DIPS AT SMALL SIZES FOR TOPOLOGICAL GRAPH OBSTRUCTION SETS

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ABSTRACT. The Graph Minor Theorem of Robertson and Seymour implies a finite set of obstructions for any minor closed graph property. We show that there are only three obstructions to knotless embedding of size 23, which is far fewer than the 92 of size 22 and the hundreds known to exist at larger sizes. We describe several other topological properties whose obstruction set demonstrates a similar dip at small size. For order ten graphs, we classify the 35 obstructions to knotless embedding and the 49 maximal knotless graphs.

1. INTRODUCTION

The Graph Minor Theorem of Robertson and Seymour [RS] implies that any minor closed graph property $\mathcal{P}$ is characterized by a finite set of obstructions. For example, planarity is determined by K_5 and $K_{3,3}$ [K, W] while linkless embeddability has seven obstructions, known as the Petersen family [RST]. However, Robertson and Seymour's proof is highly nonconstructive and it remains frustratingly difficult to identify the obstruction set, even for a simple property such as apex (see [JK]). Although we know that the obstruction set for a property $\mathcal{P}$ is finite, in practice it is often difficult to establish any useful bounds on its size.

In the absence of concrete bounds, information about the shape or distribution of an obstruction set would be welcome. Given that the number of obstructions is finite, one might anticipate a unimodal distribution with a central maximum and numbers tailing off to either side. Indeed, many of the known obstruction sets do appear to follow this pattern. In [MW, Table 2], the authors present a listing of more than 17 thousand obstructions for torus embeddings. Although, the list is likely incomplete, it does appear to follow a normal distribution both with respect to graph *order* (number of vertices) and graph *size* (number of edges), see Figure 1.

Size	18	19	20	21	22	23	$\cdots$	28	29	30	31
Count	6	19	8	123	517	2821	$\cdots$	299	8	4	1

TABLE 1. Count of torus embedding obstructions by size.

However, closer inspection (see Table 1) shows that there is a *dip*, or local minimum, in the number of obstructions at size twenty. We will say that

(A) By order

(B) By size

FIGURE 1. Distribution of torus embedding obstructions

the dip occurs at *small size* meaning it is near the end of the left tail of the size distribution.

In this paper, we will see that the knotless embedding property likewise has a dip at size 23. A *knotless embedding* of a graph is an embedding in $\mathbb{R}^3$ such that each cycle is a trivial knot. Having noticed this dip we investigated what other topological properties have a dip, or even *gap* (a size, or range of sizes, for which there is no obstruction), at small sizes. We report on what we found in the next section. In a word, the most prominent dips and gaps seem to trace back to that perennial counterexample, the Petersen graph.

In Section 3, we prove the following.

Theorem 1.1. *There are exactly three obstructions to knotless embedding of size 23.*

Since there are no obstructions of size 20 or less, 14 of size 21, 92 of size 22 and at least 156 of size 28 (see [FMMNN, GMN]), the theorem shows that the knotless embedding obstruction set has a dip at small size, 23.

The proof of Theorem 1.1 naturally breaks into two parts. We show "by hand" that the three obstruction graphs have no knotless embedding and are minor minimal for that property. To show these three are the **only** obstructions of size 23 we make use of the connection with 2-apex graphs.
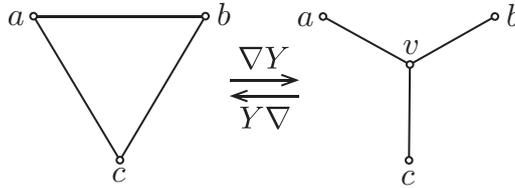
FIGURE 2. ∇Y and $Y\nabla$ moves.

This second part of the argument is novel. While our analysis depends on computer calculations, the resulting observations may be of independent interest. To describe these, we review some terminology for graph families, see [GMN]. The *family* of graph G is the set of graphs related to G by a sequence of zero or more ∇Y and $Y\nabla$ moves, see Figure 2. The graphs in G's family are *cousins* of G. We do not allow $Y\nabla$ moves that would result in doubled edges and all cousins have the same size. If a ∇Y move on G results in graph H, we say H is a *child* of G and G is a *parent* of H. The

set of graphs that can be obtained from G by a sequence of ∇Y moves are G's *descendants*. Similarly, the set of graphs that can be obtained from G by a sequence of $Y\nabla$ moves are G's *ancestors*.

To show that the three graphs are the only obstructions of size 23 relies on a careful analysis of certain graph families with respect to knotless embedding. This analysis includes progress in resolving the following question.

Question 1.2. *If G has a vertex of degree less than three, can an ancestor or descendant of G be an obstruction for knotless embedding?*

It has been fruitful to search in graph families for obstructions to knotless embedding. For example, of the 264 known obstructions described in [FMMNN], all but three occur as part of four graph families. The same paper states "It is natural to investigate the graphs obtained by adding one edge to each of ... six graphs" in the family of the Heawood graph. We carry out this investigation and classify, with respect to knotless embedding, the graphs obtained by adding an edge to a Heawood family graph; see Section 3 for details. As a first step toward a more general strategy for this type of problem, we make the following connections between sets of graphs of the form $G + e$ obtained by adding an edge to graph G.

Theorem 1.3. *If G is a parent of H, then every $G + e$ has a cousin that is an $H + e$.*

Corollary 1.4. *If G is an ancestor of H, then every $G + e$ has a cousin that is an $H + e$.*

In Section 4, we prove the following.

Theorem 1.5. *There are exactly 35 obstructions to knotless embedding of order ten.*

This depends on a classification of the maximal knotless graphs of order ten, that is the graphs that are edge maximal in the set of graphs that admit a knotless embedding, see [EFM]. In Appendix B we show that there are 49 maximal knotless graphs of order ten.

In contrast to graph size, distributions with respect to graph order generally do not have dips or gaps. In particular, Theorem 1.5 continues an increasing trend of no obstructions of order 6 or less, one obstruction of order 7 [CG], two of order 8 [CMOPRW, BBFFHL], and eight of order 9 [MMR].

In the next section we discuss some graph properties for which we know something about the obstruction set, with a focus on those that have a dip at small size. In Sections 3 and 4, we prove Theorems 1.1 and 1.5, respectively. Appendix A is a traditional proof that the graphs G_1 and G_2 are IK. In Appendix B we describe the 49 maxnik graphs of order ten. Finally, Appendix C gives graph6 [Sg] notation for the important graphs and further details of arguments throughout the paper including the structure of the large families that occur at the end of subsection 3.2.

2. Dips at small size

As mentioned in the introduction, it remains difficult to determine the obstruction set even for simple graph properties. In this section we survey

Size	15	16	17	18	19	20	21
Count	7	0	0	4	5	22	33

TABLE 2. Count of apex obstructions through size 21.

Size	21	22	23
Count	20	60	0

TABLE 3. Count of 2-apex obstructions through size 23.

some topological graph properties for which we know something about the obstruction set.

We begin by focusing on four properties that feature a prominent dip or gap at small sizes. Two are the obstructions to knotless and torus embeddings mentioned in the introduction. Although the list of torus obstructions is likely incomplete we can be confident about the dip at size 20. Like all of the incomplete sets we look at, research has focused on smaller sizes such that data on this side of the distribution is (nearly) complete. In the specific case of torus obstructions, we can compare with a 2002 study [C] that listed 16,682 torus obstructions. Of the close to one thousand graphs added to the set in the intervening decade and a half, only three are of size 23 or smaller.

Similarly, while our proof that there are three obstructions for knotless embedding depends on computer verification, it seems certain that the number of obstructions at size 23 is far smaller than the 92 of size 22 and the number for size 28, which is known to be at least 156 [GMN].

The set of apex obstructions, investigated by [JK, LMMPRTW, P], suggests one possible source for these dips. A graph is *apex* if it is planar, or becomes planar on deletion of a single vertex. As yet, we do not have a complete listing of the apex obstruction set, but Jobson and Kézdy [JK] report that there are at least 401 obstructions. Table 2 shows the classification of obstructions through size 21 obtained by Pierce [P] in his senior thesis. There is a noticeable gap at sizes 16 and 17. The seven graphs of size 15 are the graphs in the Petersen family. This is the family of the Petersen graph, which is also the obstruction set for linkless embedding. Note that, as for all our tables of size distributions, the table begins with what is known to be the smallest size in the distribution, in this case 15.

Our proof that there are only three knotless embeddding obstructions of size 23 depends on a related property, 2-apex. We say that a graph is *2-apex* if it is apex, or becomes apex on deletion of a single vertex. Table 3 shows the little that we know about the obstruction set for this family [MP]. Aside from the counts for sizes 21 and 22 and the gap at size 23, we know only that there are obstructions for each size from 24 through 30.

While it is not an explanation, these four properties with notable dips or gaps are related to one another and seem to stem from the Petersen graph, a notorious counterexample in graph theory. The most noticeable gap is for the apex property and the seven graphs to the left of the gap are precisely the Petersen family. The gap at size 23 for the 2-apex property is doubtless related. In turn, our proof of Theorem 1.1 in Section 3 relies heavily on the

Size	15	16	17	18	19	20
Count	4	7	10	10	2	2

TABLE 4. Count of projective planar obstructions by size.

strong connection between 2-apex and knotless graphs. For this reason, it is not surprising that the gap at size 23 for 2-apex obstructions results in a similar dip at size 23 for obstructions for knotless embeddings.

The connection with the dip at size 20 for torus embeddings is not as direct, but we remark that eight of the obstructions of size 19 have a minor that is either a graph in the Petersen family, or else one of those seven graphs with a single edge deleted.

In contrast, let us briefly mention some well known obstruction sets that do not have dips or gaps and are instead unimodal. There are two obstructions to planarity, one each of size nine ($K_{3,3}$) and ten (K_5). The two obstructions, K_4 and $K_{3,2}$, to outerplanarity both have size six and the seven obstructions to linkless embedding in the Petersen family [RST] are all of size 15.

Aside from planarity and linkless embedding, the most famous set is likely the 35 obstructions to projective planar embedding [A, GHW, MT]. Table 4 shows that the size distribution for these obstructions is unimodal.

3. KNOTLESS EMBEDDING OBSTRUCTIONS OF SIZE 23

In this section we prove Theorem 1.1: there are exactly three obstructions to knotless embedding of size 23. Along the way (see subsection 3.2) we provide evidence in support of a negative answer to Question 1.2 and prove Theorem 1.3 and its corollary. We also classify, with respect to knotless embedding, three graph families of size 22. These families include every graph obtained by adding an edge to a Heawood family graph.

We begin with some terminology. A graph that admits no knotless embedding is *intrinsically knotted (IK)*. In contrast, we will call the graphs that admit a knotless embedding *not intrinsically knotted (nIK)*. If G is in the obstruction set for knotless embedding we will say G is *minor minimal intrinsically knotted (MMIK)*. This reflects that, while G is IK, no proper minor of G has that property. Similarly, we will call 2-apex obstructions *minor minimal not 2-apex (MMN2A)*.

Our strategy for classifying MMIK graphs of size 23 is based on the following observation.

Lemma 3.1. [BBFFHL, OT] *If G is 2-apex, then G is not IK*

Suppose G is MMIK of size 23. By Lemma 3.1, G is not 2-apex and, therefore, G has an MMN2A minor. The MMN2A graphs through order 23 were classified in [MP]. All but eight of them are also MMIK and none are of size 23. It follows that a MMIK graph of size 23 has one of the eight exceptional MMN2A graphs as a minor. Our strategy is to construct all size 23 expansions of the eight exceptional graphs and determine which of those is in fact MMIK.

Before further describing our search, we remark that it does rely on computer support. Indeed, the initial classification of MMN2A graphs in [MP] is itself based on a computer search. We give a traditional proof that there are three size 23 MMIK graphs, which is stated as Theorem 3.2 below. We rely on computers only for the argument that there are no other size 23 MMIK graphs. Note that, even if we cannot provide a complete, traditional proof that there are no more than three size 23 MMIK graphs, our argument does strongly suggest that there are far fewer MMIK graphs of size 23 than the known 92 MMIK graphs of size 22 and at least 156 of size 28 [FMMNN, GMN]. In other words, even without computers, we have compelling evidence that there is a dip at size 23 for the obstructions to knotless embedding.

Below we give graph6 notation [Sg] and edge lists for the three MMIK graphs of size 23. See also Figures 15 and 16 in Appendix A.

G_1 J@yaig[gv@?

$[(0, 4), (0, 5), (0, 9), (0, 10), (1, 4), (1, 6), (1, 7), (1, 10), (2, 3), (2, 4), (2, 5), (2, 9),$
$\quad (2, 10), (3, 6), (3, 7), (3, 8), (4, 8), (5, 6), (5, 7), (5, 8), (6, 9), (7, 9), (8, 10)]$

G_2 JObFF'wN?{?

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (1, 5), (1, 6), (1, 7), (1, 8), (2, 6), (2, 7), (2, 8),$
$\quad (2, 9), (3, 7), (3, 8), (3, 9), (3, 10), (4, 8), (4, 9), (4, 10), (5, 9), (5, 10), (6, 10)]$

G_3 K?bAF'wN?{SO

$[(0, 4), (0, 5), (0, 7), (0, 11), (1, 5), (1, 6), (1, 7), (1, 8), (2, 7), (2, 8), (2, 9), (2, 11),$
$\quad (3, 7), (3, 8), (3, 9), (3, 10), (4, 8), (4, 9), (4, 10), (5, 9), (5, 10), (6, 10), (6, 11)]$

The graph G_1 was discovered by Hannah Schwartz [N1]. Graphs G_1 and G_2 have order 11 while G_3 has order 12. We prove the following in subsection 3.1 below.

Theorem 3.2. *The graphs G_1, G_2, and G_3 are MMIK of size 23*

We next describe the computer search that shows there are no other size 23 MMIK graphs and completes the proof of Theorem 1.1. We also describe how Question 1.2, Theorem 1.3 and its corollary fit in, along with the various size 22 families.

There are eight exceptional graphs of size at most 23 that are MMN2A and not MMIK. Six of them are in the Heawood family of size 21 graphs. The other two, H_1 and H_2, are 4-regular graphs on 11 vertices with size 22 described in [MP], listed in Appendix C, and shown in Figure 4.

It turns out that the three graphs of Theorem 3.2 are expansions of H_1 and H_2. In subsection 3.3 we show that these two graphs are MMN2A but not IK and argue that no other size 23 expansion of H_1 or H_2 is MMIK.

The Heawood family consists of twenty graphs of size 21 related to one another by ∇Y and $Y\nabla$ moves, see Figure 2. In [GMN, HNTY] two groups, working independently, verified that 14 of the graphs in the family are MMIK, and the remaining six are MMN2A and not MMIK. In subsection 3.2 below, we argue that no size 23 expansion of any of these six Heawood family graphs is MMIK. Combining the arguments of the next three subsections give a proof of Theorem 1.1.

There are two Mathematica programs written by Naimi and available at his website [N2] that we use throughout. One, isID4, is an implementation of the algorithm of Miller and Naimi [MN] and we refer the reader to that paper for details. Note that, while this algorithm can show that a particular graph is IK, it does not allow us to deduce that a graph is nIK.

Instead, we make use of a second program of Naimi, findEasyKnots, to find knotless embeddings of nIK graphs. This program determines the set of cycles Σ of a graph G. Then, given an embedding of G, for each $\sigma \in \Sigma$ the program applies $R1$ and $R2$ Reidemeister moves (see [R]) until it arrives at one of three possible outcomes: σ is the unknot; σ is an alternating (hence non-trivial) knot; or σ is a non-alternating knot (which may or may not be trivial). In this paper, we will often show that a graph is nIK by presenting a knotless embedding. In all cases, this means that when we apply findEasyKnots to the embedding, it determines that every cycle in the graph is a trivial knot.

