FORMALISING AND COMPUTING THE FOURTH HOMOTOPY GROUP OF THE 3-SPHERE IN CUBICAL AGDA

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Abstract. Brunerie's 2016 PhD thesis contains the first synthetic proof in Homotopy Type Theory (HoTT) of the classical result that the fourth homotopy group of the 3-sphere is $\mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses a lot of advanced classical algebraic topology rephrased synthetically. Furthermore, the proof is fully constructive and the main result can be reduced to the question of whether a particular "Brunerie number" β can be normalised to ± 2. The question of whether Brunerie's proof could be formalised in a proof assistant, either by computing this number or by formalising the pen-and-paper proof, has since remained open. In this paper, we present a complete formalisation in `Cubical Agda`. We do this by modifying Brunerie's proof so that a key technical result, whose proof Brunerie only sketched in his thesis, can be avoided. We also present a formalisation of a new and much simpler proof that β is ± 2. This formalisation provides us with a sequence of simpler Brunerie numbers, one of which normalises very quickly to -2 in `Cubical Agda`, resulting in a fully formalised computer-assisted proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

1. Introduction

Homotopy theory originated in algebraic topology, but is by now a central tool in many branches of modern mathematics, such as algebraic geometry and category theory. One of the central notions of study in homotopy theory is that of the *homotopy groups* of a space X, denoted $\pi_n(X)$. These groups constitute a topological invariant, making them a powerful tool for establishing whether two given spaces can or cannot be homotopy equivalent. The first two such groups of a space are easy to understand: $\pi_0(X)$ characterises the connected components of X and $\pi_1(X)$ is the fundamental group, i.e. the group of equivalence classes consisting of the loops contained in X up to homotopy. This idea generalises to higher values

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This paper is an extended version of "Formalizing $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda" published in the post-proceedings of Logic in Computer Science 2023 [LM23]. Some details about what has been added can be found in the **Outline** paragraph below.

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of n, for which $\pi_n(X)$ consists of n-dimensional loops up to homotopy. For many spaces, these groups tend to become increasingly esoteric and difficult to compute for large n. This is true also for seemingly tame spaces like spheres, for which $\pi_n(\mathbb{S}^m)$ in general is highly irregular when $n > m \geq 2$.[1] This paper concerns the first computer formalisation of the classical result that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, a result which is particularly interesting because it gives the whole first stable stem of homotopy groups of spheres, i.e. $\pi_{n+1}(\mathbb{S}^n)$ for $n \geq 3$. The fact that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ was proved already in the 1930's by Pontryagin using cobordism theory, but we instead follow the synthetic approach to homotopy theory developed in Homotopy Type Theory (HoTT) and popularised by the HoTT Book [Uni13]. In this new approach to homotopy theory, spaces are represented directly as (higher inductive) types and homotopy groups are computed using Voevodsky's univalence axiom [Voe10a]. This gives a logical approach to homotopy theory, suitable for computer formalisation in proof assistants based on type theory, while also making it possible to interpret results in any suitably structured $(\infty, 1)$-topos [Shu19].

The basis for our formalisation is the 2016 PhD thesis of Brunerie [Bru16a] which contains the first synthetic proof in HoTT that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses advanced machinery from classical algebraic topology developed synthetically, including the symmetric monoidal structure of smash products, (integral) cohomology rings, the Mayer-Vietoris and Gysin sequences, the Hopf invariant, Whitehead products, etc. The formalisation of Brunerie's proof has since remained open, primarily due to the highly technical nature of some of the details. In this paper, we will present such a formalisation in `Cubical Agda` [VMA21], a *cubical* extension of the `Agda` proof assistant [Agd24] with native support for computational univalence and higher inductive types (HITs).

In addition to being a very impressive proof in synthetic homotopy theory, Brunerie's proof is particularly interesting as it is fully constructive. The proof consists of two parts, with the first one culminating in Chapter 3 with the definition of a number $\beta : \mathbb{Z}$ such that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$. Since then, this β has been commonly referred to as the *Brunerie number*. Brunerie writes the following about it:

> *This result is quite remarkable in that even though it is a constructive proof, it is not at all obvious how to actually compute this [β]. At the time of writing, we still haven't managed to extract its value from its definition.* [Bru16a, Page 85]

In fact, [Bru16a, Appendix B] contains a complete and concise definition of β as the image of 1 under a sequence of 12 maps:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "n \mapsto \mathsf{loop}^n"] & \Omega(\mathbb{S}^1) \arrow[r, "\Omega\varphi_{\mathbb{S}^1}"] & \Omega^2(\mathbb{S}^2) \arrow[r, "\Omega^2\varphi_{\mathbb{S}^2}"] & \Omega^3(\mathbb{S}^3) \arrow[dlll, "\Omega^3 e"] & \\
\Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \arrow[r, "\Omega^3\alpha"] & \Omega^3(\mathbb{S}^2) \arrow[r, "h"] & \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \arrow[r, "\Omega^3(e^{-1})"] & \Omega^3(\mathbb{S}^3) \arrow[dlll, "e_3"] & \\
\Omega^2\|\mathbb{S}^2\|_2 \arrow[r, "\Omega\kappa_{2,\mathbb{S}^2}"] & \Omega\|\Omega(\mathbb{S}^2)\|_1 \arrow[r, "\kappa_{1,\Omega\mathbb{S}^2}"] & \|\Omega^2(\mathbb{S}^2)\|_0 \arrow[r, "e_2"] & \Omega(\mathbb{S}^1) \arrow[r, "e_1"] & \mathbb{Z}
\end{tikzcd}$$

[1] See [Bru16a, Figure 2.1] for a table of $\pi_n(\mathbb{S}^m)$ for small n and m.

