# The Limits of Limited Commitment* 

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#### Abstract

We study limited strategic leadership. A collection of subsets covering the leader's action space determine her commitment opportunities. We characterize the outcomes resulting from all possible commitment structures of this kind. If the commitment structure is an interval partition, then the leader's payoff is bounded by her Stackelberg and Cournot payoffs. However, under more general commitment structures the leader may obtain a payoff that is less than her minimum Cournot payoff. We apply our results to study information design problems in leader-follower games where a mediator communicates information about the leader's action to the follower.


## JEL: C72, D43, D82

Keywords: commitment, Stackelberg, Cournot, information design

[^0]
## 1 Introduction

We study limited strategic leadership, where one player can make early commitments, but only to a limited extent. This situation is typical in markets for innovative products like new pharmaceuticals or technologies, where the leader enjoys a temporary monopoly, often due to patents or specialized knowledge. During this monopoly phase, the leader's early strategic investments can shape - but crucially, not fully determine - future business decisions. We introduce a framework to study situations of this kind. Our framework extends the Stackelberg leadership model to account for the fact that commitment opportunities may be limited.

Our model is simple. There are two periods and two players, a leader and a follower. The model is parametrized by a collection of subsets that cover the leader's action space; we refer to this collection of subsets as the commitment structure (CST). In the first period, the leader selects an element from the CST. In the second period, leader and follower simultaneously choose one action each, the leader being restricted to pick an action from the subset which she selected in the first period. The "Stackelberg" and "Cournot" models are special cases of our model: in the former, the CST consists of singletons; in the latter, the CST comprises just one element, namely, the leader's entire action space.

We say that an outcome is plausible if it is a subgame perfect equilibrium outcome of the game described above for some CST. An outcome is simply plausible if said CST partitions the leader's action space into intervals. Our main results characterize the sets of plausible and simply-plausible outcomes.

The function $U$ mapping every action of the leader to the payoff she obtains when the follower best-responds plays a central role in our results. A subset of the leader's action space is called $U$-monotone if whenever an action is contained in it so is the upper contour set of that action with respect to $U$.

While any plausible outcome gives the leader at most her Stackelberg payoff, a natural question is whether a Cournot payoff gives a corresponding lower bound. We show that all simply-plausible outcomes give the leader at least her minimum Cournot payoff. However,
in general, the simply-plausible actions of the leader are not $U$-monotone. In particular, an outcome in which the follower best responds to the leader may fail to be simply plausible even though the corresponding payoff of the leader is greater than one of her Cournot payoffs. By contrast, the plausible actions of the leader are $U$-monotone, but an outcome may be plausible and yet be such that the leader obtains a payoff that is less than her minimum Cournot payoff.

In settings with a single Cournot outcome, both the plausible and the simply-plausible actions of the leader are $U$-monotone. But even then, some plausible outcomes give the leader less than her Cournot payoff whenever the best-response functions are sufficiently steep.

We go on to show how our results can be applied to standard leader-follower games in which information is imperfect. Specifically, we consider situations in which what the follower can observe about the action of the leader is determined by a partition of the leader's action space. For example, a mediator might communicate the partition element containing the leader's action to the follower. We are interested in the problem of a designer choosing how to partition the leader's action space in such settings. We show that the outcomes which the designer can induce are exactly the plausible outcomes we have defined. Having characterized the set of plausible outcomes thus enables us to solve problems of this kind.

We illustrate various information design problems of the kind above in a textbook duopoly setting. There, the designer's objective may be to maximize total welfare, consumer surplus, or producer surplus. We find that even simple binary partitions of the leader's action space may perform better than both the Stackelberg and Cournot CSTs.

We contribute to a literature on commitment whose starting point is that economic agents often commit to subsets of actions rather than single actions. This literature is divided in two parts. One branch of the literature posits that agents commit to subsets of actions because they are constrained to do so: even if they wanted to, agents would be unable to commit to specific actions. This branch includes Spence (1977), where an incumbent firm faces a prospective entrant and invests in productive capacity during the first period but may choose not to utilize the full capacity during the second period. It also includes Saloner (1987), Admati and Perry (1991), and Romano and Yildirim (2005), where agents can set a lower
bound on the action they will choose but retain until the last period the option to choose any action that is at least as large as this lower bound. Our paper belongs to this branch of literature. Our contribution is to consider all possible commitment structures, that is, all collections of subsets that the leader may choose from in the first period.

A second branch of literature proposes that agents commit to subsets of actions because doing so gives them a strategic advantage ${ }^{\top}$ In these models, agents could commit to single actions but typically choose to commit to subsets containing more than one action because doing so enables them to credibly threaten potential deviating players. Specifically, Bade, Haeringer and Renou (2009), Renou (2009), and Dutta and Ishii (2016), embed a strategicform game into a multi-stage game in which, early on in the game, players can freely restrict their action spaces ${ }^{2}$ A player can commit to any action of his choice, but may also choose not to commit at all. This ability to freely choose what to commit to differs from our model and has stark implications: in our setting, if the leader can choose not to commit at all, she can guarantee herself her lowest Cournot payoff, while if she can commit to any action, then she can guarantee herself her Stackelberg payoff. By contrast, we show that if the leader's ability to commit is limited, she will not only fall short of her Stackelberg payoff but may also fall short of her lowest Cournot payoff $[3$

Several other papers study what might be construed as a form of limited commitment. Some allow agents to pick specific actions but let them revise these choices later on, either at fixed times (Maskin and Tirole, 1988), stochastically (Kamada and Kandori, 2020), or by incurring various costs (Henkel, 2002; Caruana and Einav, 2008).

Finally, our paper belongs to a recent strand of papers that take a base game as given

[^1]and examine how changing the structure of this game can affect its outcome. For example, Kamenica and Gentzkow (2011), Bergemann and Morris (2016), and Makris and Renou (2023) examine the implications of changing a game's information structure. Nishihara (1997) and Gallice and Monzón (2019) study instead the effects of changing the order of moves. Salcedo (2017) and Doval and Ely (2020) allow the structure of the game to change in both of these dimensions.

## 2 The Model

### 2.1 Setup

There are two players, a leader and a follower, with action spaces $\mathcal{X}=[\underline{x}, \bar{x}]$ and $\mathcal{Y}=[\underline{y}, \bar{y}]$, respectively. A collection $K$ of non-empty subsets of $\mathcal{X}$ covers the leader's action space ${ }^{4}$ We refer to $K$ as the commitment structure (CST).

There are two periods: in period 1 , the leader publicly selects $\mathcal{X}_{i} \in K$; in period 2 , leader and follower simultaneously choose actions $x$ and $y$, with $x$ contained in $\mathcal{X}_{i}$ and $y$ contained in $\mathcal{Y}$. The resulting payoffs are $u(x, y)$ for the leader and $v(y, x)$ for the follower, where $u$ and $v$ are continuous. We further assume that $u(x, y)$ is strictly quasi-concave in $x$ for all $y \in \mathcal{Y}$, and that $v(y, x)$ is strictly quasi-concave in $y$ for all $x \in \mathcal{X}$. This game is denoted by $G(K)$.

### 2.2 Definitions and Notation

An action pair $(x, y)$ with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ is referred to as an outcome. We say that outcome $(x, y)$ is plausible if $(x, y)$ is a subgame perfect equilibrium outcome of $G(K)$, for some CST $K$. An action $x$ is plausible if it is part of a plausible outcome $(x, y)$.

Two salient commitment structures play a central role,

$$
K^{S}:=\{\{x\}: x \in \mathcal{X}\} \text { and } K^{C}:=\{\mathcal{X}\}
$$

[^2]we refer to these as the Stackelberg and Cournot CSTs, respectively. By extension, the subgame perfect equilibrium outcomes of $G\left(K^{S}\right)$ and $G\left(K^{C}\right)$ will be referred to as Stackelberg and Cournot outcomes. The Cournot actions of the leader are the actions of the leader forming part of a Cournot outcome.

A commitment structure $K$ is said to be simple if it partitions the leader's action space into intervals. For example, the Stackelberg and Cournot CSTs are simple CSTs. An outcome $(x, y)$ is simply plausible if $(x, y)$ is a subgame perfect equilibrium outcome of $G(K)$, for some simple CST $K$.

To every action $x$ of the leader corresponds a unique best response of the follower ${ }^{5}$ We denote this best response by $R_{F}(x)$, and let $U(x)$ be the payoff of the leader from taking action $x$ when the follower best-responds to $x$, that is,

$$
U(x):=u\left(x, R_{F}(x)\right)
$$

A subset $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ is called $U$-monotone if $\tilde{x} \in \tilde{\mathcal{X}}$ implies that the upper contour set of $\tilde{x}$ with respect to $U$ is contained in $\tilde{\mathcal{X}}$ as well. $b^{6}$

### 2.3 Duopoly Example

In this subsection, we illustrate the model in the context of a textbook duopoly setting. Leader and follower are two identical firms, each choosing a quantity in $\mathcal{X}=\mathcal{Y}=[0,2 /(2-r)]]^{7} \mathrm{~A}$ firm producing quantity $q$ incurs cost $3 q-r q^{2} / 2$ and sells at unit price $4-(1-d) Q-d q$, where $Q$ represents the total quantity produced by the two firms. In the previous expressions, $r<2$ measures the returns to scale, and $d \in[0,1]$ the degree of product differentiation. Letting $u(x, y)$ (respectively, $v(y, x))$ be the profit of the leader (respectively, the follower) gives $v(y, x)=u(y, x)$ and

$$
\begin{equation*}
u(x, y)=x-(1-d) x y-\left(1-\frac{r}{2}\right) x^{2} \tag{1}
\end{equation*}
$$

[^3]

Figure 1

We set for now $d=0$ and $r=4 / 5$. The (unique) Cournot and Stackelberg actions are then, respectively, $x^{C}=5 / 11$ and $x^{S}=1$.

The commitment structure is

$$
K=\left\{\left[0, \frac{3}{2}\right),\left[\frac{3}{2}, \frac{5}{3}\right]\right\} .
$$

As it partitions the leader's actions space into intervals, $K$ is a simple CST. Figure 1 illustrates this example. Any quantity in the interval $[3 / 2,5 / 3]$ is such that, whenever the follower bestresponds, the leader benefits from deviating to a smaller quantity. Hence, any subgame perfect equilibrium of $G(K)$ must be such that the leader produces $3 / 2$ in the corresponding subgame. If instead the leader picks $[0,3 / 2)$ in the first period, then each firm produces the Cournot quantity. As $U(3 / 2)>U\left(x^{C}\right)$, the unique subgame perfect equilibrium is such that in period 1 the leader chooses the upper interval.

The previous reasoning applies if $3 / 2$ is replaced by any quantity $x^{*}$ in the interval $\left[x^{C}, \overline{\bar{x}}\right]$, where $\overline{\bar{x}}$ solves $U(\overline{\bar{x}})=U\left(x^{C}\right)$. So all actions in $\left[x^{C}, \overline{\bar{x}}\right]$ are simply plausible. We will see in Section 4 that, in fact, these are the only simply-plausible actions. In contrast, we will see in Section 5 that beyond simple CSTs the Cournot payoffs generally do not bound the payoffs that the leader can obtain.

## 3 Preliminaries

Henceforth, let

$$
\eta(\tilde{x}, x):=u\left(\tilde{x}, R_{F}(x)\right)-u\left(x, R_{F}(x)\right) .
$$

In words, $\eta(\tilde{x}, x)$ measures the leader's gain from choosing $\tilde{x}$ instead of $x$ when the follower best-responds to $x$.

Consider an arbitrary CST $K$. Suppose that a subgame perfect equilibrium of $G(K)$ exists. Given $\mathcal{X}_{i} \in K$, write $\beta\left(\mathcal{X}_{i}\right)$ for the leader's action in the subgame following $\mathcal{X}_{i}$. Then $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$, and $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right) \leq 0$ for all $x \in \mathcal{X}_{i}$. The notion of admissible pair summarizes these basic properties.

Definition 1. A pair $(K, \beta)$ made up of a commitment structure $K$ and a mapping $\beta: K \rightarrow \mathcal{X}$ is said to be admissible if
(a) $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$, for all $\mathcal{X}_{i} \in K$;
(b) $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right) \leq 0$, for all $x \in \mathcal{X}_{i}$ and all $\mathcal{X}_{i} \in K$.

The following characterization of the set of plausible outcomes is immediate.

Lemma 1. An outcome $(x, y)$ is plausible if and only if there exist an admissible pair $(K, \beta)$ and $\mathcal{X}_{i} \in K$, such that
(i) $x=\beta\left(\mathcal{X}_{i}\right)$,
(ii) $U(x)=\max _{\mathcal{X}_{j} \in K} U\left(\beta\left(\mathcal{X}_{j}\right)\right)$,
(iii) $y=R_{F}(x)$.

We say that an admissible pair $(K, \beta)$ implements outcome $(x, y)$ if it satisfies conditions (i)(iii) of Lemma 1. We can then rephrase the lemma to say that an outcome $(x, y)$ is plausible if and only if some admissible pair $(K, \beta)$ implements it.

## 4 Simple Commitment Structures

This section contains the first part of our analysis; in it, we characterize the set of simplyplausible outcomes. All proofs for this section are in Appendix A.

Denote by $R_{L}(y)$ the unique best response of the leader to the follower's action $y$, and defind ${ }^{8}$

$$
\phi(x):=R_{L}\left(R_{F}(x)\right) .
$$

The fixed points of $\phi$ are thus the Cournot actions of the leader. Let $\mathcal{X}^{C}$ denote said set of Cournot actions; the notation $x_{n}^{C}$ will indicate a generic element of this set.

The following lemma characterizes the admissible pairs $(K, \beta)$ such that $K$ is a CST comprising only intervals. In essence, the content of the lemma is akin to Proposition 1 in Bade, Haeringer and Renou (2007).

Lemma 2. Let $K$ be a commitment structure comprising only intervals. Then ( $K, \beta$ ) is admissible if and only if, for all $\mathcal{X}_{i} \in K$, one of the following conditions holds:
(i) $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i} \cap \mathcal{X}^{C}$;
(ii) $\beta\left(\mathcal{X}_{i}\right)=\min \mathcal{X}_{i}$ and $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)<\beta\left(\mathcal{X}_{i}\right)$;
(iii) $\beta\left(\mathcal{X}_{i}\right)=\max \mathcal{X}_{i}$ and $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)>\beta\left(\mathcal{X}_{i}\right)$.

The intuition behind the lemma is straightforward. Consider an interval $\mathcal{X}_{i}$ forming part of a CST $K$, and a mapping $\beta: K \rightarrow \mathcal{X}$ such that $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$ for all $\mathcal{X}_{i} \in K$. Suppose $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)>\beta\left(\mathcal{X}_{i}\right)$. In this case, having chosen $\mathcal{X}_{i}$ in period 1 , the leader would like to increase her action slightly whenever the follower best-responds to $\beta\left(\mathcal{X}_{i}\right)$. This implies that for ( $K, \beta$ ) to be admissible the leader must be unable to slightly increase her action. As $\mathcal{X}_{i}$ is an interval, $\beta\left(\mathcal{X}_{i}\right)$ must be the upper bound of the interval $\mathcal{X}_{i}$. Similarly, if $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)<\beta\left(\mathcal{X}_{i}\right)$ then $\beta\left(\mathcal{X}_{i}\right)$ must be the lower bound of the interval $\mathcal{X}_{i}$.

[^4]Figure 2, panel A, illustrates Lemma 2 in the context of the duopoly example introduced in Subsection 2.3, for parameter values $d=0$ and $r=6 / 5$. The black curve represents the graph of the function $\phi$. The leader's Cournot actions are $x_{1}^{C}=0, x_{2}^{C}=5 / 9$, and $x_{3}^{C}=5 / 4$. An admissible pair $(K, \beta)$ is such that every action $\beta\left(\mathcal{X}_{i}\right)$ belonging to a region of the figure with a left-pointing arrow (respectively, right-pointing arrow) is either a Cournot action or the leftmost (respectively, rightmost) element of $\mathcal{X}$.


Figure 2

Our first theorem characterizes the set of simply-plausible outcomes.
Theorem 1. An action $x^{*}$ is simply plausible if and only if the lower contour set of $x^{*}$ with respect to $U$ contains a Cournot action $x_{n^{*}}^{C}$ such that

$$
\begin{equation*}
\left(\phi\left(x^{*}\right)-x^{*}\right)\left(x_{n^{*}}^{C}-x^{*}\right) \geq 0 . \tag{2}
\end{equation*}
$$

Theorem 1 tells us that an action $x^{*}$ at which $\phi\left(x^{*}\right)>x^{*}$ is simply plausible if and only if some Cournot action greater than $x^{*}$ belongs to the lower contour set of $x^{*}$ with respect to $U$. The if part is easy. Let $x_{n}^{C}$ be a Cournot action greater than $x^{*}$ and suppose that it belongs to the lower contour set of $x^{*}$. Now consider $K=\left\{\left[\underline{x}, x^{*}\right],\left(x^{*}, \bar{x}\right]\right\}$, and $\beta$ given by $\beta\left(\left[\underline{x}, x^{*}\right]\right)=x^{*}$ and $\beta\left(\left(x^{*}, \bar{x}\right]\right)=x_{n}^{C}$. By Lemma 2 , the pair $(K, \beta)$ is admissible; it implements $x^{*}$ since $U\left(x^{*}\right) \geq U\left(x_{n}^{C}\right)$.

The gist of the proof of the only if part is as follows. Suppose that the admissible pair $(K, \beta)$ implements $x^{*}$, where $K$ is a simple CST and $\phi\left(x^{*}\right)>x^{*}$. Let $\mathcal{X}_{i}$ be the interval of the CST $K$ containing the action $x^{*}$. Then $\beta\left(\mathcal{X}_{i}\right)=x^{*}$ and, as $\phi\left(x^{*}\right)>x^{*}$, Lemma 2 implies that $x^{*}$ must be the upper bound of $\mathcal{X}_{i}$. Now let $\mathcal{X}_{j} \in K$ be the interval containing the actions slightly greater than $x^{*}, 9$ Applying Lemma 2 once more shows that either $\beta\left(\mathcal{X}_{j}\right)$ is a Cournot action, or $\phi\left(\beta\left(\mathcal{X}_{j}\right)\right)>\beta\left(\mathcal{X}_{j}\right)$ and $\beta\left(\mathcal{X}_{j}\right)$ is the upper bound of $\mathcal{X}_{j}$. Either way, we see by induction that $\beta\left(\mathcal{X}_{k}\right)$ must be a Cournot action for some interval $\mathcal{X}_{k} \in K$ comprising actions greater than $x^{*}$; and since $(K, \beta)$ implements $x^{*}$, said Cournot action must belong to the lower contour set of $x^{*}$ with respect to $U$.

Applying Theorem 1 to the example of Figure 2 shows that the set of simply-plausible actions is equal to $\{0\} \cup[5 / 17,5 / 9] \cup[5 / 4,5 / 2]$ (illustrated in green in panel $B$ of Figure 2 2 . Firstly, Theorem 1 shows that no action in the interval $(0,5 / 17)$ is simply plausible, since all of them belong to the strict lower contour set of each Cournot action. Secondly, any $x \in(5 / 9,5 / 4)$ satisfies $\phi(x)>x$ (see panel A). The only Cournot action greater than any of these actions is $x_{3}^{C}$. As $U\left(x_{3}^{C}\right)>U(x)$ for all $x \in(5 / 9,5 / 4)$, we conclude using Theorem 1 that no action in this interval is simply plausible. Mirror arguments show that all actions in $\{0\} \cup[5 / 17,5 / 9] \cup[5 / 4,5 / 2]$ are simply plausible ${ }^{10}$

By construction, the leader's Stackelberg payoff provides an upper bound for the payoffs attainable by the leader under any CST. Theorem 1 shows that the Cournot payoffs provide

[^5]a corresponding lower bound for simple CSTs. Moreover, Theorem 1 implies that if an action is in the upper contour set of all Cournot actions, then that action must be simply plausible. The following corollary records these observations.

Corollary 1. All simply-plausible actions belong to the upper contour set of a Cournot action with respect to $U$. Furthermore, any action in the intersection of these upper contour sets is simply plausible. When there exists a unique Cournot outcome, the set of simply-plausible actions is $U$-monotone and coincides with the upper contour set of the unique Cournot action.

## 5 Beyond Simple Commitment Structures

We saw in the previous section that all simply-plausible outcomes guarantee the leader at least her lowest Cournot payoff. The following examples show that the conditions imposed on simple CSTs are crucial-an outcome may be plausible and give the leader a payoff that is smaller than her minimum Cournot payoff.

Example A: Consider the duopoly example introduced in Subsection 2.3, with parameter values $d=0$ and $r=4 / 5$. The commitment structure is

$$
K=\left\{\left(\frac{1}{8}, \frac{1}{3}\right],\left[0, \frac{1}{8}\right] \cup\left(\frac{1}{3}, \frac{5}{3}\right]\right\} .
$$

While $K$ partitions the leader's actions space, one of its elements is a non-convex set; so $K$ is not a simple CST. Figure 3 illustrates this example. The subgame following the leader's choice of $(1 / 8,1 / 3]$ possesses a unique equilibrium, in which the leader produces $1 / 3$. The other subgame has two equilibria: one yielding the Cournot outcome, $x^{C}=5 / 11$, the other involving the leader choosing quantity $1 / 8$. As $U(1 / 8)<U(1 / 3)<U\left(x^{C}\right)$, we see that $G(K)$ possesses two subgame perfect equilibria: one in which the leader produces $1 / 3$, and one in which the leader produces $x^{C}$. In the former equilibrium, the leader anticipates that if she were to select $[0,1 / 8] \cup(1 / 3,5 / 3]$ in period 1 , the follower would respond by producing a


## Figure 3

quantity larger than $x^{C}$. Consequently, the leader settles for the quantity $1 / 3$, and obtains less than the Cournot payoff $U\left(x^{C}\right)$.