Before diving into our proof of Theorem 1.1, we state a few lemmas we will use throughout. The first is about the *minimal degree* $\delta(G)$, which is the least degree among the vertices of graph G.

Lemma 3.3. *If G is MMIK, then $\delta(G) \geq 3$.*

Proof. Suppose G is IK with $\delta(G) < 3$. By either deleting, or contracting an edge on, a vertex of small degree, we find a proper minor that is also IK. $\square$
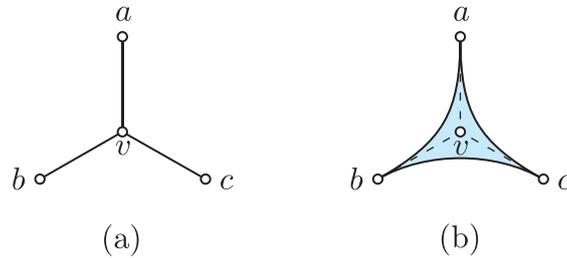
FIGURE 3. Place a triangle in a neighborhood of the Y subgraph.

Lemma 3.4. *The ∇Y move preserves IK: If G is IK and H is obtained from G by a ∇Y move, then H is also IK. Equivalently, the $Y\nabla$ move preserves nIK: if H is nIK and G is obtained from H by a $Y\nabla$ move, then G is also nIK.*

Proof. We begin by noting that the $Y\nabla$ move preserves planarity. Suppose H is planar and has an induced Y or $K_{3,1}$ subgraph with degree three vertex v adjacent to vertices a, b, c. In a planar embedding of H we can choose a neighborhood of the Y subgraph small enough that it excludes all other edges of H, see Figure 3. This allows us to place a 3-cycle a, b, c within the neighborhood which shows that the graph G that results from a $Y\nabla$ move is also planar.

Now, suppose H is nIK with degree three vertex v. Then, as in Figure 3, in a knotless embedding of H, we can find a neighborhood of the induced

Y subgraph small enough that it intersects no other edges of H. Again, place a 3-cycle a, b, c within this neighborhood. We claim that the resulting embedding of G obtained by this Y∇ move is likewise knotless. Indeed, any cycle in G that uses vertices a, b, or c has a corresponding cycle in H that differs only by a small move within the neighborhood of the Y subgraph. Since every cycle of H is unknotted, the same is true for every cycle in this embedding of G. □

Finally, we note that the MMIK property can move backwards along ∇Y moves.

Lemma 3.5. [BDLST, OT] *Suppose G is IK and H is obtained from G by a ∇Y move. If H is MMIK, then G is also MMIK.*

3.1. Proof of Theorem 3.2.

In this subsection we prove Theorem 3.2: the three graphs G_1, G_2, and G_3 are MMIK.

We first show these graphs are IK. For G_1 and G_2 we present a proof "by hand" as Appendix A. We remark that we can also verify that these two graphs are IK using Naimi's Mathematica implementation [N2] of the algorithm of Miller and Naimi [MN].

The graph G_3 is obtained from G_2 by a single ∇Y move. Specifically, using the edge list for G_2 given above:

$$[(0,2),(0,4),(0,5),(0,6),(0,7),(1,5),(1,6),(1,7),(1,8),(2,6),(2,7),(2,8),$$

$$(2,9),(3,7),(3,8),(3,9),(3,10),(4,8),(4,9),(4,10),(5,9),(5,10),(6,10)],$$

make the ∇Y move on the triangle $(0,2,6)$. Since G_2 is IK, Lemma 3.4 implies G_3 is also IK.

To complete the proof of Theorem 3.2, it remains only to show that all proper minors of G_1, G_2 and G_3 are nIK.

First we argue that no proper minor of G_1 is IK. Up to isomorphism, there are 12 minors obtained by contracting or deleting a single edge. Each of these is 2-apex, except for the MMN2A graph H_1. By Lemma 3.1, a 2-apex graph is nIK and Figure 4 gives a knotless embedding of H_1 (as we have verified using Naimi's program findEasyKnots, see [N2]). This shows that all proper minors of G_1 are nIK.
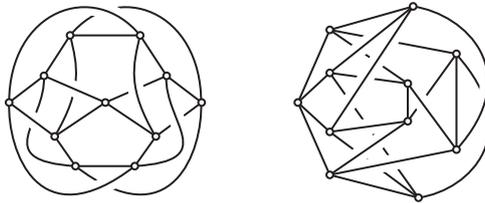
FIGURE 4. Knotless embeddings of graphs H_1 (left) and H_2 (right).

Next we argue that no proper minor of G_2 is IK. Up to isomorphism, there are 26 minors obtained by deleting or contracting an edge of G_2. Each of these is 2-apex, except for the MMN2A graph H_2. Since H_2 has a knotless

embedding as shown in Figure 4, then, similar to the argument for G_1, all proper minors of G_2 are nIK.

It remains to argue that no proper minor of G_3 is IK. Up to isomorphism, there are 26 minors obtained by deleting or contracting an edge of G_3. Each of these is 2-apex, except for the MMN2A graph H_2 which is nIK. Similar to the previous cases, all proper minors of G_3 are nIK. This completes the proof of Theorem 3.2.

3.2. Expansions of nIK Heawood family graphs.

In this subsection we argue that there are no MMIK graphs among the size 23 expansions of the six nIK graphs in the Heawood family. As part of the argument, we classify with respect to knotless embedding all graphs obtained by adding an edge to a Heawood family graph. We also discuss our progress on Question 1.2 and prove Theorem 1.3 and its corollary.

We will use the notation of [HNTY] to describe the twenty graphs in the Heawood family, which we also recall in Appendix C. For the reader's convenience, Figure 5 shows four of the graphs in the family that are central to our discussion.
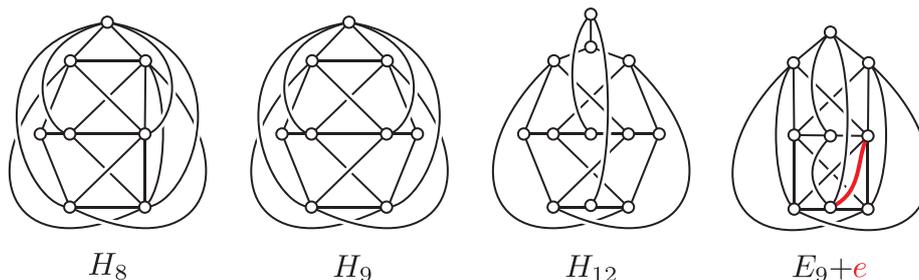
FIGURE 5. Four graphs in the Heawood family. (E_9 is in the family, but $E_9 + e$ is not.)

Kohara and Suzuki [KS] showed that 14 graphs in this family are MMIK. The remaining six, N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, and N'_{12}, are nIK [GMN, HNTY]. The graph N_9 is called E_9 in [GMN]. In this subsection we argue that no size 23 expansion of these six graphs is MMIK.

The Heawood family graphs are the cousins of the Heawood graph, which is denoted C_{14} in [HNTY]. All have size 21. We can expand a graph to one of larger size either by adding an edge or by splitting a vertex. In *splitting a vertex* we replace a graph G with a graph G' so that the order increases by one: $|G'| = |G| + 1$. This means we replace a vertex v of G with two vertices v_1 and v_2 in G' and identify the remaining vertices of G' with those of $V(G) \setminus \{v\}$. As for edges, $E(G')$ includes the edge $v_1 v_2$. In addition, we require that the union of the neigborhoods of v_1 and v_2 in G' otherwise agrees with the neighborhood of v: $N(v) = N(v_1) \cup N(v_2) \setminus \{v_1, v_2\}$. In other words, G is the result of contracting $v_1 v_2$ in G' where double edges are suppressed: $G = G'/v_1 v_2$.

Our goal is to argue that there is no size 23 MMIK graph that is an expansion of one of the six nIK Heawood family graphs, N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, and N'_{12}. As a first step, we will argue that, if there were such a size

23 MMIK expansion, it would also be an expansion of one of 29 nIK graphs of size 22.

Given a graph G, we will use $G + e$ to denote a graph obtained by adding an edge $e \notin E(G)$. As we will show, if G is a Heawood family graph, then $G + e$ will fall in one of three families that we will call the $H_8 + e$ family, the $E_9 + e$ family, and the $H_9 + e$ family. The $E_9 + e$ family is discussed in [GMN] where it is shown to consist of 110 graphs, all IK.

The $H_8 + e$ graph is formed by adding an edge to the Heawood family graph H_8 between two of its degree 5 vertices. The $H_8 + e$ family consists of 125 graphs, 29 of which are nIK and the remaining 96 are IK, as we will now argue. For this, we leverage graphs in the Heawood family. In addition to $H_8 + e$, the family includes an $F_9 + e$ graph formed by adding an edge between the two degree 3 vertices of F_9. Since H_8 and F_9 are both IK [KS], the corresponding graphs with an edge added are as well. By Lemma 3.4, $H_8 + e$, $F_9 + e$ and all their descendants are IK. These are the 96 IK graphs in the family.

The remaining 29 graphs are all ancestors of six graphs that we describe below. Once we establish that these six are nIK, then Lemma 3.4 ensures that all 29 are nIK.

We will denote the six graphs T_i, $i = 1, \ldots, 6$ where we have used the letter T since five of them have a degree two vertex ('T' being the first letter of 'two'). After contracting an edge on the degree 2 vertex, we recover one of the nIK Heawood family graphs, N_{11} or N'_{12}. It follows that these five graphs are also nIK.

The two graphs that become N_{11} after contracting an edge have the following graph6 notation[Sg]:

T_1: `KSrb'OTO?a'S` T_2: `KOtA'_LWCMSS`

The three graphs that contract to N'_{12} are:

T_3: `LSb'@OLOASASCS` T_4: `LSrbP?CO?dAIAW` T_5: `L?tBP_SODGOS_T`

The five graphs we have described so far along with their ancestors account for 26 of the nIK graphs in the $H_8 + e$ family. The remaining three are ancestors of

T_6: `KSb''OMSQSAK`

Figure 6 shows a knotless embedding of T_6. By Lemma 3.4, its two ancestors are also nIK and this completes the count of 29 nIK graphs in the $H_8 + e$ family.
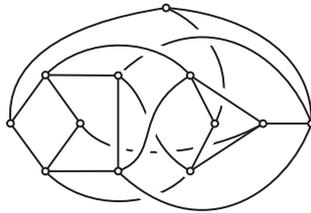
FIGURE 6. A knotless embedding of the T_6 graph.

The graph $H_9 + e$ is formed by adding an edge to H_9 between the two degree 3 vertices. There are five graphs in the $H_9 + e$ family, four of which are $H_9 + e$ and its descendants. Since H_9 is IK [KS], by Lemma 3.4, these

four graphs are all IK. The remaining graph in the family is the MMIK graph denoted G_S in [FMMNN] and shown in Figure 7. Although the graph is credited to Schwartz in that paper, it was a joint discovery of Schwartz and and Barylskiy [N1]. Thus, all five graphs in the $H_9 + e$ family are IK.
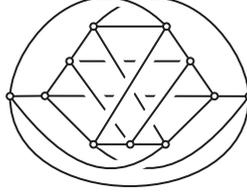
FIGURE 7. The MMIK graph G_S.

Having classified the graphs in the three families with respect to intrinsic knotting, using Corollary 3.11 below, we have completed the investigation, suggested by [FMMNN], of graphs formed by adding an edge to a Heawood family graph.

We remark that among the three families $H_8 + e$, $E_9 + e$, and $H_9 + e$, the only instances of a graph with a degree 2 vertex occur in the family $H_8 + e$, which also contains no MMIK graphs. This observation suggests the following question.

Question 1.2. *If G has minimal degree $\delta(G) < 3$, is it true that G's ancestors and descendants include no MMIK graphs?*

Initially, we suspected that such a G has no MMIK cousins at all. However, we discovered that the MMIK graph of size 26, described in Section 4 below, includes graphs of minimal degree two among its cousins. Although we have not completely resolved the question, we have two partial results.

Theorem 3.6. *If $\delta(G) < 3$ and H is a descendant of G, then H is not MMIK.*

Proof. Since $\delta(G)$ is non-increasing under the ∇Y move, $\delta(H) \leq \delta(G) < 3$ and H is not MMIK by Lemma 3.3. $\square$

As defined in [GMN] a graph has a $\bar{Y}$ if there is a degree 3 vertex that is also part of a 3-cycle. A $Y\nabla$ move at such a vertex would result in doubled edges.

Lemma 3.7. *A graph with a $\bar{Y}$ is not MMIK.*

Proof. Let G have a vertex v with $N(v) = \{a, b, c\}$ and $ab \in E(G)$. We can assume G is IK. Make a ∇Y move on triangle v, a, b to obtain the graph H. By Lemma 3.4 H is IK, as is the homeomorphic graph H' obtained by contracting an edge at the degree 2 vertex c. But $H' = G - ab$ is obtained by deleting an edge ab from G. Since G has a proper subgraph H' that is IK, G is not MMIK. $\square$

Theorem 3.8. *If G has a child H with $\delta(H) < 3$, then G is not MMIK.*

Proof. By Lemma 3.3, we can assume G is IK with $\delta(G) = 3$. It follows that G has a $\bar{Y}$ and is not MMIK by the previous lemma. $\square$

Suppose G is a size 23 MMIK expansion of one of the six nIK Heawood family graphs. We will argue that G must be an expansion of one of the graphs in the three families, $H_8 + e$, $E_9 + e$, and $H_9 + e$. However, as a MMIK graph, G can have no size 22 IK minor. Therefore, G must be an expansion of one of the 29 nIK graphs in the $H_8 + e$ family.

There are two ways to form a size 22 expansion of one of the six nIK graphs, either add an edge or split a vertex. We now show that if H is in the Heawood family, then $H + e$ is in one of the three families, $H_8 + e$, $E_9 + e$, and $H_9 + e$. We begin with a proof of a theorem and corollary, mentioned in the introduction, that describe how adding an edge to a graph G interacts with the graph's family.

Theorem 1.3. *If G is a parent of H, then every $G + e$ has a cousin that is an $H + e$.*

Proof. Let H be obtained by a ∇Y move that replaces the triangle abc in G with three edges on the new vertex v. That is, $V(H) = V(G) \cup \{v\}$. Form $G + e$ by adding the edge $e = xy$. Since $V(H) = V(G) \cup \{v\}$, then $x, y \in V(H)$ and the graph $H + e$ is a cousin of $G + e$ by a ∇Y move on the triangle abc. $\square$

Corollary 1.4. *If G is an ancestor of H, then every $G + e$ has a cousin that is an $H + e$.*

Every graph in the Heawood family is an ancestor of one of two graphs, the Heawood graph (called C_{14} in [HNTY]) and the graph H_{12} (see Figure 5).

Theorem 3.9. *Let H be the Heawood graph. Up to isomorphism, there are two $H + e$ graphs. One is in the $H_8 + e$ family, the other in the $E_9 + e$ family.*

Proof. The diameter of the Heawood graph is three. Up to isomorphism, we can either add an edge between vertices of distance two or three. If we add an edge between vertices of distance two, the result is a graph in the $H_8 + e$ family. If the distance is three, we are adding an edge between the different parts and the result is a bipartite graph of size 22. As shown in [KMO], this means it is cousin 89 of the $E_9 + e$ family. $\square$

Theorem 3.10. *Let G be formed by adding an edge to H_{12}. Then G is in the $H_8 + e$, $E_9 + e$, or $H_9 + e$ family.*

Proof. Note that H_{12} consists of six degree 4 vertices and six degree 3 vertices. Moreover, five of the degree 3 vertices are created by ∇Y moves in the process of obtaining H_{12} from K_7. Let a_i $(i = 1 \ldots 5)$ denote those five degree 3 vertices. Further assume that b_1 is the remaining degree 3 vertex and b_2, b_3, b_4, b_5, b_6 and b_7 are the remaining degree 4 vertices. Then the b_j vertices correspond to vertices of K_7 before applying the ∇Y moves.

