1 and 2 imply

$$|h_2| \simeq |h_3| \sim 10^{-2}, \ |h_4| \simeq |h_5| \sim 10^{-1},$$
 (61)

which implies that the Yukawa couplings in the charged lepton sector differ from each other by one order of magnitude for a natural explanation to the charged lepton mass hierarchy.

• For neutrino sector. Equation (37) shows that neutrino masses  $(m_{2,3} \text{ for NH and } m_{1,2} \text{ for IH})$ depend on two experimental parameters  $\Delta m_{31}^2$  and  $\Delta m_{21}^2$  which have been measured with high accuracy. In the case of  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  lie in  $3\sigma$  range [37], i.e.,  $\Delta m_{21}^2 \in (69.40, 81.40) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 3.63)10^3 \text{ meV}^2$ , we get the allowed regions for  $m_{1,2,3}, m_1 = 0, m_2 \in (8.33, 9.02) \text{ meV},$  $m_3 = (49.70, 51.30) \text{ meV}$  for NH, and  $m_1 \in (48.70, 50.30) \text{ meV}, m_2 = (49.4, 51.0) \text{ meV}, m_3 = 0$  for IH. The sum of neutrino masses are predicted to be

$$\sum m_{\nu} \,(\text{meV}) \in \begin{cases} (58.25, 60.25) \text{ for NH,} \\ (98.50, 101.0) \text{ for IH,} \end{cases}$$
(62)

which are in consistent with the limits [38]  $\sum m_{\nu} < 0.15$  eV (NH) and  $\sum m_{\nu} < 0.17$  eV (IH),  $\sum m_{\nu} < 0.14$  eV [39],  $\sum m_{\nu} < 0.152$  eV [40] (minimal  $\Lambda \text{CDM} + \sum m_{\nu}$ ),  $\sum m_{\nu} < 0.118$  eV (high-*l* polarization),  $\sum m_{\nu} < 0.101$  eV (NPDDE model),  $\sum m_{\nu} < 0.093$  eV (NPDDE+*r* model) and the most aggressive bound is  $\sum m_{\nu} < 0.078$ eV (NPDDE+*r* with the R16 prior) [40, 41],  $\sum m_{\nu} < 0.183$  eV for IH [42],  $\sum m_{\nu} < 0.13$  eV (the base dataset) and  $\sum m_{\nu} < 0.11$  eV (pol dataset) [43],  $\sum m_{\nu} < 0.19$  eV [44]. In order to determine the possible ranges of the parameters  $k_{1,2}$ ,  $n_{1,2}$ ,  $t_{1,2}$  and get predictive values for the Dirac CP viloation phase  $\delta$ , we use the observables  $\Delta m_{21}^2$ ,  $\Delta m_{31}^2$ ,  $\sin^2 \theta_{12}$ ,  $\sin^2 \theta_{23}$ and  $\sin^2 \theta_{13}$ , whose experimental values given in Table III, as input parameters.

	Best – fit point $(3\sigma \text{ range})$ (NH)	Best – fit point $(3\sigma \text{ range})$ (IH)
$\Delta m_{21}^2 \left[ \mathrm{meV}^2 \right]$	75.0(69.4  o 81.4)	75.0(69.4 o 81.4)
$\frac{ \Delta m_{31}^2  \left[\mathrm{meV}^2\right]}{10^3}$	2.55(2.47 o 2.63)	2.45(2.37 o 2.53)
$\sin^2 \theta_{12}$	$0.318(0.271 \to 0.369)$	$0.318(0.271 \to 0.369)$
$\sin^2 \theta_{23}$	$0.574(0.434\to 0.610)$	$0.578(0.433\to 0.608)$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.200(2.00\to 2.405)$	$2.225(2.018\to 2.424)$
$\delta_{CP}/\pi$	1.08(0.71  ightarrow 1.99)	1.58(1.11  o 1.96)

Table III. The global analysis of neutrino oscillation data [37]