Example B: Consider the following setting. The action spaces are $\mathcal{X}=\mathcal{Y}=[0,1]$. The payoffs of the leader are given by

$$
u(x, y)=x y+(1-x)(1-y)-\frac{1}{2}\left(x-\frac{1}{2}\right)^{2}-\frac{3}{2}\left(y-\frac{1}{2}\right)^{2}
$$

the payoffs of the follower are given by $v(y, x)=u(y, x) \sqrt{11}$ In this setting, the leader's Stackelberg actions are $0,1 / 2$, and 1 , and these are also the leader's Cournot actions.

Let $x^{*}$ denote an action in $(1 / 2,1)$; the commitment structure is

$$
K=\left\{\left[0, x^{*}\right],\left[1-x^{*}, 1\right]\right\} .
$$

As $x^{*}>1 / 2, K$ does not partition the leader's action space; so $K$ is not a simple CST. Figure 4 illustrates this example. The subgame induced by the leader's choice of $\left[0, x^{*}\right]$ has an

[^6]

Figure 4
equilibrium in which the leader chooses the action $x^{*}$. Symmetrically, the subgame induced by the leader's choice of $\left[1-x^{*}, 1\right]$ has an equilibrium in which the leader chooses the action $1-x^{*}$. Therefore, since $U\left(x^{*}\right)=U\left(1-x^{*}\right)$, both $x^{*}$ and $1-x^{*}$ are plausible. In both cases, the leader obtains less than her Cournot payoff $U(1 / 2)$.

A simple CST partitions the leader's action space into intervals. The previous examples illustrate that relaxing either of these conditions can expand the set of outcomes that are plausible. In particular, relaxing either of these conditions can induce the leader to obtain less than her minimum Cournot payoff. Subsection 5.1 characterizes the set of outcomes induced by CSTs comprising only intervals. CSTs comprising non-convex sets are examined in Subsection 5.2, we characterize the set of outcomes induced by CSTs which partition the leader's action space and show that every plausible outcome is plausible under such a CST. Subsection 5.3 discusses equilibrium refinements.

### 5.1 Commitment Structures Comprising Only Intervals

We say that an outcome $(x, y)$ is I-plausible if it is a subgame perfect equilibrium outcome of $G(K)$ for some commitment structure $K$ comprising only intervals. Every simply-plausible outcome is I-plausible. However, as Example B illustrated, an outcome may be I-plausible even though it is not simply plausible. The following theorem characterizes the set of I-plausible outcomes. All proofs for this subsection are in Appendix B.

Theorem 2. The set of I-plausible actions is $U$-monotone. An action $x^{*}$ is I-plausible if and only if the lower contour set of $x^{*}$ with respect to $U$ includes actions $x^{\prime}$ and $x^{\prime \prime}$ such that

$$
\begin{equation*}
\phi\left(x^{\prime}\right) \leq x^{\prime} \leq x^{\prime \prime} \leq \phi\left(x^{\prime \prime}\right) \tag{3}
\end{equation*}
$$

Suppose that $x$ is I-plausible, and let $(K, \beta)$ be an admissible pair that implements $x$ in which the CST $K$ contains only intervals. Pick $x^{\prime}$ in the upper contour set of $x$ with respect to $U$, and consider $K^{\prime}:=K \cup\left\{\left\{x^{\prime}\right\}\right\}$, and $\beta^{\prime}: K^{\prime} \rightarrow \mathcal{X}$ such that $\beta^{\prime}\left(\mathcal{X}_{i}\right)=\beta\left(\mathcal{X}_{i}\right)$ for all $\mathcal{X}_{i} \in K$ and $\beta^{\prime}\left(\left\{x^{\prime}\right\}\right)=x^{\prime}$. The CST $K^{\prime}$ contains only intervals. Furthermore, the pair $\left(K^{\prime}, \beta^{\prime}\right)$ is evidently admissible and implements $x^{\prime}$. We conclude that the I-plausible actions are $U$-monotone.

The if part of the theorem is straightforward. Let $x^{*}$ be such that the lower contour set of $x^{*}$ with respect to $U$ includes actions $x^{\prime}$ and $x^{\prime \prime}$ satisfying (3). Consider $K=$ $\left\{\left\{x^{*}\right\},\left[\underline{x}, x^{\prime \prime}\right],\left[x^{\prime}, \bar{x}\right]\right\}$, and $\beta$ given by $\beta\left(\left\{x^{*}\right\}\right)=x^{*}, \beta\left(\left[\underline{x}, x^{\prime \prime}\right]\right)=x^{\prime \prime}$, and $\beta\left(\left[x^{\prime}, \bar{x}\right]\right)=x^{\prime}$. By Lemma 2, the pair $(K, \beta)$ is admissible; it implements $x^{*}$ since $U\left(x^{*}\right) \geq \max \left\{U\left(x^{\prime}\right), U\left(x^{\prime \prime}\right)\right\}$.

The only if part of the theorem rests on Lemma 2. Suppose that every action $x$ in the lower contour set of $x^{*}$ for which $\phi(x) \geq x$ is strictly smaller than every action $x$ in the lower contour set of $x^{*}$ for which $\phi(x) \leq x$, and pick $x^{\dagger}$ in between these two subsets of actions. Now let $(K, \beta)$ be an admissible pair such that $K$ is a CST comprising only intervals. By Lemma 2 , if $\mathcal{X}_{i} \in K$ is an interval containing $x^{\dagger}$ then either $\beta\left(\mathcal{X}_{i}\right) \geq x^{\dagger}$ and $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right) \geq \beta\left(\mathcal{X}_{i}\right)$, or $\beta\left(\mathcal{X}_{i}\right) \leq x^{\dagger}$ and $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right) \leq \beta\left(\mathcal{X}_{i}\right)$. Yet every action $x$ in the lower contour set of $x^{*}$ for which $\phi(x) \geq x$ is strictly smaller than $x^{\dagger}$, while every action $x$ in the lower contour set of $x^{*}$
for which $\phi(x) \leq x$ is strictly greater than $x^{\dagger}$. We conclude that $\beta\left(\mathcal{X}_{i}\right)$ must belong to the strict upper contour set of $x^{*}$. So $(K, \beta)$ cannot implement $x^{*}$.

Applying Theorem 2 to the example of Figure 2 shows that the I-plausible actions are $\{0\} \cup[5 / 17,5 / 2]$. In particular, actions in the interval $(5 / 9,5 / 4)$ are I-plausible but are not simply plausible ${ }^{12}$

If some Cournot action $x_{n}^{C}$ belongs to the lower contour set of $x^{*}$, then setting $x^{\prime}=x^{\prime \prime}=x_{n}^{C}$ in Theorem 2 proves that $x^{*}$ is I-plausible. We thus obtain the following corollary:

Corollary 2. All actions in the upper contour set of a Cournot action with respect to $U$ are I-plausible.

The previous analysis has shown that an outcome may be I-plausible even though it is not simply plausible and that I-plausible outcomes can give the leader a smaller payoff than her lowest Cournot payoff. However, in certain prominent cases, every I-plausible outcome does guarantee the leader at least her minimum Cournot payoff. For example, consider a setting with a unique Cournot action $x^{C}$. We have in this case $\{x: \phi(x) \geq x\}=\left[\underline{x}, x^{C}\right]$ and $\{x: \phi(x) \leq x\}=\left[x^{C}, \bar{x}\right]$. Applying Theorem 2 thus shows that all I-plausible actions must belong to the upper contour set of $x^{C}$ with respect to $U$. As all actions in the upper contour set of $x^{C}$ are simply plausible (Corollary 1), we conclude that in this case an action is I-plausible if and only if it is in the upper contour set of $x^{C}$ with respect to $U$. A similar result holds when $U$ is either quasi-convex or quasi-concave.

Proposition 1. If there exists a unique Cournot outcome, or if $U$ is either quasi-convex or quasi-concave, then an action is I-plausible if and only if it belongs to the union of the upper contour sets of the Cournot actions with respect to $U$.

[^7]
### 5.2 Partitional Commitment Structures

We say that an outcome $(x, y)$ is $P$-plausible if it is a subgame perfect equilibrium outcome of $G(K)$ for some commitment structure $K$ which partitions the leader's action space. Every simply-plausible outcome is P-plausible. However, as Example A illustrated, an outcome may be P-plausible even though it is not simply plausible. We start this subsection by establishing that any plausible outcome is, in fact, P-plausible. $\sqrt{13}$ All proofs for this section are in Appendix C.

Proposition 2. An outcome is plausible if and only if it is P-plausible. Moreover, the set of $P$-plausible outcomes is $U$-monotone.

The proof that the set of P-plausible outcomes is $U$-monotone is similar to the proof that the set of I-plausible outcomes is $U$-monotone. The if part of the proposition is trivial. The gist of the proof of the only if part is as follows. Let $(x, y)$ be plausible, and $(K, \beta)$ be an admissible pair that implements $(x, y)$. Suppose that $K=\left\{\mathcal{X}_{1}, \cdots, \mathcal{X}_{n}\right\}$, and ${ }^{14}$

$$
\begin{equation*}
\beta\left(\mathcal{X}_{i}\right) \notin \bigcup_{j \neq i} \mathcal{X}_{j}, \quad \text { for } i=1, \cdots, n . \tag{4}
\end{equation*}
$$

Now let $\mathcal{X}_{i}^{\prime}=\mathcal{X}_{i} \backslash \bigcup_{j<i} \mathcal{X}_{j}$, for $i=1, \cdots, n$, and $K^{\prime}=\left\{\mathcal{X}_{i}^{\prime}, \cdots, \mathcal{X}_{n}^{\prime}\right\}$. Then $K^{\prime}$ partitions $\mathcal{X}$ and (4) implies $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}^{\prime}$, for $i=1, \cdots, n$. Letting $\beta^{\prime}\left(\mathcal{X}_{i}^{\prime}\right)=\beta\left(\mathcal{X}_{i}\right)$ for $i=1, \cdots, n$, we see that $\left(K^{\prime}, \beta^{\prime}\right)$ is admissible, since $\mathcal{X}_{i}^{\prime} \subseteq \mathcal{X}_{i}$. Finally, $\left(K^{\prime}, \beta^{\prime}\right)$ evidently implements $(x, y)$, so $(x, y)$ is P-plausible.

Proposition 2 suggests that the set of P-plausible outcomes may be very large. To make progress, we restrict attention in the rest of this subsection to settings that satisfy the following three regularity conditions:
(RC1) there exists a unique and interior Cournot outcome;
(RC2) all externalities are either strictly positive or strictly negative $\sqrt{15}$

[^8](RC3) payoffs are either strictly supermodular or strictly submodular. ${ }^{16}$

For instance, the duopoly example of Subsection 2.3 satisfies all three conditions as long as the returns to scale are not too large $(r<d+1)$. When $u$ and $v$ are twice differentiable, conditions (RC2) and (RC3) become $u_{2} v_{2}>0$ and $u_{12} v_{12}>0$, respectively. Slightly abusing notation, whether or not the payoffs are differentiable, $u_{2}>0$ will indicate positive externalities, and $u_{2}<0$ negative externalities. Similarly, $u_{12}>0$ will indicate supermodular payoffs, and $u_{12}<0$ submodular payoffs.

For every $x \in \mathcal{X}$, the function $\eta(\cdot, x)$ is strictly quasi-concave and satisfies $\eta(x, x)=0$. It follows that $\eta(\tilde{x}, x)=0$ for at most one action $\tilde{x}$ different from $x$. We can thus define $\gamma: \mathcal{X} \rightarrow \mathcal{X}$ as follows: if $\eta(\tilde{x}, x)=0$ for some $\tilde{x} \neq x$, set $\gamma(x)=\tilde{x}$; otherwise, set ${ }^{17}$

$$
\gamma(x)= \begin{cases}\bar{x} & \text { if } x<x^{C} \\ x^{C} & \text { if } x=x^{C} \\ \underline{x} & \text { if } x>x^{C}\end{cases}
$$

The interpretation is straightforward: in cases where such an action exists, $\gamma(x)$ is the action making the leader indifferent between choosing $x$ or $\gamma(x)$ when the follower best-responds to $x$.

Next, let

$$
\mathcal{S}:= \begin{cases}\left\{x: x \leq \gamma(x) \leq x^{C}\right\} & \text { if } u_{2} u_{12}>0 \\ \left\{x: x^{C} \leq \gamma(x) \leq x\right\} & \text { if } u_{2} u_{12}<0\end{cases}
$$

Note that, as $\gamma$ is continuous, the set $\mathcal{S}$ is compact. ${ }^{18}$ Moreover, this set evidently contains $x^{C}$. We are now ready to characterize the set of P-plausible outcomes.

[^9]Theorem 3. Suppose ( $R C 1$ ) $-(R C 3)$ hold. The set of P-plausible actions coincides with the upper level set of $\underline{U}:=\min _{x \in \mathcal{S}} U(\gamma(x))$ with respect to $U{ }^{19}$

For an action $x$ to be plausible, the leader must face a credible punishment for choosing any action from the strict upper contour set of $x$ with respect to $U$. The characterization of the plausible actions in Theorem 3 rests on the identification of all possible threats for the leader. To understand the logic of the theorem, consider the case in which $u_{2}>0$ and $u_{12}>0$, that is, players' actions have positive externalities and are strategic complements. In this case, the function $U$ is increasing in the range $\left[\underline{x}, x^{C}\right]{ }^{20}$ Now let $\hat{x}$ be an action of the leader such that $\hat{x}<\gamma(\hat{x})<x^{C}$, and $x^{*}$ an action in the upper contour set of $\gamma(\hat{x})$ with respect to $U$. We will argue that $x^{*}$ is P-plausible. Let $\mathcal{X}_{1}$ be the subset of actions comprising $\hat{x}$ as well as all actions in the strict upper contour set of $x^{*}$. Note that, apart from $\hat{x}$, all actions in $\mathcal{X}_{1}$ have to be at least as large as $\gamma(\hat{x})$, since $U\left(x^{*}\right) \geq U(\gamma(\hat{x}))$ and $U$ is increasing over $[\underline{x}, \gamma(\hat{x})]$. Next, let $K$ be the partition of the leader's action space containing $\mathcal{X}_{1}$ and in which all other subsets are singletons. Lastly, let $\beta\left(\mathcal{X}_{1}\right)=\hat{x}$ and $\beta(\{x\})=x$ for all $x \in \mathcal{X} \backslash \mathcal{X}_{1}$. We claim that the pair $(K, \beta)$ is admissible. Indeed, $\hat{x}<\gamma(\hat{x})$, whence $\eta(x, \hat{x}) \leq 0$ for all $x \geq \gamma(\hat{x})$ (recall, $\eta(\cdot, \hat{x})$ is strictly quasi-concave and $\eta(\gamma(\hat{x}), \hat{x})=0$, by definition of $\gamma(\hat{x}))$. Yet we saw above that all actions in $\mathcal{X}_{1}$ other than $\hat{x}$ are at least as large as $\gamma(\hat{x})$, so $\eta(x, \hat{x}) \leq 0$ for all $x \in \mathcal{X}_{1}$. Finally, we claim that $(K, \beta)$ implements $x^{*}$. Indeed, all actions in the complement of $\mathcal{X}_{1}$ belong to the lower contour set of $x^{*}$. Moreover, $U(\hat{x})<U(\gamma(\hat{x})) \leq U\left(x^{*}\right)$; the first inequality follows from $\hat{x}<\gamma(\hat{x})$ and the fact that $U$ is increasing over $[\underline{x}, \gamma(\hat{x})]$; the second inequality is immediate, since $x^{*}$ is in the upper contour set of $\gamma(\hat{x})$.

Figure 5 illustrates Theorem 3 in the context of the duopoly example from Subsection 2.3 with parameter values $d=0$ and $r=4 / 5$. In panel A , the black curve represents the graph of the function $\phi$, which crosses the 45 -degree line at $x^{C}=5 / 11$. In this example $u_{2}<0$ and $u_{12}<0$, so $\mathcal{S}=\left\{x: x \leq \gamma(x) \leq x^{C}\right\}$. The gray curve represents the graph of the function $\gamma$ :

[^10]we see that $\mathcal{S}=\left[0, x^{C}\right]$ and $\gamma(\mathcal{S})=\left[5 / 18, x^{C}\right]$. Panel B depicts the graph of the function $U$. Minimizing $U(\gamma(x))$ over $\mathcal{S}$ shows that $\underline{U}=U(\gamma(0))=U(5 / 18) \cdot{ }^{21}$ Because here externalities are negative and the players' actions are strategic substitutes, having the follower believe that the leader played action 0 constitutes the worst possible threat to the leader. However, for this threat to be credible, the partition element containing action 0 must not contain any action in $(0, \gamma(0))$. Hence, actions greater or equal to $\gamma(0)$ are plausible, whereas those less than $\gamma(0)$ are not.


Figure 5

In the previous example, $\underline{U}$ is less than the leader's Cournot payoff. The question remains as to whether we can find conditions that guarantee $\underline{U}<U\left(x^{C}\right)$. We show in Appendix C

[^11]that when the payoff functions are twice differentiable, a simple sufficient condition is given by $\gamma^{\prime}\left(x^{C}\right)>0$. Calculations relegated to Appendix Cestablish that $\gamma^{\prime}\left(x^{C}\right)>0$ if and only if $R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)>1 / 2$. We thus obtain:

Proposition 3. Suppose $u$ and $v$ are twice differentiable and ( $R C 1$ ) $-(R C 3)$ hold. If $R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)>$ $1 / 2$ then $\underline{U}<U\left(x^{C}\right)$.

### 5.3 Equilibrium Refinements

We saw that the Stackelberg and Cournot payoffs provide the bounds of the payoffs attainable by the leader under any simple CST. A natural question is whether some equilibrium refinement ensures that the Stackelberg and Cournot payoffs provide the bounds of the payoffs attainable by the leader under arbitrary CSTs.

Forward induction type of arguments eliminate some, but not all, subgame perfect equilibria giving the leader less than her Cournot payoffs ${ }^{22}$ For instance, consider the setting of Example B at the beginning of this section, but this time with commitment structure

$$
\left\{\left[\frac{1}{9}, \frac{4}{9}\right),\left[0, \frac{1}{9}\right) \cup\left[\frac{4}{9}, 1\right)\right\}
$$

Figure 6 depicts the graph of $U$. The subgame induced by the leader's choice of $[1 / 9,4 / 9$ ) has a unique equilibrium, in which the leader chooses the action $1 / 9$. The other subgame has an equilibrium in which the leader picks $4 / 9$. As $U(4 / 9)>U(1 / 9)$, a subgame perfect equilibrium exists in which the leader chooses $4 / 9$. Yet, $U(4 / 9)<U\left(x_{n}^{C}\right)$, so the leader obtains a payoff smaller than her Cournot payoff. Since the subgame off the equilibrium path possesses a unique equilibrium, forward induction type of arguments have no bite.

One alternative is to restrict attention to subgame perfect equilibria that select, in every period-2 subgame, the best continuation equilibrium from the perspective of the leader. In this case, any subgame induced by the leader's period-1 choice of a subset containing a Cournot action must give the leader a payoff at least as large as that Cournot payoff. Consequently,

[^12]

Figure 6
any such subgame perfect equilibrium ensures that the leader obtains at least her maximum Cournot payoff.

## 6 Applications

In this section, we examine leader-follower games in which the follower imperfectly observes the action of the leader. When the follower's information about the leader's action is designed (e.g., by a mediator), the outcomes that the designer can induce are exactly the plausible outcomes defined in Section 2. This enables us to use Theorem 3 to solve such information design problems. We explore various design problems of this kind in the context of oligopolies.

### 6.1 Leader-Follower Games with Imperfect Information

The leader-follower games that we consider are as follows. The players' action spaces are $\mathcal{X}=[\underline{x}, \bar{x}]$ for the leader and $\mathcal{Y}=[\underline{y}, \bar{y}]$ for the follower. A partition $K$ of the leader's action space $\mathcal{X}$ is exogenously fixed. The leader is the first mover, choosing $x \in \mathcal{X}$. The follower then observes the partition element containing $x$ and chooses $y \in \mathcal{Y}$. The resulting payoffs
are $u(x, y)$ for the leader and $v(y, x)$ for the follower, where $u$ and $v$ are continuous, $u(x, y)$ is strictly quasi-concave in $x$ for all $y \in \mathcal{Y}$, and $v(y, x)$ is strictly quasi-concave in $y$ for all $x \in \mathcal{X}$. The game described above is denoted by $\widehat{G}(K)$.