First suppose that G is obtained from H_{12} by adding an edge which connects two b_j vertices. Since these seven vertices are the vertices of K_7 before using ∇Y moves, there is exactly one vertex among the a_i, say a_1, that is adjacent to the two endpoints of the added edge. Let G' be the graph obtained from G by applying $Y \nabla$ moves at a_2, a_3, a_4 and a_5. Then G' is isomorphic to $H_8 + e$. Therefore G is in the $H_8 + e$ family.

Next suppose that G is obtained from H_{12} by adding an edge which connects two a_i vertices. Let a_1 and a_2 be the endpoints of the added edge. We assume that G' is obtained from G by using $Y\nabla$ moves at a_3, a_4 and a_5. Then there are two cases: either G' is obtained from H_9 or F_9 by adding an edge which connects two degree 3 vertices. In the first case, G' is isomorphic to $H_9 + e$. Thus G is in the $H_9 + e$ family. In the second case, G' is in the $H_8 + e$ or $E_9 + e$ family by Corollary 1.4 and Theorem 3.9. Thus G is in the $H_8 + e$ or $E_9 + e$ family.

Finally suppose that G is obtained from H_{12} by adding an edge which connects an a_i vertex and a b_j vertex. Let a_1 be a vertex of the added edge. We assume that G' is the graph obtained from G by using $Y\nabla$ moves at a_2, a_3, a_4 and a_5. Since G' is obtained from H_8 by adding an edge, G' is in the $H_8 + e$ or $E_9 + e$ family by Corollary 1.4 and Theorem 3.9. Therefore G is in the $H_8 + e$ or $E_9 + e$ family. $\qquad\square$

Corollary 3.11. *If H is in the Heawood family, then $H + e$ is in the $H_8 + e$, $E_9 + e$, or $H_9 + e$ family.*

Proof. The graph H is either an ancestor of the Heawood graph or H_{12}. Apply Corollary 1.4 and Theorems 3.9 and 3.10. $\qquad\square$

Corollary 3.12. *If H is in the Heawood family and $H + e$ is nIK, then $H + e$ is one of the 29 nIK graphs in the $H_8 + e$ family*

Lemma 3.13. *Let H be a nIK Heawood family graph and G be an expansion obtained by splitting a vertex of H. Then either G has a vertex of degree at most two, or else it is in the $H_8 + e$, $E_9 + e$, or $H_9 + e$ family.*

Proof. Note that $\Delta(H) \leq 5$. If G has no vertex of degree at most two, then the vertex split produces a vertex of degree three. A $Y\nabla$ move on the degree three vertex produces G' which is of the form $H + e$. $\qquad\square$

Corollary 3.14. *Suppose G is nIK and a size 22 expansion of a nIK Heawood family graph. Then either G has a vertex of degree at most two or G is in the $H_8 + e$ family.*

Theorem 3.15. *Let G be size 23 MMIK with a minor that is a nIK Heawood family graph. Then G is an expansion of one of the 29 nIK graphs in the $H_8 + e$ family.*

Proof. There must be a size 22 graph G' intermediate to G and the Heawood family graph H. That is, G is an expansion of G', which is an expansion of H. By Corollary 3.14, we can assume G' has a vertex v of degree at most two.

By Lemma 3.3, a MMIK graph has minimal degree, $\delta(G) \geq 3$. Since G' expands to G by adding an edge or splitting a vertex, we conclude v has degree two exactly and G is G' with an edge added at v. Since $\delta(H) \geq 3$, this means G' is obtained from H by a vertex split.

In G', let $N(v) = \{a, b\}$ and let cv be the edge added to form G. Then $H = G'/av$ and we recognize $H + ac$ as a minor of G. We are assuming G is MMIK, so $H + ac$ is nIK and, by Corollary 3.12, one of the 29 nIK graphs in the $H_8 + e$ family. Thus, G is an expansion of $H + ac$, which is one of these 29 graphs, as required. $\qquad\square$

It remains to study the expansions of the 29 nIK graphs in the $H_8 + e$ family. We will give an overview of the argument, leaving many of the details to Appendix C.

The size 23 expansions of the 29 size 22 nIK graphs fall into one of eight families, which we identify by the number of graphs in the family: $\mathcal{F}_9$, $\mathcal{F}_{55}$ $\mathcal{F}_{174}$, $\mathcal{F}_{183}$, $\mathcal{F}_{547}$, $\mathcal{F}_{668}$, $\mathcal{F}_{1229}$, and $\mathcal{F}_{1293}$. We list the graphs in each family in Appendix C.

Theorem 3.16. *If G is a size 23 MMIK expansion of a nIK Heawood family graph, then G is in one of the eight families, $\mathcal{F}_9$, $\mathcal{F}_{55}$ $\mathcal{F}_{174}$, $\mathcal{F}_{183}$, $\mathcal{F}_{547}$, $\mathcal{F}_{668}$, $\mathcal{F}_{1229}$, and $\mathcal{F}_{1293}$.*

Proof. By Theorem 3.15, G is an expansion of G', which is one of the 29 nIK graphs in the $H_8 + e$ family. As we have seen, these 29 graphs are ancestors of the six graphs $T_1, \ldots, T_6$. By Corollary 1.4, we can find the $G' + e$ graphs by looking at the six graphs. Given the family listings in Appendix C, it is straightforward to verify that each $T_i + e$ is in one of the eight families. This accounts for the graphs G obtained by adding an edge to one of the 29 nIK graphs in the $H_8 + e$ family.

If instead G is obtained by splitting a vertex of G', we use the strategy of Lemma 3.13. By Lemma 3.3, $\delta(G) \geq 3$. Since $\Delta(G') \leq 5$, the vertex split must produce a vertex of degree three. Then, a $Y\nabla$ move on the degree three vertex produces G'' which is of the form $G' + e$. Thus G is a cousin of $G' + e$ and must be in one of the eight families. □

To complete our argument that there is no size 23 MMIK graph with a nIK Heawood family minor, we argue that there are no MMIK graphs in the eight families $\mathcal{F}_9, \mathcal{F}_{55}, \ldots, \mathcal{F}_{1293}$. In large part our argument is based on two criteria that immediately show a graph G is not MMIK.

(1) $\delta(G) < 3$, see Lemma 3.3.
(2) By deleting an edge, there is a proper minor $G - e$ that is an IK graph in the $H_8 + e$, $E_9 + e$, or $H_9 + e$ families. In this case G is IK, but not MMIK.

By Lemma 3.5, if G has an ancestor that satisfies criterion 2, then G is also not MMIK. By Lemma 3.4, if G has a nIK descendant, then G is also not nIK.

Theorem 3.17. *There is no MMIK graph in the $\mathcal{F}_9$ family.*

Proof. Four of the nine graphs satisfy the first criterion, $\delta(G) = 2$, and these are not MMIK by Lemma 3.3. The remaining graphs are descendants of a graph G that is IK but not MMIK. Indeed, G satisfies criterion 2: by deleting an edge, we recognize $G - e$ as an IK graph in the $H_9 + e$ family (see Appendix C for details). By Lemma 3.5, G and its descendants are also not MMIK. □

Theorem 3.18. *There is no MMIK graph in the $\mathcal{F}_{55}$ family.*

Proof. All graphs in this family have $\delta(G) \geq 3$, so none satisfy the first criterion. All but two of the graphs in this family are not MMIK by the second criterion. The remaining two graphs have a common parent that is IK but not MMIK. By Lemma 3.5, these last two graphs are also not MMIK. See Appendix C for details. □

We remark that $\mathcal{F}_{55}$ is the only one of the eight families that has no graph with $\delta(G) < 3$.

Theorem 3.19. *There is no MMIK graph in the $\mathcal{F}_{174}$ family.*

Proof. All but 51 graphs are not MMIK by the first criterion. Of the remaining graphs, all but 17 are not MMIK by the second criterion. Of these, 11 are descendants of a graph G that is IK but not MMIK by the second criterion. By Lemma 3.5, these 11 are also not MMIK. This leaves six graphs. For these we find two nIK descendants. By Lemma 3.4 the remaining six are also not MMIK. Both descendants have a degree 2 vertex. On contracting an edge of the degree 2 vertex, we obtain a homeomorphic graph that is one of the 29 nIK graphs in the $H_8 + e$ family. $\qquad\square$

Theorem 3.20. *There is no MMIK graph in the $\mathcal{F}_{183}$ family.*

Proof. All graphs in this family have a vertex of degree two or less and are not MMIK by the first criterion. $\qquad\square$

Theorem 3.21. *There is no MMIK graph in the $\mathcal{F}_{547}$ family.*

Proof. All but 229 of the graphs in the family are not MMIK by criterion one. Of those, all but 52 are not MMIK by criterion two. Of those, 25 are ancestors of one of the graphs meeting criterion two and are not MMIK by Lemma 3.5. For the remaining 27 graphs, all but five have a nIK descendant and are not IK by Lemma 3.4. For the remaining five, three are ancestors of one of the five. In Figure 8 we give knotless embeddings of the other two graphs. Using Lemma 3.4, all five graphs are nIK, hence not MMIK. $\qquad\square$
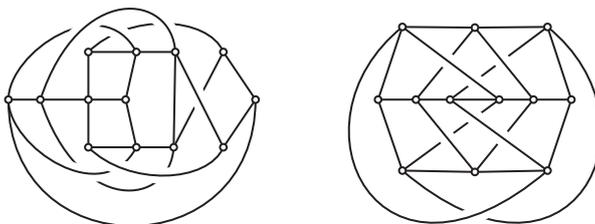
FIGURE 8. Knotless embeddings of two graphs in $\mathcal{F}_{547}$.

Theorem 3.22. *There is no MMIK graph in the $\mathcal{F}_{668}$ family.*

Proof. All but 283 of the graphs in the family are not MMIK by criterion one. Of those, all but 56 are not MMIK by criterion two. Of those, 23 are ancestors of one of the graphs meeting criterion two and are not MMIK by Lemma 3.5. For the remaining 33 graphs all but three have a nIK descendant and are not IK by Lemma 3.4. Of the remaining three, two are ancestors of the third. Figure 9 is a knotless embedding of the common descendant. By Lemma 3.4 all three of these graphs are nIK, hence not MMIK. $\qquad\square$

Theorem 3.23. *There is no MMIK graph in the $\mathcal{F}_{1229}$ family.*
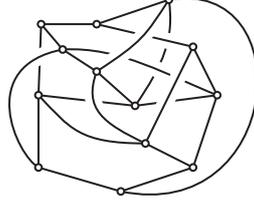
FIGURE 9. Knotless embedding of a graph in $\mathcal{F}_{668}$.

Proof. There are 268 graphs in the family that are not MMIK by criterion one. Of the remaining 961 graphs, all but 140 are not MMIK by criterion two. Of those, all but three are ancestors of one of the graphs meeting criterion two and are not MMIK by Lemma 3.5. The remaining three graphs have an IK minor by contracting an edge and are, therefore, not MMIK. □

Theorem 3.24. *There is no MMIK graph in the $\mathcal{F}_{1293}$ family.*

Proof. There are 570 graphs in the family that are not MMIK by criterion one. Of the remaining 723 graphs, all but 99 are not MMIK by criterion two. Of those, all but 12 are ancestors of one of the graphs meeting criterion two and are not MMIK by Lemma 3.5. The remaining 12 graphs have an IK minor by contracting an edge and are, therefore, not MMIK. □

3.3. **Expansions of the size 22 graphs H_1 and H_2.** We have argued that a size 23 MMIK graph must have a minor that is either one of six nIK graphs in the Heawood family, or else one of two $(11, 22)$ graphs that we call H_1: `J?B@xzoyEo?` and H_2: `J?bFF'wN?{?` (see Figure 4). We treated expansions of the Heawood family graphs in the previous subsection. In this subsection we show that G_1, G_2, and G_3 are the only size 23 MMIK expansions of H_1 and H_2. Recall that these two graphs were shown MMN2A (minor minimal for 2-apex) in [MP]. The unknotted embeddings of Figure 4 demonstrate that they are nIK.

By Lemma 3.3, if a vertex split of H_1 results in a vertex of degree less than three, the resulting graph is not MMIK. Since H_1 is 4-regular, the only other way to make a vertex split produces adjacent degree 3 vertices. Then, a $Y\nabla$ move on one of the degree three vertices yields an $H_1 + e$. Thus, a size 23 MMIK expansion of H_1 must be in the family of a $H_1 + e$.

Up to isomorphism, there are six $H_1 + e$ graphs formed by adding an edge to H_1. These six graphs generate families of size 6, 2, 2, and 1. Three of the six graphs are in the family of size 6 and there is one each in the remaining three families.

All graphs in the family of size six are ancestors of three graphs. In Figure 10 we provide knotless embeddings of those three graphs. By Lemma 3.4, all graphs in this family are nIK, hence not MMIK.

In a family of two graphs, there is a single ∇Y move. In Figure 11 we give knotless embeddings of the children in these two families. By Lemma 3.4, all graphs in these two families are nIK, hence not MMIK.

The unique graph in the family of size one is G_1. In subsection 3.1 we show that this graph is MMIK. Using the edge list of G_1 given above near
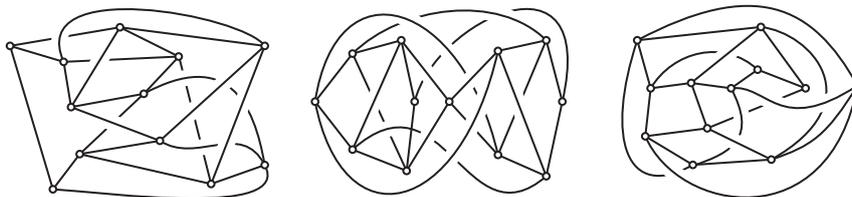
FIGURE 10. Knotless embeddings of three graphs in the size six family from H_1.
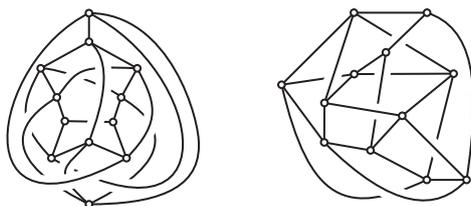
FIGURE 11. Knotless embeddings of two graphs in the size two families from H_1.

the beginning of Section 3:

$$[(0,4),(0,5),(0,9),(0,10),(1,4),(1,6),(1,7),(1,10),(2,3),(2,4),(2,5),(2,9),$$
$$(2,10),(3,6),(3,7),(3,8),(4,8),(5,6),(5,7),(5,8),(6,9),(7,9),(8,10)],$$

we recover H_1 by deleting edge $(2,5)$.

Again, since H_2 is 4-regular, MMIK expansions formed by vertex splits (if any) will be in the families of $H_2 + e$ graphs. Up to isomorphism, there are three $H_2 + e$ graphs. These produce a family of size four and another of size two.
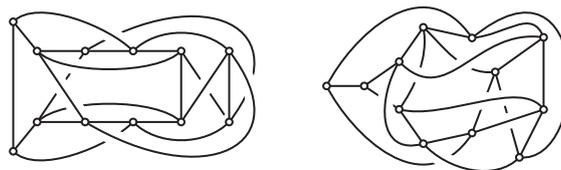
FIGURE 12. Knotless embeddings of two graphs in the size four family from H_2.

The family of size four includes two $H_2 + e$ graphs. All graphs in the family are ancestors of the two graphs that are each shown to have a knotless embedding in Figure 12. By Lemma 3.4, all graphs in this family are nIK, hence not MMIK.