By implementing this number in a proof assistant with computational support for univalence and HITs, one should be able to normalise it using a computer to establish that $\beta = \pm 2$ and hence that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. In 2016, by the time Brunerie was finishing his thesis, there were some experimental proof assistants based on the cubical type theory of [CCHM18], but these were too slow to perform such a complex computation. So, instead of relying on normalisation, Brunerie spends the second part of the thesis (Chapters 4–6) to prove, using a lot of the advanced machinery mentioned above, that $|\beta|$ is propositionally equal to 2. However, if one were instead able to compute the number automatically in a proof assistant, this equality would hold definitionally—effectively reducing the complexity and length of the proof by an order of magnitude.

The intriguing possibility of a computer assisted formal proof made many people interested and countless attempts to normalise Brunerie's β have been made using increasingly powerful computers. However, to date, no one has succeeded and it is still unclear whether it is normalisable in a reasonable amount of time. In light of this, it is natural to wonder whether it is possible to simplify Brunerie's number in order to be able to compute it. For example, Brunerie's original definition only involves 1-HITs, as the status of higher HITs was still quite understudied at the time. With a better understanding of higher HITs developed in [LS20, CHM18, CH19], one quickly sees that the first 3 maps can be combined into one sending 1 to the 3-cell of $\mathbb{S}^3$ defined as a 3-HIT and not as an iterated suspension as in Brunerie's thesis. Unfortunately, simple optimisations like this do not seem to reduce the complexity of the computation enough and all attempts to run it have thus far failed.

After several unsuccessful attempts at optimising the computation, we instead decided to formalise the second half of Brunerie's thesis. However, this is by no means straightforward. The first issue appears already in the beginning of Chapter 4, a chapter concerning smash products of spheres. The main result of the section is Proposition 4.1.2, which says that the smash product is a 1-coherent symmetric monoidal product on pointed types. However, the proof of this result is just a sketch and Brunerie writes the following about it:

> *The following result is the main result of this section even though we essentially admit it.* [Bru16a, Page 90]

Unfortunately, this result is then used to construct integral cohomology rings, $H^*(X)$, whose cup product, $\smile$, appears in the definition of the so-called Hopf invariant which is crucially used to prove that $|\beta|$ is 2. While one might be convinced that Brunerie's informal proof sketch is correct, it is not obvious how one convinces a proof assistant of this. A complete formalisation would either have to fill in the holes in the sketch or find an alternative construction which avoids Proposition 4.1.2. In fact, Brunerie tried very hard to fill these holes using `Agda` metaprogramming [Bru18]. However, he never managed to typecheck his computer generated proof of the pentagon identity. Hence, this approach also seems infeasible with current proof assistant technology.

Luckily, Brunerie, Ljungström and Mörtberg [BLM22] recently gave an alternative synthetic definition of the cup product on $H^*(X)$ which completely avoids smash products. This has allowed us to completely skip the problematic Chapter 4 and, in particular, Proposition 4.1.2, while still following the proofs in Chapters 5 and 6. Having a strategy for a formal proof, we were then able to embark on able to embark on the ambitious project of formalising Brunerie's proof. Even though we do not need any theory about smash products, there was still a lot left to formalise and our final formalisation closely follows Brunerie's proof, except for various smaller simplifications and adjustments which we discuss in the paper.

In addition to this, we have also formalised a new proof by Ljungström [Lju22] which completely circumvents Chapters 4–6. This major simplification builds on manually calculating the image of the element $\eta : \pi_3(\mathbb{S}^2)$, corresponding to β under the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, by dividing this isomorphism into several maps, tracing η in each step. In particular, the new proof is completely elementary and does not rely on advanced tools such as cohomology. The elements that one obtains while tracing η are all new "Brunerie numbers" that should normalise to ± 2. In fact, one of these normalises, in just under 4 seconds on a regular laptop, to -2 in `Cubical Agda` at the time of writing. So, despite still not being able to compute the original β, this work can be seen as an alternative solution to Brunerie's conjecture about obtaining a computational proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ which relies on simplifying the Brunerie number until it becomes effectively computable.

Outline. The paper closely follows the structure of Brunerie's proof. In section 2, we discuss key results from HoTT that we will need and their formalisation in `Cubical Agda`. section 3, which roughly corresponds to Chapter 2 of Brunerie's thesis, contains some first results on homotopy groups of spheres—e.g. the computation of $\pi_n(\mathbb{S}^m)$ for $n \leq m$. We then give Brunerie's definition of β and prove that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$, the formalisation of which involves the James construction and Whitehead products. The remainder of the paper is then devoted to the formalisation of the different proofs that $\beta = \pm 2$. We first discuss the formalisation of Chapters 4–6 of Brunerie's proof in section 5. This involves a lot of technical machinery like cohomology, the Hopf invariant, etc. We then, in section 6, turn our attention to the new elementary proof that $\beta = \pm 2$ and the new Brunerie number which quickly normalises to -2 in `Cubical Agda`. Here, we also present some result concerning joins of spheres and the vanishing of Whitehead products. We conclude in section 7 with a discussion and comparison of the different formal proofs, as well as some directions for future work.

Compared to the previous publication on which the current paper is based, [LM23], the main differences are the following.