For $\mathcal{X}_{i} \in K$, let $\beta\left(\mathcal{X}_{i}\right)$ represent the follower's belief concerning the action of the leader when the leader chooses $x \in \mathcal{X}_{i}$. In a pure-strategy sequential equilibrium, ${ }^{23}$

$$
\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}, \quad \forall \mathcal{X}_{i} \in K,
$$

that is, when the follower observes $\mathcal{X}_{i}$, the follower must believe that the leader chose an action in $\mathcal{X}_{i}$ whenever this is true. Yet, off the equilibrium path, the notion of sequential equilibrium imposes no further restrictions on the follower's beliefs. In particular, the follower could believe that the leader played $\beta\left(\mathcal{X}_{i}\right)$ even though some other actions in $\mathcal{X}_{i}$ were to give the leader a payoff strictly larger than playing action $\beta\left(\mathcal{X}_{i}\right)$. The notion of proper sequential equilibrium refines that of sequential equilibrium by ruling out such beliefs.

Definition 2. A pure-strategy sequential equilibrium of $\widehat{G}(K)$ is said to be proper if the follower's beliefs $\beta: K \rightarrow \mathcal{X}$ are such tha $\underline{L}^{24}$

$$
u\left(\beta\left(\mathcal{X}_{i}\right), R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right) \geq u\left(x, R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right), \quad \text { for all } x \in \mathcal{X}_{i} \text { and all } \mathcal{X}_{i} \in K
$$

In the setting that we consider, the set of sequential equilibria is also the set of perfect equilibria. In the spirit of Myerson (1978), our notion of proper sequential equilibrium effectively refines that of perfect equilibrium by making "worse actions" infinitely less likely to result from players' trembles than "better actions". In fact, we show in the online appendix F that our proper sequential equilibria are isomorphic to Myerson's proper equilibria.

[^13]
### 6.2 Information Design

We now consider the problem of a designer choosing the information disclosed to the follower concerning the action of the leader. Letting $W(x, y)$ denote the payoff of the designer when the leader chooses action $x$ and the follower chooses action $y$, the problem of the designer is

$$
\begin{equation*}
\max _{K} W(x, y) \quad \text { s.t. }(x, y) \text { is a proper sequential equilibrium outcome of } \widehat{G}(K) . \tag{DP}
\end{equation*}
$$

To solve this problem, we use the following lemma:
Lemma 3. An action pair $(x, y)$ is a proper sequential equilibrium outcome of $\widehat{G}(K)$ if and only if $(x, y)$ is a subgame perfect equilibrium outcome of $G(K)$.

Proof: Suppose $(x, y)$ is a proper sequential equilibrium outcome of $\widehat{G}(K)$, and pick an abritrary equilibrium with outcome $(x, y)$. Let $\beta\left(\mathcal{X}_{i}\right)$ be the follower's belief when the follower observes $\mathcal{X}_{i} \in K$. By definition of a proper sequential equilibrium: firstly, $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$; secondly, $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right) \leq 0$ for all $x \in \mathcal{X}_{i}$. So $(K, \beta)$ constitutes an admissible pair. Furthermore, conditions (i)-(iii) of Lemma 1 evidently hold. Thus, $(x, y)$ is a subgame perfect equilibrium outcome of $G(K)$. The proof of the converse is similar.

Coupling Lemma 3 with Proposition 2 enables us to rewrite (DP) as ${ }^{25}$

$$
\begin{equation*}
\max _{(x, y)} W(x, y) \quad \text { s.t. }(x, y) \text { is plausible. } \tag{DP'}
\end{equation*}
$$

This equivalence enables us to use our results concerning plausible outcomes as a building block for solving the problem of the designer.

### 6.3 Information Design in Oligopolies

In this subsection, we study design problems in the context of the duopoly example presented in Subsection 2.3. For expository convenience, we rewrite the problem of the designer in terms

[^14]of the leader's action $x$ :
\[

$$
\begin{equation*}
\max W\left(x, R_{F}(x)\right) \quad \text { s.t. } x \text { is plausible. } \tag{DP"}
\end{equation*}
$$

\]

We first examine situations where the designer is one of the two firms (i.e., where $W=u$ or $W=v$ ). The Stackelberg outcome is plainly the best plausible outcome from the perspective of the leader. On the other hand, since $v_{2}$ is here negative, the optimal plausible outcome from the perspective of the follower involves the leader producing as little as plausibly possible. The proposition which follows summarizes these observations. In the rest of this section, $\underline{\underline{x}}$ (respectively, $\overline{\bar{x}}$ ) indicates the smallest (respectively, largest) plausible action of the leader.

Proposition 4. Suppose $u$ is given by (1) and $v(y, x)=u(y, x)$. If $W=u$, the unique solution of (DP") is $x^{S}$. If $W=v$, the unique solution of $(\mathrm{DP}$ is $\underline{\underline{x}}$.

The Stackelberg CST is optimal for the leader. The Cournot CST is optimal for the follower if and only if $r \notin\left(r^{*}(d), d+1\right)$, where $r^{*}(d):=2-\sqrt{2}(1-d)$. For $r \in\left(r^{*}(d), d+1\right)$, the CST

$$
\{(0, \gamma(0)],\{0\} \cup(\gamma(0), \bar{x}]\}
$$

is optimal for the follower. The latter CST is such that the leader either commits to producing a quantity in the interval $(0, \gamma(0)]$, or commits to producing a quantity outside of this interval.

We next examine situations in which the designer aims to maximize either consumer surplus, producer surplus, or total welfare (i.e., the sum of producer and consumer surplus). We follow Singh and Vives (1984) and define the consumer surplus generated by an outcome ( $x, y$ ) $a ⿷^{26}$

$$
C S(x, y)=\frac{(x+y)^{2}}{2}-d x y
$$

Producer surplus is defined as

$$
P S(x, y)=u(x, y)+v(y, x) .
$$

[^15]Proposition 5. Suppose $u$ is given by (1) and $v(y, x)=u(y, x)$.
(i) If $W=C S$, the unique solution of (DP") is $\overline{\bar{x}}$.
(ii) If $W=P S$, the unique solution of (DP") is $x^{C}$ if $r<r^{\dagger}(d)$, and $x^{S}$ if $r^{\dagger}(d)<r<d+1,{ }^{27}$ if $r>d+1$ then the solutions are $x_{3}^{C}$ and $0{ }^{28}$
(iii) If $W=C S+P S$, the unique solution of (DP" is $\overline{\bar{x}}$.

Part (i) of Proposition 5 is explained as follows. Firstly, we show that consumer surplus is a convex function of the quantity which the leader produces. The problem of the designer therefore reduces to choosing between $\underline{\underline{x}}$ and $\overline{\bar{x}}$. Inducing the leader to produce $\overline{\bar{x}}$ instead of $\underline{\underline{x}}$ is optimal because in this way the designer can exploit the strategic motive to produce large quantities which arises from commitment. With multiple Cournot actions, or if there exists a single Cournot action and $\gamma(0) \geq x^{C}$, the binary partition $\{[\underline{x}, \overline{\bar{x}}),[\overline{\bar{x}}, \bar{x}]\}$ is consumer-optimal. Otherwise, the CST

$$
\{(0, \gamma(0)],\{0\} \cup(\gamma(0), \overline{\bar{x}}),[\overline{\bar{x}}, \bar{x}]\}
$$

is optimal for the consumer. The latter CST is such that the leader either commits to producing a quantity in the interval $(0, \gamma(0)]$, or commits to producing a quantity outside of this interval; in the latter case, the leader either commits to producing a quantity at least as large as $\overline{\bar{x}}$, or commits to producing less than this.

Part (ii) of Proposition 5 is straightforward. With decreasing returns to scale, producer surplus is maximized by inducing both firms to produce the same quantity; in this case, the Cournot CST is producer-optimal. By contrast, with large returns to scale, producer surplus is maximized by letting one firm acquire a bigger market share than the other. In particular, for very large returns to scale, producer surplus is maximized by letting one firm act as a monopolist. Consequently, the Cournot CST is producer-optimal for extreme returns to scale, whereas the Stackelberg CST is producer-optimal for sufficiently large returns to scale.
${ }^{27} r^{\dagger}(d):=2-\left(\frac{\sqrt[3]{3(9-\sqrt{78})}}{3}+\frac{1}{\sqrt[3]{3(9-\sqrt{78})}}\right)(1-d)$.
${ }^{28} x_{3}^{C}=\frac{1}{2-r}$ is the highest of the three Cournot equilibria.

Part (iii) of Proposition 5 follows from the fact that producer surplus tends to be less sensitive than consumer surplus to the quantity which the leader produces. So maximizing total welfare implies maximizing consumer surplus.

## 7 Richer and Finer Commitment

In this section, we explore the intuitive notion that two commitment structures might give different "degrees" of commitment to the leader.

Two natural partial orders on the set of commitment structures emerge from our analysis. Firstly, we say that a CST $K^{\prime}$ is richer than a CST $K$ if $K \subseteq K^{\prime}$. Secondly, we say that a CST $K^{\prime}$ is finer than a CST $K$ if the following conditions hold:
(i) each element $\mathcal{X}_{i}^{\prime}$ of the CST $K^{\prime}$ is a subset of some element $\mathcal{X}_{i}$ of the CST $K$,
(ii) each element of $K$ can be written as the union of elements of $K^{\prime}$.

In other words, $K^{\prime}$ is finer than $K$ if $K^{\prime}$ can be obtained from $K$ by replacing each element $\mathcal{X}_{i}$ of $K$ by some cover of $\mathcal{X}_{i}{ }^{29}$

Finally, given a CST $K$ such that the set of subgame perfect equilibria of $G(K)$ is nonempty, say that a CST $K^{\prime}$ is worse than $K$ if some subgame perfect equilibrium of $G\left(K^{\prime}\right)$ gives the leader a strictly lower payoff than every subgame perfect equilibrium of $G(K)$.

It is not hard to see that, starting from a given CST, either enriching this CST or refining it can make the leader better off. It is equally clear that a CST cannot be both richer and worse than another one. By contrast, our analysis reveals that a CST may be finer and worse than another CST. Indeed, every CST is finer than the Cournot CST. A corollary of our analysis is thus that refining the Cournot CST can yield a CST that is worse.

A second corollary of our analysis is that every CST that is both finer and worse than the Cournot CST must be non-simple. Therefore, a natural question is whether, by restricting

[^16]attention to simple CSTs, we ensure that a CST which is finer than another is not also worse. The following example shows that the answer is no.

Consider a more general version of Example B from Section 5, in which the payoffs of the leader are given by

$$
u(x, y)=x y+(1-x)(1-y)-\frac{1}{2}\left(x-\frac{1}{2}\right)^{2}-\frac{3(1+a)}{2}\left(y-\frac{1}{2}\right)^{2}
$$

for some parameter $a$. Suppose $a$ is a small but positive number ${ }^{30}$ There are then three Cournot actions ( $0,1 / 2$, and 1 ), but only one Stackelberg action ( $1 / 2$ ). Next, consider the simple CST

$$
K=\{\{0\},(0,1),\{1\}\} .
$$

Let $\nu$ be a small positive number, and denote by $K^{\prime}$ the CST comprising $[\nu, 1-\nu]$ and all singletons $\{x\}$ where $x \in \mathcal{X} \backslash[\nu, 1-\nu]$. Notice that $K^{\prime}$ is a finer partition than $K$. The game $G(K)$ has a unique subgame perfect equilibrium outcome: $(1 / 2,1 / 2)$. However, the game $G\left(K^{\prime}\right)$ has three subgame perfect equilibrium outcomes, namely, $(1 / 2,1 / 2),(0,0)$, and $(1,1)$. As $U(0)=U(1)<U(1 / 2), K^{\prime}$ is worse than $K$. We show in Appendix E that Theorems 1 and 2 yield a method for checking whether a CST can be refined by some worse CST.

## 8 Conclusion

The Stackelberg leadership model assumes that the leader can commit to any action she might choose. Our paper takes a different view: we only assume that the leader can commit not to take certain subsets of actions.

We provide a tractable model of commitment that encompasses the Stackelberg and Cournot models as special cases but also enables us to capture situations of limited commitment. We characterize the set of outcomes resulting from all possible commitment structures, and shed light thereby on the "limits of commitment". Our results highlight that, more than

[^17]commitment, what matters is the precise form that commitment takes. For instance, we show that whereas the Stackelberg and Cournot payoffs provide the bounds of the payoffs attainable by the leader under some appropriately defined class of "simple" commitment structures, this property fails to hold more generally.

Lastly, our results make it possible to study new information design problems in leaderfollower games, where a mediator communicates to the follower information about the action of the leader.

## A Appendix of Section 4

Proof of Lemma 2: We prove the only if part of the lemma; the proof of the other part is similar. Suppose that $(K, \beta)$ constitutes an admissible pair. Reason by contradiction, and suppose that we can find $\mathcal{X}_{i} \in K$ such that $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)<\beta\left(\mathcal{X}_{i}\right)$ while $\beta\left(\mathcal{X}_{i}\right) \neq \min \mathcal{X}_{i}$. The function $\eta\left(\cdot, \beta\left(\mathcal{X}_{i}\right)\right)$ is strictly quasi-concave, maximized at $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)$, and satisfies $\eta\left(\beta\left(\mathcal{X}_{i}\right), \beta\left(\mathcal{X}_{i}\right)\right)=0$. So $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right)>0$ for all $x \in\left[\phi\left(\beta\left(\mathcal{X}_{i}\right)\right), \beta\left(\mathcal{X}_{i}\right)\right)$. Since $\mathcal{X}_{i}$ is an interval, $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$, and $\beta\left(\mathcal{X}_{i}\right) \neq \min \mathcal{X}_{i}$, we can find $\varepsilon>0$ such that $\left(\beta\left(\mathcal{X}_{i}\right)-\varepsilon, \beta\left(\mathcal{X}_{i}\right)\right) \subset \mathcal{X}_{i}$. Coupling the previous remarks shows the existence of $x \in \mathcal{X}_{i}$ such that $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right)>0$; this contradicts the assumption that $(K, \beta)$ is admissible. Hence, $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)<\beta\left(\mathcal{X}_{i}\right)$ implies $\beta\left(\mathcal{X}_{i}\right)=\min \mathcal{X}$. Analogous arguments show that $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)>\beta\left(\mathcal{X}_{i}\right)$ implies $\beta\left(\mathcal{X}_{i}\right)=\max \mathcal{X}_{i}$.

Proof of Theorem 1: The if part of the theorem was proven in the text; we prove here the converse. In the rest of the appendix, the upper contour set of $x$ with respect to $U$ will be denoted by $\mathcal{Q}_{\geq}(x)$, that is,

$$
\mathcal{Q}_{\geq}(x):=\{\tilde{x}: U(\tilde{x}) \geq U(x)\} .
$$

The sets $\mathcal{Q}_{<}(x), \mathcal{Q}_{\leq}(x)$, and $\mathcal{Q}_{>}(x)$ are similarly defined.
Pick an arbitrary simply-plausible action $x^{*}$. We aim to prove the existence of a Cournot action $x_{n^{*}}^{C} \in \mathcal{Q}_{\leq}\left(x^{*}\right)$ such that (2) holds. If $\phi\left(x^{*}\right)=x^{*}$, just take $x_{n^{*}}^{C}=x^{*}$; we treat below the case in which $\phi\left(x^{*}\right)>x^{*}$ (the remaining case is analogous). Reason by contradiction, and suppose that

$$
\begin{equation*}
\mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right] \cap \mathcal{Q}_{\leq}\left(x^{*}\right)=\varnothing . \tag{5}
\end{equation*}
$$

Let $(K, \beta)$ be an admissible pair that implements $x^{*}$, with $K$ a simple CST. We will show that $K$ cannot be finite. By Berge's maximum theorem, both $R_{F}$ and $R_{L}$ are continuous, thus $\phi$ is continuous as well. As $\phi\left(x^{*}\right)>x^{*}$ and $\phi(\bar{x}) \leq \bar{x}$, the intermediate value theorem shows that

$$
\mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right] \neq \varnothing
$$

Note that the continuity of the function $\phi$ implies the compactness of $\mathcal{X}^{C}$. So $\mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right]=$ $\mathcal{X}^{C} \cap\left[x^{*}, \bar{x}\right]$ possesses a smallest element, that we denote by $x_{1}^{C}$. Let $\mathcal{X}_{1}$ be the member of $K$
containing $x_{1}^{C}$. Then Lemma 2 combined with (5) gives

$$
\beta\left(\mathcal{X}_{1}\right) \in\left(x_{1}^{C}, \bar{x}\right] \cap\{x: \phi(x)>x\} .
$$

Now let $x_{2}^{C}$ be the smallest Cournot action greater than $\beta\left(\mathcal{X}_{1}\right)$, and denote by $\mathcal{X}_{2}$ the member of $K$ containing $x_{2}^{C}$. The same logic as above gives $\beta\left(\mathcal{X}_{2}\right) \in\left(x_{2}^{C}, \bar{x}\right] \cap\{x: \phi(x)>x\}$, and so on. If $K$ were finite, the previous iteration would have to end after finitely many steps, say $m$. But then $\beta\left(\mathcal{X}_{m}\right)=\bar{x}$ and $\beta\left(\mathcal{X}_{m}\right) \in\{x: \phi(x)>x\}$, giving $\phi(\bar{x})>\bar{x}$. The previous contradiction proves that $K$ cannot be finite.

We proceed to show that $K$ cannot be infinite either. The function $U$ is continuous and, by (5), $U\left(x_{n}^{C}\right)>U\left(x^{*}\right)$ for all $x_{n}^{C} \in \mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right]$. Furthermore, as already pointed out above, $\mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right]$ is a compact set. Therefore,

$$
\begin{equation*}
\Delta:=\min _{x_{n}^{C} \in \mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right]} U\left(x_{n}^{C}\right)-U\left(x^{*}\right)>0 . \tag{6}
\end{equation*}
$$

Next, $U$ being continuous and $\mathcal{X}$ compact, the function $U$ is uniformly continuous on $\mathcal{X}$. We can thus find $\delta>0$ such that $\left|U\left(x^{\prime}\right)-U(x)\right|<\Delta$ whenever $\left|x^{\prime}-x\right|<\delta$. By (6), we thus have

$$
\begin{equation*}
U(x)>U\left(x^{*}\right), \text { for all } x \text { such that }\left|x-x_{n}^{C}\right|<\delta, x_{n}^{C} \in \mathcal{X}^{C} \cap\left(x^{*}, \bar{x}\right] . \tag{7}
\end{equation*}
$$

Now, since $(K, \beta)$ implements $x^{*}$, we must have $U\left(\beta\left(\mathcal{X}_{i}\right)\right) \leq U\left(x^{*}\right)$ for all $\mathcal{X}_{i} \in K$. So (7) shows that each member of the sequence $\mathcal{X}_{1}, \mathcal{X}_{2}, \ldots$ defined in the first part of the proof must have a length $\delta$ or more. This in turn implies that said sequence can have no more than $\frac{\bar{x}-x^{*}}{\delta}$ terms. Yet we showed previously that this sequence cannot be finite. This contradiction completes the proof of the theorem.

## B Appendix of Subsection 5.1

Proof of Theorem 2: We prove here the only if part of the theorem; the rest was proven in the text. Pick an arbitrary action $x^{*}$ of the leader. Suppose that $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \leq x\}=\varnothing$. Applying Lemma 2 shows that any admissible pair $(K, \beta)$ in which the CST $K$ contains only
intervals must be such that $\beta\left(\mathcal{X}_{i}\right) \in\{x: \phi(x) \leq x\}$ for every $\mathcal{X}_{i} \in K$ containing $\bar{x}$. This, in turn, implies that every I-plausible action belongs to $\mathcal{Q}_{>}\left(x^{*}\right)$, whence $x^{*}$ cannot be I-plausible. A similar argument shows that $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \geq x\}=\varnothing$ implies that $x^{*}$ is not I-plausible. Next, suppose that $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \leq x\}$ and $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \geq x\}$ are non-empty. Both $\phi$ and $U$ being continuous, the min and max of (3) are in this case well defined (since $\mathcal{X}$ is a compact set). Suppose that max $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \geq x\}<\min \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \leq x\}$, and pick

$$
\begin{equation*}
x^{\dagger} \in\left(\max \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \geq x\}, \min \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \leq x\}\right) \tag{8}
\end{equation*}
$$

Applying Lemma 2 shows that any admissible pair ( $K, \beta$ ) comprising an interval CST must be such that, for every $\mathcal{X}_{i} \in K$ containing $x^{\dagger}$, either (i) $\beta\left(\mathcal{X}_{i}\right) \in\left\{x \geq x^{\dagger}: \phi(x) \geq x\right\}$ or (ii) $\beta\left(\mathcal{X}_{i}\right) \in\left\{x \leq x^{\dagger}: \phi(x) \leq x\right\}$. So (8) gives $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{Q}_{>}\left(x^{*}\right)$. It ensues that $x^{*}$ cannot be I-plausible.

Proof of Proposition 1: By Corollary 2, an action that belongs to the upper contour set of some Cournot action is I-plausible. Below we show that the converse is true too if $U$ is either quasi-convex or quasi-concave.

Suppose that $U$ is quasi-convex, and consider an action $x^{*}$ in the strict lower contour set of every Cournot action. Then $\mathcal{Q}_{\leq}\left(x^{*}\right)$ is a convex set, and $\phi(x) \neq x$ for all $x \in \mathcal{Q}_{\leq}\left(x^{*}\right)$. The intermediate value theorem shows that either $x<\phi(x)$ for all $x \in \mathcal{Q}_{\leq}\left(x^{*}\right)$, or $x>\phi(x)$ for all $x \in \mathcal{Q}_{\leq}\left(x^{*}\right)$. Either way, Theorem 2 shows that $x^{*}$ cannot be I-plausible.