The family of size two consists of the graphs G_2 and G_3. In subsection 3.1 we show that these two graphs are MMIK. Using the edge list for G_2 given above near the beginning of Section 3:

$$[(0,2),(0,4),(0,5),(0,6),(0,7),(1,5),(1,6),(1,7),(1,8),(2,6),(2,7),(2,8),$$
$$(2,9),(3,7),(3,8),(3,9),(3,10),(4,8),(4,9),(4,10),(5,9),(5,10),(6,10)],$$

we recover H_2 by deleting edge $(0, 2)$. As for G_3:

$$[(0, 4), (0, 5), (0, 7), (0, 11), (1, 5), (1, 6), (1, 7), (1, 8), (2, 7), (2, 8), (2, 9), (2, 11),$$

$$(3, 7), (3, 8), (3, 9), (3, 10), (4, 8), (4, 9), (4, 10), (5, 9), (5, 10), (6, 10), (6, 11)],$$

contracting edge $(6, 11)$ leads back to H_2.

4. Knotless embedding obstructions of order ten.

In this section, we prove Theorem 1.5: there are exactly 35 obstructions to knotless embedding of order ten. As in the previous section, we refer to knotless embedding obstructions as MMIK graphs. We first describe the 26 graphs given in [FMMNN, MNPP] and then list the 9 new graphs unearthed by our computer search.

4.1. **26 previously known order ten MMIK graphs.** In [FMMNN], the authors describe 264 MMIK graphs. There are three sporadic graphs (none of order ten), the rest falling into four graph families. Of these, 24 have ten vertices and they appear in the families as follows.

There are three MMIK graphs of order ten in the Heawood family [KS, GMN, HNTY]: H_{10}, F_{10}, and E_{10}.

In [GMN], the authors study the other three families. All 56 graphs in the $K_{3,3,1,1}$ family are MMIK. Of these, 11 have order ten: Cousins 4, 5, 6, 7, 22, 25, 26, 27, 28, 48, and 51. There are 33 MMIK graphs in the family of $E_9 + e$. Of these seven have order ten: Cousins 3, 28, 31, 41, 44, 47, and 50. Finally, the family of $G_{9,28}$ includes 156 MMIK graphs. Of these, there are three of order ten: Cousins 2, 3, and 4.

The other two known MMIK graphs of order ten are described in [MNPP], one having size 26 and the other size 30. We remark that the family for the graph of size 26 includes both MMIK graphs and graphs with $\delta(G) = 2$. However, no ancestor or descendant of a $\delta(G) = 2$ graph is MMIK. This is part of our motivation for Question 1.2

4.2. **Nine new MMIK graphs of order ten.** In this subsection we list the nine additional MMIK graphs that we found after a computer search described following the list. In each case, we use the program of [MN] to verify that the graph we found is IK. We use the Mathematica implementation of the program available at Ramin Naimi's website [N2]. To show that the graph is MMIK, we must in addition verify that each minor formed by deleting or contracting an edge is nIK. Many of these minors are 2-apex and not IK by Lemma 3.1. There remain 21 minors and below we discuss how we know that those are also nIK.

First we list the nine new MMIK graphs of order ten, including size, graph6 format [Sg], and an edge list.

(1) Size: 25; graph6 format: `ICrfbp{No`

$$[(0, 3), (0, 4), (0, 5), (0, 6), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8),$$

$$(2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 6), (3, 7), (3, 8),$$

$$(3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 9), (7, 9)]$$

(2) Size: 25; graph6 format: `ICrbrrqNg`

$$[(0,3),(0,4),(0,5),(0,8),(1,4),(1,5),(1,6),(1,7),(1,8),$$
$$(2,5),(2,6),(2,7),(2,8),(2,9),(3,6),(3,7),(3,8),$$
$$(3,9),(4,6),(4,7),(4,9),(5,9),(6,8),(6,9),(8,9)]$$

(3) Size: 25; graph6 format: `ICrbrriVg`

$$[(0,3),(0,4),(0,5),(0,8),(1,4),(1,5),(1,6),(1,7),(1,8),$$
$$(1,9),(2,5),(2,6),(2,7),(2,8),(3,6),(3,7),(3,9),$$
$$(4,6),(4,7),(4,8),(4,9),(5,9),(6,8),(6,9),(8,9)]$$

(4) Size: 25; graph6 format: `ICrbrriNW`

$$[(0,3),(0,4),(0,5),(0,8),(1,4),(1,5),(1,6),(1,7),(1,8),$$
$$(2,5),(2,6),(2,7),(2,8),(2,9),(3,6),(3,7),(3,9),$$
$$(4,6),(4,7),(4,8),(4,9),(5,9),(6,8),(7,9),(8,9)]$$

(5) Size: 27; graph6 format: `ICfvRzwfo`

$$[(0,3),(0,4),(0,5),(0,6),(0,8),(0,9),(1,5),(1,6),(1,7),$$
$$(1,8),(2,5),(2,6),(2,7),(2,8),(3,4),(3,5),(3,7),(3,8),$$
$$(3,9),(4,6),(4,7),(4,8),(4,9),(5,7),(5,9),(6,9),(7,9)]$$

(6) Size: 29; graph6 format: `ICfvRr^vo`

$$[(0,3),(0,4),(0,5),(0,6),(0,8),(0,9),(1,5),(1,6),(1,7),(1,8),$$
$$(1,9),(2,5),(2,6),(2,7),(3,4),(3,5),(3,7),(3,8),(3,9),(4,6),$$
$$(4,7),(4,8),(4,9),(5,8),(5,9),(6,8),(6,9),(7,8),(7,9)]$$

(7) Size: 30; graph6 format: `IQjuvrm^o`

$$[(0,2),(0,4),(0,5),(0,6),(0,7),(0,8),(1,3),(1,5),(1,6),(1,7),$$
$$(1,8),(1,9),(2,4),(2,5),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),$$
$$(3,9),(4,6),(4,7),(4,8),(4,9),(5,8),(5,9),(6,8),(6,9),(7,9)]$$

(8) Size: 31; graph6 format: `IQjur~m^o`

$$[(0,2),(0,4),(0,5),(0,6),(0,8),(1,3),(1,5),(1,6),(1,7),(1,8),(1,9),$$
$$(2,4),(2,5),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),(3,9),(4,6),$$
$$(4,7),(4,8),(4,9),(5,7),(5,8),(5,9),(6,7),(6,8),(6,9),(7,9)]$$

(9) Size: 32; graph6 format: `IEznfvm|o`

$$[(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(1,3),(1,4),(1,5),(1,6),$$
$$(1,7),(1,8),(1,9),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(3,6),(3,7),$$
$$(3,9),(4,5),(4,7),(4,8),(5,8),(5,9),(6,7),(6,8),(6,9),(7,9)]$$
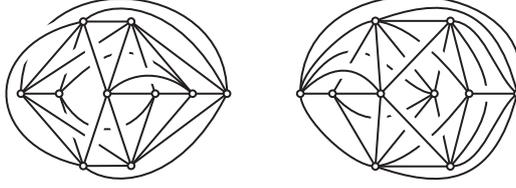
FIGURE 13. Knotless embeddings of two order ten graphs.

To complete our argument, it remains to argue that the 21 non 2-apex minors are nIK. Each of these minors is formed by deleting or contracting an edge in one of the nine graphs just listed. Of these minors, 19 have a 2-apex descendant and are nIK by Lemmas 3.1 and 3.4. In Figure 13 we give knotless embeddings of the remaining two minors showing that they are also nIK.

Let us describe our computer search. A MMIK graph G of order ten must be connected, have $\delta(G) \leq 3$ (by Lemma 3.3) and have size (number of edges) between 21 and 35. For, the lower bound on size, see [M]. The upper bound follows as Mader [Ma] showed that a graph with $n \geq 7$ vertices and $5n - 14$ or more edges has a K_7 minor. Since K_7 is IK (see [CG]), this means that a graph of order ten with 36 or more edges cannot be MMIK as it has a proper IK minor, K_7. There remain just under 5 million graphs to consider.

We next sieve out any graph that has one of the 26 known MMIK graphs of order ten (see subsection 4.1) as a subgraph or any of the known MMIK graphs of order less than ten as a minor. We also discard any 2-apex graph, which must be nIK by Lemma 3.1. We test the remaining graphs using Ramin Naimi's implementation [N2] of the algorithm of Miller and Naimi [MN]. Those which were found to be IK led us to the nine new MMIK graphs listed above. Those graphs that we could not otherwise classify led us to a list of 35 graphs that we subsequently showed to be maxnik: following [EFM], we say that a graph G is *maximal knotless* or *maxnik* if G is nIK, but every $G + e$ is IK. In Appendix B we provide a classification of the 49 maxnik graphs of order ten. In addition to the 35 graphs just mentioned, there are 14 maximal 2-apex graphs.

Once we determined the 35 MMIK obstructions and the 35 maxnik (and not 2-apex) graphs of order ten, we were in a position to show that the remaining nearly 5 million candidates G are not MMIK for at least one of the following three reasons: 1) G is 2-apex and nIK by Lemma 3.1, 2) G has a proper minor that is IK and is therefore not MMIK, or 3) G is a subgraph of one of the 35 maxnik graphs and is therefore nIK.

Having determined the 49 maxnik graphs of order ten (see Appendix B), we can update some observations of [EFM], which catalogs the maxnik graphs through order nine. The fourteen maximally 2-apex graphs are of the form $K_2 * T_8$, the join of K_2 and a planar triangulation on eight vertices. As such, each has 35 edges and $\Delta(G) = 9$. The remaining 35 maxnik graphs range between size 23 and 34. In particular, we still know of no maxnik graph on 22 edges. To continue Tables 1 and 2 of [EFM], the least $|E|/|V|$ among maxnik graphs of order ten is 23/10 and we have $2 \leq \delta(G) \leq 6$ and

$5 \leq \Delta(G) \leq 9$. Although we still have no regular maxnik graph, there are several examples where $\Delta(G) - \delta(G)$ is only one.

APPENDIX A. PROOF BY HAND THAT G_1 AND G_2 ARE IK

In this section, we give a traditional proof (ie one that does not rely on computers) that G_1 and G_2 are IK. For this we use a lemma due, independently, to two groups [F, TY]. Let $\mathcal{D}_4$ denote the multigraph of Figure 14 and, for $i = 1, 2, 3, 4$, let C_i be the cycle of edges e_{2i-1}, e_{2i}. For any given embedding of $\mathcal{D}_4$, let σ denote the mod 2 sum of the Arf invariants of the 16 Hamiltonian cycles in $\mathcal{D}_4$ and $\text{lk}(C, D)$ the mod 2 linking number of cycles C and D. Since the Arf invariant of the unknot is zero, an embedding of $\mathcal{D}_4$ with $\sigma \neq 0$ must have a knotted cycle.
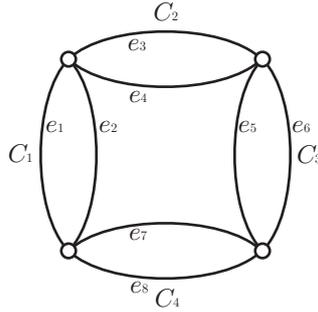
FIGURE 14. The $\mathcal{D}_4$ graph.

Lemma A.1. [F, TY] *Given an embedding of $\mathcal{D}_4$, $\sigma \neq 0$ if and only if $lk(C_1, C_3) \neq 0$ and $lk(C_2, C_4) \neq 0$.*

A.1. G_1 is IK.

In this subsection, we show that G_1 is IK. To use Lemma A.1 we need pairs of linked cycles. We first describe three sets of pairs that we will call A_i's, B_i's, and C_i's

Step I: Define A_i pairs.

To find pairs of linked cycles, we use minors of G_1 that are members of the Petersen family of graphs as these are the obstructions to linkless embedding [RST]. Our first example is based on contracting the edges $(3, 6)$, $(5, 7)$, and $(6, 9)$ in Figure 15 to produce a $K_{4,4}$ minor with $\{3, 4, 5, 10\}$ and $\{0, 1, 2, 8\}$ as the two parts. For convenience, we use the smallest vertex label to denote the new vertex obtained when contracting edges. Thus, we denote by 3 the vertex obtained by identifying 3, 6, and 9 of G_1. Further deleting the edge $(1, 3)$ we identify the Petersen family graph $K_{4,4}^-$ as a minor of G_1.
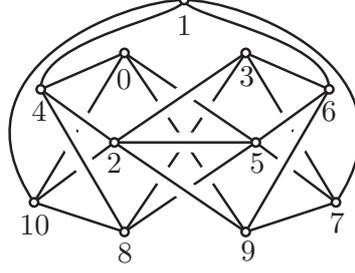
FIGURE 15. The graph G_1.

There are nine pairs of disjoint cycles in $K_{4,4}^-$ and we denote these pairs as A_1 through A_9. In Table 5, we first give the cycle pair in the $K_{4,4}^-$ and then the corresponding pair in G_1.

A_1	0,4,1,5 − 2,3,8,10	0,4,1,7,5 − 2,3,8,10
A_2	0,4,1,10 − 2,3,8,5	0,4,1,10 − 2,3,8,5
A_3	1,4,2,5 − 0,3,8,10	1,4,2,5,7 − 0,9,6,3,8,10
A_4	1,4,2,10 − 0,3,8,5	1,4,2,10 − 0,9,6,3,8,5
A_5	1,4,8,5 − 0,3,2,10	1,4,8,5,7 − 0,9,6,3,2,10
A_6	1,4,8,10 − 0,3,2,5	1,4,8,10 − 0,9,6,3,2,5
A_7	1,5,2,10 − 0,3,8,4	1,7,5,2,10 − 0,9,6,3,8,4
A_8	0,5,1,10 − 2,3,8,4	0,5,7,1,10 − 2,3,8,4
A_9	1,5,8,10 − 0,3,2,4	1,7,5,8,10 − 0,9,6,3,2,4

TABLE 5. Nine pairs of cycles in G_1 called $A_1, \ldots, A_9$.

Step II: Find pairs B_i and C_i.

Similarly, we will describe a $K_{3,3,1}$ minor that gives pairs of cycles B_1 through B_9. Contract edges $(1,7)$, $(3,7)$, and $(7,9)$. Delete vertex 6 and edge $(2,9)$. The result is a $K_{3,3,1}$ minor with parts $\{0,2,8\}$, $\{4,5,10\}$, and $\{1\}$. In Table 6 we give the nine pairs of cycles, first in $K_{3,3,1}$ and then in G_1.

B_1	0,1,4 − 2,5,8,10	0,4,1,7,9 − 2,5,8,10
B_2	0,1,5 − 2,4,8,10	0,5,7,9 − 2,4,8,10
B_3	0,1,10 − 2,4,8,5	0,9,7,1,10 − 2,4,8,5
B_4	1,2,4 − 0,5,8,10	1,4,2,3,7 − 0,5,8,10
B_5	1,2,5 − 0,4,8,10	2,3,7,5 − 0,4,8,10
B_6	1,2,10 − 0,4,8,5	2,3,7,1,10 − 0,4,8,5
B_7	1,4,8 − 0,5,2,10	1,4,8,3,7 − 0,5,2,10
B_8	1,5,8 − 0,4,2,10	3,7,5,8 − 0,4,2,10
B_9	1,8,10 − 0,4,2,5	1,7,3,8,10 − 0,4,2,5

TABLE 6. Nine pairs of cycles in G_1 called $B_1, \ldots, B_9$.

Another $K_{4,4}^-$ minor of G_1 will give our last set of nine cycle pairs. Contract edges $(0,9)$, $(2,5)$, and $(3,8)$ to obtain a $K_{4,4}$ with parts $\{0,1,2,8\}$

and $\{4, 6, 7, 10\}$. Then delete edge $(1, 4)$ to make a $K_{4,4}^-$ minor. Table 7 lists the nine pairs of cycles, first in the $K_{4,4}^-$ minor and then in G_1.