At the best-fit values of the lepton mixing angles[37],  $\sin^2 \theta_{12} = 0.318$  and  $\sin^2 \theta_{13} = 2.200 \times 10^{-2}$ for NH while  $\sin^2 \theta_{12} = 0.318$  and  $\sin^2 \theta_{13} = 2.225 \times 10^{-2}$  for IH,  $s_{\delta}, k_{1,2}, n_{1,2}$  and  $t_{1,2}$  depend on two parameters  $c_{\theta}$  and  $s_{\psi}$ . The Dirac CP violating phase  $\delta$  (more precisely,  $s_{\delta}$ ) as a function of two parameters  $c_{\theta}$  and  $s_{\psi}$ , with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for both IH and NH, is plotted in Fig. 3, which implies that

$$s_{\delta} \in (-0.95, -0.50), \text{ i.e., } \delta^{\circ} \in (288.20, 330.00) \text{ (NH and IH).}$$
 (63)



Figure 3.  $s_{\delta}$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for both NH and IH.



Figure 4.  $k_1$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).



Figure 5.  $k_2$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).



Figure 6.  $n_1$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).



Figure 7.  $n_2$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).

These figures imply:

$$k_1 \in \begin{cases} (-1.54, -1.42) & \text{for NH,} \\ (-0.215, -0.170) & \text{for IH,} \end{cases} \quad k_2 \in \begin{cases} (-0.25, 0.10) & \text{for NH,} \\ (-4.60, -3.20) & \text{for IH,} \end{cases}$$
(64)

$$n_1 \in \begin{cases} (0.70, 0.875) & \text{for NH,} \\ (-4.50, -2.75) & \text{for IH,} \end{cases} \quad n_2 \in \begin{cases} (0.20, 0.80) & \text{for NH,} \\ (-3.00, -1.60) & \text{for IH,} \end{cases}$$
(65)

$$t_1 \in \begin{cases} (0.30, 1.00) & \text{for NH,} \\ (0.90, 1.20) & \text{for IH,} \end{cases} \quad t_2 \in \begin{cases} (-5.00, -1.50) & \text{for NH,} \\ (-1.50, -0.90) & \text{for IH.} \end{cases}$$
(66)



Figure 8.  $t_1$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).



Figure 9.  $t_2$  versus  $c_{\theta}$  and  $s_{\psi}$  with  $c_{\theta} \in (0.29, 0.31)$  and  $s_{\psi} \in (0.25, 0.65)$  for NH (left panel) and IH (right panel).

Similarly, to determine the possible ranges of the parameters  $A, B_{1,2}, C_{1,2,3}, \langle m_{ee} \rangle$  and  $m_{\beta}$  we fix  $\sin^2 \theta_{12}$ ,  $\sin^2 \theta_{23}$  and  $\sin^2 \theta_{13}$  at their best-fit points [37] and  $c_{\theta} = 0.30 \ (\theta = 72.54^{\circ})$  and  $s_{\psi} = 0.40 \ (\psi = 23.58^{\circ})$  for both IH and NH, and  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  take the values in their 3  $\sigma$  ranges [37],  $\Delta m_{21}^2 \in (69.4, 81.4) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63)10^3 \text{ meV}^2$  (NH) while  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  (IH). The dependence of  $A, B_{1,2}, C_{1,2,3}, \langle m_{ee} \rangle$  and  $m_{\beta}$  on two parameters  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  are presented in Figs. 10, 11,12, 13,14, 15, 16 and 17, respectively.



Figure 10. A (meV) versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).



Figure 11.  $B_1 \text{ (meV)}$  versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).



 $B_2 \,(\mathrm{meV})$  versus  $\Delta m^2_{21}$  and  $\Delta m^2_{31}$  with  $\Delta m^2_{21} \in (69.4, 81.4) \,\mathrm{meV}^2$  and  $\Delta m^2_{31} \in C^2$ Figure 12.  $(2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).



 $C_1 \,(\mathrm{meV})$  versus  $\Delta m^2_{21}$  and  $\Delta m^2_{31}$  with  $\Delta m^2_{21} \in (69.4, 81.4) \,\mathrm{meV}^2$  and  $\Delta m^2_{31} \in C_1 \,(\mathrm{meV})$ Figure 13.  $(2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).