Next, suppose that $U$ is quasi-concave, and consider an action $x^{*}$ in the strict lower contour set of every Cournot action. Then $\mathcal{Q}_{>}\left(x^{*}\right)$ is a convex set, and $\phi(x) \neq x$ for all $x \in \mathcal{Q}_{\leq}\left(x^{*}\right)$. This implies that, given $x \in \mathcal{Q}_{\leq}\left(x^{*}\right)$, either (i) $\phi(x)>x$ and $x<x_{n}^{C}$ for all $x_{n}^{C} \in \mathcal{X}^{C}$, or (ii) $\phi(x)<x$ and $x>x_{n}^{C}$ for all $x_{n}^{C} \in \mathcal{X}^{C}$. We conclude using Theorem 2 that $x^{*}$ is not I-plausible.

## C Appendix of Subsection 5.2

Proof of Proposition 2; The if part of the proposition is evident. We prove here the only if part. Let $\left(x^{*}, y^{*}\right)$ be plausible, and $(K, \beta)$ be an admissible pair that implements $\left(x^{*}, y^{*}\right)$. Let $B$ denote the image of the function $\beta$, that is, $B:=\beta(K)$. Then, for each action $x \in \mathcal{X} \backslash B$, let

$$
B_{x}:=\left\{\beta\left(\mathcal{X}_{i}\right) \mid x \in \mathcal{X}_{i}\right\} .
$$

Since $(K, \beta)$ is admissible, note that

$$
\begin{equation*}
\tilde{x} \in B_{x} \Rightarrow \eta(\tilde{x}, x) \leq 0 \tag{9}
\end{equation*}
$$

Next, by the axiom of choice, there exists a function $\rho: \mathcal{X} \backslash B \rightarrow B$. For each $x \in B$ let

$$
[x]:=\{x\} \cup \rho^{-1}(x)
$$

Finally, define

$$
K^{\prime}:=\{[x] \mid x \in B\}
$$

and $\beta^{\prime}: K^{\prime} \rightarrow \mathcal{X}$ such that $\beta^{\prime}([x])=x$, for all $x \in B$. By (9), the pair $\left(K^{\prime}, \beta^{\prime}\right)$ is admissible. Moreover, it implements $\left(x^{*}, y^{*}\right)$ since $\beta^{\prime}\left(K^{\prime}\right)=B=\beta(K)$.

Lemma C.1. Suppose (RC1)-(RC3) hold. If $u_{2} u_{12}>0$, then $U$ is increasing over $\left[\underline{x}, x^{C}\right]$. If $u_{2} u_{12}<0$, then $U$ is decreasing over $\left[x^{C}, \bar{x}\right]$.

Proof: We show the proof for the case in which $u_{2}>0$ and $u_{12}>0$; the other cases are similar ${ }^{31}$ Pick an arbitrary $x<x^{C}$, and $\varepsilon>0$ sufficiently small that $u\left(x+\varepsilon, R_{F}(x)\right)>$ $u\left(x, R_{F}(x)\right){ }^{32}$ Then, $R_{F}$ being non-decreasing (since $v_{12}>0$ ) and $u_{2}>0$ :

$$
U(x+\varepsilon)=u\left(x+\varepsilon, R_{F}(x+\varepsilon)\right) \geq u\left(x+\varepsilon, R_{F}(x)\right)>u\left(x, R_{F}(x)\right)=U(x)
$$

[^18]Lemma C.2. Suppose ( $R C 1$ )-( $R C 3$ ) hold. Then

$$
\begin{equation*}
\mathcal{S}=\left\{x: \eta\left(x^{C}, x\right) \leq 0\right\} \cap\left\{x: u\left(x^{C}, R_{F}(x)\right) \leq U\left(x^{C}\right)\right\} . \tag{10}
\end{equation*}
$$

Proof: We show the proof of the lemma for the case $u_{2}>0$ and $u_{12}>0$ (the other cases are similar). Recall that in this case $\mathcal{S}:=\left\{x: x \leq \gamma(x) \leq x^{C}\right\}$.

The function $R_{F}$ being in this case non-decreasing (and, indeed, increasing in a neighborhood of $x^{C}$ since $\left.y^{C} \in \operatorname{int}(\mathcal{Y})\right)$ and $u_{2}>0$, notice that

$$
u\left(x^{C}, R_{F}(x)\right)>u\left(x^{C}, R_{F}\left(x^{C}\right)\right)=U\left(x^{C}\right), \text { for all } x>x^{C} .
$$

So $u\left(x^{C}, R_{F}(x)\right) \leq U\left(x^{C}\right)$ implies $x \leq x^{C}$. Now consider $x \leq x^{C}$ such that $\eta\left(x^{C}, x\right) \leq 0$. We will show that $x \in \mathcal{S}$. If $x=x^{C}$ the previous claim is immediate, so pick $x<x^{C}$. The function $\eta(\cdot, x)$ is strictly quasi-concave, and maximized at $\phi(x)>x{ }^{33}$ As $\eta(x, x)=0 \geq \eta\left(x^{C}, x\right)$, we see by definition of $\gamma(x)$ that $x<\gamma(x) \leq x^{C}$. The right-hand side of 10 is thus contained in the set $\mathcal{S}$. The proof of the reverse inclusion is analogous.

Lemma C.3. Suppose ( $R C 1$ ) $-(R C 3)$ hold, and $\mathcal{S}=\left\{x^{C}\right\}$. Then all plausible actions belong to the upper contour set of $x^{C}$ with respect to $U$.

Proof: Reason by contradiction, and suppose that some action $x^{*} \in \mathcal{Q}_{<}\left(x^{C}\right)$ is plausible. Let $(K, \beta)$ be an admissible pair that implements $x^{*}$. Choose an element $\mathcal{X}_{i}$ of the CST $K$ such that $x^{C} \in \mathcal{X}_{i}$. Using Lemma 1 yields $\beta\left(\mathcal{X}_{i}\right) \in\left\{x: \eta\left(x^{C}, x\right) \leq 0\right\} \cap \mathcal{Q}_{\leq}\left(x^{*}\right)$, and, since $x^{*} \in \mathcal{Q}_{<}\left(x^{C}\right)$,

$$
\begin{equation*}
\beta\left(\mathcal{X}_{i}\right) \in\left\{x: \eta\left(x^{C}, x\right) \leq 0\right\} \cap \mathcal{Q}_{<}\left(x^{C}\right) \tag{12}
\end{equation*}
$$

In turn, (12) yields

$$
\begin{equation*}
u\left(x^{C}, R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right) \leq u\left(\beta\left(\mathcal{X}_{i}\right), R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right)=U\left(\beta\left(\mathcal{X}_{i}\right)\right)<U\left(x^{C}\right) \tag{13}
\end{equation*}
$$

[^19]Coupling (12) and (13) gives

$$
\beta\left(\mathcal{X}_{i}\right) \in\left\{x: \eta\left(x^{C}, x\right) \leq 0\right\} \cap\left\{x: u\left(x^{C}, R_{F}(x)\right)<U\left(x^{C}\right)\right\} .
$$

Applying Lemma C.2, we obtain $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{S} \backslash\left\{x^{C}\right\}$, contradicting $\mathcal{S}=\left\{x^{C}\right\}$.
Lemma C.4. Suppose (RC1)-(RC3) hold. Assume $u_{12}>0$ and $u_{2}>0$. Consider an admissible pair $(K, \beta)$ which implements some action $x^{*}$. Then, if $x \in \mathcal{Q}_{>}\left(x^{*}\right)$, we have $\beta\left(\mathcal{X}_{i}\right)<x$ for every $\mathcal{X}_{i} \in K$ which contains $x$.

Proof: Let $x \in \mathcal{Q}_{>}\left(x^{*}\right)$, and pick an arbitrary $\mathcal{X}_{i} \in K$ containing $x$. Reason by contradiction, and suppose that $\beta\left(\mathcal{X}_{i}\right) \geq x$. Then, $R_{F}$ being non-decreasing (since $v_{12}>0$ ) and $u_{2}>0$, we obtain

$$
\begin{equation*}
u\left(x, R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right) \geq u\left(x, R_{F}(x)\right)>u\left(x^{*}, R_{F}\left(x^{*}\right)\right) \tag{14}
\end{equation*}
$$

Since $(K, \beta)$ is admissible, we also have

$$
\begin{equation*}
u\left(\beta\left(\mathcal{X}_{i}\right), R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right) \geq u\left(x, R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right) \tag{15}
\end{equation*}
$$

Coupling (14) and (15) yields

$$
u\left(\beta\left(\mathcal{X}_{i}\right), R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)\right)>u\left(x^{*}, R_{F}\left(x^{*}\right)\right)
$$

The previous inequality contradicts the assumption that $(K, \beta)$ implements $x^{*}$.
Lemma C.5. Suppose (RC1) holds. Let $(K, \beta)$ be an admissible pair. If $\beta\left(\mathcal{X}_{i}\right)<\min \left\{x^{C}, x\right\}$ for some $\mathcal{X}_{i} \in K$ which contains $x$, then $\gamma\left(\beta\left(\mathcal{X}_{i}\right)\right) \in\left(\beta\left(\mathcal{X}_{i}\right), x\right]$.

Proof: Pick $x \in \mathcal{X}$, and $\mathcal{X}_{i} \in K$ containing $x$. Since $(K, \beta)$ is admissible:

$$
\begin{equation*}
\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right) \leq 0 . \tag{16}
\end{equation*}
$$

Now suppose that $\beta\left(\mathcal{X}_{i}\right)<\min \left\{x^{C}, x\right\}$. In this case, the strictly concave function $\eta\left(\cdot, \beta\left(\mathcal{X}_{i}\right)\right)$ attains (by virtue of (11)) a maximum at $\phi\left(\beta\left(\mathcal{X}_{i}\right)\right)>\beta\left(\mathcal{X}_{i}\right)$. From (16) and the fact that $\beta\left(\mathcal{X}_{i}\right)<x$ we obtain (by definition of $\gamma$ ) $\beta\left(\mathcal{X}_{i}\right)<\gamma\left(\beta\left(\mathcal{X}_{i}\right)\right) \leq x$.

Proof of Theorem 3: Start with the case $\mathcal{S}=\left\{x^{C}\right\}$. Combining Corollary 1, Proposition 2, and Lemma C. 3 shows that the set of simply-plausible actions, the set of I-plausible actions, the set of P-plausible actions, and the set of plausible actions all coincide with the upper contour set of $x^{C}$.

The remainder of the proof deals with the case $\mathcal{S} \supsetneq\left\{x^{C}\right\}$. Below, assume $u_{12}>0$ and $u_{2}>0$ (the other cases are analogous). Recall that in this case $\mathcal{S}:=\left\{x: x \leq \gamma(x) \leq x^{C}\right\}$. The function $\gamma$ being continuous, $\mathcal{S}$ is a compact set. By Lemma C.1, we can thus find $\hat{x} \in \mathcal{S}$ with $\hat{x}<x^{C}$ and

$$
\begin{equation*}
U(\gamma(\hat{x}))=\min _{x \in \mathcal{S}} U(\gamma(x)) \tag{17}
\end{equation*}
$$

To shorten notation, let $\hat{\gamma}:=\gamma(\hat{x})$; as $\hat{x}<x^{C}$, note that, by definition of $\gamma$,

$$
\begin{equation*}
\hat{x}<\hat{\gamma} \leq x^{C} \tag{18}
\end{equation*}
$$

We proceed to show that (a) all actions in $\mathcal{Q}_{\geq}(\hat{\gamma})$ are P-plausible, and (b) any plausible action belongs to $\mathcal{Q}_{\geq}(\hat{\gamma})$.

All actions in $\mathcal{Q}_{\geq}(\hat{\gamma})$ are P-plausible. We know by Corollary 1 that all actions in $\mathcal{Q}_{\geq}\left(x^{C}\right)$ are simply plausible. So pick an action $x^{*} \in \mathcal{Q}_{\geq}(\hat{\gamma}) \backslash \mathcal{Q}_{\geq}\left(x^{C}\right)$ (if there exists none, we are done). Define

$$
\mathcal{X}_{1}:=\{\hat{x}\} \cup \mathcal{Q}_{>}\left(x^{*}\right),
$$

and let $K$ denote the partition of $\mathcal{X}$ made up of $\mathcal{X}_{1}$, and only singletons besides $\mathcal{X}_{1}$. Lastly, let $\beta: K \rightarrow \mathcal{X}$ be given by $\beta\left(\mathcal{X}_{1}\right)=\hat{x}$ and $\beta(\{x\})=x$ for all $x \in \mathcal{X} \backslash \mathcal{X}_{1}$. We now show that $(K, \beta)$ constitutes an admissible pair; notice that this amounts to showing that

$$
\begin{equation*}
\eta(\tilde{x}, \hat{x}) \leq 0, \quad \text { for all } \tilde{x} \in \mathcal{X}_{1} \tag{19}
\end{equation*}
$$

As $x^{*} \in \mathcal{Q}_{\geq}(\hat{\gamma})$, any $\tilde{x} \in \mathcal{Q}_{>}\left(x^{*}\right)$ belongs to $\mathcal{Q}_{\geq}(\hat{\gamma})$. On the other hand, since $\hat{\gamma} \leq x^{C}$ (see (18)), Lemma C. 1 shows that every $\tilde{x} \in \mathcal{Q}_{>}\left(x^{*}\right)$ satisfies $\tilde{x} \geq \hat{\gamma}$. Now, the function $\eta(\cdot, \hat{x})$ is strictly quasi-concave, with $\eta(\hat{x}, \hat{x})=\eta(\hat{\gamma}, \hat{x})=0$; it thus follows from (18) that $\eta(\tilde{x}, \hat{x}) \leq 0$ for all $\tilde{x} \geq \hat{\gamma}$. Combining the previous observations establishes (19); so (K, $\beta$ ) is admissible.

Finally, coupling (18) and Lemma C.1 yields $U(\hat{\gamma})>U(\hat{x})$, giving in turn $U\left(x^{*}\right)>U(\hat{x})=$ $U\left(\beta\left(\mathcal{X}_{1}\right)\right)$ (since $\left.x^{*} \in \mathcal{Q}_{\geq}(\hat{\gamma})\right)$. We conclude that $(K, \beta)$ implements $x^{*}$, since $\mathcal{X} \backslash \mathcal{X}_{1} \subset \mathcal{Q}_{\leq}\left(x^{*}\right)$.

All plausible actions belong to $\mathcal{Q}_{\geq}(\hat{\gamma})$. Reason by contradiction, and suppose that some plausible action $x^{*}$ belongs to $\mathcal{Q}_{<}(\hat{\gamma})$. Combining (18), Lemma C.1, and the fact that $U$ is continuous shows that we can find an action, say $x^{\dagger}$, such that:

$$
\begin{equation*}
x^{\dagger}<\hat{\gamma} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
x^{\dagger} \in \mathcal{Q}_{>}\left(x^{*}\right) \cap \mathcal{Q}_{<}(\hat{\gamma}) . \tag{21}
\end{equation*}
$$

Now consider a pair $(K, \beta)$ which implements $x^{*}$, and $\mathcal{X}_{i}$ an element of the CST $K$ containing $x^{\dagger}$. By virtue of (21), applying Lemma C. 4 shows that

$$
\begin{equation*}
\beta\left(\mathcal{X}_{i}\right)<x^{\dagger} . \tag{22}
\end{equation*}
$$

On the other hand, (18) and (20) show that

$$
x^{\dagger}<\hat{\gamma} \leq x^{C}
$$

Hence, Lemma C. 5 gives

$$
\begin{equation*}
\beta\left(\mathcal{X}_{i}\right)<\gamma\left(\beta\left(\mathcal{X}_{i}\right)\right) \leq x^{\dagger}<\hat{\gamma} \leq x^{C} \tag{23}
\end{equation*}
$$

We thus obtain, firstly,

$$
\begin{equation*}
\beta\left(\mathcal{X}_{i}\right) \in \mathcal{S}, \tag{24}
\end{equation*}
$$

and, secondly (using Lemma C.1),

$$
\begin{equation*}
U\left(\gamma\left(\beta\left(\mathcal{X}_{i}\right)\right)\right)<U(\hat{\gamma}) \tag{25}
\end{equation*}
$$

The combination of (24) and (25) contradicts (17). Therefore, every plausible action must belong to $\mathcal{Q}_{\geq}(\hat{\gamma})$.

Proof of Proposition 3: By definition of $\gamma: \eta(\gamma(x), x)=0$ for all $x$ in some neighborhood $O$ of $x^{C}$. We thus have

$$
u\left(\gamma(x), R_{F}(x)\right)=u\left(x, R_{F}(x)\right), \quad \forall x \in O
$$

Differentiating the previous expression with respect to $x$ yields

$$
u_{1}\left(\gamma(x), R_{F}(x)\right) \gamma^{\prime}(x)+u_{2}\left(\gamma(x), R_{F}(x)\right) R_{F}^{\prime}(x)=u_{1}\left(x, R_{F}(x)\right)+u_{2}\left(x, R_{F}(x)\right) R_{F}^{\prime}(x),
$$

and, therefore,

$$
\begin{equation*}
\gamma^{\prime}(x)=\frac{u_{1}\left(x, R_{F}(x)\right)+R_{F}^{\prime}(x)\left[u_{2}\left(x, R_{F}(x)\right)-u_{2}\left(\gamma(x), R_{F}(x)\right)\right]}{u_{1}\left(\gamma(x), R_{F}(x)\right)}, \quad \forall x \in O \backslash\left\{x^{C}\right\} . \tag{26}
\end{equation*}
$$

The numerator and denominator on the right-hand side of 26 tend to 0 as $x \rightarrow x^{C}$. Then, by virtue of L'Hospital's rule and using the fact that $\gamma(x) \rightarrow x^{C}$ as $x \rightarrow x^{C}$ :

$$
\begin{equation*}
\lim _{x \rightarrow x^{C}} \gamma^{\prime}(x)=\lim _{x \rightarrow x^{C}} \frac{u_{11}\left(x, R_{F}(x)\right)+2 u_{12}\left(x, R_{F}(x)\right) R_{F}^{\prime}(x)-u_{12}\left(x, R_{F}(x)\right) R_{F}^{\prime}(x) \gamma^{\prime}(x)}{u_{11}\left(\gamma(x), R_{F}(x)\right) \gamma^{\prime}(x)+u_{12}\left(\gamma(x), R_{F}(x)\right) R_{F}^{\prime}(x)} . \tag{27}
\end{equation*}
$$

On the other hand, in a neighborhood of $y=y^{C}$ :

$$
R_{L}^{\prime}(y)=\frac{-u_{12}\left(R_{L}(y), y\right)}{u_{11}\left(R_{L}(y), y\right)} .
$$

Therefore,

$$
\begin{equation*}
R_{L}^{\prime}\left(y^{C}\right)=\frac{-u_{12}\left(x^{C}, y^{C}\right)}{u_{11}\left(x^{C}, y^{C}\right)}=\lim _{x \rightarrow x^{C}} \frac{-u_{12}\left(x, R_{F}(x)\right)}{u_{11}\left(x, R_{F}(x)\right)}=\lim _{x \rightarrow x^{C}} \frac{-u_{12}\left(\gamma(x), R_{F}(x)\right)}{u_{11}\left(\gamma(x), R_{F}(x)\right)} . \tag{28}
\end{equation*}
$$

Combining (28) with (27) gives

$$
\gamma^{\prime}\left(x^{C}\right)=\frac{1-2 R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)+R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right) \gamma^{\prime}\left(x^{C}\right)}{\gamma^{\prime}\left(x^{C}\right)-R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)} .
$$

So $\gamma^{\prime}\left(x^{C}\right)$ is a solution of

$$
Z(Z-2 \alpha)=1-2 \alpha
$$

where $\alpha:=R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)$. So either $\gamma^{\prime}\left(x^{C}\right)=1$ or $\gamma^{\prime}\left(x^{C}\right)=2 \alpha-1$, whence $\gamma^{\prime}\left(x^{C}\right)>0$ if $R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)>1 / 2$.

Now suppose that $u_{12} u_{2}>0$ (the other case is similar), so that $\mathcal{S}=\left\{x: x \leq \gamma(x) \leq x^{C}\right\}$. If $R_{L}^{\prime}\left(y^{C}\right) R_{F}^{\prime}\left(x^{C}\right)>1 / 2$, then $\gamma^{\prime}\left(x^{C}\right)>0$. This in turn implies the existence of $x<x^{C}$ such that $x<\gamma(x)<x^{C}$. Such an $x$ belongs to $\mathcal{S}$, so Lemma C.1 enables us to conclude that $\underline{U}<U\left(x^{C}\right)$.

## D Appendix of Section 6

All the results in this appendix refer to the duopoly example presented in Subsection 2.3. Subsection D. 1 characterizes the sets of plausible quantities. Subsection D. 2 proves Proposition 5.

In this appendix, we denote the set of all CSTs by $\mathcal{K}$. We use $\mathcal{K}^{I}$ to denote the set of all CSTs consisting only of intervals and $\mathcal{K}^{P}$ to denote the set of all CSTs that are partitions. Finally, the set of all simple CSTs (i.e., interval partitions) is denoted by $\mathcal{K}^{I P}$. For each of these four classes of CSTs, $\mathcal{K}^{z}$, we denote all plausible leader's actions as $\mathcal{X}^{\mathcal{K}^{z}}$. Whenever this set has a minimum (respectively, a maximum) we denote it $\underline{x}^{\mathcal{K}^{z}}$, (resp. $\left.\bar{x}^{\mathcal{K} z}\right)$. For example, $\underline{x}^{\mathcal{K}^{I P}}$ denotes the smallest simply-plausible quantity, and $\bar{x}^{\mathcal{K}}$ denotes the largest plausible quantity.