C_1	0,6,1,7 – 2,4,8,10	9,6,1,7 – 2,4,8,10
C_2	0,6,1,10 – 2,4,8,7	0,9,6,1,10 – 2,4,8,3,7,5
C_3	1,6,2,7 – 0,4,8,10	1,6,5,7 – 0,4,8,10
C_4	1,6,2,10 – 0,4,8,7	1,6,5,2,10 – 0,4,8,3,7,9
C_5	1,6,8,7 – 0,4,2,10	1,6,,3,7 – 0,4,2,10
C_6	1,6,8,10 – 0,4,2,7	1,6,3,8,10 – 0,4,2,5,7,9
C_7	0,7,1,10 – 2,4,8,6	0,9,7,1,10 – 2,4,8,3,6,5
C_8	1,7,2,10 – 0,4,8,6	1,7,5,2,10 – 0,4,8,3,6,9
C_9	1,7,8,10 – 0,4,2,6	1,7,3,8,10 – 0,4,2,5,6,9

TABLE 7. Nine pairs of cycles in G_1 called $C_1, \ldots, C_9$.

As shown by Sachs [Sc], in any embedding of $K_{4,4}^-$ or $K_{3,3,1}$, at least one pair of the nine disjoint cycles in each graph has odd linking number. We will simply say the cycles are *linked* if the linking number is odd. Fix an embedding of G_1. Our goal is to show that the embedding must have a knotted cycle.

We will argue by contradiction. For a contradiction, assume that there is no knotted cycle in the embedding of G_1. We leverage Lemma A.1 to deduce that certain pairs of cycles are not linked. Eventually, we will conclude that none of $B_1, \ldots, B_9$ are linked. This is a contradiction as these correspond to cycles in a $K_{3,3,1}$ and we know that every embedding of this Petersen family graph must have a pair of linked cycles [Sc]. The contradiction shows that there must in fact be a knotted cycle in the embedding of G_1. As the embedding is arbitrary, this shows that G_1 is IK.

Step III: Eliminate A_2 by combining with each B_i.

We illustrate our strategy by first focusing on the pair $A_2 = 0,4,1,10$ – 2,3,8,5. Combine A_2 with each B_i. In each case we form a $\mathcal{D}_4$ graph as in Figure 14. Since the B_i are pairs of cycles in $K_{3,3,1}$, a Petersen family graph, at least one pair is linked [Sc]. If A_2 is also linked, then Lemma A.1 implies that the embedding of G_1 has a knotted cycle, in contradiction to our assumption. Therefore, we conclude that A_2 is not linked (i.e., does not have odd linking number).

B_1	$\{0, 1, 4\}, \{2, 5, 8\}, \{3, 7\}, \{10\}$
B_2	$\{0\}, \{1, 4, 10\}, \{2, 3, 8\}, \{5\}$
B_3	$\{0, 1, 10\}, \{2, 5, 8\}, \{3, 7\}, \{4\}$
B_4	$\{0, 10\}, \{1, 4\}, \{2, 3\}, \{5, 8\}$
B_5	$\{0, 4, 10\}, \{1\}, \{2, 3, 5\}, \{8\}$
B_6	$\{0, 4\}, \{1, 10\}, \{2, 3\}, \{5, 8\}$
B_7	$\{0, 10\}, \{1, 4\}, \{2, 5\}, \{3, 8\}$
B_8	$\{0, 4, 10\}, \{1, 7\}, \{2\}, \{3, 8, 5\}$
B_9	$\{0, 4\}, \{1, 10\}, \{2, 5\}, \{3, 8\}$

TABLE 8. Pairing A_2 with $B_1, \ldots, B_9$.

In Table 8, we list the vertices in G_1 that are identified to give each of the four vertices of $\mathcal{D}_4$. Let us examine the pairing with B_2 as an example to see how this results in a $\mathcal{D}_4$. We identify $\{1, 4, 10\}$ as a single vertex by contracting edges $(1, 4)$ and $(1, 10)$. Similarly contract $(2, 3)$ and $(3, 8)$ to make a vertex of the $\mathcal{D}_4$ from vertices $\{2, 3, 8\}$ of G_1. In this way, the cycle 0,4,1,10 of A_2 in G_1 becomes cycle C_1 of the $\mathcal{D}_4$ (see Figure 14) between $\{1, 4, 10\}$ and $\{0\}$ and the cycle 2,3,8,5 becomes cycle C_3 between $\{2, 3, 8\}$ and $\{5\}$. Similarly 0,5,7,9 of B_2 becomes homeomorphic to the cycle C_2 between $\{0\}$ and $\{5\}$. For the final cycle of B_2, 2,4,8,10, we observe that, in homology, 2,4,8,10 = 1,4,2,10 ∪ 1,4,8,10. Assuming lk$((0,5,7,9), (2,4,8,10)) \neq 0$, then one of lk$((0,5,7,9), (1,4,2,10))$ and lk$((0,5,7,9), (1,4,8,10))$ is also nonzero. Whichever it is, 1,4,2,10 or 1,4,8,10, that will be our C_4 cycle in the $\mathcal{D}_4$ of Figure 14.

To summarize, we have argued that A_2 forms a $\mathcal{D}_4$ with each pair $B_1, \ldots, B_9$. Since at least one of the B_i's is linked, then, assuming A_2 is linked, these two pairs make a $\mathcal{D}_4$ that has a knotted cycle. Therefore, by way of contradiction, going forward, we may assume A_2 is not linked.

Step IV: Argue A_6 is not linked.

We next argue that A_6 is not linked. For a contradiction, assume instead that A_6 is linked. Pairing with the B_i's again, the vertices for each $\mathcal{D}_4$ are

$B_1\{0, 9\}, \{1, 4\}, \{2, 5\}, \{8, 9\}$ 　　　 $B_2\{0, 5, 9\}, \{1, 7\}, \{2\}, \{4, 8, 10\}$

$B_3\{0, 9\}, \{1, 10\}, \{2, 5\}, \{4, 8\}$ 　　　 $B_4\{0, 5\}, \{1, 4\}, \{2, 3\}, \{8, 10\}$

$B_5\{0\}, \{1, 7\}, \{2, 3, 5\}, \{4, 8, 10\}$ 　　 $B_6\{0, 5\}, \{1, 10\}, \{2, 3\}, \{4, 8\}$

$B_7\{0, 2, 5\}, \{1, 4, 8\}, \{3\}, \{10\}$ 　　　 $B_9\{0, 2, 5\}, \{1, 8, 10\}, \{3\}, \{4\}.$

For $B_8 = 3,7,5,8 - 0,4,2,10$, we first split one of the A_6 cycles: 0,5,2,3,6,9 = 0,5,2,9 ∪ 2,3,6,9. One of the two summands must link with the other A_2 cycle 1, 4, 8, 10. If lk$((0,5,2,9), (1,4,8,10)) \neq 0$, then, by a symmetry of G_1, lk$((8,5,2,3), (1,4,0,10)) \neq 0$. But this last is the pair A_2, and we have already argued that A_2 is not linked.

Therefore, it must be that lk$((2,3,6,9), (1,4,8,10)) \neq 0$. Next, we split a cycle of B_8: 0,4,2,10 = 1,4,2,10 ∪ 0,4,1,10. If lk$((3,7,5,8), (1,4,2,10)) \neq 0$, form a $\mathcal{D}_4$ with vertices $\{1, 4, 10\}$, $\{2\}$, $\{3\}$, and $\{8\}$. On the other hand, If lk$((3,7,5,8), (0,4,1,10)) \neq 0$, form a $\mathcal{D}_4$ with vertices $\{0, 9\}$, $\{1, 4, 10\}$, $\{3\}$, and $\{8\}$.

For every choice of B_i we can make a $\mathcal{D}_4$ with A_6. We know that at least one B_i is a linked pair. If A_6 is also linked, then, by Lemma A.1, this embedding of G_1 has a knotted cycle. Therefore, by way of contradiction, we may assume the pair of A_6 is not linked.

Step V: Eliminate B_2 and B_8.

We next eliminate B_2 by pairing it with each A_i. As we are assuming A_2 and A_6 are not linked, it must be some other A_i pair that is linked. Here

are the vertices of the $\mathcal{D}_4$'s in each case:

$A_1\{0,5,7\},\{2,8,10\},\{3,6,9\},\{4\}$ $A_3\{0,9\},\{2,4\},\{5,7\},\{8,10\}$

$A_4\{0,5,9\},\{1,7\},\{2,4,10\},\{8\}$ $A_5\{0,9\},\{2,10\},\{4,8\},\{5,7\}$

$A_7\{0,9\},\{2,10\},\{4,8\},\{5,7\}$ $A_8\{0,5,7\},\{2,4,8\},\{3,6,9\},\{10\}$

$A_9\{0,9\},\{2,4\},\{5,7\},\{8,10\}$.

This shows B_2 is not linked. Since B_8 is the same as B_2 by a symmetry of G_1, B_8 is likewise not linked. Ultimately, we will show that no B_i is linked. So far, we have this for B_2 and B_8.

Step VI: Eliminate C_1, C_5, and C_3.

Our next step is to argue C_1 is not linked by pairing it with the remaining A_i's:

$A_1\{1,7\},\{2,8,10\},\{3,6\},\{4\}$ $A_3\{1,7\},\{2,4\},\{6,9\},\{8,10\}$

$A_4\{1\},\{2,4,10\},\{6,9\},\{8\}$ $A_5\{1,7\},\{2,10\},\{4,8\},\{6,9\}$

$A_7\{0,4,8\},\{1,5,7\},\{6\},\{10\}$ $A_8\{1,7\},\{2,4,8\},\{3,6\},\{10\}$

$A_9\{1,7\},\{2,4\},\{6,9\},\{8,10\}$.

By a symmetry of G_1, C_5 is also not linked.

Now we argue that C_3 is not linked, again by pairing with A_i's:

$A_1\{0,4\},\{1,5,7\},\{3,6\},\{8,10\}$ $A_3\{0,8,10\},\{1,5,7\},\{4\},\{6\}$

$A_5\{0,10\},\{1,5,7\},\{4,8\},\{6\}$ $A_7\{0,4,8\},\{1,5,7\},\{6\},\{10\}$

$A_8\{0,10\},\{1,5,7\},\{4,8\},\{6\}$ $A_9\{0,4\},\{1,5,7\},\{6\},\{8,10\}$.

For A_4 we split the second cycle: 0,9,6,3,8,5 = 0,9,6,5 ∪ 6,3,8,5. Suppose that it is 0,9,6,5 that is linked with 1,4,2,10. In this case we also split that cycle: 1,4,2,10 = 1,4,8,10 ∪ 2,4,8,10. To get a $\mathcal{D}_4$ when pairing 0,9,6,5 – 1,4,8,10 with C_3 we use vertices $\{0\}$, $\{1\}$, $\{4,8,10\}$, and $\{5,6\}$ and for 0,9,6,5 – 2,4,8,10 with C_3, $\{0\}$, $\{2,3,7\}$, $\{4,8,10\}$, and $\{5,6\}$.

On the other hand, if it is 6,3,8,5 that is linked with 1,4,2,10, we write 1,4,2,10 = 0,4,1,10 ∪ 0,4,2,10. Then when C_3 is paired with 6,3,8,5 — 0,4,1,10, we have a $\mathcal{D}_4$ using vertices $\{0,4,10\}$, $\{1\}$, $\{5,6\}$, and $\{8\}$ while if C_3 is paired with 6,3,8,5 – 0,4,2,10 the vertices are $\{0,4,10\}$, $\{2,7,9\}$, $\{5,6\}$, and $\{8\}$. This completes the argument that C_3 is not linked.

Step VII: Eliminate A_8 and A_1.

We will show that A_8 is not linked by pairing with the remaining C_i's. For C_8, the vertices would be $\{0\}$, $\{1,5,7,10\}$, $\{2\}$, and $\{3,4,8\}$. The remaining cases involve splitting cycles.

For C_2 we write 2,4,8,3,7,5 = 3,7,5,8 ∪ 2,4,8,5. If it is 3,7,5,8 – 0,9,6,1,10 that is linked, we use vertices $\{0,1,10\}$, $\{2,9\}$, $\{3,8\}$, and $\{5,7\}$ and if, instead, 2,4,8,5 – 0,9,6,10 is linked, we have $\{0,1,10\}$, $\{2,4,8\}$, $\{3,6\}$, and $\{5\}$.

For C_4, it is a cycle of A_8 that we rewrite: 0,5,7,1,10 = 0,9,7,1,10 ∪ 0,9,7,5. In either case, we use the same vertices: $\{0, 7, 9\}$, $\{1, 5, 6, 10\}$, $\{2\}$, and $\{3, 4, 8\}$.

For C_6, 0,4,2,5,7,9 = 0,4,2,9 ∪ 2,5,7,9. When 0,4,2,9 is linked with 1,6,3,8,10 the vertices are $\{0, 9\}$, $\{1, 10\}$, $\{2, 4\}$, and $\{3, 8\}$ while if 2,5,7,9 links 1,6,3,8,10, we use $\{1, 10\}$, $\{2\}$, $\{3, 8\}$, and $\{5, 7\}$.

Continuing with C_7, 2,4,8,3,6,5 = 3,6,5,8 ∪ 2,4,8,5. The vertices for 3,6,5,8 – 0,9,7,1,0 are $\{0, 1, 7, 10\}$, $\{2, 9\}$, $\{3, 8\}$, and $\{5\}$ and for 2,4,8,5 – 0,9,7,1,10, use $\{0, 1, 7, 10\}$, $\{2, 4, 8\}$, $\{3, 6, 9\}$, and $\{5\}$.

Finally, in the case of C_9, write 0,4,2,5,6,9 = 0,4,2,9 ∪ 2,5,6,9. When 0,4,2,9 – 1,7,3,8,10 is linked, the vertices are $\{0\}$, $\{1, 7, 10\}$, $\{2, 4\}$, and $\{3, 8\}$ and for 2,5,6,9 – 1,7,3,8,10 use $\{1, 7, 10\}$, $\{2\}$, $\{3, 8\}$, and $\{5\}$. This completes the argument for A_8. By a symmetry of G_1, A_1 is also not linked. In other words, going forward, we will assume it is one of A_3, A_4, A_5, A_7, or A_9 that is linked.

Step VIII: Eliminate B_1, B_3, and B_5.

Next, we will argue that B_1 is not linked by comparing with the remaining A_i's:

$A_3\{0, 9\}, \{1, 4, 7\}, \{2, 5\}, \{8, 10\}$ $A_4\{0, 9\}, \{1, 4\}, \{2, 10\}, \{5, 8\}$

$A_5\{0, 9\}, \{1, 4, 7\}, \{2, 10\}, \{5, 8\}$ $A_7\{0, 4, 9\}, \{1, 7\}, \{2, 5, 10\}, \{8\}$

$A_9\{0, 4, 9\}, \{1, 7\}, \{2\}, \{5, 8, 10\}$.

By a symmetry of G_1 we also assume B_3 is not linked.

Now, by pairing with the remaining A_i's, we show B_5 is not linked:

$A_3\{0, 8, 10\}, \{2, 5, 7\}, \{3\}, \{4\}$ $A_5\{0, 10\}, \{2, 3\}, \{4, 8\}, \{5, 7\}$

$A_7\{0, 4, 8\}, \{2, 5, 7\}, \{3\}, \{10\}$ $A_9\{0, 4\}, \{2, 3\}, \{5, 7\}, \{8, 10\}$.

For A_4 we employ several splits. First, 0,9,6,3,8,5 = 0,9,6,5 ∪ 6,3,8,5. In the case that 0,9,6,5 – 1,4,2,10 is linked, write 1,4,2,10 = 2,4,8,10 ∪ 1,4,8,10. Pairing 0,9,6,5 – 2,4,8,10 with B_5, the vertices are $\{0\}$, $\{2\}$, $\{4, 8, 10\}$, and $\{5\}$. If instead it is 0,9,6,5 – 1,4,8,10 that is linked, we use $\{0\}$, $\{1, 7\}$, $\{4, 8, 10\}$, and $\{5\}$.