Figures 10 and 15 imply that:

 $A \in$ 

$$A \in \begin{cases} (3.700, 3.925) \text{ meV for NH}, \\ (48.00, 49.40) \text{ meV for IH}, \end{cases} B_{1} \in \begin{cases} (3.90, 4.25) \text{ meV for NH}, \\ (-4.775, -4.60) \text{ meV for IH}, \end{cases} (67)$$
$$B_{2} \in \begin{cases} (-7.15, -6.80) \text{ meV for NH}, \\ (-5.75, -5.575) \text{ meV for IH}, \end{cases} C_{1} \in \begin{cases} (38.40, 39.60) \text{ meV for NH}, \\ (27.40, 28.20) \text{ meV for IH}. \end{cases} (68)$$

$$C_{2} \in \begin{cases} (16.00, 16.60) \text{ meV for NH}, \\ (23.00, 23.70) \text{ meV for IH}, \end{cases} C_{3} \in \begin{cases} (-19.00, -18.20) \text{ meV for NH}, \\ (-24.70, -24.00) \text{ meV for IH}. \end{cases}$$
(69)



Figure 14.  $C_2 \text{ (meV)}$  versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).



Figure 15.  $C_3 \,(\text{meV})$  versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \,\text{meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63) 10^3 \,\text{meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37) 10^3 \,\text{meV}^2$  for IH (right panel).

Figures 16 and 17 show the predictive regions of the effective neutrino-masses:

$$\langle m_{ee} \rangle \in \begin{cases} (3.700, 3.925) \text{ meV for NH,} \\ (48.00, 49.40) \text{ meV for IH,} \end{cases} m_{\beta} \in \begin{cases} (8.75, 9.10) \text{ meV for NH,} \\ (48.40, 49.80) \text{ meV for IH,} \end{cases}$$
(70)

which are below the upper limits for  $\langle m_{ee} \rangle$  from KamLAND-Zen [45]  $\langle m_{ee} \rangle < 61 \div 165 \text{ meV}$ , GERDA [46]  $\langle m_{ee} \rangle < 104 \div 228 \text{ meV}$  and CUORE [47]  $\langle m_{ee} \rangle < 75 \div 350 \text{ meV}$ , and the constraints for  $m_{\beta}$ with 8.5 meV  $\langle m_{\beta} < 1.1 \text{ eV}$  for NH and 48 meV  $\langle m_{\beta} < 1.1 \text{ eV}$  for IH [1],  $m_{\beta} \in (8.90 \div 12.60) \text{ eV}$ [48], and  $m_{\beta} < 0.8 \text{ eV}$  [49].



Figure 16.  $\langle m_{ee} \rangle$  (meV) versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \text{ meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63)10^3 \text{ meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37)10^3 \text{ meV}^2$  for IH (right panel).



Figure 17.  $m_{\beta} \,(\text{meV})$  versus  $\Delta m_{21}^2$  and  $\Delta m_{31}^2$  with  $\Delta m_{21}^2 \in (69.4, 81.4) \,\text{meV}^2$  and  $\Delta m_{31}^2 \in (2.47, 2.63) 10^3 \,\text{meV}^2$  for NH (left panel) and  $\Delta m_{31}^2 \in (-2.53, -2.37) 10^3 \,\text{meV}^2$  for IH (right panel).