- Many proofs which were omitted because of page constraints in [LM23] have been added or extended throughout the paper. In particular, the proofs in [LM23, Section VI] have been substantially expanded with many details added in subsection 6.3.
- In section 6, many results from [LM23, Section VI] have also been generalised, e.g. the alternative definition of homotopy groups in terms of joins of spheres, π_n^*, is now studied in general and not just for $n = 3$.
- As part of the expansion and generalisation of [LM23, Section VI] in section 6, a new subsection 6.1 on joins and smash products of spheres, a new subsection 6.2 on homotopy groups in terms of joins and a new subsection 6.4 on the possibility of a stand-alone proof of Brunerie's theorem have been added.

Formalisation. All results in the paper have been formalised in `Cubical Agda` and are part of the `agda/cubical` library, available at `https://github.com/agda/cubical/`. The code in the paper is mainly literal `Agda` code taken verbatim from the library, but we have taken some liberties when typesetting, e.g. shortening notations and omitting some universe levels. A `Cubical Agda` summary file linking the formalisation and paper can be found at: `https://github.com/agda/cubical/blob/master/Cubical/Papers/Pi4S3-JournalVersion.agda`

The development typechecks with `Agda`'s `--safe` flag, which ensures that there are no admitted goals or postulates.

2. Homotopy Type Theory in `Cubical Agda`

In this section, we concisely summarise the key HoTT concepts needed for the proofs and their formalisation in `Cubical Agda`. This roughly corresponds to [Bru16a, Chapter 1]. For a more in-depth introduction, see the HoTT Book [Uni13] which also serves as a reference for the formal language "Book HoTT". In this paper, we will present many things with cubical notations, but almost all of the results also hold with minor changes in Book HoTT where paths are represented using Martin-Löf's inductive Id-types [ML75] instead of cubical path types. In section 7 we discuss in more detail which proofs crucially rely on cubical features.

All of the results presented in this section were already part of the `agda/cubical` library before we began our formalisation and, while useful as a resource for our notations, experts on HoTT and `Cubical Agda` can safely skim this section.

2.1. **Elementary HoTT notions and `Cubical Agda` notations.** We write $(x : A) \to B\,x$ for dependent function types and denote the identity function by $\mathsf{id}_A : A \to A$. We write $\Sigma_{x:A}(B\,x)$ for the dependent pair type and fst and snd for its projection maps. In what follows, we mean by a *pointed type* a dependent pair $(A, \star_A)$ consisting of a type A and a fixed basepoint $\star_A : A$. For ease of notation, we will often omit the basepoint and simply write A for the pointed type $(A, \star_A)$. Given two pointed types A and B, the type of *pointed functions* $A \to_\star B$ consists of pairs $(f, \star_f)$ where $f : A \to B$ and $\star_f : f\,\star_A \equiv \star_B$ witnesses basepoint preservation. Again, we simply write $f : A \to_\star B$ and take $\star_f$ implicit.

HoTT supports inductive types, i.e. types inductively generated by their constructors/points. We write Bool for the type of booleans and $\mathbb{1}$ for the unit/singleton type with a single point $\star_{\mathbb{1}}$. A defining feature of HoTT, as opposed to plain Martin-Löf type theory [ML84], is the existence of *higher inductive types* (HITs). This is a generalisation of inductive types where we are not only allowed to specify the generating points of the type in question, but also identifications between these points (and possibly identifications of these identifications, and so on). This is useful for defining quotient types, but also for defining spaces when working in the *types-as-spaces* interpretation of HoTT (see e.g. [Uni13, Table 1] and [AW09]). `Cubical Agda` natively supports HITs and a type representing the circle can be defined as follows:

```
data 𝕊¹ : Type where
  base : 𝕊¹
  loop : base ≡ base
```

Here, $\mathsf{base} \equiv \mathsf{base}$ denotes the type of identifications of base with itself. This is interpreted as the type of *paths* from base to itself when viewing $\mathbb{S}^1$ as a space. Hence, the above HIT captures precisely the representation of the circle as a cell complex with one 0-cell (base) and one 1-cell (loop). We always take $\mathbb{S}^1$ to be pointed by base. In order to discuss the induction principle for $\mathbb{S}^1$, we need to discuss paths in more detail. Cubically, paths correspond to functions out of the unit interval, just like in traditional topology. In `Cubical Agda`, there is a primitive interval type[2] I with endpoints $\mathsf{i0}$ and $\mathsf{i1}$. A path of type $x \equiv y$ between two points $x, y : A$ is a function $p : \mathsf{I} \to A$ such that $p\,\mathsf{i0} = x$ and $p\,\mathsf{i1} = y$ judgmentally. For instance, refl, the constant path at a point x, is defined by:

$$\mathsf{refl} : (x : A) \to x \equiv x$$
$$\mathsf{refl}\ x = \lambda\ i \to x$$

Note that we use "=" for definitional/judgmental equality and "$\equiv$" for `Cubical Agda`'s path-equality. This can be contrasted with the HoTT Book [Uni13] which uses the opposite convention where "=" is propositional/typal equality and "≡" definitional/judgmental equality.