So we assume that 6,3,8,5 – 1,4,2,10 is linked and rewrite a B_5 cycle: 2,3,7,5 = 2,3,6,5 ∪ 3,6,5,7. In case 2,3,6,5 – 0,4,8,10 is linked, we make a further split: 1,4,2,10 = 1,7,9,2,10 ∪ 1,4,9,2,10. Thus, assuming 2,3,6,5 – 0,4,8,10 and 1,7,9,2,10 – 6,3,8,5 are both linked, we have a $\mathcal{D}_4$ with vertices $\{2\}$, $\{3, 5, 6\}$, $\{8\}$, and $\{10\}$. If instead it is 2,3,6,5 – 0,4,8,10 and 1,4,9,2,10 – 6,3,8,5 that are linked, use $\{2\}$, $\{3, 5, 6\}$, $\{4\}$, and $\{8\}$. This leaves the case where 3,6,5,7 – 0,4,8,10 is linked. We must split a final time: 1,4,2,10 = 0,4,1,10 ∪ 0,4,2,10. If 3,6,5,7 – 0,4,8,10 and 0,4,1,10 – 6,3,8,5 are both linked, the vertices are $\{0, 4, 10\}$, $\{1, 7\}$, $\{3, 5, 6\}$, and $\{8\}$. On the other hand, when 3,6,5,7 – 0,4,8,10 and 0,4,2,10 – 6,3,8,5 are linked, use $\{0, 4, 10\}$, $\{2, 7, 9\}$, $\{3, 5, 6\}$, and $\{8\}$. This completes the argument for B_5, which we have shown is not linked.

Step IX: Eliminate B_9. Only B_4 and B_6 remain.

Next we turn to B_9 which we compare with the remaining A_i's:

$A_3\{0\}, \{1, 7\}, \{2, 4, 5\}, \{3, 8, 10\}$ $A_4\{0, 5\}, \{1, 10\}, \{2, 4\}, \{3, 8\}$

$A_7\{0, 4\}, \{1, 7, 10\}, \{2, 5\}, \{3, 8\}$ $A_9\{0, 2, 4\}, \{1, 7, 8, 10\}, \{3\}, \{5\}.$

This leaves A_5 for which we rewrite: 0,9,6,3,2,10 = 0,9,2,10 ∪ 2,3,6,9. If 0,9,2,10 – 1,4,8,5,7 is linked, then observe that, by a symmetry of G_1, this is the same as A_8, which is not linked. Thus, we can assume 2,3,6,9 – 1,4,8,5,7 is linked and rewrite: 1,4,8,5,7 = 1,4,0,5,7 ∪ 0,4,8,5. Pairing 2,3,6,9 - 1,4,0,5,7 with B_9 gives a $\mathcal{D}_4$ on vertices $\{0, 4, 5\}$, $\{1, 7\}$, $\{2\}$, and $\{3\}$. If it is 2,3,6,9 – 0,4,8,5 that is linked, then, pairing with B_9, use $\{0, 4, 5\}$, $\{2\}$, $\{3\}$, and $\{8\}$. This completes the argument for B_9 and, by a symmetry of G_1, also for B_7.

Recall that our goal is to argue, for a contradiction, that no B_i is linked. At this stage we are left only with B_4 and B_6 as pairs that could be linked.

Step X: Eliminate A_4, A_5, and A_9 leaving only A_3 and A_7.

Before providing the argument for the remaining two B_i's, we first eliminate a few more A_i's, starting with A_4, which we compare with the two remaining B_i's:

$B_4\{0, 5, 8\}, \{1, 2, 4\}, \{3\}, \{10\}$ $B_6\{0, 5, 8\}, \{1, 2, 10\}, \{3\}, \{4\}.$

Next A_5, again by pairing with B_4 and B_6:

$B_4\{0, 10\}, \{1, 4, 7\}, \{2, 3\}, \{5, 8\}$ $B_6\{0\}, \{1, 7\}, \{2, 3, 10\}, \{4, 8, 5\}.$

Since A_9 agrees with A_5 under a symmetry of G_1, this leaves only A_3 and A_7 as pairs that may yet be linked among the A_i's.

Step XI: Eliminate C_6 and C_9, leaving C_2, C_4, C_7, and C_8.

As a penultimate step, we show that C_6 is not linked by comparing with these two remaining A_i's:

$A_3\{0, 9\}, \{1\}, \{2, 4, 5, 7\}, \{3, 6, 8, 10\}$ $A_7\{0, 4, 9\}, \{1, 10\}, \{2, 5, 7\}, \{3, 6, 8\}.$

By a symmetry of G_1, we can also assume that C_9 is not linked. This leaves only four C_i's that may be linked: C_2, C_4, C_7, and C_8.

Step XII: Eliminate the remaining two B_i's (B_6 and B_4) to complete the argument.

Finally, compare B_6 with the remaining C_i's:

$C_4\{0, 4, 8\}, \{1, 2, 10\}, \{3, 7\}, \{5\}$ $C_8\{0, 4, 8\}, \{1, 2, 7, 10\}, \{3\}, \{5\}.$

For C_2 we rewrite: 2,4,8,3,7,5 = 2,3,8,4 ∪ 2,3,7,5. If 2,3,8,4 – 0,9,6,1,10 is linked, then pairing with B_6, we have vertices $\{0\}$, $\{1, 10\}$, $\{2, 3\}$, and $\{4, 8\}$. On the other hand pairing B_6 with 2,3,7,5 – 0,9,6,1,10 we will have a $\mathcal{D}_4$ using the vertices $\{0\}$, $\{1, 10\}$, $\{2, 3, 7\}$, and $\{5\}$.

For C_7: 2,4,8,3,6,5 = 2,3,8,4 ∪ 2,3,6,5. Then 2,3,8,4 – 0,9,7,1,10 with B_6 gives a $\mathcal{D}_4$ for vertices $\{0\}$, $\{1, 7, 10\}$, $\{2, 3\}$, and $\{4, 8\}$. On the other

hand, 2,3,6,5 – 0,9,7,1,10 pairs with B_6 using vertices $\{0\}$, $\{1, 7, 10\}$, $\{2, 3\}$, and $\{5\}$.

This completes the argument for B_6. By a symmetry of G_1 we see that B_4 is also not linked.

In this way, the assumption that there is no knotted cycle in the given embedding of G_1 forces us to conclude that no pair $B_1, \ldots, B_9$ is linked. However, these correspond to the cycles of a $K_{3,3,1}$. As Sachs [Sc] has shown, any embedding of $K_{3,3,1}$ must have a pair of cycles with odd linking number. The contradiction shows that there can be no such knotless embedding and G_1 is IK. This completes the proof that G_1 is MMIK.

A.2. G_2 is MMIK.

In this subsection, we show that G_2 is IK. The argument is similar to that for G_1 above. To use Lemma A.1, we need pairs of linked cycles in G_2.

Step I: Define pairs A_i, B_i, C_i, and D_i.

We begin by identifying four ways that the Petersen family graph P_9, of order nine, appears as a minor of graph G_2. Using the vertex labelling of Figure 16, on contracting edge $(0, 4)$ and deleting vertex 8, the resulting graph has P_9 as a subgraph.
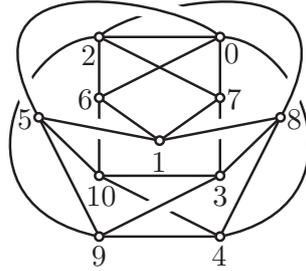
FIGURE 16. The graph G_2.

There are seven pairs of disjoint cycles in P_9. we denote these pairs as A_1 through A_7. In Table 9, we give the pairs in G_2.

$$
\begin{array}{c|l}
A_1 & 0,4,10,5 - 1,6,2,9,3,7 \\
A_2 & 0,2,6,10,4 - 1,5,9,3,7 \\
A_3 & 0,2,9,5 - 1,6,10,3,7 \\
A_4 & 0,5,1,7 - 2,6,10,3,9 \\
A_5 & 0,4,10,3,7 - 1,5,9,2,6 \\
A_6 & 1,5,10,6 - 0,2,9,3,7 \\
A_7 & 3,9,5,10 - 0,2,6,1,7
\end{array}
$$

TABLE 9. Seven pairs of cycles in G_2 called $A_1, \ldots, A_7$.

Similarly, if we contract edge $(3, 10)$ and delete vertex 7 in G_2, the resulting graph has a P_9 subgraph. We will call these seven cycles $B_1, \ldots B_7$ as in Table 10

Contracting edge $(4, 10)$ and deleting vertex 6, we have the seven cycles of Table 11.

B_1	0,2,6 − 1,5,9,3,10,4,8
B_2	0,2,8,4 − 1,5,9,3,10,6
B_3	0,2,9,5 − 1,6,10,4,8
B_4	0,4,10,6 − 1,5,9,2,8
B_5	0,5,1,6 − 2,8,4,10,3,9
B_6	1,6,2,8 − 0,4,10,3,9,5
B_7	2,6,10,3,9 − 0,4,8,1,5

TABLE 10. Seven pairs of cycles in G_2 called $B_1, \ldots, B_7$.

C_1	3,8,4,10 − 0,2,9,5,1,7
C_2	3,8,1,7 − 0,2,9,5,10,4
C_3	2,8,3,9 − 0,4,10,5,1,7
C_4	0,4,10,3,7 − 1,5,9,2,8
C_5	3,9,5,10 − 0,2,8,1,7
C_6	1,5,10,4,8 − 0,2,9,3,7
C_7	0,2,8,4 − 1,5,9,3,7

TABLE 11. Seven pairs of cycles in G_2 called $C_1, \ldots, C_7$.

Finally, if we contract edge $(3, 9)$ and delete vertex 7 in G_2, the resulting graph has a P_9 subgraph. We will call these seven cycles $D_1, \ldots D_7$ as in Table 12.

D_1	2,8,3,9 − 0,4,10,6,1,5
D_2	0,2,9,5 − 1,6,10,4,8
D_3	0,2,8,4 − 1,5,9,3,10,6
D_4	1,5,9,3,8 − 0,2,6,10,4
D_5	1,6,2,8 − 0,4,10,3,9,5
D_6	3,8,4,10 − 0,2,6,1,5
D_7	2,6,10,3,9 − 0,4,8,1,5

TABLE 12. Seven pairs of cycles in G_2 called $D_1, \ldots, D_7$.

Step II: Eliminate A_5.

We will need to introduce two more Petersen family graph minors later, but let us begin by ruling out some of the pairs we already have. As in our argument for G_1, we assume that we have a knotless embedding of G_2 and step by step argue that various cycle pairs are not linked (i.e. do not have odd linking number) using Lemma A.1. Eventually, this will allow us to deduce that all seven pairs $B_1, \ldots, B_7$ are not linked. This is a contradiction since Sachs [Sc] showed that in any embedding of P_9, there must be a pair of cycles with odd linking number. The contradiction shows that there is no such knotless embedding and G_2 is IK.

We will see that A_5 is not linked by showing it results in a $\mathcal{D}_4$ with every pair $B_1, \ldots, B_7$. Indeed the vertices of the $\mathcal{D}_4$ are formed by contracting the following vertices

$B_1\{0\}, \{1,5,9\}, \{2,6\}, \{3,4,10\}$ $B_2\{0,4\}, \{1,5,6,9\}, \{2\}, \{3,10\}$
$B_3\{0\}, \{1,6\}, \{2,5,9\}, \{4,10\}$ $B_4\{0,4,10\}, \{1,2,5,9\}, \{3,8\}, \{6\}$
$B_5\{0\}, \{1,5,6\}, \{2,9\}, \{3,4,10\}$ $B_7\{0,4\}, \{1,5\}, \{2,6,9\}, \{3,10\}$.

For $B_6 = 1,6,2,8 - 0,4,10,3,9,5$, we first split one of the A_5 cycles: 0,4,10,3,7 = 0,4,8,3,7 $\cup$ 3,8,4,10. One of the two summands must link with the other A_5 cycle 1,5,9,2,6. If lk$((3,8,4,10),(1,5,9,2,6)) \neq 0$, then, by contracting edges, we form a $\mathcal{D}_4$ with A_5 whose vertices are $\{1,2,6\}, \{3,4,10\}, \{5,9\}, \{8\}$. On the other hand, if lk$((0,4,8,3,7),(1,5,9,2,6)) \neq 0$, then we will split the B_6 cycle 0,4,10,3,9,5 = 0,4,10,5 $\cup$ 3,9,5,10. When lk$((1,6,2,8),(0,4,10,5)) \neq 0$, we have a $\mathcal{D}_4$ with vertices $\{0,4\}, \{1,2,6\}, \{5\}, \{8\}$ and when lk$((1,6,2,8),(3,9,5,10)) \neq 0$, the $\mathcal{D}_4$ is on $\{1,2,6\}, \{3\}, \{5,9\}, \{8\}$.

We have shown that, for each B_i, we must have a $\mathcal{D}_4$ with A_5. Sachs [Sc] showed that in every embedding of P_9, there is a pair of linked cycles. Thus, in our embedding of G_2, at least one B_i is linked. If A_5 were also linked, that would result in a $\mathcal{D}_4$ with a knotted cycle by Lemma A.1. This contradicts our assumption that we have a knotless embedding of G_2. Therefore, going forward, we can assume A_5 is not linked.

Step III: Eliminate B_7 and A_4.

We next argue that B_7 is not linked by comparing with $C_1, \ldots, C_7$. For C_4, C_6, and C_7, we immediately form a $\mathcal{D}_4$ as follows

$C_4\{0,4\}, \{1,5,8\}, \{2,9\}, \{3,7\}$ $C_6\{0\}, \{1,4,5,8\}, \{2,3,9\}, \{10\}$
$C_7\{0,4,8\}, \{1,5\}, \{2\}, \{3,9\}$.

For the remaining pairs, we will split a cycle of the C_i. For C_1, write 0,2,9,5,1,7 = 0,2,7 $\cup$ 1,5,9,2,7. In the first case, where lk$((3,8,4,10),(0,2,7)) \neq 0$, the $\mathcal{D}_4$ is on $\{0\}, \{2\}, \{3,10\}, \{4,8\}$. In the second case, lk$((3,8,4,10),(1,5,9,7)) \neq 0$, we have $\{1,5\}, \{2,9\}, \{3,10\}, \{4,8\}$.

For C_2, split 0,2,9,5,10,4 = 0,2,9,5 $\cup$ 0,4,10,5 with, in the first case, a $\mathcal{D}_4$ on $\{0,5\}, \{1,8\}, \{2,9\}, \{3\}$ and, in the second, on $\{0,4,5\}, \{1,8\}, \{3\}, \{10\}$. For C_5, 0,2,8,1,7 = 0,2,7 $\cup$ 1,7,2,8, the first case is $\{0\}, \{2\}, \{3,9,10\}, \{5\}$ and the second has $\{1,8\}, \{2\}, \{3,9,10\}, \{5\}$.

Finally, for C_3 split 0,4,10,5,1,7 = 1,6,7 $\cup$ 1,6,7,0,4,10,5. In the first case, there is a $\mathcal{D}_4$ with vertices $\{1\}, \{2,3,9\}, \{6\}, \{8\}$. In the second case, we split the same cycle a second time: 1,6,7,0,4,10,5 = 1,5,10,6 $\cup$ 0,4,10,6,7. In the first subcase, we have a $\mathcal{D}_4$ with vertices $\{1,5\}, \{2,3,9\}, \{6,10\}, \{8\}$ and in the second subcase, $\{0,4\}, \{2,3,9\}, \{6,10\}, \{8\}$.

Going forward, we can assume that B_7 is not linked.