#### VI. CONCLUSIONS

We have constructed a gauge B - L model with  $D_4 \times Z_4 \times Z_2$  symmetry that can explain the quark and lepton mass hierarchies and their mixing patterns with the realistic CP phases via the type-I seesaw mechanism. Six quark mases, three quark mixing angles and CP phase in the quark sector can get the central values and Yukawa couplings in the quark sector are diluted a range of three orders of magnitude difference by the perturbation theory at the first order. For neutrino sector, the smallness of neutrino mass is achieved by the Type-I seesaw mechanism. Both inverted and normal neutrino mass hierarchies are in consistent with the experimental data. The prediction for the sum of neutrino masses is  $58.25 \text{meV} \leq \sum m_{\nu} \leq 60.25 \text{ meV}$  for normal hierarchy and  $98.50 \text{meV} \leq \sum m_{\nu} \leq 101.00 \text{ meV}$  for inverted hierarchy which are well consistent with all the recent limits. In addition, the Dirac CP phase is predicted to be  $288.20 \leq \delta(^{\circ}) \leq 330.00$  within the  $3\sigma$  range of experimental constraint. The effective neutrino masses are predicted to be  $3.700 \text{ meV} \leq \langle m_{ee} \rangle \leq 3.925 \text{ meV}$ ,  $8.75 \text{ meV} \leq m_{\beta} \leq 9.10 \text{ meV}$  for normal hierarchy and  $48.00 \text{ meV} \leq \langle m_{ee} \rangle \leq 49.40 \text{ meV}$  and  $48.40 \text{ meV} \leq m_{\beta} \leq 49.80 \text{ meV}$  for inverted hierarchy which are in consistence with the recent constraints.

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## Appendix A: Forbidden terms under the model's symmetries