This type of notational conventions is not the only difference between `Cubical Agda` and Book HoTT. Many proofs that are complicated in Book HoTT become remarkably direct using the direct treatment of equality using path types. For instance, function extensionality and its inverse funExt^{-} are one-liners that just flip the arguments:

$$\mathsf{funExt} : ((x : A) \to f\ x \equiv g\ x) \to f \equiv g$$
$$\mathsf{funExt}\ p\ i\ x = p\ x\ i$$

$$\mathsf{funExt}^{-} : f \equiv g \to ((x : A) \to f\ x \equiv g\ x)$$
$$\mathsf{funExt}^{-}\ p\ x\ i = p\ i\ x$$

In Book HoTT, however, funExt is typically proved as a consequence of the univalence axiom using a rather ingenious proof [Lic14] while its inverse follows from path induction. Another elementary example of a proof involving $\_\equiv\_$ is cong (called `ap` in Book HoTT), which applies a function to a path:

$$\mathsf{cong} : (f : A \to B)\ (p : x \equiv y) \to f\ x \equiv f\ y$$
$$\mathsf{cong}\ f\ p\ i = f\ (p\ i)$$

Although the treatment of paths in `Cubical Agda` differs somewhat from Book HoTT, we may still prove *path induction*: for any dependent type $B : (y : A)\,(p : x \equiv y) \to \mathsf{Type}$, all dependent functions $f : (y : A)\,(p : x \equiv y) \to B\,x\,p$ are uniquely determined by $f\,x\,(\mathsf{refl}\,x)$. In Book HoTT, this can be used, among other things, to define the notion of a *dependent* path, which formalises the situation when two points $a : A$ and $b : B$ are equal up to a path $p : A \equiv B$. In `Cubical Agda`, however, the type of dependent paths is primitive:

$$\mathsf{PathP} : (A : \mathsf{I} \to \mathsf{Type}) \to A\ \mathsf{i0} \to A\ \mathsf{i1} \to \mathsf{Type}$$

In fact, $\_\equiv\_$ is just the special case of PathP where the line of paths is constant:

$$\_\equiv\_ : A \to A \to \mathsf{Type}$$
$$x \equiv y = \mathsf{PathP}\ (\lambda\ \_ \to A)\ x\ y$$

We are now ready to describe the induction principle of $\mathbb{S}^1$. A dependent function $f : (x : \mathbb{S}^1) \to B\,x$ is determined by a point $b : B\,\mathsf{base}$ and a loop $\ell : \mathsf{PathP}(\lambda i \to B(\mathsf{loop}\,i))\,b\,b$. In `Cubical Agda`, this would be written using pattern matching, as in the left-most definition below, which is introduced side-by-side with the way it would commonly be written in informal HoTT (as in Brunerie's thesis):

[2]For technical reasons, this is actually just a "pre-type" in `Cubical Agda`.

```
f base = b
f (loop i) = ℓ i
```

$$f(\mathsf{base}) = b$$
$$\mathsf{ap}_f(\mathsf{loop}) = \ell$$

2.2. **More higher inductive types.** Let us now introduce the remaining HITs used in [Bru16a]. These come equipped with induction principles analogous to that of $\mathbb{S}^1$. To define higher spheres, we need suspensions:

```
data Susp (A : Type) : Type where
  north : Susp A
  south : Susp A
  merid : A → north ≡ south
```

We always take suspensions to be pointed by north. We may now define the n-sphere, for $n \geq 1$, by $\mathbb{S}^n = \mathsf{Susp}^{n-1}\,\mathbb{S}^1$ where Susp^{n-1} denotes $(n-1)$-fold suspension. We also define $\mathbb{S}^{-1} = \bot$ (the empty type) and $\mathbb{S}^0 = \mathsf{Bool}$. We remark that we could equivalently have defined $\mathbb{S}^1$ as the suspension of $\mathbb{S}^0$ as is done in [Bru16a]. Our reason for not doing so is that certain functions using $\mathbb{S}^1$ appear to compute better with the base/loop definition. Furthermore, this is the definition used in already existing code in the `agda/cubical` library.

We may also capture the (homotopy) pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$ by the HIT:

```
data Pushout (f : A → B) (g : A → C) : Type where
  inl : B → Pushout f g
  inr : C → Pushout f g
  push : (a : A) → inl (f a) ≡ inr (g a)
```

Diagrammatically this corresponds to:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\mathsf{inr}"] \\ B \arrow[r, "\mathsf{inl}"'] & \mathsf{Pushout}\,f\,g \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We use pushouts to define the wedge sum of two pointed types, denoted $A \vee B$, the join of two types, denoted $A * B$, and the cofibre of a map $f : A \to B$, denoted $\mathsf{cofib}\,f$:

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \vee B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[r, "\mathsf{snd}"] \arrow[d, "\mathsf{fst}"'] & B \arrow[d] \\ A \arrow[r] & A * B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathbb{1} \arrow[r] & \mathsf{cofib}\,f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Two particularly important functions out of wedge sums are

```
∇ : A ∨ A → A
∇ (inl x) = x
∇ (inr x) = x
∇ (push ⋆₁ i) = ⋆_A
```

and

```
i∨ : A ∨ B → A × B
i∨ (inl a) = (a , ⋆_B)
```

```
i∨ (inr b) = (⋆A , b)
i∨ (push ⋆1 i) = (⋆A , ⋆B)
```

2.3. **Truncation levels and n-truncations.** An important concept in HoTT is that of Voevodsky's *h-levels* [Voe10b], which gives rise to the notion of an *n-type*. Since types in HoTT are interpreted as spaces (or rather, as homotopy types), they are not only determined by their points but also by which higher paths they may contain. We say that a type A is an n-type if all $(n+1)$-dimensional structure of A is trivial. Formally, this is captured by an inductive definition. We say that A is a (-2)-type if it is contractible, i.e. consisting of a single point, as captured by $\mathsf{isContr}\, A = \Sigma_{a_0:A}((a : A) \to a_0 \equiv a)$. We inductively say that A is an $(n+1)$-type if for any $x, y : A$, the type $x \equiv y$ is an n-type. We call (-1)-types *propositions* and 0-types *sets*.