We define the following functions:

$$
\begin{aligned}
r^{*}(d) & :=2-\sqrt{2}(1-d) ; \\
r^{* *}(d) & :=2-\left(\sqrt[3]{\frac{\sqrt{57}}{9}+1}\right)(1-d)-\frac{2(1-d)}{3 \sqrt[3]{\frac{\sqrt{57}}{9}+1}} ; \\
r^{* * *}(d) & :=\frac{1}{2}(3-\sqrt{5}+(1+\sqrt{5}) d) ; \\
r^{\dagger}(d) & :=2-\left(\frac{\sqrt[3]{3(9-\sqrt{78})}}{3}+\frac{1}{\sqrt[3]{3(9-\sqrt{78})}}\right)(1-d) \\
r^{\dagger \dagger}(d) & :=2-\sqrt{3}(1-d) ; \\
r^{\dagger \dagger}(d) & :=2+\left(\frac{1-\sqrt[3]{80-9 \sqrt{79}}}{3}-\frac{1}{3 \sqrt[3]{80-9 \sqrt{79}}}\right)(1-d) .
\end{aligned}
$$

A firm acting as a monopolist would choose quantity $x^{M}:=\frac{1}{2-r}$.

## D. 1 Plausible Quantities

The unique best response of the follower to $x$, and the leader payoff from $x$ when the follower best-responds to $x$ are given, respectively, by

$$
R_{F}(x)=\left\{\begin{array}{ll}
\frac{1-(1-d) x}{2-r} & \text { if } x \leq \frac{1}{1-d}, \\
0 & \text { if } x>\frac{1}{1-d},
\end{array} \text { and } U(x)= \begin{cases}\frac{2(1-r+d) x-\left((2-r)^{2}-2(1-d)^{2}\right) x^{2}}{2(2-r)} & \text { if } x \leq \frac{1}{1-r} \\
x-\left(1-\frac{r}{2}\right) x^{2} & \text { if } x>\frac{1}{1-d}\end{cases}\right.
$$

Function $\phi$ takes the form:

$$
\phi(x)= \begin{cases}0 & \text { if } x \leq \frac{r-(d+1)}{(1-d)^{2}} \\ \frac{d+1-r+(1-d)^{2} x}{(2-r)^{2}} & \text { if } \frac{r-(d+1)}{(1-d)^{2}}<x<\frac{1}{1-d} \\ x^{M} & \text { if } x \geq \frac{1}{1-d}\end{cases}
$$

We characterize next the Cournot and the Stackelberg quantities.
Lemma D.1. The set of Cournot quantities is as follows:

$$
\mathcal{X}^{C}= \begin{cases}\left\{\frac{1}{3-r-d}\right\} & \text { if } r<d+1 \\ {\left[0, x^{M}\right]} & \text { if } r=d+1 \\ \left\{0, \frac{1}{3-r-d}, x^{M}\right\} & \text { if } r>d+1\end{cases}
$$

## Proof:

(i) If $r<d+1$, then

$$
\frac{r-(d+1)}{(1-d)^{2}}<0 \text { and } \frac{1}{1-d}>\frac{2}{2-r}
$$

hence $\mathcal{X}^{C}=\left\{x^{*}\right\}$ where $x=x^{*}$ solves

$$
\begin{equation*}
\frac{d+1-r+(1-d)^{2} x}{(2-r)^{2}}=x \tag{29}
\end{equation*}
$$

(ii) If $r=d+1$, then $\phi(x)=x \Longleftrightarrow x \leq \frac{1}{1-d}$, and $x^{M}=\frac{1}{1-d}$.
(iii) If $r>d+1$, then

$$
\frac{2}{2-r}>\frac{1}{1-d}>\frac{r-(d+1)}{(1-d)^{2}}>0
$$

hence set $\mathcal{X}^{C}$ includes only $0, x^{M}$, and the solution to (29).

In this appendix, $x_{1}^{C}=0, x_{2}^{C}=\frac{1}{3-r-d}, x_{3}^{C}=x^{M}$ and $x^{C}=x_{2}^{C}$.
Lemma D.2. The Stackelberg quantity, denoted $x^{S}$, is as follows:

$$
x^{S}= \begin{cases}\frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}} & \text { if } r<r^{* * *}(d) \\ \frac{1}{1-d} & \text { if } r^{* * *}(d) \leq r \leq d+1 \\ x^{M} & \text { if } r>d+1\end{cases}
$$

Proof: If $r \leq d+1$, then $U^{\prime}(x)<0$ for any $x>\frac{1}{1-d}$, hence $x^{S}=\left[0, \frac{1}{1-d}\right]$. Note that (i) $U$ is a quadratic function over this interval, (ii) $U^{\prime}(0)>0$, and (iii) $U^{\prime}\left(\frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}}\right)=0$. Thus,

$$
\arg \max _{x \in \mathcal{X}} U(x) \in\left\{\frac{1}{1-d}, \frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}}\right\}
$$

A few steps of algebra yield:

$$
U\left(\frac{1}{1-d}\right) \geq U\left(\frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}}\right) \Longleftrightarrow r \geq r^{* * *}(d)
$$

One can also check that: $r \in\left[0, r^{* * *}(d)\right] \Rightarrow \frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}} \in\left[0, \frac{1}{1-d}\right]$. Thus, $x^{S}=\frac{d+1-r}{(2-r)^{2}-2(1-d)^{2}}$ for $r<r^{* * *}(d)$ and $x^{S}=\frac{1}{1-d}$ for $r \in\left[r^{* * *}(d), d+1\right]$. Finally, if $r>d+1$ then $R_{F}\left(x^{M}\right)=0$ and therefore $\arg \max _{x \in \mathcal{X}} U(x)=x^{M}$.

Next, we characterize the sets of plausible quantities.

Lemma D.3. The set of simply-plausible quantities is as follows:

$$
\mathcal{X}^{\mathcal{K}^{I P}}= \begin{cases}{\left[x^{C}, \frac{(2-r)^{2}}{(-r-d+3)\left((2-r)^{2}-2(1-d)^{2}\right)}\right]} & \text { if } r<r^{* *}(d), \\ {\left[x^{C}, \frac{\sqrt{(1-d)(-2 r-d+5)}-r-d+3}{(2-r)(-r-d+3)}\right]} & \text { if } r^{* *}(d) \leq r<d+1, \\ \mathcal{X} & \text { if } r=d+1, \\ \left\{x_{1}^{C}\right\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, x_{2}^{C}\right] \cup\left[x_{3}^{C}, \frac{2}{2-r}\right] & \text { if } r>d+1 .\end{cases}
$$

## Proof:

(i) If $r<d+1$, then $\mathcal{X}^{C}=\left\{x^{C}\right\}$, hence Proposition 1 ensures $\mathcal{X}^{\mathcal{K}^{I P}}=\mathcal{Q}_{\geq}\left(x^{C}\right)$. For $r<d+1$, then (i) $U^{\prime}(x)<0$ for any $x \geq \frac{1}{1-d}$, and (ii) over the interval $\left[0, \frac{1}{1-d}\right]$, function $U$ is either non decreasing or concave, or both. Function $U$ is thus quasi-concave. As $U^{\prime}\left(x^{C}\right)>0$, then $\mathcal{Q}_{\geq}\left(x^{C}\right)=\left[x^{C}, \bar{x}^{\mathcal{K}^{I P}}\right]$, where $\bar{x}^{I^{I P}}$ satisfies $\bar{x}^{\mathcal{K}^{I P}}>x^{C}$ and $U\left(\bar{x}^{\mathcal{K}^{I P}}\right)=$ $U\left(x^{C}\right)$. It is easy to verify that $x^{C}<(1+d)^{-1}$, while $r>r^{* *}(d) \Longleftrightarrow \bar{x}^{K^{I P}}>(1+d)^{-1}$. A few steps of algebra thus yield the expressions for $\bar{x}^{\mathcal{K}^{I P}}$.
(ii) Lemma D. 1 ensures that if $r=d+1$, then $\mathcal{X}^{C}=\left[0, x^{M}\right]$. For all $x>x^{M}$, it is the case that $x>\phi(x)=x^{M}$. Theorem 1 thus ensures $\mathcal{X}^{\mathcal{K}^{I P}}=\mathcal{X}$.
(iii) If $r>d+1$, the characterization of the set $\mathcal{X}^{\mathcal{K}^{I P}}$ follows directly from Theorem 1 and properties of $\phi$. Note in particular that

- if $x^{*} \in\left\{x_{1}^{C}\right\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, x_{2}^{C}\right] \cup\left[x_{3}^{C}, \frac{2}{2-r}\right]$, then $\left(\phi\left(x^{*}\right)-x^{*}\right)\left(x_{1}^{C}-x^{*}\right) \geq 0$ hence $x^{*} \in \mathcal{X}^{\mathcal{K}^{I P}} ;$
- if instead $x^{*} \notin\left\{x_{1}^{C}\right\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, x_{2}^{C}\right] \cup\left[x_{3}^{C}, \frac{2}{2-r}\right]$, then $\left(\phi\left(x^{*}\right)-x^{*}\right)\left(x_{i}^{C}-x^{*}\right)<0$ for $i=1,2$ and 3 , hence $x^{*} \notin \mathcal{X}^{\mathcal{K}^{I P}}$.

Lemma D.4. The set of I-plausible quantities is as follows:

$$
\mathcal{X}^{\mathcal{K}^{I}}= \begin{cases}\mathcal{X}^{\mathcal{K}^{I P}} & \text { if } r \leq d+1 \\ \{0\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, \frac{2}{2-r}\right] & \text { if } r>d+1\end{cases}
$$

## Proof:

(i) If $r<d+1$, conditions (RC1)-(RC3) hold, hence $\mathcal{X}^{\mathcal{K}^{I}}=\mathcal{X}^{\mathcal{K}^{I P}}$ by Proposition 1 .
(ii) If $r=d+1$, then $\mathcal{X}^{\mathcal{K}^{I P}}=\mathcal{X}$ (Lemma D.3). As $\mathcal{X} \supseteq \mathcal{X}^{\mathcal{K}^{I}}$ and $\mathcal{X}^{\mathcal{K}^{I}} \supseteq \mathcal{X}^{\mathcal{K}^{I P}}$, then $\mathcal{X}^{\mathcal{K}^{I}}=\mathcal{X}^{\mathcal{K}^{I P}}$.
(iii) Suppose $r>d+1$. If $x^{*} \in\left(0, \frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}\right)$, then $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x: \phi(x) \geq x\}=\varnothing$; Theorem 2 ensures $x^{*} \notin \mathcal{X}^{\mathcal{K}^{I}}$. If instead $x^{*} \in\{0\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, \frac{2}{2-r}\right]$, then $x^{*} \in$ $\mathcal{Q}_{\geq}\left(x_{1}^{C}\right)$; Corollary 2 ensures $x^{*} \in \mathcal{X}^{\mathcal{K}^{I}}$.

Lemma D.5. The set of plausible quantities is as follows:

$$
\mathcal{X}^{\mathcal{K}}= \begin{cases}{\left[\frac{2(d+1-r)}{(2-r)^{2}}, \frac{(2-r)^{2}+\sqrt{(2-r)^{4}-8(1-d)^{2}(d+1-r)^{2}}}{(2-r)^{3}}\right]} & \text { if } r^{*}(d) \leq r<d+1 \\ \mathcal{X}^{\mathcal{K}^{I}} & \text { otherwise }\end{cases}
$$

## Proof:

(i) If $r<d+1$, conditions (RC1)-(RC3) hold, and Theorem 3 applies. In particular, if $r<r^{*}(d)$, then $\mathcal{S}=\left\{x^{C}\right\}$, and therefore $\mathcal{X}^{\mathcal{K}}=\mathcal{X}^{\mathcal{K}^{I P}}$, which in turn implies $\mathcal{X}^{\mathcal{K}}=\mathcal{X}^{\mathcal{K}^{I}}$. If instead $r \geq r^{*}(d)$, then $\mathcal{S}=\left[0, x^{C}\right]$. Note that $x^{C}<\frac{1}{1-d}$, hence

$$
\gamma\left(x^{*}\right)=\frac{2(1+d-r)-x(2-r)^{2}+2 x(1-d)^{2}}{(2-r)^{2}}, \text { for all } x^{*} \in\left[0, x^{C}\right]
$$

One can then verify that $0=\arg \min _{x \in\left[0, x^{C}\right]} U(\gamma(x))$, and $\gamma(0)=\frac{2(d+1-r)}{(2-r)^{2}}$. Solving the equation $U(x)=U(\gamma(0))$, and noting that $U$ is quasi-concave, yields

$$
\mathcal{X}^{\mathcal{K}}=\left[\gamma(0), \frac{(2-r)^{2}+\sqrt{(2-r)^{4}-8(1-d)^{2}(d+1-r)^{2}}}{(2-r)^{3}}\right] .
$$

(ii) If $r=d+1$, then $\mathcal{X}^{\mathcal{K}^{I}}=\mathcal{X}$ (see Lemma D.4. As $\mathcal{X}^{\mathcal{K}} \supseteq \mathcal{X}^{\mathcal{K}^{I}}$ and $\mathcal{X} \supseteq \mathcal{X}^{\mathcal{K}}$, we conclude that $\mathcal{X}^{\mathcal{K}^{I}}=\mathcal{X}^{\mathcal{K}}$.
(iii) If $r>d+1$, Lemma D. 4 ensures that $\mathcal{Q}_{\geq}(0)=\mathcal{X}^{\mathcal{K}^{I}}$. As $u(0, y)=0$ for any $y \in \mathcal{X}$, clearly $\mathcal{Q}_{<}(0) \notin \mathcal{X}^{\mathcal{K}}$; thus, $\mathcal{X}^{\mathcal{K}}=\mathcal{X}^{\mathcal{K}^{I}}$.

The following remark is easy to verify.
Remark D.1. If $r>d+1$, then $\mathcal{Q}_{\geq}(0)=\{0\} \cup\left[\frac{2(r-d-1)}{2(1-d)^{2}-(2-r)^{2}}, \frac{2}{2-r}\right]$. If instead $r \leq d+1$, then $\mathcal{Q}_{\geq}(0)=\mathcal{X}$.

By Proposition 2, the set of all plausible outcomes coincides with the set of P-plausible outcomes. We conclude with an immediate corollary of Lemma D. 5 that will prove useful in the next subsection.

Corollary D.1. The smallest and the largest plausible actions correspond to:
$\left\{\underline{x}^{\mathcal{K}}, \bar{x}^{\mathcal{K}}\right\}= \begin{cases}\left\{\underline{x}^{\mathcal{K}^{I P}}, \bar{x}^{\mathcal{K}^{I P}}\right\}=\left\{0, \frac{2}{2-r}\right\} & \text { if } r \geq d+1, \\ \left\{\underline{x}^{\mathcal{K}^{P}}, \bar{x}^{\mathcal{K}^{P}}\right\}=\left\{\frac{2(d+1-r)}{(2-r)^{2}}, \frac{(2-r)^{2}+\sqrt{(2-r)^{4}-8(1-d)^{2}(d+1-r)^{2}}}{(2-r)^{3}}\right\} & \text { if } r^{*}(d)<r<d+1, \\ \left\{\underline{x}^{\mathcal{K}^{I P}}, \bar{x}^{\mathcal{K}^{I P}}\right\}=\left\{x^{C}, \frac{\sqrt{(1-d)(-2 r-d+5)}-r-d+3}{(2-r)(-r-d+3)}\right\} & \text { if } r^{* *}(d)<r<r^{*}(d), \\ \left\{\underline{x}^{\mathcal{K}^{I P}}, \bar{x}^{\mathcal{K}^{I P}}\right\}=\left\{x^{C}, \frac{(2-r)^{2}}{\left((-r-d+3)\left((2-r)^{2}-2(1-d)^{2}\right)\right.}\right\} & \text { if } r \leq r^{* *}(d) .\end{cases}$

## D. 2 The Designer Problem

We prove each of the three parts of Proposition 5 separately. To prove the first part, we need the next two lemmata, where we characterize the solution the following problems

$$
\begin{equation*}
\max x+y \quad \text { s.t. }(x, y) \text { is plausible, } \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\min x y \quad \text { s.t. }(x, y) \text { is plausible. } \tag{31}
\end{equation*}
$$

Lemma D.6. The unique solution of (30) is $\left(\bar{x}^{\mathcal{K}}, R_{F}\left(\bar{x}^{\mathcal{K}}\right)\right)$.
Proof: Outcome $(x, y)$ is plausible only if $y=R_{F}(x)$, and

$$
x+R_{F}(x)= \begin{cases}\frac{1+(1+d-r) x}{2-r}, & \text { if } x<\frac{1}{1-d} \\ x, & \text { if } x \geq \frac{1}{1-d}\end{cases}
$$

If $r \leq d+1$, then $x+R_{F}(x)$ is non-decreasing in $x$, and therefore $\bar{x}^{\mathcal{K}} \in \arg \max _{x \in \mathcal{X}^{\mathcal{K}}}\left\{x+R_{F}(x)\right\}$. If $r>d+1$, then: (i) $x+R_{F}(x)$ is quasi-convex in $x$, (ii) $\bar{x}^{\mathcal{K}}=\frac{2}{2-r}$ (Corollary D.1), and (iii) $0+R_{F}(0)=\frac{1}{2-r}<\bar{x}^{\mathcal{K}} \leq \bar{x}^{\mathcal{K}}+R_{F}\left(\bar{x}^{\mathcal{K}}\right)$. The lemma follows.

Lemma D.7. If $r \geq 2 d$, the unique solution of (31) is $\left(\bar{x}^{\mathcal{K}}, R_{F}\left(\bar{x}^{\mathcal{K}}\right)\right)$.

Proof: If $r \geq d+1$ then $\bar{x}^{\mathcal{K}}=\frac{2}{2-r}$. Note that $r \geq 2 d \Longleftrightarrow>\frac{2}{2-r} \geq \frac{1}{1-d} \Longleftrightarrow R_{F}\left(\frac{2}{2-r}\right)=0$. The proof of Lemma D.3 shows that $\bar{x}^{\mathcal{K}^{I P}} \geq \frac{1}{1-d}$ if $r \in\left(r^{* *}(d), d+1\right)$. As $\bar{x}^{\mathcal{K}} \geq \bar{x}^{\mathcal{K}{ }^{I P}}$, then $r \in\left(r^{* *}(d), d+1\right) \Rightarrow R_{F}\left(\bar{x}^{\mathcal{K}}\right)=0$. Let $f(x):=x R_{F}(x)$. As $f(x) \geq 0$ for all $x \in \mathcal{X}$, we conclude that $\bar{x}^{\mathcal{K}}=\arg \min _{x \in \mathcal{X}^{\mathcal{K}}} f(x)$ for $r>r^{* *}(d)$. Finally, if $r \leq r^{* *}(d)$, then

$$
\left\{\underline{x}^{\mathcal{K}}, \bar{x}^{\mathcal{K}}\right\}=\left\{x^{C}, \frac{(2-r)^{2}}{(3-r-d)\left((2-r)^{2}-2(1-d)^{2}\right)}\right\} .
$$

Function $f$ is convex, and

$$
f\left(\frac{(2-r)^{2}}{(3-r-d)\left((2-r)^{2}-2(1-d)^{2}\right)}\right) \geq f\left(x^{C}\right) \Longleftrightarrow r \geq 2 d
$$

The lemma follows.

Proof of Proposition 5, part (i). Any plausible quantity $x$ is associated with consumer surplus:

$$
C S\left(x, R_{F}(x)\right)=\frac{\left(x+R_{F}(x)\right)^{2}}{2}-d x R_{F}(x)
$$

Let $g(x):=C S\left(x, R_{F}(x)\right)$. If $r \geq 2 d$, then Lemmata D. 6 and D. 7 together ensure that $\bar{x}^{\mathcal{K}}=\arg \max _{x \in \mathcal{X}^{\mathcal{K}}} g(x)$.