Yukawa terms	Forbidden by
$(\overline{\psi}_{\alpha L} l_{\alpha R})_{1_{+-}} \widetilde{H}, (\overline{\psi}_{\alpha L} l_{\alpha R})_{1_{-+}} \widetilde{H'}; (\overline{\psi}_{1 L} \nu_{\alpha R})_2 (H\rho^*)_2, (\overline{\psi}_{1 L} \nu_{\alpha R})_2 (H'\rho^*)_2;$	
$(\overline{\psi}_{\alpha L}\nu_{\alpha R})_{1-+}(\widetilde{H}\varphi)_{1-+}, (\overline{\psi}_{\alpha L}\nu_{\alpha R})_{1+-}(\widetilde{H'}\varphi)_{1+-}; (Q_{1L}u_{1R})_{1++}(H\phi)_{1++},$	
$(Q_{\alpha L}u_{\alpha R})_{1++}(H\phi)_{1++}, (Q_{\alpha L}u_{\alpha R})_{1+-}H, (Q_{\alpha L}u_{\alpha R})_{1-+}H', (Q_{\alpha L}u_{\alpha R})_{1}(H'\phi)_{1};$	
$(Q_{1L}u_{\alpha R})_2(H\rho)_2, (Q_{1L}u_{\alpha R})_2(H'\rho)_2, (Q_{\alpha L}u_{1R})_2(H\rho^*)_2, (Q_{\alpha L}u_{1R})_2(H'\rho^*)_2,$	$U(1)_Y$
$(Q_{1L}d_{1R})_{1_{++}}(\widetilde{H}\phi)_{1_{++}}, (Q_{\alpha L}d_{\alpha R})_{1_{++}}(\widetilde{H}\phi)_{1_{++}}, (Q_{\alpha L}d_{\alpha R})_{1_{+-}}\widetilde{H}, (Q_{\alpha L}d_{\alpha R})_{1_{-+}}\widetilde{H'},$	
$(Q_{\alpha L}d_{\alpha R})_{1}(\widetilde{H'}\phi)_{1}; (Q_{1L}d_{\alpha R})_2(\widetilde{H}\rho)_2, (Q_{1L}d_{\alpha R})_2(\widetilde{H'}\rho)_2,$	
$(Q_{\alpha L}d_{1R})_2(\widetilde{H} ho^*)_2, (Q_{\alpha L}d_{1R})_2(\widetilde{H'} ho^*)_2$	
$(\overline{\nu}_{1R}^C \nu_{1R})_{1++} (\phi \chi^*)_{1++}, (\overline{\nu}_{1R}^C \nu_{1R})_{1++} (\rho^2)_{1++}, (\overline{\nu}_{1R}^C \nu_{1R})_{1++} (\rho^{*2})_{1++};$	
$(\overline{\nu}_{\alpha R}^{C}\nu_{\alpha R})_{1++}(\phi\chi^{*})_{1++},(\overline{\nu}_{\alpha R}^{C}\nu_{\alpha R})_{1++}(\rho^{2})_{1++},(\overline{\nu}_{\alpha R}^{C}\nu_{\alpha R})_{1++}(\rho^{*2})_{1++};$	$U(1)_{B-L}$
$(\overline{\nu}_{\alpha R}^C \nu_{\alpha R})_{1_{+-}} \chi^*; (\overline{\psi}_{1L} \psi_{1L}^C)_{1_{++}} \widetilde{H^2}, (\overline{\psi}_{1L} \psi_{1L}^C)_{1_{++}} \widetilde{H'^2}.$	
$(\overline{\psi}_{1L}l_{1R})_{1_{++}}H, (\overline{\psi}_{1L}l_{1R})_{1_{++}}H', (\overline{\psi}_{1L}l_{1R})_{1_{++}}(H'\phi)_{1_{}}; (\overline{\psi}_{1L}\nu_{1R})_{1_{+-}}(H\rho^*)_2,$	
$(\overline{\psi}_{1L}\nu_{1R})_{1+-}(H'\rho^*)_2; (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H}\varphi)_{1-+}, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H'}\varphi)_{1+-}; (\overline{\nu}_{1R}^C\nu_{1R})_{1++}\chi;$	
$(\overline{\nu}_{1R}^C \nu_{\alpha R})_2 \chi, (\overline{\nu}_{1R}^C \nu_{\alpha R})_2 (\phi \chi)_{1_{++}}; (Q_{1L} u_{1R})_{1_{++}} \widetilde{H}, (Q_{1L} u_{1R})_{1_{++}} \widetilde{H'},$	ת
$\left  (Q_{1L}u_{1R})_{1++}(\widetilde{H'}\phi)_{1}, (Q_{\alpha L}u_{\alpha R})_{1++}\widetilde{H}, (Q_{\alpha L}u_{\alpha R})_{1++}\widetilde{H'}, (Q_{\alpha L}u_{\alpha R})_{1++}(\widetilde{H'}\phi)_{1}, \right  $	$D_4$
$(Q_{1L}d_{1R})_{1++}H, (Q_{1L}d_{1R})_{1++}H', (Q_{1L}d_{1R})_{1++}(H'\phi)_{1},$	
$(Q_{\alpha L}d_{\alpha R})_{1_{++}}H, (Q_{\alpha L}d_{\alpha R})_{1_{++}}H', (Q_{\alpha L}d_{\alpha R})_{1_{++}}(H'\phi)_{1_{}}$	
$(\overline{\psi}_{1L}\nu_{\alpha R})_2(\widetilde{H}\rho)_2, (\overline{\psi}_{1L}\nu_{\alpha R})_2(\widetilde{H'}\rho)_2, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H}\rho)_2, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H}\rho^*)_2, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H}\rho^*)_2, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H}\rho)_2, (\overline{\psi}_{1R})_2(\widetilde{H}\rho)_2, (\overline{\psi}_{1R})_2, (\overline{\psi}_{1R})_2, (\overline{\psi}_{1R})_2, (\overline{\psi}_{1R})_2, (\psi$	
$(\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H'}\rho)_2, (\overline{\psi}_{\alpha L}\nu_{1R})_2(\widetilde{H'}\rho^*)_2, (Q_{1L}u_{\alpha R})_2(H\rho^*)_2, (Q_{1L}u_{\alpha R})_2(H'\rho^*)_2, (Q_{1L}u_{\alpha R})_2(H$	7
$(Q_{\alpha L}u_{1R})_2(H\rho)_2, (Q_{\alpha L}u_{1R})_2(H'\rho)_2; (Q_{1L}d_{\alpha R})_2(\widetilde{H}\rho^*)_2, (Q_{1L}d_{\alpha R})_2(\widetilde{H'}\rho^*)_2,$	$\mathbb{Z}_4$
$(Q_{\alpha L}d_{1R})_2(\widetilde{H}\rho)_2, (Q_{\alpha L}d_{1R})_2(\widetilde{H'}\rho)_2$	
$(\overline{\psi}_{1L}l_{\alpha R})_2(H\rho)_{2}, (\overline{\psi}_{1L}l_{\alpha R})_2(H'\rho)_2, (\overline{\psi}_{\alpha L}l_{1R})_2(H\rho^*)_2, (\overline{\psi}_{\alpha L}l_{1R})_2(H'\rho^*)_2$	$Z_2$