We can turn any type A into an n-type by *n-truncation*, denoted $\| A \|_n$. For instance, the (-1)-truncation may be directly defined using the following HIT:

```
data ∥_∥₋₁ (A : Type) : Type where
  |_| : A → ∥ A ∥₋₁
  squash : (x y : ∥ A ∥₋₁) → x ≡ y
```

We often use direct definitions like this of (-1)- and 0-truncation in our formalisation, and similar constructions work for any fixed value of n, but not when n is arbitrary. For higher n we rely on the hub-and-spoke construction [Uni13, Section 7.3].

```
data ∥_∥ (A : Type) (n : ℕ₋₁) : Type where
  |_| : A → ∥ A ∥ n
  hub : (f : 𝕊 n → ∥ A ∥ n) → ∥ A ∥ n
  spoke : (f : 𝕊 n → ∥ A ∥ n) (x : 𝕊 n) → hub f ≡ f x
```

One caveat with truncations is that a map $f : A \to B$ does *not*, in general, induce a map $f : \| A \|_n \to B$. This is, however, the case when B is an n-type. In particular, f always induces a function $\| f \|_n : \| A \|_n \to \| B \|_n$.

2.4. **Univalence, loop spaces, and H-spaces.** In order to introduce Voevodsky's univalence principle [Voe10a], we need to define the (homotopy) fibre of a function. Given a function $f : A \to B$ and a point $b : B$, we define the fibre of f over b by $\mathsf{fib}\, f\, b = \Sigma_{x:A}(f\, a \equiv b)$. We say that $f : A \to B$ is an equivalence, written $f : A \simeq B$, if $\mathsf{fib}\, f\, b$ is contractible for all $b : B$. In order to prove that a function $f : A \to B$ is an equivalence, it suffices to provide an inverse $f^- : B \to A$ and two paths $f \circ f^- \equiv \mathsf{id}_B$ and $f^- \circ f \equiv \mathsf{id}_A$. If f is also pointed, we write $f : A \simeq_\star B$.

Univalence states that the canonical map $A \equiv B \to A \simeq B$, defined by path induction, is an equivalence. In particular, we get a map $\mathsf{ua} : A \simeq B \to A \equiv B$ promoting equivalences to paths. This provides us with a useful method for transferring proofs between equivalent types which extends to *structured* types and are then referred to as the *structure identity principle* [Uni13, Section 9.8].

Transferring proofs is, however, not the only use case of univalence in HoTT. It can also be used to characterise *loop spaces* of HITs. This is often done using the *encode-decode method* [Uni13, Section 8.1.4], a type theoretic analogue of proofs by contractibilty of total spaces of fibrations. In HoTT, we define the loop space of a pointed type A, by

$\Omega\, A = (\star_A \equiv \star_A)$. This is again pointed by $\mathsf{refl}\, \star_A$, so we may iterate this definition to get the nth loop space of A, denoted $\Omega^n\, A$. Loop spaces belong to a particularly important class of types called *H-spaces*. These consist of a pointed type B equipped with a unital magma structure

$$\begin{aligned}
&\mu : B \times B \to B \\
&\mu_l : (b : B) \to \mu(\star_B, b) \equiv b \\
&\mu_r : (b : B) \to \mu(b, \star_B) \equiv b
\end{aligned}$$

satisfying $\mu_l\, \star_B \equiv \mu_r\, \star_B$. Another particularly important H-space for our purposes is $\mathbb{S}^1$, for which we will use $+$ to denote its binary operation. $\mathbb{S}^1$ also comes equipped with a notion of inversion which we will denote by $-$. In fact, $\mathbb{S}^1$ is a commutative and associative H-space.

3. First results on homotopy groups of spheres

In this section, we cover [Bru16a, Chapter 2], which introduces some elementary results on the homotopy groups of spheres. All of these results can also be found in the HoTT Book [Uni13]. Before even stating them, we need homotopy groups:

Definition 3.1 (Homotopy groups)**.** For $n : \mathbb{N}$, we define the nth *homotopy group* of a pointed type A by:

$$\pi_n(A) = \|\, \mathbb{S}^n \to_\star A \,\|_0$$

The name homotopy *group* should be taken with a grain of salt: it, in general, only has a group structure when $n \geq 1$ (abelian when $n \geq 2$). The structure may be defined, much like in [BHF18, Section 5], by considering the equivalence $(\mathbb{S}^n \to_\star A) \simeq (\mathbb{S}^{n-1} \to_\star \Omega\, A)$, where the latter type has a multiplication given by pointwise path composition. An alternative definition of $\pi_n(A)$ is via loop spaces. There is an equivalence $\omega_n : \Omega^n\, A \simeq (\mathbb{S}^n \to_\star A)$ and, hence, we could equivalently have defined $\pi_n(A)$ by setting $\pi_n(A) = \|\, \Omega^n\, A \,\|_0$. This makes the group structure on $\pi_n(A)$ more transparent: it is simply path composition. This is the definition used in the HoTT Book [Uni13]. Brunerie uses both definitions in his thesis and often passes between the two without comment.

An elementary but crucial result for the computation of homotopy groups is the existence of the *long exact sequence of homotopy groups*. Its proof is usually phrased using the loop space definition of homotopy groups as in e.g. [Uni13, Theorem 8.4.6]. For ease of notation, let us write $\mathsf{fib}\, f$ for the fibre of a pointed function $f : A \to_\star B$ over the basepoint of B.