We next argue that A_4 is not linked by comparing with $D_1, \ldots, D_7$. For four links, we immediately give the vertices of the $\mathcal{D}_4$:

$D2\{0,5\}, \{1\}, \{2,9\}, \{6,10\}$ $\qquad$ $D3\{0\}, \{1,5\}, \{2\}, \{3,6,9,10\}$

$D4\{0\}, \{1,5\}, \{2,6,10\}, \{3,9\}$ $\qquad$ $D5\{0,5\}, \{1\}, \{2,6\}, \{3,9,10\}$.

For D_1, split the second cycle of A_4: 2,6,10,3,9 $= 3,9,4,10 \cup 2,9,4,10,6$. In the first case, the verices of the $\mathcal{D}_4$ are $\{0,1,5\}, \{2,7\}, \{3,9\}, \{4,10\}$ and in the second, $\{0,1,5\}, \{2,9\}, \{3,7\}, \{4,6,10\}$. For D_6, split the second cycle: 0,2,6,1,5 $= 0,2,9,5 \cup 1,5,9,2,6$. In the first case, we have vertices $\{0,5\}, \{1,8\}, \{2,9\}, \{3,10\}$ and in the second, $\{0,4\}, \{1,5\}, \{2,6,9\}, \{3,10\}$. Finally, D_7 is the same pair of cycles as B_7, which we have assumed is not linked. Going forward, we will assume A_4 is not linked.

Step IV: Introduce pairs E_i to eliminate A_2 and A_3. This leaves only A_1, A_6, and A_7.

We have already argued that we can assume A_4 and A_5 are not linked. In this step we eliminate A_2 and A_3, leaving only three A_i that could be linked. For this we use another Petersen graph minor. Using the labelling of Figure 16, partition the vertices as $\{0, 8, 9, 10\}$ and $\{2, 3, 4, 5\}$. Contract edges $(0, 7)$ and $(2, 6)$. This resulting graph has a $K_{4,4}^-$ subgraph, where $(5, 8)$ is the missing edge. As in Table 13, we will call the resulting nine pairs of cycles $E_1, \ldots, E_9$.

E_1	2,8,4,9 − 0,5,10,3,7
E_2	0,4,10,5 − 2,8,3,9
E_3	0,2,6,10,5 − 3,8,4,9
E_4	0,2,4,8 − 3,9,5,10
E_5	4,9,5,10 − 0,2,8,3,7
E_6	0,4,8,3,7 − 2,6,10,5,9
E_7	0,5,9,3,7 − 2,6,10,4,8
E_8	0,4,9,5 − 2,6,10,3,8
E_9	0,2,9,5 − 3,8,4,10

TABLE 13. Nine pairs of cycles in G_2 called $E_1, \ldots, E_9$.

We will use $E_1, \ldots, E_9$ to show that A_2 may be assumed unlinked. Except for E_1, we list the vertices of the $\mathcal{D}_4$:

$E_2\{0, 4, 10\}, \{2\}, \{3, 9\}, \{5\}$ $\qquad$ $E_3\{0, 2, 6, 10\}, \{3, 9\}, \{4\}, \{5\}$

$E_4\{0, 2, 4\}, \{1, 8\}, \{3, 5, 9\}, \{10\}$ $\qquad$ $E_5\{0, 2\}, \{3, 7\}, \{4, 10\}, \{5, 9\}$

$E_6\{0, 4\}, \{2, 6, 10\}, \{3, 7\}, \{5, 9\}$ $\qquad$ $E_7\{0\}, \{1, 8\}, \{2, 4, 6, 10\}, \{3, 5, 7, 9\}$

$E_8\{0, 4\}, \{2, 6, 10\}, \{3\}, \{5, 9\}$ $\qquad$ $E_9\{0, 2\}, \{3\}, \{4, 10\}, \{5, 9\}$.

The argument for E_1 is a little involved. Let us split the second cycle 0,5,10,3,7 $= 0,5,10,6,1,7 \cup 1,6,10,3,7$.

Case 1: Suppose that $\text{lk}((2,8,4,9), (0,5,10,6,1,7)) \neq 0$. Next split the first cycle of A_2: 0,2,6,10,4 $= 0,2,6 \cup 0,4,10,6$.

Case 1a): Suppose that lk$((0,2,6),(1,5,9,3,7)) \neq 0$. We split the second E_1 cycle again: 0,5,10,6,1,7 = 0,5,10,6 ∪ 0,6,1,7 In the first case, where lk$((2,8,4,9),(0,5,10,6)) \neq 0$, we have a $\mathcal{D}_4$ with vertices $\{0,6\}, \{2\}, \{5\}, \{9\}$. In the second case, lk$((2,8,4,9),(0,6,1,7)) \neq 0$, the $\mathcal{D}_4$ vertices are $\{0,6\}, \{1,7\}, \{2\}, \{9\}$.

Case 1b): Suppose that lk$((0,4,10,6),(1,5,9,3,7)) \neq 0$. Split the second E_1 cycle again: 0,5,10,6,1,7 = 0,5,10,6 ∪ 0,6,1,7. In the first case, where lk$((2,8,4,9),(0,5,10,6)) \neq 0$, the $\mathcal{D}_4$ vertices are $\{0,6,10\}, \{4\}, \{5\}, \{9\}$. In the second case, where lk$((2,8,4,9),(0,6,1,7)) \neq 0$, we have vertices $\{0,6\}, \{1,7\}, \{4\}, \{9\}$.

Case 2: Suppose that lk$((2,8,4,9),(1,6,10,3,7)) \neq 0$. We split the fist cycle of A_2: 0,2,6,10,4 = 0,2,6 ∪ 0,4,10,6. In the first case, the $\mathcal{D}_4$ has vertices $\{1,3,7\}, \{2\}, \{6\}, \{9\}$ and in the second $\{1,3,7\}, \{4\}, \{6,10\}, \{9\}$.

This completes the argument for A_2, which we henceforth assume is not linked.

Next we again use $E_1, \ldots, E_9$ to see that A_3 is also not linked. Except for E_8 and E_9 we immediately have a $\mathcal{D}_4$:

$E_1 \{0,5\}, \{1,8\}, \{2,9\}, \{3,7,10\}$ $E_2 \{0,5\}, \{2,9\}, \{3\}, \{10\}$

$E_3 \{0,2,5\}, \{3\}, \{6\}, \{9\}$ $E_4 \{0,2\}, \{1,8\}, \{3,10\}, \{5,9\}$

$E_5 \{0,2\}, \{3,7\}, \{5,9\}, \{10\}$ $E_6 \{0\}, \{2,5,9\}, \{3,7\}, \{6,10\}$

$E_7 \{0,5,9\}, \{2\}, \{3,7\}, \{6,10\}$

For E_8, split the first cycle 0,4,9,5 = 0,5,1,7 ∪ 0,4,9,5,1,7. In the first case, we have a $\mathcal{D}_4$ with vertices $\{0,5\}, \{1,7\}, \{2\}, \{3,6,10\}$. In the second case, we further split the first cycle of A_3: 0,2,9,5 = 2,8,4,9 ∪ 0,2,8,4,9,5 leading to two subcases. In the first subcase the $\mathcal{D}_4$ is $\{1,7\}, \{2,8\}, \{3,6,10\}, \{4,9\}$ and in the second $\{0,4,5,9\}, \{1,7\}, \{2,8\}, \{3,6,10\}$.

For E_9 split the first cycle 0,2,9,5 = 0,2,6,1,5 ∪ 2,6,1,5,9 with a $\mathcal{D}_4$ on first $\{0,2,5\}, \{1,6\}, \{3,10\}, \{4,9\}$ and then $\{0,4\}, \{1,6\}, \{2,5,9\}, \{3,10\}$.

We will not need $E_1, \ldots, E_9$ in the remainder of the argument. At this stage, we can assume that it is one of A_1, A_6, and A_7 that is linked.

Step V: Eliminate B_5, B_2, and B_3, leaving only B_1, B_4, and B_6.

Our next step is to argue that we can assume B_5 is not linked by comparing with the three remaining A_i's. For the first two, we immediately recognize a $\mathcal{D}_4$:

$A_1 \{0,5\}, \{1,6\}, \{2,3,9\}, \{4,10\}$ $A_6 \{0\}, \{1,5,6\}, \{2,3,9\}, \{10\}$

For A_7, split the second cycle: 0,2,6,1,7 = 0,2,7 ∪ 2,6,1,7 so that the $\mathcal{D}_4$ has vertices $\{0\}, \{2\}, \{3,9,10\}, \{5\}$ in the first case and then $\{1,6\}, \{2\}, \{3,9,10\}, \{5\}$. Going forward, we assume that B_5 is not linked.

Next we will eliminate B_2 and B_3. For B_3, we have a $\mathcal{D}_4$ with each of the remaining A_i's:

$A_1\{0,5\}, \{1,6\}, \{2,9\}, \{4,10\}$ $A_6\{0,2,9\}, \{1,6,10\}, \{3,8\}, \{5\}$

$A_7\{0,2\}, \{1,6\}, \{5,9\}, \{10\}$

For B_2, notice first that, as we are assuming A_3 is not linked, by a symmetry of G_2, we can assume that 0,2,8,4 - 1,6,10,3,7 is also not linked. Again, since B_3 is unlinked, the symmetric pair 0,2,8,4 - 1,5,9,3,7 is also not linked. Since 1,5,9,3,10,6 = 1,5,9,3,7 ∪ 1,6,10,3,7 we conclude that B_2 is not linked. Having eliminated four of the B_i's, going forward, we can assume that it is one of B_1, B_4, and B_6 that is linked. Recall that our ultimate goal is to argue that none of the B_i are linked and thereby force a contradiction.

Step VI: Eliminate C_1 and A_7. This leaves only A_1 and A_6 among the A_i pairs.

Our next step is to argue that C_1 is not linked by comparing with the remaining three A_i's. For A_1 split the second cycle of C_1 0,2,9,5,1,7 = 0,2,9,5 ∪ 0,5,1,7, yielding first a $\mathcal{D}_4$ on $\{0,5\}, \{2,9\}, \{3\}, \{4,10\}$ and then on $\{0,5\}, \{1,7\}, \{3\}, \{4,10\}$. For A_6, we have $\mathcal{D}_4$ with vertices $\{0,2,7,9\}, \{1,5\}, \{3\}, \{10\}$.

For A_7, split the second cycle 0,2,6,1,7 = 0,2,8,1,7 ∪ 1,6,2,8. In the first case, we have a $\mathcal{D}_4$ with vertices $\{0,1,2,7\}, \{3,10\}, \{5,9\}, \{8\}$. In the second case, split the second cycle of C_1 0,2,9,5,1,7 = 0,2,9,5 ∪ 0,5,1,7, resulting in a $\mathcal{D}_4$ either on $\{2\}, \{3,10\}, \{5,9\}, \{8\}$ or $\{1\}, \{3,10\}, \{5\}, \{8\}$.

Now we can eliminate A_7. Using a symmetry of G_2 we will instead argue that the pair $A_7' = 3,8,4,10 - 0,2,6,1,7$ is not linked. Note that this resembles C_1, which we just proved unlinked. Since 0,2,6,1,7 ∪ 0,2,9,5,1,7 = 1,5,9,2,6 it will be enough to show that 3,8,4,10 - 1,5,9,2,6 is not linked by comparing with the remaining A_i's:

$A_1\{1,2,6,9\}, \{3\}, \{4,10\}, \{5\}$ $A_6\{1,5,6\}, \{2,9\}, \{3\}, \{10\}$

$A_7\{0,4\}, \{1,2,6\}, \{3,10\}, \{5,9\}$

Thus we can assume that it is A_1 or A_6 that is the linked pair in our embedding of G_2.

Step VII: Eliminate B_1 and B_6, leaving only B_4.

As for the B_i's, only three candidates remain. We next eliminate B_1 by comparing with the remaining two A_i's. For A_1, split the second cycle of B_1: 1,5,9,3,10,4,8 = 1,5,10,4,8 ∪ 3,9,10,5 giving a $\mathcal{D}_4$ on either $\{0\}, \{1\}, \{2,6\}, \{4,5,10\}$ or $\{0\}, \{2,6\}, \{3,9\}, \{5,10\}$. For A_6, using the same split of the second cycle of B_1 the vertices are either $\{0,2\}, \{1,5,10\}, \{4,9\}, \{6\}$ or $\{0,2\}, \{3,9\}, \{5,10\}, \{6\}$.

To proceed, we will argue that C_5 is unlinked. In fact, we will show that it is $D_6 = C_5'$, the result of applying the symmetry of G_2, that is not linked by comparing with the two remaining A_i's:

$A_1\{0,5\}, \{1,2,6\}, \{3\}, \{4,10\}$ $A_6\{0,2\}, \{1,5,6\}, \{3\}, \{10\}$

We can now eliminate B_6 by comparing with the C_i's. This will leave only B_4, which, therefore, must be linked. We have already argued that C_1

and C_5 are not linked. Also, by a symmetry of G_2, since B_7 is not linked, C_4 is also not linked. For C_3 and C_7, we immediately see a $\mathcal{D}_4$:

$$C_3\{0,4,5,10\},\{1\},\{2,6\},\{3,9\} \qquad C_7\{0,4\},\{1\},\{2,8\},\{3,9,5\}.$$

For C_2, split the second cycle: 0,2,9,5,10,4 = 0,2,6,10,4 ∪ 2,6,10,5,9. In the first case, we have a $\mathcal{D}_4$ on $\{0,4,10\},\{1,8\},\{2,6\},\{3\}$. In the second case, split the second cycle of B_6: 0,4,10,3,9,5 = 0,4,10,3,7 ∪ 0,5,9,3,7 giving a $\mathcal{D}_4$ with vertices $\{1,8\},\{2,6\},\{3,7\},\{10\}$ in the first subcase and $\{1,8\},\{2,6\},\{3,7\},\{5,9\}$ in the second.

For C_6, we split the second cycle of B_6: 0,4,10,3,9,5 = 0,4,10,5 ∪ 3,9,5,10 giving, first, a $\mathcal{D}_4$ on $\{0\},\{1,8\},\{2\},\{4,5,10\}$ and, second, a $\mathcal{D}_4$ on $\{1,8\},\{2\},\{3,9\},\{5,10\}$.

Step VIII: Introduce F_i pairs to eliminate B_4 and complete the argument.

Since $B_1,\ldots,B_7$, represent the pairs of cycles in an embedding of P_9, we know by [Sc] that at least one pair must have odd linking number. We have just argued that all but B_4 are not linked, so we can conclude that it is B_4 that has odd linking number in our embedding of G_2. We will now derive a contradiction by using a final Petersen family graph minor.

$$
\begin{array}{c|l}
F_1 & 0,5,1,7 - 2,6,10,3,8 \\
F_2 & 0,4,8,1,5 - 2,6,10,3,7 \\
F_3 & 0,2,8,4 - 1,6,10,3,7 \\
F_4 & 0,2,7 - 1,6,10,3,8 \\
F_5 & 1,5,10,6 - 2,7,3,8 \\
F_6 & 1,7,3,8 - 0,2,6,10,5 \\
F_7 & 1,6,2,8 - 0,5,10,3,7 \\
F_8 & 1,6,2,7 - 0,4,8,3,10,5
\end{array}
$$

TABLE 14. Eight pairs of cycles in G_2 called $F_1,\ldots,F_8$.

Our last set of cycles comes from a P_8 minor. This is the Petersen family graph on eight vertices that is not $K_{4,4}^-$. Using the labelling of G_2 in Figure 16, contracting edges $(0,4)$ and $(0,5)$ and deleting vertex 9 results in a graph with a P_8 subgraph. This graph has eight pairs of cycles shown in Table 14.