Table IV. Yukawa terms forbidden by the model's symmetries

### as functions of quark masses and quark mixing matrix elements

The explicit expressions of  $a_{1u,d}, a_{2u,d}, a_{3u,d}, b_{u,d}, c_{1u,d}, c_{2u,d}, c_{3u}$  and  $c_{4u,d}$  are:

$$\begin{split} a_{1u} &= m_u, \ a_{2u} = \frac{m_c + m_t}{2}, \ a_{3u} = \frac{m_c - m_t}{2}, \\ a_{1d} &= m_d, \ a_{2d} = \frac{m_s + m_b}{2}, \ a_{3d} = \frac{m_s - m_b}{2}, \\ c_{1u} &= -c_{3u} + \frac{(m_d - m_s)(m_c - m_u)(1 - V_{11}^{exp})}{2c_{1d}^*}, \\ c_{2u} &= \frac{(m_u - m_c)(m_d - m_s)}{c_{2d}^* + c_{4s}^*} \left[ \frac{4b_d^* b_u}{(m_b - m_s)(m_c - m_t)} + \frac{c_{4u}(c_{2d}^* + c_{4d}^*)}{(m_c - m_u)(m_d - m_s)} + V_{22}^{exp} - 1 \right], \\ c_{3u} &= \frac{m_u - m_t}{2} \left[ \frac{(V_{11}^{exp} - 1)(m_c - m_u)(m_d - m_s)}{2c_{1d}^* (m_t - m_u)} + \frac{c_{2d}^* - c_{4d}^*}{m_b - m_d} + V_{13}^{exp} \right], \\ 2c_{4u} &= \frac{4b_d^* b_u(m_c - m_u)(m_s - m_d)}{(c_{2d}^* + c_{4d}^*)(m_b - m_s)(m_c - m_t)} - \frac{V_{33}^{exp}(m_b - m_d)(m_t - m_u)}{c_{2d}^* - c_{4d}^*} \\ &+ \frac{(m_b - m_d)(m_t - m_u)}{c_{2d}^* - c_{4d}^*} + \frac{V_{22}^{exp}(m_u - m_c)(m_d - m_s)}{(c_{2d}^* + c_{4d}^*)} + \frac{(m_c - m_u)(m_d - m_s)}{(c_{2d}^* + c_{4d}^*)}, \quad (B1) \\ b_u &= \frac{(m_b - m_s)(m_c - m_t)\{(c_{2d}^* + c_{4d}^*)[2c_{1d}^* + (m_d - m_s)V_{21}^{exp}] + (m_d - m_s)^2(1 - V_{22}^{exp})\}}{4b_d^*(m_d - m_s)^2} \\ b_d^* &= \frac{(m_b - m_d)(m_b - m_s)\{(1 - V_{11}^{exp})(m_d - m_s)^2 + 2c_{1a}^* [c_{2d}^* + c_{4d}^* + (m_s - m_d)V_{12}^{exp}]\}}{4c_{1d}^*(m_d - m_s)[c_{4d}^* - c_{2d}^* + V_{13}^{exp}(m_d - m_b)]} \\ c_{2d}^* &= \frac{c_{4d}^* V_{31}^{exp} + (V_{33}^{exp} - 1)(m_d - m_b)}{V_{31}^{exp}} , \quad c_{4d}^* &= \{2c_{1d}^*(m_d - m_s)[m_b \mathbf{F}_q + m_d \mathbf{G}_q + m_s \mathbf{H}_q] \\ + (m_d - m_s)^3 \mathbf{T}_q + 4c_{1d}^* (V_{33}^{exp} - 1)[V_{13}^{exp}(m_b - m_d) + V_{12}^{exp}(m_d - m_s)]\}/\{4c_{1d}^* V_{31}^{exp} - V_{21}^{exp}V_{32}^{exp}} \\ + (m_d - m_s)(V_{13}^{exp}V_{21}^{exp} - V_{23}^{exp})]\}, \\ c_{1d}^* &= \frac{(m_s - m_d)\sqrt{\mathbf{K}_{1q}} + (m_s - m_d)\mathbf{P}_{1q} + V_{13}^{exp}(m_d - m_s)[(V_{22}^{exp} - 1)V_{31}^{exp} - V_{21}^{exp}V_{32}^{exp}} ]}{4V_{13}^{exp}V_{32}^{exp} - 4V_{12}^{exp}V_{33}^{exp}}, \end{aligned}$$