Proposition 3.2 (LES of homotopy groups)**.** *For any pointed map $f : A \to_\star B$, there is a long exact sequence*

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{n+1}(B) \arrow[dll] \\
\pi_n(\mathsf{fib}\, f) \arrow[r] & \pi_n(A) \arrow[r] & \pi_n(B) \arrow[dll] \\
\pi_{n-1}(\mathsf{fib}\, f) \arrow[r] & \cdots &
\end{tikzcd}$$

where the horizontal maps are induced by the functorial action of π_n on $\mathsf{fst} : \mathsf{fib}\, f \to A$ and $f : A \to B$.

Above, we have implicitly taken the kernel and image of a group homomorphism $\phi : G \to H$ to be defined by

$$\begin{aligned} \mathsf{ker}\,\phi &= \mathsf{fib}\,\phi\,0_H \\ \mathsf{im}\,\phi &= \Sigma_{h:H} \| \, \Sigma_{g:G}(\phi(g) \equiv h) \, \|_{-1} \end{aligned}$$

When analysing loop spaces and homotopy groups of suspensions, the following function is of great importance. It will be used in many constructions to come.

Definition 3.3 (The suspension map)**.** Given a pointed type A, there is a canonical map $\sigma : A \to \Omega\,(\mathsf{Susp}\,A)$ given by

$$\sigma\,x = \mathsf{merid}\ x \cdot (\mathsf{merid}\ \star_A)^{-1}$$

This induces a homomorphism on homotopy groups by post-composition:

$$\pi_n(A) \xrightarrow{\sigma_*} \pi_n(\Omega\,(\mathsf{Susp}\,A)) \xrightarrow{\cong} \pi_{n+1}(\mathsf{Susp}\,A)$$

We will often, with some abuse of notation, simply write σ_* for this composition. We also define $\sigma_n : \| \, A \, \|_n \to \Omega \, \| \, \mathsf{Susp}\,A \, \|_{n+1}$ by

$$\sigma_n |\, x \,| = \mathsf{cong}\ |\_|\ (\sigma\,x)$$

We will soon see the suspension map in action, but first we need the following elementary result.

Proposition 3.4 (Join of spheres)**.** $\mathbb{S}^n * \mathbb{S}^m \simeq \mathbb{S}^{n+m+1}$.

In fact, as we will see in section 6, there is more to say about this equivalence. We make a forwards reference to Proposition 6.4 and the preceding discussion for a detailed account of its construction.

In particular, Proposition 3.4 gives us an equivalence $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$. Using this fact, we define the following map, which will play a crucial role in the analysis of $\pi_4(\mathbb{S}^3)$.

Definition 3.5 (Hopf map)**.** We define $\mathsf{hopf} : \mathbb{S}^3 \to \mathbb{S}^2$ by the composition $\mathbb{S}^3 \xrightarrow{\sim} \mathbb{S}^1 * \mathbb{S}^1 \xrightarrow{\mathsf{h}} \mathbb{S}^2$ where h is given by

```
h : S¹ * S¹ → S²
h (inl x) = north
h (inr y) = north
h (push (x , y) i) = σ (y - x) i
```

where $-$ is defined using the H-space and inversion structure on $\mathbb{S}^1$.

It turns out that the following is true [Uni13, Theorem 8.5.1].

Proposition 3.6 (The fibre of the Hopf map)**.** *The fibre of* hopf *is equivalent to* $\mathbb{S}^1$*, i.e.* $\mathsf{fib}\,\mathsf{hopf} \simeq \mathbb{S}^1$.

Proposition 3.6 gives us a fibration sequence $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ which, in particular, will allow us to connect homotopy groups of $\mathbb{S}^2$ with those of $\mathbb{S}^3$ and $\mathbb{S}^1$. For this, we need to introduce the notion of *connectedness*. We say that a type A is n-connected if $\| \, A \, \|_n$ is contractible. Similarly, we say that a function $f : A \to B$ is n-connected if all of its fibres are n-connected. This means, in particular, that the induced function $\| \, f \, \|_n : \| \, A \, \|_n \to \| \, B \, \|_n$ is an equivalence. The following is an immediate consequence of the definition of n-truncations.

Lemma 3.7 (Connectedness of spheres)**.** *For $n \geq -1$, $\mathbb{S}^n$ is $(n-1)$-connected.*

Using Lemma 3.7, we can easily prove the following:

Proposition 3.8 ([Bru16a, Proposition 2.4.1])**.** *For $n < m$, the group $\pi_n(\mathbb{S}^m)$ is trivial.*

For the sake of completeness, let us take the liberty of mentioning some results from [Bru16a, Chapter 3] already here, since they also concern low-dimensional homotopy groups of spheres. A crucial result is the following theorem [Uni13, Theorem 8.6.4]:

Theorem 3.9 (Freudenthal suspension theorem)**.** *Given an n-connected and pointed type A, the map $\sigma : A \to \Omega\,(\mathsf{Susp}\,A)$ is $2n$-connected.*

On can easily deduce from Theorem 3.9 that, in particular, $\sigma_n : \|\, A \,\|_n \to \|\, \Omega\,(\mathsf{Susp}\,A) \,\|_n$ is an equivalence. This allows us to prove the following result:

Corollary 3.10. *For $n \geq 1$, we have $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$. Furthermore, $\pi_n(\mathbb{S}^n)$ is generated by $i_n = |\, \mathsf{id}_{\mathbb{S}^n} \,|$.*