Suppose that $r<2 d$, so that $\frac{2}{2-r}<\frac{1}{1-d}$. We now prove that $g(\cdot)$ is increasing over the set $\mathcal{X}^{\mathcal{K}}$. First note that, in this parameter region, $g(x)=a_{0}+a_{1} x+a_{2} x^{2}$, where

$$
a_{0}:=\frac{1}{2(2-r)^{2}}, \quad a_{1}:=\frac{(1-r)(1-d)}{(2-r)^{2}}, \text { and } \quad a_{2}:=\frac{(d+1-r)^{2}+2(2-r)(1-d) d}{2(2-r)^{2}} .
$$

Function $g$ is then convex, and $\arg \min _{x} g(x)=\frac{-a_{1}}{2 a_{2}}$. As $2 d<r^{* *}(d)$, then $r<2 d$ implies $\underline{x}^{\mathcal{K}}=x^{C}$. Note that

$$
x^{C}>\frac{-a_{1}}{2 a_{2}} \Longleftrightarrow \frac{(2-r)(2-d)(d+1-r)}{(3-r-d)\left((d+1-r)^{2}+2(2-r)(1-d) d\right)}>0 .
$$

This inequality holds, hence $g(\cdot)$ is increasing over the set $\mathcal{X}^{\mathcal{K}}$.
To prove the second part of Proposition 5 we need the following lemma.
Lemma D.8. For any $d \in[0,1)$,

$$
2 d<r^{\dagger \dagger}(d)<r^{\dagger \dagger}(d)<r^{\dagger}(d)<r^{*}(d)<d+1
$$

Proof: Functions $2 d, r^{\dagger \dagger}(d), r^{\dagger \dagger}(d), r^{\dagger}(d), r^{* * *}(d)$, and $1+d$ are linear and take value 2 for $d=1$. To prove the lemma it is therefore sufficient to verify that their slopes are ordered appropriately. The slopes are shown in Table 1

| Function | Slope |
| :---: | ---: |
| $2 d$ | 2 |
| $r^{\dagger \dagger}(d)$ | $\sqrt{3} \approx 1.732$ |
| $r^{\dagger \dagger}(d)$ | $\frac{1}{3} \sqrt[3]{80-9 \sqrt{79}}-\frac{1}{3}+\frac{1}{3 \sqrt[3]{80-9 \sqrt{79}}} \approx 1.538$ |
| $r^{\dagger}(d)$ | $\frac{\sqrt[3]{3(9-\sqrt{78)}}}{3}+\frac{1}{\sqrt[3]{3(9-\sqrt{78})}} \approx 1.518$ |
| $r^{*}(d)$ | $\sqrt{2} \approx 1.414$ |
| $d+1$ | 1 |

Table 1: Slopes of functions from Lemma D. 8

Proof of Proposition 5, part (ii). Any plausible quantity $x$ is associated with producer surplus

$$
\begin{align*}
P S\left(x, R_{F}(x)\right) & =\left(x+R_{F}(x)\right)-\left(1-\frac{r}{2}\right)\left(x+R_{F}(x)\right)^{2}-(r-2 d) x R_{F}(x) \\
& = \begin{cases}\frac{1-2 r x+4 d x-x^{2}+4 r x^{2}-r^{2} x^{2}-6 d x^{2}+3 d^{2} x^{2}}{2(2-r)} & \text { if } x<\frac{1}{1-d} \\
x-\left(1-\frac{r}{2}\right) x^{2} & \text { if } x \geq \frac{1}{1-d} .\end{cases} \tag{32}
\end{align*}
$$

Let $h(x):=P S\left(x, R_{F}(x)\right)$. If $r>d+1$, then $x_{1}^{C} \in \mathcal{X}^{\mathcal{K}}, x_{3}^{C} \in \mathcal{X}^{\mathcal{K}}, R_{F}\left(x_{3}^{C}\right)=x_{1}^{C}=0$ and $R_{F}\left(x_{1}^{C}\right)=x_{3}^{C}$. As

$$
x_{3}^{C}+R_{F}\left(x_{3}^{C}\right)=x_{1}^{C}+R_{F}\left(x_{1}^{C}\right)=\arg \max _{x \in \mathcal{X}} x-\left(1-\frac{r}{2}\right) x^{2},
$$

and $x_{3}^{C} R_{F}\left(x_{3}^{C}\right)=x_{1}^{C} R_{F}\left(x_{1}^{C}\right)=0$, we conclude that both $x_{1}^{C}$ and $x_{3}^{C}$ maximize producer surplus among plausible quantities. The argument can be extended to the case $r=d+1$.

Suppose now that $r<d+1$. It is easy to check that $h(\cdot)$ is decreasing over the interval $\left[\frac{1}{1-d}, \frac{2}{2-r}\right]$. Note that $\frac{1}{1-d} \in \mathcal{X}^{\mathcal{K}}$. For $x \in\left[0, \frac{1}{1-d}\right]$ instead, $g(x)=a_{0}+a_{1} x+a_{2} x^{2}$, where

$$
a_{0}:=\frac{1}{2(2-r)}, \quad a_{1}:=-\frac{r-2 d}{2-r}<0, \text { and } \quad a_{2}:=\frac{-r^{2}+4 r+3 d^{2}-6 d-1}{2(2-r)} .
$$

Specifically, $a_{2}>0$ if and only if $r>r^{\dagger \dagger}(d)$. Therefore for $r \in\left[r^{\dagger \dagger}(d), 1+d\right)$, the function $g$ takes the highest value either at $\frac{1}{1-d}$, or at $\underline{x}^{\mathcal{K}}$. Note that $g\left(\frac{1}{1-d}\right)=P S^{1}:=\frac{r-2 d}{2(1-d)^{2}}$. In order to characterize $g\left(\underline{x}^{\mathcal{K}}\right)$, we distinguish two cases. If $r<r^{*}(d)$, then $\underline{x}^{\mathcal{K}}=x^{C}$. Note that $g\left(\frac{1}{1-d}\right)>g\left(x^{C}\right) \Longleftrightarrow r>r^{\dagger}(d)$. Lemma D. 8 ensures that $r^{\dagger \dagger}(d)<r^{\dagger}(d)$. If instead $r \geq r^{*}(d)$, then $\underline{x}^{\mathcal{K}}=\frac{2(d+1-r)}{(2-r)^{2}}$, and

$$
\begin{aligned}
& g\left(\frac{1}{1-d}\right)>g\left(\frac{2(d+1-r)}{(2-r)^{2}}\right) \Longleftrightarrow \\
& A \cdot\left(r^{3}+r^{2} d-7 r^{2}-4 r d+16 r-6 d^{3}+18 d^{2}-14 d-6\right)>0
\end{aligned}
$$

where

$$
A:=\frac{(d+1-r)\left(r^{2}-2 r d-2 r+2 d^{2}+2\right)}{2(2-r)^{5}(1-d)^{2}}>0
$$

This inequality holds in the interval $\left[r^{\dagger \dagger \dagger}(d), 1+d\right]$. As $r^{*}(d)>r^{\dagger \dagger \dagger}(d)$ (Lemma D.8), we conclude that

$$
\frac{1}{1-d}=\arg \max _{x \in \mathcal{X}} g(x) \text { for } r \in\left[r^{\dagger}(d), 1+d\right],
$$

and

$$
x^{C}=\arg \max _{x \in \mathcal{X}^{\mathcal{K}}} g(x) \text { for } r \in\left[r^{\dagger \dagger}(d), r^{\dagger}(d)\right] .
$$

Consider next $r \in\left[2 d, r^{\dagger \dagger}(d)\right)$. For these parameter values the function $g$ is concave over the interval $\left[0, \frac{1}{1-d}\right]$. The global maximum obtains at $x=-a_{1} / 2 a_{2} \leq 0$. Therefore $\arg \max _{x \in \mathcal{X}^{\mathcal{K}}} g(x)=\underline{x}^{\mathcal{K}}$. As $r^{\dagger \dagger}(d)<r^{*}(d)$, then $\underline{x}^{\mathcal{K}}=x^{C}$.

Finally, consider the case $r<2 d$. For these parameter values, the function $g$ is concave over the interval $x \in\left[0, \frac{1}{1-d}\right]$, and reaches its maximum at

$$
\frac{-a_{1}}{2 a_{2}}=\frac{-(r-2 d)}{r^{2}-4 r-3 d^{2}+6 d+1}>0 .
$$

As $r<2 d$, then (i) $\underline{x}^{\mathcal{K}}=x^{C}$, and (ii) $x^{C}>\frac{-a_{1}}{2 a_{2}} \Longleftrightarrow r<r^{\dagger \dagger}(d)$. Noting that $r^{\dagger \dagger}(d)>2 d$ (Lemma D.8) concludes the proof.

Proof of Proposition 5, part (iii): Any plausible quantity $x$ is associated with total welfare

$$
W\left(x, R_{F}(x)\right)=C S\left(x, R_{F}(x)\right)+P S\left(x, R_{F}(x)\right)=Q(x)-\frac{1-r}{2} Q(x)^{2}-(r-d) x R_{F}(x)
$$

where $Q(x)=x+R_{F}(x)$ is the total quantity.
Let us first consider the case $r \geq 2 d$. Define $f(Q):=Q-\frac{1}{2}(1-r) Q^{2}$. Whenever $r \geq 0$, the function $f$ is increasing over the interval $\mathcal{X}$. To see this, note that (i) if $r>1$ then $f$ is convex and $\arg \min f=(1-r)^{-1}<0$; (ii) if $r=1$, then $f$ is increasing for all $Q$; (iii) if $r<1$ then, $f$ is concave and $\arg \max f=(1-r)^{-1}>\frac{2}{2-r}$. Therefore for $r \geq 2 d \geq 0$ the function $f$ is increasing in total quantity $Q$ for any $Q(x) \in \mathcal{X}$. It is easy to verify that $Q(x) \in \mathcal{X}$ for any $x \in \mathcal{X}$. By Lemma D.6, arg $\max _{x \in \mathcal{X} \mathcal{K}} f(Q(x))=\bar{x}^{\mathcal{K}}$. Moreover, by Lemma D.7, when $r \geq 2 d$, then $\arg \min _{x \in \mathcal{X}^{\mathcal{K}}} x R_{F}(x)=\bar{x}^{\mathcal{K}}$. We conclude that $\arg \max _{x \in \mathcal{X} \mathcal{K}} W(x)=\bar{x}^{\mathcal{K}}$, for $r \geq 2 d$.

Suppose that instead $r<2 d$. In this case $\frac{1}{1-d}>\frac{2}{2-r}$, hence for all $x \in \mathcal{X}$ it is the case that $R_{F}(x)>0$ and $W(x)=a_{0}+a_{1} x+a_{2} x^{2}$, where

$$
\begin{aligned}
& a_{0}:=\frac{3-r}{2(2-r)^{2}} ; \quad a_{1}:=1-\frac{(3-r)(1-d)}{(2-r)^{2}} ; \text { and } \\
& a_{2}:=\frac{(d+1-r)^{2}+(2-r)(1-d)(3-d)-(2-r)^{3}}{2(2-r)^{2}} .
\end{aligned}
$$

There are three cases, depending on the sign of $a_{2}$.
(i) Consider the case $a_{2}=0$. This happens if and only if

$$
d=d^{*}(r):=1-\frac{(2-r) \sqrt{(2-r)^{2}-1}}{3-r} .
$$

Note that (i) $d^{*}(r)$ is strictly increasing over the interval $[0,1]$, (ii) $d^{*}(1)=1$, and (iii) $2 d^{*}(r)=r \Longleftrightarrow r=1 / 3$. So $a_{2}=0$ requires that $r \in(1 / 3,1]$ and $d=d^{*}(r)$. Replacing $d$ with $d^{*}(r)$ in $a_{1}$ gives:

$$
a_{1}=1-\sqrt{1-\frac{1}{(2-r)^{2}}}>0
$$

Therefore $\arg \max _{x \in \mathcal{X}^{\mathcal{K}}} W(x)=\bar{x}^{\mathcal{K}}$.
(ii) Suppose that $a_{2}>0$. Note that (i) $a_{2}>0$ if and only if $d<d^{*}(r)$, and (ii) for $a_{2}>0$ function $W(x)$ is convex and reaches a minimum at $\frac{-a_{1}}{a_{2}}$. We distinguish two cases.
(a) If $r \leq 1$, then $a_{1} \geq 0$. To see this, note that (i) $a_{1}$ is increasing in $d$, so $a_{1}$ for $d=r / 2$ is strictly smaller than for any $d \in\left(r / 2, d^{*}(r)\right)$, and (ii) evaluating $a_{1}$ for
$d=r / 2$, gives

$$
\frac{1-r}{2(2-r)} \geq 0
$$

As $\frac{-a_{1}}{a_{2}} \leq 0$, then $\arg \max _{x \in \mathcal{X} \mathcal{K}} W(x)=\bar{x}^{\mathcal{K}}$.
(b) Let $r>1$. As $r<2 d$, Corollary D.1 and Lemma D. 8 together ensure that $\underline{x}^{\mathcal{K}}=x^{C}$.

Now,

$$
x^{C}-\frac{-a_{1}}{2 a_{2}}=\frac{A(r, d)}{B(r, d)},
$$

where

$$
\begin{aligned}
& A(r, d):=\frac{(2-r)(d+1-r)}{(3-r-d)} \\
& B(r, d):=(1-r+d)^{2}+(2-r)(1-d)(3-d)-(2-r)^{3} .
\end{aligned}
$$

Clearly $A(r, d)>0$ for the relevant values of $r$ and $d$. We show that $B(r, d)>0$. To see this, note that (i) $B(r, d)$ is convex in $d$, with minimum at $d=1$, therefore $B(r, d)$ decreasing in $d \in[0,1]$, and (ii) $B(r, 1)=(2-r)^{2}(r-1)>0$ for $r>1$. Again, as $W(x)$ is increasing in $x$ over plausible values, it is maximized by $\bar{x}^{\mathcal{K}}$.
(iii) Finally, suppose that $a_{2}<0$. In this region $W(\cdot)$ is concave and reaches a maximum at $\frac{-a_{1}}{2 a_{2}}$. As parameters satisfy $\min \{2 d, 1\}>r, d>d^{*}(r)$, to conclude the proof it suffices to show that

$$
\frac{-a_{1}}{2 a_{2}} \geq \bar{x}^{\mathcal{K}}, \quad \forall r<1 \text { and } \forall d>d^{* *}(r)
$$

where

$$
d^{* *}(r):= \begin{cases}0 & \text { if } r \leq 0 \\ \frac{r}{2} & \text { if } 0<r \leq \frac{1}{3} \\ d^{*}(r) & \text { if } r>\frac{1}{3}\end{cases}
$$

Simple algebra shows that $2 d<r^{* *}(d)$ for all $d \in[0,1]$, hence $r<2 d$ ensures $r<r^{* *}(d)$. Corollary D. 1 and Lemma D. 8 together thus ensure that

$$
\bar{x}^{\mathcal{K}}=\frac{(2-r)^{2}}{(3-r-d)\left((2-r)^{2}-2(1-d)^{2}\right)} .
$$

Therefore

$$
\frac{-a_{1}}{2 a_{2}}-\bar{x}^{\mathcal{K}}=\frac{F(r, d)}{D(r, d) E(r, d)(3-r-d)}
$$

where

$$
\begin{aligned}
D(r, d) & :=(2-r)^{3}-(1-r+d)^{2}-(2-r)(1-d)(3-d) \\
E(r, d) & :=(2-r)^{2}-2(1-d)^{2} ; \\
F(r, d) & :=(2-r)(3-r-d)\left(3 r-3 r^{2}+r^{3}-2 d+r d-r^{2} d+2 d^{2}\right) \\
& +\left(2-4 r+r^{2}+4 d-2 d^{2}\right)\left((2-r)^{3}-(2-r)(1-d)(3-d)-(1-r+d)^{2}\right) .
\end{aligned}
$$

Clearly $3-r-d>0$ for all $(r, d)$ such that $r<1$, and $d>d^{* *}(r)$. We show next that for these parameter values $D>0$ and $E>0$. Note that both $D$ and $E$ are concave functions of $d$, and they both reach a maximum at $d=1$. We conclude that both $D$ and $E$ are increasing functions of $d$ for all $d \in[0,1]$. We consider, in turn, cases $r \leq 0$, $r \in(0,1 / 3]$ and $r \in(1 / 3,1)$.
(a) If $r \leq 0$, then $d^{* *}(r)=0$. We just established that $D(r, d) \geq D(r, 0)$ for all $d \in[0,1]$. As $D_{1}(r, 0)<0$, then $D(r, d) \geq D(r, 0) \geq D(0,0)=1$. Similarly, $E(r, d) \geq E(r, 0)$ for all $d \in[0,1]$. As $E_{1}<0$, then $E(r, d) \geq E(r, 0) \geq E(0,0)=2$.
(b) If $r \in\left(0, \frac{1}{3}\right]$, then $d^{* *}(r)=\frac{r}{2}$, and $D(r, d) \geq D\left(r, \frac{r}{2}\right)=1 / 4(2-r)^{2}(1-3 r) \geq 0$, while $E(r, d) \geq E\left(r, \frac{r}{2}\right)=1 / 2(2-r)^{2}>0$.
(c) If $r \in\left(\frac{1}{3}, 1\right)$ : then $d^{* *}(r)=d^{*}(r)$, and $D(r, d) \geq D\left(r, d^{*}(r)\right)=0$, while $E(r, d) \geq$ $E\left(r, d^{*}(r)\right)=\frac{(2-r)^{2}(r+1)}{3-r}>0$.

In the rest of the proof we show that $F(r, d) \geq 0$ for all $r \leq 1$ and $d \in[0,1]$.
For any $d \in[0,1]$, the function $F(r, d)$ is a 4th degree polynomial function of $r$. To prove that it is non-negative for all $r \leq 1$ and $d \in[0,1]$, it suffices to show that for all $d \in[0,1]$ : (i) $F(1, d)>0$ and (ii) $F(\cdot, d)$ does not have any real roots in $(-\infty, 1)$. To prove the first claim, note that:

$$
F(1, d)=4-17 d+28 d^{2}-18 d^{3}+4 d^{4}
$$

All four roots of this polynomial are complex, and, for example, $F(1,1)=1>0$. Therefore $F(1, d)>0$ for all $d \in[0,1]$.

To prove the second claim, we use Sturm's theorem. For any $d \in[0,1]$, let: $p_{0}(r):=$ $F(r, d), p_{1}(r):=F_{1}(r, d), p_{2}(r)=-\operatorname{rem}\left(p_{0}(r), p_{1}(r)\right), p_{3}(r)=-\operatorname{rem}\left(p_{1}(r), p_{2}(r)\right)$ and $p_{4}(r)=$ $-\operatorname{rem}\left(p_{3}(r), p_{4}(r)\right)$, where $\operatorname{rem}(a, b)$ is the remainder of the Euclidean division of $a$ by $b$. So

$$
\begin{aligned}
& p_{1}(r)=4 r^{3}+6 r^{2} d^{2}-15 r^{2} d-15 r^{2}-24 r d^{2}+60 r d+12 r-2 d^{4}+10 d^{3}+6 d^{2}-46 d ; \\
& p_{2}(r)=-\frac{1}{16}(1-d)^{2}\binom{32+60 r-27 r^{2}-102 d-72 r d+36 r^{2} d}{+76 d^{2}+24 r d^{2}-12 r^{2} d^{2}-22 d^{3}+4 d^{4}} \\
& p_{3}(r)=\frac{32(1-d)^{2}}{3(2 d-3)^{4}}\binom{16 r d^{4}-104 r d^{3}+212 r d^{2}-126 r d-24 r}{-52 d^{4}+314 d^{3}-600 d^{2}+335 d+55} ; \\
& p_{4}(r)=\frac{(1-d)^{4}(2 d-3)^{4}\left(64 d^{6}-672 d^{5}+2340 d^{4}-2984 d^{3}+252 d^{2}+1560 d+197\right)}{64\left(8 d^{4}-52 d^{3}+106 d^{2}-63 d-12\right)^{2}} .
\end{aligned}
$$

Sturm's theorem ensures that the number of real roots of $F(\cdot, d)$ in $(-\infty, 1]$ is equal to $V(-\infty)-V(1)$, where $V(r)$ denote the number of sign changes at $r$. We prove below that $V(-\infty)=V(1)=2$, so that indeed the theorem ensures that $F(\cdot, d)$ does not have any real roots in $(-\infty, 1)$.

First, we establish that $V(-\infty)=2$. To see this note that, at $r \rightarrow-\infty$ the sign of the polynomial are

- positive for $p_{0}(r)$ (a 4th degree polynomial with leading coefficient 1 );
- negative for $p_{1}(r)$ (a 3rd degree polynomial with leading coefficient 4);
- positive for $p_{2}(r)$ (a 2 nd degree polynomial with leading coefficient $\frac{3}{4}(1-d)^{2}\left(\frac{3}{2}-d\right)^{2}>$ 0 );
- positive for $p_{3}(r)$ (a linear function with negative slope for all $d \in[0,1]$ ).
- positive for $p_{4}(r)$ (a positive constant).

The number of sign changes is therefore 2 . Next, we establish that $V(1)=2$. To see this note that, at $r=1$ the sign of the polynomial are

- positive for $p_{0}(r), p_{3}(r)$ and $p_{4}(r)$;
- positive if $d<a_{1}$ and negative if $d>a_{1}$, where $a_{1} \approx 0.278$ for $p_{1}(r)$;
- is negative if $d<a_{2}$ and positive if $d>a_{2}$, where $a_{2} \approx 0.845$ for $p_{2}(r){ }_{4}^{34}$

For any $d \in[0,1]$, the number of sign changes is indeed 2 .

## E Appendix of Subsection 7

In this appendix, we first show a method for checking whether a simple CST can be refined by some worse simple CST. Then, we show a method for checking whether a CST that satisfies Property I can be refined by some worse CST that satisfies Property I.

## E. 1 Simple Commitment Structures

Definition E.1. Let CST K satisfy Property I, and let $\tilde{\mathcal{X}}$ be an interval corresponding to the union of some elements of $K$. We define with $G_{\tilde{\mathcal{X}}}(K)$ a game that differs from $G(K)$ only in that, in period 1, the leader has to select some $\mathcal{X}_{i}$ such that $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$.

Definition E.2. Let $\tilde{\mathcal{X}}$ be an interval. An outcome $\left(x^{*}, y^{*}\right)$ is said to be simply plausible with respect to $\tilde{\mathcal{X}}$ if there exists a simple CST, denoted $K$, such that (i) $\tilde{\mathcal{X}}$ corresponds to the union of some elements of $K$, and (ii) outcome $\left(x^{*}, y^{*}\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$.