where

$$\begin{split} \mathbf{F}_{q} &= (\mathbf{V}_{33}^{\exp} - 1)(\mathbf{V}_{13}^{\exp}\mathbf{V}_{21}^{\exp} - \mathbf{V}_{23}^{\exp}), \ \mathbf{G}_{q} = \mathbf{V}_{33}^{\exp}\left[\mathbf{V}_{11}^{\exp} + \left(\mathbf{V}_{12}^{\exp} - \mathbf{V}_{13}^{\exp}\right)\mathbf{V}_{21}^{\exp} - \mathbf{V}_{22}^{\exp} + \mathbf{V}_{23}^{\exp}\right] \\ &+ \mathbf{V}_{22}^{\exp} - \mathbf{V}_{23}^{\exp} - \mathbf{V}_{11}^{\exp} - \mathbf{V}_{12}^{\exp}(\mathbf{V}_{21}^{\exp} + \mathbf{V}_{23}^{\exp}\mathbf{V}_{31}^{\exp}) + \mathbf{V}_{13}^{\exp}\left[\mathbf{V}_{21}^{\exp} + \left(\mathbf{V}_{22}^{\exp} - 1\right)\mathbf{V}_{31}^{\exp}\right], \\ \mathbf{H}_{q} &= \left(\mathbf{V}_{11}^{\exp} + \mathbf{V}_{22}^{\exp}\right)(1 - \mathbf{V}_{33}^{\exp}) + \mathbf{V}_{12}^{\exp}(\mathbf{V}_{21}^{\exp} - \mathbf{V}_{21}^{\exp}\mathbf{V}_{33}^{\exp} + \mathbf{V}_{23}^{\exp}\mathbf{V}_{31}^{\exp}) + \mathbf{V}_{13}^{\exp}\mathbf{V}_{31}^{\exp}(1 - \mathbf{V}_{22}^{\exp}), \\ \mathbf{T}_{q} &= (1 - \mathbf{V}_{11}^{\exp})\left[\mathbf{V}_{21}^{\exp}(1 - \mathbf{V}_{33}^{\exp}) + \mathbf{V}_{23}^{\exp}\mathbf{V}_{31}^{\exp}\right], \\ \mathbf{K}_{1q} &= \left[\left(\mathbf{V}_{11}^{\exp} + \mathbf{V}_{12}^{\exp}\mathbf{V}_{21}^{\exp}\right)\mathbf{V}_{33}^{\exp} - \mathbf{V}_{12}^{\exp}\mathbf{V}_{23}^{\exp}\mathbf{V}_{31}^{\exp} - \mathbf{V}_{13}^{\exp}\mathbf{V}_{32}^{\exp} + \left(\mathbf{V}_{22}^{\exp} - 1\right)\mathbf{V}_{13}^{\exp}\mathbf{V}_{31}^{\exp} \\ &- \mathbf{V}_{22}^{\exp}\mathbf{V}_{33}^{\exp} + \mathbf{V}_{23}^{\exp}\mathbf{V}_{32}^{\exp}\right]^{2} + 4\left(\mathbf{V}_{11}^{\exp} - 1\right)\left(\mathbf{V}_{13}^{\exp}\mathbf{V}_{32}^{\exp} - \mathbf{V}_{12}^{\exp}\mathbf{V}_{33}^{\exp} - \mathbf{V}_{23}^{\exp}\mathbf{V}_{31}^{\exp}\right), \\ \mathbf{P}_{1q} &= \left(\mathbf{V}_{22}^{\exp} - \mathbf{V}_{11}^{\exp} - \mathbf{V}_{12}^{\exp}\mathbf{V}_{21}^{\exp}\right)\mathbf{V}_{33}^{\exp} + \left(\mathbf{V}_{12}^{\exp}\mathbf{V}_{31}^{\exp} - \mathbf{V}_{32}^{\exp}\right)\mathbf{V}_{23}^{\exp}. \end{aligned}$$