Proof. The synthetic proof of the classical result that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ is due to Licata and Shulman [LS13]. The fact that $\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1)$ is given by the LES associated to the Hopf fibration combined with Proposition 3.8. The fact that $\pi_{n+1}(\mathbb{S}^{n+1}) \cong \pi_n(\mathbb{S}^n)$ is an immediate consequence of Theorem 3.9. The second statement follows by induction on n, using that suspension is functorial and thereby preserves the identity map. □

We have now analysed all homotopy groups $\pi_n(\mathbb{S}^m)$ with $n \leq m$. This yields the following:

Proposition 3.11. *Post-composition by hopf induces an isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$.*

Proof. By Proposition 3.2 and Proposition 3.6, we get an exact sequence

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \xrightarrow{\mathsf{hopf}_*} \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1)$$

as $\pi_n(\mathbb{S}^1)$ vanishes for $n > 1$, hopf_* is an isomorphism. □

Corollary 3.12. *There is an isomorphism $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Furthermore, $\pi_3(\mathbb{S}^2)$ is generated by hopf.*

Proof. By Corollary 3.10 we know that $\pi_3(\mathbb{S}^3)$ is generated by the identity function on $\mathbb{S}^3$. We know that the isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$ is given by post-composition by hopf and thus the generator of $\pi_3(\mathbb{S}^3)$ is mapped to hopf. □

3.1. **Formalisation of Brunerie's Chapter 2.** Most of these results have already been added to `agda/cubical` by Mörtberg & Pujet [MP20], Ljungström [Lju20], and Brunerie, Ljungström & Mörtberg [BLM22]. The Freudenthal suspension theorem was formalised in `Cubical Agda` by Cavallo [Cav20], using a direct cubical proof following [Uni13, Theorem 8.6.4]. Corollary 3.10 was given a direct proof, following the computation of cohomology groups of spheres in [BLM22].

There were some technical difficulties related to the equivalence $\omega_n : \Omega^n\,A \simeq (\mathbb{S}^n \to_\star A)$, which is used to show that the two different definitions of homotopy groups are equivalent. In several proofs, it is more natural to work on the left-hand-side of ω_n. At the same time, working on the right-hand-side often makes constructing elements easier (compare, for instance, an explicit description of the generator of $i_3 : \pi_3(\mathbb{S}^3)$ described as a 3-loop in

$\mathbb{S}^3$ to the very compact definition $i_3 = |\,\mathsf{id}_{\mathbb{S}^3}\,|$). This means that we often have to translate between the two definitions. One particularly important example is the LES of homotopy groups associated to a function $A \to_\star B$. On each level, the maps are given as follows:

$$\Omega^n\,(\mathsf{fib}\,f) \xrightarrow{\Omega^n\,\mathsf{fst}} \Omega^n\,A \xrightarrow{\Omega^n\,f} \Omega^n\,B$$

This is then transported to the definition of homotopy groups as maps from spheres via ω_n. For the proof of e.g. Corollary 3.12, we need to know that the maps in the sequence are given as follows:

$$\pi_n(\mathsf{fib}\,f) \xrightarrow{\mathsf{fst}_*} \pi_n(A) \xrightarrow{f_*} \pi_n(B)$$

What we need is then more than just an equivalence $\omega_n : \Omega^n\,A \simeq (\mathbb{S}^n \to_\star A)$ – we need to show that this equivalence is functorial. This is implicitly assumed in Brunerie's thesis, but, in `Cubical Agda`, we need to make it precise. Formalising this fact is not entirely trivial. First, we need a tractable definition of the equivalence in question. It can be described inductively with base case $\omega_1 : \Omega\,A \to (\mathbb{S}^1 \to_\star A)$ given by:

$$\begin{aligned} \omega_1\,p\,\mathsf{base} &= \star_A \\ \omega_1\,p\,(\mathsf{loop}\,i) &= p\,i \end{aligned}$$

which we take to be pointed by refl. It is easy to verify that this is an equivalence. We define ω_{n+1} by the composition:

$$\Omega^{n+1}\,A = \Omega\,(\Omega^n\,A) \xrightarrow{\Omega\,\omega_n} \Omega\,(\mathbb{S}^n \to_\star A) \xrightarrow{\mathsf{funExt}_\star^-} (\mathbb{S}^n \to_\star \Omega\,A) \to (\mathbb{S}^{n+1} \to_\star A)$$

where the last arrow comes from the adjunction $\mathsf{Susp} \dashv \Omega$. This is a composition of equivalences, and hence an equivalence. We then need to verify that the following commutes

$$\begin{tikzcd}
\Omega^n\,A \arrow[r, "\omega_n"] \arrow[d, "\Omega^n\,f"'] & (\mathbb{S}^n \to_\star A) \arrow[d, "f_*"] \\
\Omega^n\,B \arrow[r, "\omega_n"'] & (\mathbb{S}^n \to_\star B)
\end{tikzcd}$$

This can be proved inductively. The base case is easy and the inductive step is given by the following diagram

$$\begin{tikzcd}
& & \Omega\,(\mathbb{S}^n \to_\star A) \arrow[dl, "\simeq"] \arrow[ddd, "\Omega\,f_*"] \\
\Omega^{n+1}\,A \arrow[urr, "\Omega\,\omega_n"] \arrow[r, "\omega_{n+1}"'] \arrow[d, "\Omega^{n+1}\,f"'] & (\mathbb{S}^{n+1} \to_\star A) \arrow[d, "f_*"] & \\
\Omega^{n+1}\,B \arrow[r, "\omega_{n+1}"] \arrow[drr, "\Omega\,\omega_n"'] & (\mathbb{S}^{n+1} \to_\star B) & \\
& & \Omega\,(\mathbb{S}^n \to_\star B) \arrow[ul, "\simeq"']
\end{tikzcd}$$

where the commutativity of the outer square comes from the base case paired with the inductive hypothesis, the triangles from the definition of ω_{n+1} and the right-most square from a straightforward argument.