Accordingly, an action $x^{*}$ is said to be simply plausible with respect to $\tilde{\mathcal{X}}$ if it forms part of an outcome simply plausible with respect to $\tilde{\mathcal{X}}$.

Proposition E.1. Let $\tilde{\mathcal{X}}$ be an interval. An action $x^{*} \in \tilde{\mathcal{X}}$ is simply plausible with respect to $\tilde{\mathcal{X}}$ if and only if either the set

$$
\tilde{\mathcal{X}} \cap \mathcal{X}^{C} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid\left(x-x^{*}\right)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0\right\}
$$

[^20]is not empty, or else the set
$$
\tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid(\phi(x)-x)\left(\phi\left(x^{*}\right)-x^{*}\right)>0\right\}
$$
includes every element of a sequence $\left(x_{k}\right)_{k=1}^{\infty}$ such that $\lim _{k \rightarrow \infty} x_{k}=\tilde{x}$, for some action $\tilde{x}$ such that $(\tilde{x}-x)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0$ for every $x \in \tilde{\mathcal{X}}$, or both.

Proof: We prove first the if part of the proposition. Consider an action $x^{*} \in \tilde{\mathcal{X}}$. If $x^{*}=\phi\left(x^{*}\right)$, the argument is trivial. Let $x^{*}<\phi\left(x^{*}\right)$. We consider two cases.

Case 1: $\tilde{\mathcal{X}} \cap \mathcal{X}^{C} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid\left(x-x^{*}\right)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0\right\} \neq \varnothing$.
As $x^{*}<\phi\left(x^{*}\right)$, then $\tilde{\mathcal{X}} \cap \mathcal{X}^{C} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid x>x^{*}\right\} \neq \varnothing$. It ensues that outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$, for some simple CST $K$ such that $\{x \mid x \in \tilde{\mathcal{X}}, x \leq$ $\left.x^{*}\right\} \in K$, and $\left\{x \mid x \in \tilde{\mathcal{X}}, x>x^{*}\right\} \in K$.

Case 2: The set $\tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid(\phi(x)-x)\left(\phi\left(x^{*}\right)-x^{*}\right)>0\right\}$ includes every element of a sequence $\left(x_{k}\right)_{k=1}^{\infty}$ such that $\lim _{k \rightarrow \infty} x_{k}=\tilde{x}$, for some action $\tilde{x}$ such that $(\tilde{x}-x)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0$ for every $x \in \tilde{\mathcal{X}}$.

As $x^{*}<\phi\left(x^{*}\right)$, then the set $\tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x \mid \phi(x)>x\}$ includes a sequence $\left(x_{k}\right)_{k=1}^{\infty}$ such that $\lim _{k \rightarrow \infty} x_{k}=\sup (\tilde{\mathcal{X}})$. Suppose that this sequence includes a strictly increasing subsequence. Denote the subsequence $\left(x_{k}^{\prime}\right)_{k=1}^{\infty}$. Consider a simple CST $K^{\prime}$ such that $\left\{x \mid x \leq x^{*}, x \in \tilde{\mathcal{X}}\right\} \in K$, $\left(x^{*}, x_{1}^{\prime}\right] \in K$, and $\left(x_{i}^{\prime}, x_{i+1}^{\prime}\right] \in K$ for $i \in\{1, \ldots, \infty\}$. Outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}\left(K^{\prime}\right)$. Suppose instead that the aforementioned sequence $\left(x_{k}\right)_{k=1}^{\infty}$ does not include any strictly increasing subsequence. Then the sequence must include a subsequence $\left(x_{k}^{\prime}\right)_{k=1}^{\infty}$ such that every element satisfies $x_{k}^{\prime}=\sup (\tilde{\mathcal{X}})$. We conclude that

$$
\sup (\tilde{\mathcal{X}}) \in \tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\{x \mid \phi(x)>x\}
$$

Outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is in this case a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$ for some simple CST $K$ such that $\left\{x \mid x \leq x^{*}, x \in \tilde{\mathcal{X}}\right\} \in K$ and $\left(x^{*}, \sup (\tilde{\mathcal{X}})\right] \in K$. The proof for the case $x^{*}>\phi\left(x^{*}\right)$ is analogous.

We prove now only if part of the proposition. Consider action $x^{*}$ in $\tilde{\mathcal{X}}$. If $x^{*}=\phi\left(x^{*}\right)$, then

$$
\tilde{\mathcal{X}} \cap \mathcal{X}^{C} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid\left(x-x^{*}\right)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0\right\} \neq \varnothing
$$

Suppose instead that $x^{*}<\phi\left(x^{*}\right)$. Let action $x^{*}$ be simply plausible with respect to $\tilde{\mathcal{X}}$. Let $K^{\prime}$ denote a generic CST that satisfies:
(i) $K^{\prime}$ is simple;
(ii) $\tilde{\mathcal{X}}$ is equal to the union of some elements of $K^{\prime}$;
(iii) $\left\{x \mid x \leq x^{*}, x \in \tilde{\mathcal{X}}\right\} \in K^{\prime}$.

Suppose some $K^{\prime}$ that satisfies (i)-(iii) also satisfies:
(iv) an equilibrium of $G_{\tilde{\mathcal{X}}}\left(K^{\prime}\right)$ exists in which the leader's equilibrium action is $x^{*}$, and in the subgame corresponding to some $\mathcal{X}_{i} \in \tilde{\mathcal{X}}$ the leader's action belongs to $\mathcal{X}^{C}$.

Then

$$
\tilde{\mathcal{X}} \cap \mathcal{X}^{C} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid\left(x-x^{*}\right)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0\right\} \neq \varnothing
$$

Suppose instead that every $K^{\prime}$ that satisfies (i)-(iii) violates (iv). Then, a SPE of $G_{\tilde{\mathcal{X}}}\left(K^{\prime}\right)$ for some $K^{\prime}$ that satisfies (i)-(iii) exists in which the leader selects action $x^{*}$ on path, and an action $x_{i} \notin \mathcal{X}^{C}$ for every interval $\mathcal{X}_{i} \in K^{\prime}$ such that $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$. Standard arguments ensure that the leader picks an action $x_{i}<\phi\left(x_{i}\right)$ for every interval $\mathcal{X}_{i} \in K^{\prime}$ such that $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$ (call this Remark 1).

We distinguish two cases. In the first case, $\sup (\tilde{\mathcal{X}}) \in \tilde{\mathcal{X}}$. Remark 1 ensures that $\sup (\tilde{\mathcal{X}})<$ $\phi(\sup (\tilde{\mathcal{X}}))$ and $\sup (\tilde{\mathcal{X}}) \in \mathcal{Q}_{\leq}\left(x^{*}\right)$. In this case, the set

$$
\tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid(\phi(x)-x)\left(\phi\left(x^{*}\right)-x^{*}\right)>0\right\}
$$

includes a sequence $\left(x_{k}\right)_{k=1}^{\infty}$ such that $\lim _{k \rightarrow \infty} x_{k}=\tilde{x}$, where $(\tilde{x}-x)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0$ for every $x \in \tilde{\mathcal{X}}$ (consider, for instance, the sequence in which every element satisfies $x_{k}=\sup (\tilde{\mathcal{X}})$ ). In the second case, $\sup (\tilde{\mathcal{X}}) \notin \tilde{\mathcal{X}}$. We can then construct a monotonically increasing sequence
including only actions $x_{i}$ as described in Remark 1. Such sequence converges to $\sup (\tilde{\mathcal{X}})$ and every element of the sequence satisfies $x_{i}<\phi\left(x_{i}\right)$ and $x_{i} \in\left\{x \mid x \geq x^{*}, x \in \tilde{\mathcal{X}}\right\} \cap \mathcal{Q}_{\leq}\left(x^{*}\right)$. Thus, also in this case the set $\tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right) \cap\left\{x \mid(\phi(x)-x)\left(\phi\left(x^{*}\right)-x^{*}\right)>0\right\}$ includes a sequence $\left(x_{k}\right)_{k=1}^{\infty}$ such that $\lim _{k \rightarrow \infty} x_{k}=\tilde{x}$, where $(\tilde{x}-x)\left(\phi\left(x^{*}\right)-x^{*}\right) \geq 0$ for every $x \in \tilde{\mathcal{X}}$. This concludes the only if part of the proof for $x^{*}<\phi\left(x^{*}\right)$. The only if part of the proof for $x^{*}>\phi\left(x^{*}\right)$ is analogous.

The procedure to check whether a simple CST $K$ can be refined by some worse simple CST $K^{\prime}$ has two steps:

Step 1: for every $\mathcal{X}_{i} \in K$ find the set of actions that are simply plausible with respect to $\mathcal{X}_{i}$.
Step 2: a worse CST that refines $K$ exists if and only if there exists a utility level $\bar{u}$ such that
(i) $u\left(x^{*}, y^{*}\right)>\bar{u}$ for every SPE outcome $\left(x^{*}, y^{*}\right)$ of $G(K)$;
(ii) for every $\mathcal{X}_{i} \in K$ there exist an action $x^{* *}$ simply plausible with respect to $\mathcal{X}_{i}$ such that $U\left(x^{* *}\right) \leq \bar{u}$, and the inequality holds as an equality for at least one $\mathcal{X}_{i}$.

## E. 2 Commitment Structures that Satisfy Property I

Definition E.3. Let $\tilde{\mathcal{X}}$ be an interval. An outcome $\left(x^{*}, y^{*}\right)$ is said to be I-plausible with respect to $\tilde{\mathcal{X}}$ if there exists a CST, denoted $K$, such that (i) $K$ satisfies Property $I$, (ii) $\tilde{\mathcal{X}}$ corresponds to the union of some elements of $K$, and (iii) outcome $\left(x^{*}, y^{*}\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$.

Accordingly, an action $x^{*}$ is said to be I-plausible with respect to $\tilde{\mathcal{X}}$ if it forms part of an outcome I-plausible with respect to $\tilde{\mathcal{X}}$.

Definition E.4. Let $\tilde{\mathcal{X}} \subseteq \mathcal{X}$. Define $\tilde{\mathcal{X}} \geq:=\{x \mid x \in \tilde{\mathcal{X}}, \phi(x) \geq x\}$, and $\tilde{\mathcal{X}} \leq:=\{x \mid x \in$ $\tilde{\mathcal{X}}, \phi(x) \leq x\}$.

Proposition E.2. Let $\tilde{\mathcal{X}}$ be an interval. Action $x^{*} \in \tilde{\mathcal{X}}$ is I-plausible with respect to $\tilde{\mathcal{X}}$ if and only if at least one of these conditions holds:
(i) the set $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap \tilde{\mathcal{X}} \geq$ includes a sequence $\left(x_{k}\right)_{k=1}^{\infty}$, where $\lim _{k \rightarrow \infty} x_{k}=\sup (\tilde{\mathcal{X}})$;
(ii) the set $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap \tilde{\mathcal{X}} \leq$ includes a sequence $\left(x_{k}\right)_{k=1}^{\infty}$, where $\lim _{k \rightarrow \infty} x_{k}=\inf (\tilde{\mathcal{X}})$;
(iii) the set $\mathcal{Q}_{\leq}\left(x^{*}\right)$ includes two actions, denoted $x^{\prime}$ and $x^{\prime \prime}$, such that (i) $x^{\prime} \in \tilde{\mathcal{X}} \leq$, (ii) $x^{\prime \prime} \in \tilde{\mathcal{X}} \geq$ and (iii) $x^{\prime} \leq x^{\prime \prime}$.

Proof: We prove the if part of the proposition by construction. Let $x^{*} \in \tilde{\mathcal{X}}$. Suppose that the set $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap \tilde{\mathcal{X}} \geq$ includes every element of a sequence $\left(x_{k}\right)_{k=1}^{\infty}$, such that $\lim _{k \rightarrow \infty} x_{k}=\sup (\tilde{\mathcal{X}})$. Continuity of $U$ then ensures that $\sup (\tilde{\mathcal{X}}) \in \mathcal{Q}_{\leq}\left(x^{*}\right)$, while continuity of $\phi$ ensures that $\sup (\tilde{\mathcal{X}}) \leq \phi(\sup (\tilde{\mathcal{X}}))$. If $\sup (\tilde{\mathcal{X}}) \in \tilde{\mathcal{X}}$, then outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$ for any CST $K$ such that if $\mathcal{X}_{i} \in K$ and $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$, then $\mathcal{X}_{i} \in\left\{\tilde{\mathcal{X}},\left\{x^{*}\right\}\right\}$. If instead $\sup (\tilde{\mathcal{X}}) \notin \tilde{\mathcal{X}}$, then outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$ for any CST $K$ such that if $\mathcal{X}_{i} \in K$ and $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$, then

$$
\mathcal{X}_{i} \in\left\{\left\{x^{*}\right\},\left\{x \mid x \in \mathcal{X}_{i}, x \leq x_{k}\right\}_{k=1}^{\infty}\right\} .
$$

An analogous argument holds if the set $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap \tilde{\mathcal{X}} \leq$ includes every element of a sequence $\left(x_{k}\right)_{k=1}^{\infty}$, such that $\lim _{k \rightarrow \infty} x_{k}=\inf (\tilde{\mathcal{X}})$.

Suppose instead that $\mathcal{Q}_{\leq}\left(x^{*}\right)$ includes two actions, respectively denoted $x^{\prime}$ and $x^{\prime \prime}$, such that $x^{\prime} \in \tilde{\mathcal{X}} \leq, x^{\prime \prime} \in \tilde{\mathcal{X}}^{\geq}$and $x^{\prime} \leq x^{\prime \prime}$. Outcome $\left(x^{*}, R_{F}\left(x^{*}\right)\right)$ is then a SPE outcome of $G_{\tilde{\mathcal{X}}}(K)$ for any CST $K$ such that if $\mathcal{X}_{i} \in K$ and $\mathcal{X}_{i} \subseteq \tilde{\mathcal{X}}$, then

$$
\mathcal{X}_{i} \in\left\{\left\{x \mid x \in \tilde{\mathcal{X}}, x \leq x^{\prime \prime}\right\},\left\{x \mid x \in \tilde{\mathcal{X}}, x \geq x^{\prime}\right\},\left\{x^{*}\right\}\right\}
$$

We prove now the only if part of the proposition. Let action $x^{*} \in \tilde{\mathcal{X}}$ be I-plausible with respect to $\tilde{\mathcal{X}}$. If $x^{*}=\phi\left(x^{*}\right)$, then the set $\mathcal{Q}_{\leq}\left(x^{*}\right)$ includes a pair of actions, denoted $x^{\prime}$ and $x^{\prime \prime}$, such that $x^{\prime}=x^{\prime \prime}=x^{*}, x^{\prime} \in \tilde{\mathcal{X}} \leq, x^{\prime \prime} \in \tilde{\mathcal{X}} \geq$ and $x^{\prime} \leq x^{\prime \prime}$. Suppose instead that $x^{*}<\phi\left(x^{*}\right)$. Action $x^{*}$ being I-plausible with respect to $\tilde{\mathcal{X}}$, for every action $x \in \tilde{\mathcal{X}}$ there exits an action $\tilde{x} \in \tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right)$ such that either $\phi(\tilde{x}) \geq \tilde{x} \geq x$, or $x>\tilde{x} \geq \phi(\tilde{x})$. Suppose that there does not exist a pair of actions $\left\{x^{\prime}, x^{\prime \prime}\right\} \in \mathcal{Q}_{\leq}\left(x^{*}\right)$, such that $x^{\prime} \in \tilde{\mathcal{X}} \leq, x^{\prime \prime} \in \tilde{\mathcal{X}}^{\geq}$and $x^{\prime} \leq x^{\prime \prime}$. Then either $x^{*}=\sup (\tilde{\mathcal{X}})$, or else for any action $x \in \tilde{\mathcal{X}}$ such that $x>x^{*}$ there exists an action $\tilde{x} \in \tilde{\mathcal{X}} \cap \mathcal{Q}_{\leq}\left(x^{*}\right)$ such that $\phi(\tilde{x}) \geq \tilde{x} \geq x$. In either case, the set $\mathcal{Q}_{\leq}\left(x^{*}\right) \cap \tilde{\mathcal{X}} \geq$ includes
every element of a sequence $\left(x_{k}\right)_{k=1}^{\infty}$, such that $\lim _{k \rightarrow \infty} x_{k}=\sup (\tilde{\mathcal{X}})$. The argument in case $x^{*}>\phi\left(x^{*}\right)$ is analogous.

The procedure to check whether a CST $K$ that satisfies Property I can be refined by some worse CST $K^{\prime}$ that satisfies Property I resembles the procedure for a simple CST illustrated in Subsection E.1.

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## F Online Appendix

In this appendix, we consider a finite actions version of the leader-follower games introduced in Subsection 6.1 and show that the pure-strategy proper equilibrium outcomes of these games (as defined by Myerson (1978)) coincide with the plausible outcomes of our paper.

All definitions and notations from the main text carry over to this appendix. We keep in line with the notation and terminology used in Myerson (1978) wherever possible.

To simplify notation, we work in a symmetric setting. That is, we assume that $\mathcal{Y}=\mathcal{X}$ and that $v=u$. Furthermore, we suppose that $\mathcal{X}$ is finite, and that $(x, y) \neq\left(x^{\prime}, y^{\prime}\right) \Rightarrow u(x, y) \neq$ $u\left(x^{\prime}, y^{\prime}\right){ }^{35}$ Throughout, $K$ is some fixed partition of $\mathcal{X}$ with cardinality $m$, and $\mathcal{X}_{1}, \ldots, \mathcal{X}_{m}$ represent the elements of the partition $K$. The $m$-tuples of $\mathcal{X}^{m}$ will be represented by bold letters, e.g., $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right)$.

A pure (respectively, mixed and totally mixed) strategy of the leader is an element of $\mathcal{X}$ (resp., $\Delta \mathcal{X}$ and $\Delta^{0} \mathcal{X}$ ); a pure strategy of the follower is an element of $\mathcal{X}^{m}$ (resp., $\Delta \mathcal{X}^{m}$ and $\left.\Delta^{0} \mathcal{X}^{m}\right)$. Given an arbitrary strategy set $S$ and $s \in S$, we sometimes slightly abuse notation and write $s$ for the element of $\Delta S$ attaching probability 1 to strategy $s$.

Given pure strategies $x$ and $\mathbf{y}=\left(y_{1}, \ldots, y_{m}\right)$ of the leader and follower, define (throughout, the subscript $\ell$ stands for "leader" and $f$ for "follower")

$$
v_{\ell}(x, \mathbf{y}):=u\left(x, y_{k}\right), \text { whenever } x \in \mathcal{X}_{k},
$$

and

$$
v_{f}(x, \mathbf{y}):=v\left(y_{k}, x\right), \text { whenever } x \in \mathcal{X}_{k} .
$$

Together, the players (leader and follower), their strategy spaces, and the above payoff functions define a (normal-form) game $G$. In the remainder, we aim to show that the pure-strategy proper equilibrium outcomes of $G$ are the plausible outcomes of our setting.

Let $V_{j}\left(s_{j} \mid \sigma_{-j}\right)$ denote the expected payoff of player $j$ (i.e., $\left.j \in\{\ell, f\}\right)$ from playing pure strategy $s_{j}$ when the other player plays the mixed strategy $\sigma_{-j}$. Say that $\sigma_{-j}$ induces a strict

[^21]ranking over the pure strategies in $S_{j}$ if $V_{j}\left(s_{j} \mid \sigma_{-j}\right) \neq V_{j}\left(s_{j}^{\prime} \mid \sigma_{-j}\right)$ whenever $s_{j} \neq s_{j}^{\prime}$. Say that a mixed strategy $\sigma_{j}$ satisfies the $\varepsilon$-criterion against $\sigma_{-j}$ if
$$
V_{j}\left(s_{j} \mid \sigma_{-j}\right)<V_{j}\left(s_{j}^{\prime} \mid \sigma_{-j}\right) \Rightarrow \sigma_{j}\left(s_{j}\right) \leq \varepsilon \sigma_{j}\left(s_{j}^{\prime}\right) .
$$

A pair $\left(\sigma_{\ell}, \sigma_{f}\right)$ constitutes a proper equilibrium of $G$ (cf. Myerson (1978)) if there exist sequences of totally mixed strategies $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ and $\left\{\sigma_{f}^{n}\right\}_{n \in \mathbb{N}}$, as well as a sequence of positive scalars $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ with $\lim _{n \rightarrow \infty} \varepsilon_{n}=0$, such that, for every $j \in\{\ell, f\}$ and all $n \in \mathbb{N}$ : $\sigma_{j}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{-j}^{n}$, and $\lim _{n \rightarrow \infty} \sigma_{j}^{n}=\sigma_{j}$.