# Appendix C: The explicit expressions of $k_{1,2}, n_{1,2}$ and $t_{1,2}$ as functions of $a_D, b_D, c_D, f_D, g_D, a_R, b_R$ and $c_R$

The explicit expressions of  $k_{1,2}, n_{1,2}$  and  $t_{1,2}$  are:

$$\begin{aligned} k_1 &= \frac{c_D - d_D}{a_D + b_D}, \ k_2 &= \frac{a_D - b_D}{c_D + d_D}, \end{aligned} \tag{C1} \\ n_1 &= \left\{ a_D^2(b_D - a_D)(b_R + c_R) - b_D \left[ (c_R - b_R)(b_D^2 + 2c_D d_D) + (c_D^2 + d_D^2)c_R \right] \right. \\ &- a_D \left[ b_D^2(b_R - c_R) + 2c_D d_D(b_R + c_R) + (c_D^2 + d_D^2)c_R \right] + (a_D - b_D)\sqrt{\Delta} \right\} \\ &/ \left\{ (c_D - d_D) \left\{ b_D^2(c_R - b_R) + a_D^2(b_R + c_R) + \left[ (c_D + d_D)^2 - 2a_D b_D \right]c_R \right\} \right\}, \end{aligned} \tag{C2} \\ n_2 &= \left\{ (c_D + d_D) \left\{ \left[ (a_D + b_D)^2 + (c_D - d_D)^2 \right]c_R^2 + (a_D^2 - b_D^2)c_R b_R + 2(c_D d_D - a_D b_D)b_R^2 \right. \\ &- b_R\sqrt{\Delta} \right\} \right\} / \left\{ (c_D - d_D) \left[ b_D^2 b_R(c_R - b_R) + a_D^2 b_R(b_R + c_R) + b_R c_R(c_D^2 + d_D^2) - c_R\sqrt{\Delta} \right] \right\}, \end{aligned} \tag{C3} \\ t_1 &= \left\{ a_D^2(b_D - a_D)(b_R + c_R) + b_D \left[ (b_D^2 + 2c_D d_D)(b_R - c_R) - (c_D^2 + d_D^2)c_R \right] \right. \\ &- a_D \left[ b_D^2(b_R - c_R) + c_D^2 c_R + 2c_D d_D(b_R + c_R) + c_R d_D^2 \right] + (b_D - a_D)\sqrt{\Delta} \right\} \\ &/ \left\{ (c_D - d_D) \left[ b_D^2(c_R - b_R) - 2a_D b_D c_R + a_D^2(b_R + c_R) + c_R(c_D + d_D)^2 \right] \right\}, \end{aligned} \tag{C4} \\ t_2 &= \left\{ (c_D + d_D) \left\{ \left[ (a_D + b_D)^2 + (c_D - d_D)^2 \right]c_R^2 + (a_D^2 - b_D^2)c_R b_R + 2(c_D d_D - a_D b_D)b_R^2 \right. \\ &+ b_R\sqrt{\Delta} \right\} \right\} / \left\{ (c_D - d_D) \left[ b_D^2 b_R(c_R - b_R) + a_D^2 b_R(b_R + c_R) + b_R c_R(c_D^2 + d_D^2) + c_R\sqrt{\Delta} \right] \right\}, \end{aligned} \tag{C5}$$

where

$$\Delta = a_D^4 (b_R + c_R)^2 + \left[ b_D^2 b_R - (b_D^2 + c_D^2) c_R \right]^2 + 8a_D b_D c_D d_D (c_R^2 - b_R^2) + c_R^2 d_D^4 + 2 \left[ 2b_R^2 c_D^2 - b_D^2 b_R c_R + (b_D^2 - c_D^2) c_R^2 \right] d_D^2 + 2a_D^2 (b_R + c_R) \left[ b_D^2 (b_R - c_R) + c_R (c_D^2 + d_D^2) \right].$$
(C6)

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