Using B_4 will derive a $\mathcal{D}_4$ with each F_i. For the first five F_i we immediately find a $\mathcal{D}_4$:

$$F_1\{0\},\{1,2,6\},\{3\},\{4,10\} \qquad F_2\{0,2\},\{1,5,6\},\{3\},\{10\}$$
$$F_3\{0,5\},\{1,2,6\},\{3\},\{4,10\} \qquad F_4\{0,2\},\{1,5,6\},\{3\},\{10\}$$
$$F_5\{0,5\},\{1,2,6\},\{3\},\{4,10\}.$$

Since B_4 is linked, we deduce that the pair $B_4' = 2,7,3,9 - -0,4,8,1,5$, obtained by the symmetry of G_2, is also linked. Using B_4' we have a $\mathcal{D}_4$ with the remaining cycles of F:

$F_6\{0,5\},\{1,8\},\{2\},\{3,7\}$ $\qquad$ $F_7\{0,5\},\{1,8\},\{2\},\{3,7\}$

$F_8\{0,4,5,8\},\{1\},\{2,7\},\{3\}$

We have shown that there is a $\mathcal{D}_4$ with each F_i using the pairs B_4 or B_4', both of which must be linked. Since the $F_1,\ldots,F_8$ represent the cycle pairs of a P_8 minor, at least one of them has odd linking number [Sc]. By Lemma A.1, our embedding of G_2 has a knotted cycle. This contradicts our assumption that we were working with a knotless embedding. The contradiction shows that there is no such knotless embedding and G_2 is IK. This completes the proof that G_2 is MMIK.

Appendix B. Maxnik graphs of order ten

In this section we describe the maxnik graphs of order ten. Recall (see [EFM]) that a maximal 2-apex graph is maxnik. There are 14 maximal 2-apex graphs formed as the join $K_2 * T_8$ of K_2 with one of the 14 triangulations on eight vertices, see Bowen and Fisk [BF]. We list the 14 maximal 2-apex graphs of order ten in Appendix C. Aside from those 14 there are 35 additional maxnik graphs. Our computer search, described in Section 4 above, shows that there are no other maxnik graphs beyond these 49. In this section we argue that the 35 non 2-apex graphs that we have found are indeed maxnik. The graphs are listed below.

To show a graph is maxnik requires two things. Using Naimi's implementation [N2] of Miller and Naimi's [MN] algorithm, we have verified for each graph G that whenever we add an edge $e \notin E(G)$, the graph $G + e$ is IK. It remains to show that each of the graphs is nIK. We divide the graphs into three bins. For the first 14 graphs we argue the graph is nIK by demonstrating a 2-apex child. We handle the next three graphs using two lemmas from [EFM]. For the remaining 18 graphs, we give a knotless embedding.

B.1. **Graphs with a 2-apex child.** In this subsection we list the first 14 of the 35 maxnik graphs of order ten that are not 2-apex. We show these graphs are nIK by demonstrating a 2-apex child.

For each graph G in this list of 14, we provide the size, graph6 format [Sg], edge list, and the vertices of a triangle in the graph. Making a ∇Y move on that triangle results in a child H that is 2-apex. In each case, H becomes planar on deleting vertex 9 and the new degree 3 vertex. By Lemma 3.4, since G has a 2-apex (hence nIK) child, G is also nIK.

(1) Size: 33; graph6 format: `ICf^f\~~w`; triangle: $0, 3, 6$

$[(0,3),(0,4),(0,5),(0,6),(0,7),(0,9),(1,5),(1,6),(1,7),(1,8),(1,9),$

$(2,6),(2,7),(2,8),(2,9),(3,4),(3,5),(3,6),(3,8),(3,9),(4,5),(4,7),$

$(4,8),(4,9),(5,7),(5,8),(5,9),(6,7),(6,8),(6,9),(7,8),(7,9),(8,9)]$

(2) Size: 33; graph6 format: `ICxu|~{~w`; triangle: $0, 3, 8$

$[(0,3),(0,4),(0,6),(0,7),(0,8),(0,9),(1,4),(1,5),(1,6),(1,8),(1,9),$

$(2,4),(2,5),(2,7),(2,8),(2,9),(3,5),(3,6),(3,7),(3,8),(3,9),(4,6),$

$(4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 9), (7, 9), (8, 9)]$

(3) Size: 33; graph6 format: `ICvbm~}~w`; triangle $0, 5, 7$

$[(0, 3), (0, 4), (0, 5), (0, 7), (0, 8), (0, 9), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 5), (2, 6), (2, 8), (2, 9), (3, 4), (3, 6), (3, 7), (3, 8), (3, 9), (4, 7),$
$(4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(4) Size: 33; graph6 format: `IEirt~}~w`; triangle $0, 4, 7$

$[(0, 3), (0, 4), (0, 5), (0, 7), (0, 8), (0, 9), (1, 3), (1, 6), (1, 8), (1, 9), (2, 4),$
$(2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(5) Size: 33; graph6 format: `IEhuV~}~w`; triangle $0, 3, 7$

$[(0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 7), (3, 8), (3, 9), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(6) Size: 33; graph6 format: `IEhvVn}~w`; triangle $0, 3, 7$

$[(0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(7) Size: 33; graph6 format: `IEh~f]}~w`; triangle $0, 4, 7$

$[(0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 9),$
$(2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 8), (3, 9), (4, 5),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(8) Size: 33; graph6 format: `IQjnex~~w`; triangle $0, 2, 6$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 6), (3, 7), (3, 8), (3, 9), (4, 5),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)]$

(9) Size: 33; graph6 format: `IQzTuz}~w`; triangle $3, 6, 8$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 9), (8, 9)]$

(10) Size: 33; graph6 format: `IQzTvz]~w`; triangle $3, 6, 8$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 7), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 9), (8, 9)]$

(11) Size: 33; graph6 format: `IQzTu~]~w`; triangle $0, 5, 7$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 9), (8, 9)]$

(12) Size: 33; graph6 format: `IQyuvx}~w`; triangle $0, 4, 6$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 4), (1, 6), (1, 7), (1, 8),$

$(1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$

$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 9), (8, 9)]$

(13) Size: 33; graph6 format: `IQyvux|~w`; triangle $1, 3, 6$

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 4), (1, 6), (1, 7), (1, 8),$

$(1, 9), (2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$

$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 9), (7, 8), (7, 9), (8, 9)]$

(14) Size: 33; graph6 format: `IUZurzm~w`; triangle $0, 3, 6$

$[(0, 2), (0, 3), (0, 5), (0, 6), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7),$

$(1, 8), (1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 9),$

$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 9), (8, 9)]$

B.2. **Lemmas from** [EFM].

In this subsection, we argue that graphs 15, 16, and 17 (listed below) are nIK using ideas from [EFM].

Graphs 16 and 17 have a degree 2 vertex and we can recognize them as clique sums. In both cases, the graph is the sum of K_3 and the Heawood family graph E_9 over K_2. Since K_3 and E_9 are both nIK, by [EFM, Lemma 3.1] graphs 16 and 17 are nIK.

Graph 15 has a degree 3 vertex and is the clique sum over K_3 of K_4 and E_9. In the embedding of E_9 in [EFM, Figure 2] the K_3 is given by the vertices a, b, c, which bound a disk whose interior is disjoint from the graph. By [EFM, Lemma 3.4], Graph 15 is also nIK.

(15) Size: 24; graph6 format: `ICRffQmn_`

$[(0, 3), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 4), (1, 5), (1, 6), (1, 7), (2, 5), (2, 6),$

$(2, 7), (2, 8), (2, 9), (3, 6), (3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 8), (6, 9)]$

(16) Size: 23; graph6 format: `I?qtdo}^_`

$[(0, 4), (0, 5), (0, 6), (0, 7), (1, 4), (1, 9), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5),$

$(3, 6), (3, 7), (3, 8), (3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 8), (6, 9)]$

(17) Size: 23; graph6 format: `I?qtfo}N_`

$[(0, 4), (0, 5), (0, 6), (0, 7), (1, 4), (1, 7), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5),$

$(3, 6), (3, 7), (3, 8), (3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 8), (6, 9)]$
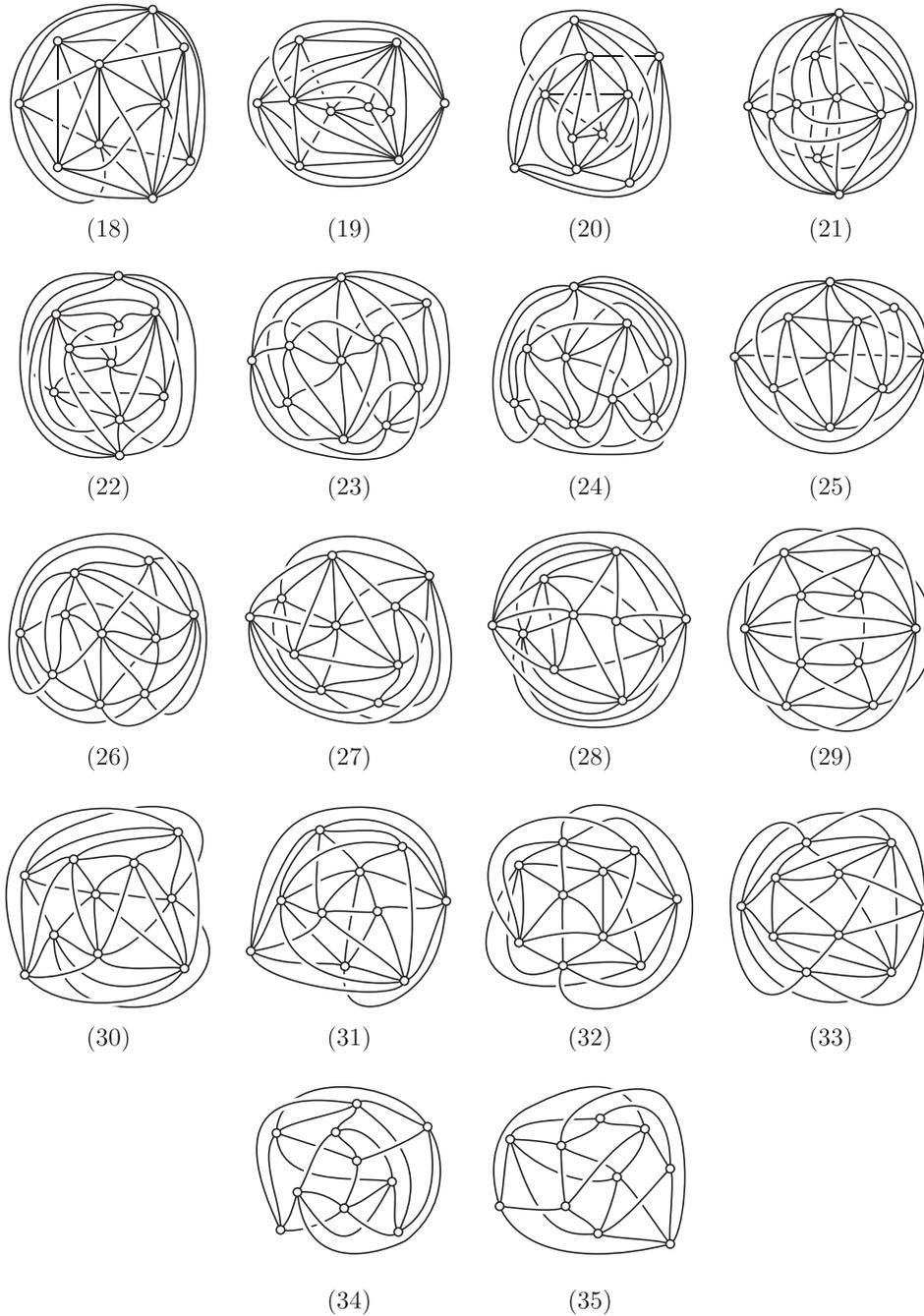
FIGURE 17. Knotless embeddings of 18 graphs.

B.3. **Knotless embeddings.** We show that the remaining 18 graphs (listed below) are nIK by presenting knotless embeddings in Figure 17. Recall that this means when we apply Naimi's findEasyKnots program [N2] to the embedding, it confirms that every cycle in the graph is an unknot.

(18) Size: 34; graph6 format: `IQjuz~nnw`

$[(0,2),(0,4),(0,5),(0,6),(0,8),(0,9),(1,3),(1,5),(1,6),(1,7),(1,8),(2,4),$

$(2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 9), (4, 6), (4, 7), (4, 8),$
$(4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)]$

(19) Size: 33; graph6 format: `IQjUnzz~w`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)]$

(20) Size: 33; graph6 format: `IQjne~^^w`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 9), (3, 6), (3, 7), (3, 8), (3, 9), (4, 5), (4, 7),$
$(4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)]$

(21) Size: 33; graph6 format: `IQjne|~~W`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 6), (3, 7), (3, 8), (3, 9), (4, 5),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (7, 8), (7, 9), (8, 9)]$

(22) Size: 33; graph6 format: `IQjne|~zw`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 6), (3, 7), (3, 8), (4, 5), (4, 7),$
$(4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9), (8, 9)]$

(23) Size: 33; graph6 format: `IQzTvz^~o`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 7), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)]$

(24) Size: 33; graph6 format: `IQzTvv^~o`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 7), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9)]$

(25) Size: 33; graph6 format: `IQzTu~^^o`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7),$
$(1, 8), (1, 9), (2, 4), (2, 6), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 9), (7, 8), (7, 9)]$

(26) Size: 33; graph6 format: `IQyuz~{~o`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 8), (0, 9), (1, 3), (1, 4), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 9), (7, 9)]$

(27) Size: 33; graph6 format: `IQyuz~{zw`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 8), (0, 9), (1, 3), (1, 4), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 9), (7, 9), (8, 9)]$

(28) Size: 33; graph6 format: `IQyuz~{vw`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 8), (0, 9), (1, 3), (1, 4), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 6),$
$(4, 7), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 9), (7, 9), (8, 9)]$

(29) Size: 32; graph6 format: `IQzTrj~~o`

$[(0, 2), (0, 4), (0, 5), (0, 6), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 6), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9),$
$(4, 6), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)]$

(30) Size: 32; graph6 format: `IUZuvzmno`

$[(0, 2), (0, 3), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (1, 3), (1, 4), (1, 5), (1, 6),$
$(1, 7), (1, 8), (2, 4), (2, 5), (2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 9),$
$(4, 6), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 9)]$

(31) Size: 31; graph6 format: `IEivux~zo`

$[(0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 9), (1, 3), (1, 6), (1, 7), (1, 8), (1, 9),$
$(2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 8), (4, 6), (4, 7),$
$(4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)]$

(32) Size: 31; graph6 format: `IEhvuzn^o`

$[(0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9),$
$(2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 5), (3, 6), (3, 7), (3, 9), (4, 6), (4, 7),$
$(4, 8), (4, 9), (5, 7), (5, 8), (5, 9), (6, 8), (6, 9), (7, 8), (7, 9)]$

(33) Size: 31; graph6 format: `IEnb~jm}W`

$[(0, 3), (0, 4), (0, 5), (0, 7), (0, 8), (0, 9), (1, 3), (1, 5), (1, 6), (1, 7), (1, 8),$
$(1, 9), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (2, 9), (3, 4), (3, 6), (3, 7), (3, 9),$
$(4, 6), (4, 8), (4, 9), (5, 6), (5, 7), (5, 8), (6, 8), (7, 9), (8, 9)]$

(34) Size: 23; graph6 format: `ICpVbrkN_`

$[(0, 3), (0, 4), (0, 6), (0, 8), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 6), (2, 7), (2, 8),$
$(2, 9), (3, 5), (3, 6), (3, 7), (3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 9)]$

(35) Size: 23; graph6 format: `ICpvbqkN_`

$[(0, 3), (0, 4), (0, 6), (0, 8), (1, 4), (1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7), (2, 8),$
$(2, 9), (3, 5), (3, 6), (3, 7), (3, 9), (4, 7), (4, 8), (4, 9), (5, 8), (5, 9), (6, 9)]$

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