Given $\mathbf{x} \in \mathcal{X}_{1} \times \cdots \times \mathcal{X}_{m}$, say that a sequence of totally mixed strategies of the leader $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $x_{i}$ via $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right)$ if
(i) $\sigma_{\ell}^{n} \underset{n \rightarrow \infty}{\longrightarrow} x_{i}$;
(ii) for $k=1, \ldots, m$ :

$$
\frac{\sigma_{\ell}^{n}\left(x_{k}\right)}{\sum_{\tilde{x} \in \mathcal{X}_{k}} \sigma_{\ell}^{n}(\tilde{x})} \underset{n \rightarrow \infty}{\longrightarrow} 1
$$

For any pure strategy $\mathbf{y}$ of the follower, let $\Lambda_{k}(\mathbf{y})$ be the unique maximizer of $u\left(\cdot, y_{k}\right)$ over $\mathcal{X}_{k}$, that is,

$$
\begin{equation*}
\arg \max _{x \in \mathcal{X}_{k}} u\left(x, y_{k}\right)=\left\{\Lambda_{k}(\mathbf{y})\right\}, \tag{33}
\end{equation*}
$$

and $\boldsymbol{\Lambda}(\mathbf{y}):=\left(\Lambda_{1}(\mathbf{y}), \ldots, \Lambda_{m}(\mathbf{y})\right)$. Letting $k^{*}$ denote the index maximizing $u\left(\Lambda_{k}(\mathbf{y}), y_{k}\right)$, define also $\Lambda^{*}(\mathbf{y}):=\Lambda_{k^{*}}(\mathbf{y})$.

Finally, given $\mathbf{x} \in \mathcal{X}_{1} \times \cdots \times \mathcal{X}_{m}$, let $\mathbf{R}(\mathbf{x})$ denote the strategy of the follower given by $R_{k}(\mathbf{x})=R\left(x_{k}\right)$ for $k=1, \ldots, m$.

Lemma F.1. Let $\mathbf{y}$ be a pure strategy of the follower, $\left\{\sigma_{f}^{n}\right\}_{n \in \mathbb{N}}$ a sequence of mixed strategies with $\lim _{n \rightarrow \infty} \sigma_{f}^{n}=\mathbf{y}$, and $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ a sequence of positive scalars with $\lim _{n \rightarrow \infty} \varepsilon_{n}=0$. Suppose that, for all $n, \sigma_{\ell}^{n}$ is a totally mixed strategy of the leader which satisfies the $\varepsilon_{n}$-criterion against $\sigma_{f}^{n}$. Then $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $\Lambda^{*}(\mathbf{y})$ via $\boldsymbol{\Lambda}(\mathbf{y})$.

Proof: Notice to begin with that $\mathbf{y}$ induces a strict ranking over the pure strategies of the leader. Furthermore, for all sufficiently large $n$, the mixed strategy $\sigma_{f}^{n}$ induces the same strict
ranking as $\mathbf{y}$. As $\sigma_{\ell}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{f}^{n}$, we conclude from (33) that, for all sufficiently large $n$ and every $k \in\{1, \ldots, m\}$ :

$$
\sigma_{\ell}^{n}(\tilde{x}) \leq \varepsilon_{n} \sigma_{\ell}^{n}\left(\Lambda_{k}(\mathbf{y})\right), \quad \forall \tilde{x} \in \mathcal{X}_{k} \backslash\left\{\Lambda_{k}(\mathbf{y})\right\}
$$

Similarly, letting $k^{*}$ denote the index maximizing $u\left(\Lambda_{k}(\mathbf{y}), y_{k}\right)$, we have, for all sufficiently large $n$,

$$
\sigma_{\ell}^{n}\left(\Lambda_{k}(\mathbf{y})\right) \leq \varepsilon_{n} \sigma_{\ell}^{n}\left(\Lambda_{k^{*}}(\mathbf{y})\right), \quad \forall k \neq k^{*}
$$

Taking the limits as $n$ tends to infinity of these highlighted expressions establishes that $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $\Lambda^{*}(\mathbf{y})$ via $\Lambda(\mathbf{y})$.

Lemma F.2. Let $\mathrm{x} \in \mathcal{X}_{1} \times \cdots \times \mathcal{X}_{m},\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ a sequence of totally mixed strategies which tends to $x_{i}$ via $\mathbf{x}$, and $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ a sequence of positive scalars with $\lim _{n \rightarrow \infty} \varepsilon_{n}=0$. Suppose that, for all $n, \sigma_{f}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{\ell}^{n}$. Then $\sigma_{f}^{n} \underset{n \rightarrow \infty}{\longrightarrow} \mathbf{R}(\mathbf{x})$.

Proof: Consider two strategies of the follower, $\mathbf{y}$ and $\mathbf{y}^{\prime}$, with $y_{j}^{\prime}=R\left(x_{j}\right) \neq y_{j}$ and $y_{k}^{\prime}=y_{k}$ for all $k \neq j$. Then

$$
\begin{aligned}
V_{f}\left(\mathbf{y} \mid \sigma_{\ell}^{n}\right)-V_{f}\left(\mathbf{y}^{\prime} \mid \sigma_{\ell}^{n}\right) & =\sum_{\tilde{x} \in \mathcal{X}_{j}} \sigma_{\ell}^{n}(\tilde{x})\left[u\left(y_{j}, \tilde{x}\right)-u\left(y_{j}^{\prime}, \tilde{x}\right)\right] \\
& =\left(\sum_{\tilde{x} \in \mathcal{X}_{j}} \sigma_{\ell}^{n}(\tilde{x})\right) \sum_{\tilde{x} \in \mathcal{X}_{j}} \frac{\sigma_{\ell}^{n}(\tilde{x})}{\sum_{\tilde{x} \in \mathcal{X}_{j}} \sigma_{\ell}^{n}(\tilde{x})}\left[u\left(y_{j}, \tilde{x}\right)-u\left(y_{j}^{\prime}, \tilde{x}\right)\right] .
\end{aligned}
$$

As $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $x_{i}$ via $\mathbf{x}$, we conclude that, for all sufficiently large $n, V_{f}\left(\mathbf{y} \mid \sigma_{\ell}^{n}\right)<V_{f}\left(\mathbf{y}^{\prime} \mid\right.$ $\left.\sigma_{\ell}^{n}\right)$. We then find by induction that, for all sufficiently large $n$,

$$
V_{f}\left(\mathbf{y} \mid \sigma_{\ell}^{n}\right)<V_{f}\left(\mathbf{R}(\mathbf{x}) \mid \sigma_{\ell}^{n}\right), \quad \forall \mathbf{y} \neq \mathbf{R}(\mathbf{x})
$$

Since $\sigma_{f}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{\ell}^{n}$, the previous inequality implies that, for all sufficiently large $n$,

$$
\sigma_{f}^{n}(\mathbf{y})<\varepsilon_{n} \sigma_{f}^{n}(\mathbf{R}(\mathbf{x})), \quad \forall \mathbf{y} \neq \mathbf{R}(\mathbf{x})
$$

As $\lim _{n \rightarrow \infty} \varepsilon_{n}=0$, we obtain $\sigma_{f}^{n} \underset{n \rightarrow \infty}{\longrightarrow} \mathbf{R}(\mathbf{x})$.

Proposition F.1. Let $(x, y)$ be a pure-strategy proper equilibrium of $G$, with $x \in \mathcal{X}{ }_{i}$. Then $\left(x, y_{i}\right)$ constitutes a plausible outcome.

Proof: The fact that $(x, y)$ is a pure-strategy proper equilibrium of $G$ tells us that there exist sequences of totally mixed strategies $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ and $\left\{\sigma_{f}^{n}\right\}_{n \in \mathbb{N}}$ with $\lim _{n \rightarrow \infty} \sigma_{\ell}^{n}=x$ and $\lim _{n \rightarrow \infty} \sigma_{f}^{n}=\mathbf{y}$, as well as a sequence of positive scalars $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ with $\lim _{n \rightarrow \infty} \varepsilon_{n}=0$, such that $\sigma_{j}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{-j}^{n}$ (for every $j \in\{\ell, f\}$ and all $n \in \mathbb{N}$ ). Then, by virtue of Lemma F.1, $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $\Lambda^{*}(\mathbf{y})$ via $\boldsymbol{\Lambda}(\mathbf{y})$. Thus,

$$
\begin{equation*}
x=\Lambda^{*}(\mathbf{y}), \tag{34}
\end{equation*}
$$

and, by virtue of LemmaF.2,

$$
\begin{equation*}
\mathbf{y}=\mathbf{R}(\boldsymbol{\Lambda}(\mathbf{y})) \tag{35}
\end{equation*}
$$

Now define $\beta: K \rightarrow \mathcal{X}$ such that, for $k=1, \ldots, m$ :

$$
\begin{equation*}
\beta\left(\mathcal{X}_{k}\right)=\Lambda_{k}(\mathbf{y}) . \tag{36}
\end{equation*}
$$

The combination of (33) and (35) shows that $(K, \beta)$ constitutes an admissible pair. Finally, we claim that $(K, \beta)$ implements $x$. By (34), it is enough to show that $\Lambda^{*}(\mathbf{y})$ maximizes $u(\tilde{x}, R(\tilde{x}))$ over all $\tilde{x} \in \beta(\mathcal{X})=\left\{\Lambda_{1}(\mathbf{y}), \ldots, \Lambda_{m}(\mathbf{y})\right\}$. Yet, by definition, $u\left(\Lambda_{k^{*}}(\mathbf{y}), y_{k^{*}}\right)>u\left(\Lambda_{k}(\mathbf{y}), y_{k}\right)$ for all $k \neq k^{*}$. So (35) finishes to show that $(K, \beta)$ implements $x$. Then $\left(x, y_{k^{*}}\right)$ is a plausible outcome since, using (34) and (35):

$$
R(x)=R\left(\Lambda^{*}(\mathbf{y})\right)=R\left(\Lambda_{k^{*}}(\mathbf{y})\right)=y_{k^{*}}
$$

Proposition F.2. Let $\left(x^{*}, y^{*}\right)$ be a plausible outcome, with $x^{*} \in \mathcal{X}_{i}$. Then there exists $\mathbf{y}$, with $y_{i}=y^{*}$, such that $\left(x^{*}, \mathbf{y}\right)$ is a pure-strategy proper equilibrium of $G$.

Proof: Let $(K, \beta)$ be an admissible pair that implements $\left(x^{*}, y^{*}\right)$. Let $\boldsymbol{\beta} \in \mathcal{X}_{1} \times \cdots \times \mathcal{X}_{m}$ be given by $\beta_{k}=\beta\left(\mathcal{X}_{k}\right)$, for $k=1, \ldots, m$. Then $\beta_{k}$ maximizes $u\left(\cdot, R\left(\beta_{k}\right)\right)$ over $\mathcal{X}_{k}$. In view of (33), we conclude that

$$
\begin{equation*}
\boldsymbol{\beta}=\boldsymbol{\Lambda}(\mathbf{R}(\boldsymbol{\beta})) . \tag{37}
\end{equation*}
$$

Furthermore, by definition,

$$
u\left(\Lambda_{k^{*}}(\mathbf{R}(\boldsymbol{\beta})), R\left(\beta_{k^{*}}\right)\right)>u\left(\Lambda_{k}(\mathbf{R}(\boldsymbol{\beta})), R\left(\beta_{k}\right)\right), \text { for all } k \neq k^{*}
$$

By (37), this implies that $\Lambda_{k^{*}}(\mathbf{R}(\boldsymbol{\beta}))$ maximizes $u(\tilde{x}, R(\tilde{x}))$ over all $\tilde{x} \in\left\{\beta_{1}, \ldots, \beta_{m}\right\}$. Recalling that $\Lambda^{*}(\mathbf{R}(\boldsymbol{\beta}))=\Lambda_{k^{*}}(\mathbf{R}(\boldsymbol{\beta}))$ then yields

$$
\begin{equation*}
x^{*}=\Lambda^{*}(\mathbf{R}(\boldsymbol{\beta})) . \tag{38}
\end{equation*}
$$

Next, let $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ be a sequence of positive scalars converging to 0 , and pick a sequence $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ of totally mixed strategies of the leader such that, for every $n, \sigma_{\ell}^{n}$ satisfies the $\varepsilon_{n}$ criterion against $\mathbf{R}(\boldsymbol{\beta})$. By Lemma F.1. $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $\Lambda^{*}(\mathbf{R}(\boldsymbol{\beta}))$ via $\boldsymbol{\Lambda}(\mathbf{R}(\boldsymbol{\beta}))$, that is, using (37) and (38), $\left\{\sigma_{\ell}^{n}\right\}_{n \in \mathbb{N}}$ tends to $x^{*}$ via $\boldsymbol{\beta}$.

Now let $\left\{\sigma_{f}^{n}\right\}_{n \in \mathbb{N}}$ be a sequence of totally mixed strategies of the follower such that, for each $n, \sigma_{f}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\sigma_{\ell}^{n}$. By Lemma $\mathrm{F} .2, \sigma_{f}^{n} \underset{n \rightarrow \infty}{\longrightarrow} \mathbf{R}(\boldsymbol{\beta})$. Since the strategy $\mathbf{R}(\boldsymbol{\beta})$ of the follower induces a strict ranking over the pure strategies of the leader, we conclude that, for all sufficiently large $n, \sigma_{f}^{n}$ induces the same strict ranking as $\mathbf{R}(\boldsymbol{\beta})$. As $\sigma_{\ell}^{n}$ satisfies the $\varepsilon_{n}$-criterion against $\mathbf{R}(\boldsymbol{\beta})$, the previous remark implies that, for all sufficiently large $n, \sigma_{\ell}^{n}$ will satisfy the $\varepsilon_{n}$-criterion against $\sigma_{f}^{n}$. In sum, $\sigma_{\ell}^{n}$ tends to $x^{*}$ via $\boldsymbol{\beta}, \sigma_{f}^{n}$ tends to $\mathbf{R}(\boldsymbol{\beta})$, and, for all sufficiently large $n$, each of these strategies satisfies the $\varepsilon_{n}$-criterion against the other. So $\left(x^{*}, \mathbf{R}(\boldsymbol{\beta})\right)$ is a proper equilibrium of $G$.


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[^1]:    ${ }^{1}$ For a study of more general commitment devices that include commitments contingent on other players' commitments, see Kalai, Kalai, Lehrer and Samet (2010).
    ${ }^{2}$ For a multi-stage model of commitment where the base-game itself is an extensive-form game, see Arieli, Babichenko and Tennenholtz (2017).
    ${ }^{3}$ Relatedly, Pei (2016) studies a setting where a player can restrict the actions of her opponent. Pei (2016) shows that when a player has limited commitment ability, i.e., cannot reduce her opponent's action set to a singleton, then it is sometimes strictly optimal not to restrict the opponent's actions at all.

[^2]:    ${ }^{4}$ That is, (i) for all $\mathcal{X}_{i} \in K: \mathcal{X}_{i} \subseteq \mathcal{X}$; (ii) for all $x \in \mathcal{X}: x \in \mathcal{X}_{i}$ for some $\mathcal{X}_{i} \in K$.

[^3]:    ${ }^{5}$ Recall, the follower's action space is compact, and $v(y, x)$ is strictly quasi-concave in $y$.
    ${ }^{6}$ The upper contour set of an action $x$ with respect to $U$ is the set of actions $\tilde{x}$ such that $U(\tilde{x}) \geq U(x)$.
    ${ }^{7}$ Quantities larger than $2 /(2-r)$ would lead to negative profits no matter what.

[^4]:    ${ }^{8}$ The leader's action space being compact and $u_{11}$ negative, to every $y \in \mathcal{Y}$ corresponds a unique best response of the leader.

[^5]:    ${ }^{9}$ Of course, $K$ could be infinite, in which case the argument sketched in the text needs to be modified. See Appendix A
    ${ }^{10} \mathrm{An}$ action $x^{*} \in[5 / 17,5 / 9] \cup[5 / 4,5 / 2]$ is for instance implemented by the pair $(K, \beta)$ where $K=$ $\left\{\left[0, x^{*}\right),\left[x^{*}, 5 / 2\right]\right\}, \beta\left(\left[0, x^{*}\right)\right)=0$, and $\beta\left(\left[x^{*}, 5 / 2\right]\right)=x^{*}$.

[^6]:    ${ }^{11}$ This setting might capture a situation in which two firms with complementary production processes choose the locations of their plants. The first two terms of the function $u$ capture the firms' desire to be close to each other. The remaining terms capture intrinsic features specific to the different locations.

[^7]:    ${ }^{12} \mathrm{An}$ action $x^{*} \in(5 / 9,5 / 4)$ is for instance implemented by the pair $(K, \beta)$ where $K=\left\{[0,5 / 2],\left[0, x^{*}\right]\right\}$, $\beta([0,5 / 2])=0$, and $\beta\left(\left[0, x^{*}\right]\right)=x^{*}$. To see that no $x^{*} \in(0,5 / 17)$ is I-plausible, notice that the intersection between $\{x: \phi(x) \geq x\}$ and the lower contour set of $x^{*}$ with respect to $U$ is empty.

[^8]:    ${ }^{13}$ We thank an anonymous referee for pointing this out to us.
    ${ }^{14}$ See the Appendix $C$ for the general case.
    ${ }^{15}$ Formally, externalities are strictly positive (respectively, negative) if $u$ and $v$ are strictly increasing (respectively, decreasing) in their second arguments.

[^9]:    ${ }^{16}$ Formally, payoffs are strictly supermodular (respectively, submodular) if $u\left(x^{\prime}, y^{\prime}\right)+u(x, y)>u\left(x^{\prime}, y\right)+$ $u\left(x, y^{\prime}\right)$ (respectively, $\left.u\left(x^{\prime}, y^{\prime}\right)+u(x, y)<u\left(x^{\prime}, y\right)+u\left(x, y^{\prime}\right)\right)$ for all $x^{\prime}>x$ and $y^{\prime}>y$. Supermodular payoffs capture strategic complementarities; submodular payoffs capture strategic substitutabilities.
    ${ }^{17}$ Here $x^{C}$ denotes the Cournot action.
    ${ }^{18}$ The continuity of $\gamma$ is inherited from the continuity of $u$ and $R_{F}$.

[^10]:    ${ }^{19}$ The upper level set of $\underline{U}$ with respect to $U$ is defined as $\{x: U(x) \geq \underline{U}\}$.
    ${ }^{20}$ Intuitively, increasing the action of the leader induces the follower to increase her action too (due to strategic complementarities), and this benefits the leader (since externalities are positive).

[^11]:    ${ }^{21}$ The upper level set of $\underline{U}$ corresponds to $\left[5 / 18, \hat{x}_{2}\right]$. Since the upper contour set of $x^{C}$ with respect to $U$ is $\left[x^{C}, \hat{x}_{1}\right]$, actions in the intervals $\left[5 / 18, x^{C}\right)$ and $\left(\hat{x}_{1}, \hat{x}_{2}\right]$ are P-plausible but are not simply plausible.

[^12]:    ${ }^{22}$ See Myerson (1997) for a discussion of the merits and flaws of forward induction.

[^13]:    ${ }^{23}$ We extend the standard definition of sequential equilibria Kreps and Wilson, 1982 to our model as follows: a pure-strategy sequential equilibrium of $\widehat{G}(K)$ is a triple $\left(x^{*}, f, \beta\right)$ with $x^{*} \in \mathcal{X}, f: K \rightarrow \mathcal{Y}$, and $\beta: K \rightarrow \mathcal{X}$ such that, letting $\mathcal{X}_{i^{*}}$ denote the element of $K$ comprising $x^{*}:(i) u\left(x^{*}, f\left(\mathcal{X}_{i^{*}}\right)\right) \geq u\left(x, f\left(\mathcal{X}_{i}\right)\right)$ for all $x \in \mathcal{X}$ and $\mathcal{X}_{i} \in K$ comprising $x$, (ii) $f\left(\mathcal{X}_{i}\right)=R_{F}\left(\beta\left(\mathcal{X}_{i}\right)\right)$, (iii) $\beta\left(\mathcal{X}_{i}\right) \in \mathcal{X}_{i}$ for all $\mathcal{X}_{i} \in K$.
    ${ }^{24}$ This condition is equivalent to $\eta\left(x, \beta\left(\mathcal{X}_{i}\right)\right) \leq 0$, for all $x \in \mathcal{X}_{i}$ and all $\mathcal{X}_{i} \in K$.

[^14]:    ${ }^{25}$ As Lemma 3 holds "partition by partition", this observation can be generalized to any restricted class of partitions, such as simple CSTs.

[^15]:    ${ }^{26}$ The expression for consumer surplus is based on the representative consumer utility function, given by $4(x+y)+d x y-(x+y)^{2} / 2$.

[^16]:    ${ }^{29}$ Intuitively, a richer CST means that the leader is able to commit to a smaller set of actions (but does not have to), whereas a finer CST means that the leader must commit to a smaller set of actions.

[^17]:    ${ }^{30}$ Example B has $a=0$. Intuitively, the effect of $a>0$ is that coordination towards the middle outcome $(1 / 2,1 / 2)$ is slightly better for both players.

[^18]:    ${ }^{31}$ Recall, $u_{2}>0$ is shorthand notation for positive externalities and $u_{12}>0$ for supermodular payoffs.
    ${ }^{32}$ The function $u\left(\cdot, R_{F}(x)\right)$ being strictly quasi-concave and maximized at $\phi(x)$, it ensues that $x<x^{C}$ implies $\eta(x+\varepsilon, x)>0$ for all sufficiently small $\varepsilon>0$.

[^19]:    ${ }^{33} \mathrm{As} \phi$ is continuous, notice that

    $$
    \begin{cases}\phi(x)>x & \text { for } x<x^{C}  \tag{11}\\ \phi(x)<x & \text { for } x>x^{C}\end{cases}
    $$

[^20]:    ${ }^{34}$ The exact values of $a_{1}$ and $a_{2}$ do not change the conclusions.

[^21]:    ${ }^{35}$ Note that the latter condition is satisfied "generically".