4. The Brunerie number

Here we give an overview of the first half of Brunerie's proof. This corresponds to [Bru16a, Chapter 3] and culminates in the isomorphism $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$ for an at this point unknown "Brunerie number" $\beta : \mathbb{Z}$. We also discuss the formalisation of this part of the proof and various simplifications found during the formalisation.

4.1. The James construction.

To define β, Brunerie uses the *James construction* [Jam55], which he introduced in HoTT and partially formalised in [Bru19].

Proposition 4.1 (James construction)**.** *For a $(k \geq 0)$-connected pointed type A, there are types $\mathsf{J}_n\, A$ with inclusions*

$$\mathsf{J}_0\, A \xhookrightarrow{j_0} \mathsf{J}_1\, A \xhookrightarrow{j_1} \mathsf{J}_2\, A \xhookrightarrow{j_2} \cdots$$

such that its sequential colimit $\mathsf{J}_\infty\, A \simeq \Omega\,(\mathsf{Susp}\, A)$. Furthermore, $j_n : \mathsf{J}_n\, A \hookrightarrow \mathsf{J}_{n+1}\, A$ is $(n(k+1) + (k-1))$-connected.

A consequence of Proposition 4.1 is the following fact

Proposition 4.2. *Given a $(k \geq 0)$-connected type A, there is a $(3k+1)$-connected map $\mathsf{J}_2\, A \to \Omega\,(\mathsf{Susp}\, A)$.*

The proof of Proposition 4.2 uses that $\mathsf{J}_\infty\, A$, the sequential colimit of the sequence in Proposition 4.1, can be shown to be equivalent to $\Omega\,(\mathsf{Susp}\, A)$. This, paired with some results on the connectivity of sequential colimits, gives the statement. A key consequence of this is the following result which allows us to express $\pi_4(\mathbb{S}^3)$ as $\pi_3(\mathsf{J}_2\,\mathbb{S}^2)$ – a group which turns out to be quite a bit easier to reason about.

Theorem 4.3. $\pi_4(\mathbb{S}^3) \cong \pi_3(\mathsf{J}_2\,\mathbb{S}^2)$

Proof. Because $\mathbb{S}^2$ is 1-connected, Proposition 4.2 tells us that there is a 4-connected map

$$\mathsf{J}_2\,\mathbb{S}^2 \to \Omega\,(\mathsf{Susp}\,\mathbb{S}^2) = \Omega\,(\mathbb{S}^3)$$

In particular, it is 3-connected and induces an equivalence $\|\,\mathsf{J}_2\,\mathbb{S}^2\,\|_3 \simeq \|\,\Omega\,\mathbb{S}^3\,\|_3$. We get:

$$\pi_4(\mathbb{S}^3) \cong \pi_3(\Omega\,\mathbb{S}^3) \cong \pi_3(\mathsf{J}_2\,\mathbb{S}^2) \qquad \square$$

4.2. **Formalisation of the James construction.** This is a particularly technical part of Brunerie's thesis, primarily due to the high number of higher coherences which need to be verified in the proof of Proposition 4.1. While this has, subsequent to our efforts, been formalised in its entirety by Kang [Kan22a], we have taken a shortcut by giving a direct proof of Theorem 4.3, which means we do not in fact need the full James construction. Consequently, we instead give direct definitions of $\mathsf{J}_n\, A$ for $n \leq 2$ for a pointed type A.

Definition 4.4 (Low dimensional James construction)**.** We define $\mathsf{J}_0\, A = \mathbb{1}$ and $\mathsf{J}_1\, A = A$. The type $\mathsf{J}_2\, A$ is defined as the pushout:

$$\begin{array}{ccc} A \vee A & \xrightarrow{\nabla} & A \\ {\scriptstyle i^\vee}\downarrow & \ulcorner & \downarrow \\ A \times A & \longrightarrow & \mathsf{J}_2\, A \end{array}$$

We remark that the construction in Definition 4.4 is not definitionally the same as Brunerie's; in his thesis, these constructions are theorems rather than definitions. Here we take them as definitions. With $\mathsf{J}_n\, A$ defined this way, the map $j_0 : \mathsf{J}_0\, A \to \mathsf{J}_1\, A$ is just the constant pointed map and $j_1 : \mathsf{J}_1\, A \to \mathsf{J}_2\, A$ is inr.

Before we continue, let us temporarily redefine $\mathbb{S}^2$ to be the following equivalent HIT. This will make some of the following constructions more compact.

```
data S² : Type where
  base : S²
  surf : refl base ≡ refl base
```

The next lemma will be crucial. It is a special case of the *Wedge Connectivity Lemma* [Uni13, Lemma 8.6.2], of which we have formalised a version of the proof of the sphere case in [BLM22, Lemma 8]. From the point of view of formalisation, this proof is easier to work with since it gives more useful definitional equalities.

Lemma 4.5 (Wedge connectivity for $\mathbb{S}^2$)**.** *Let $P : \mathbb{S}^2 \times \mathbb{S}^2 \to 2\text{-}\mathsf{Type}$. Any function $f : (x : \mathbb{S}^2 \times \mathbb{S}^2) \to P\, x$ is induced by the following data:*

$$f_l : (x : \mathbb{S}^2) \to P(x\,,\mathsf{base})$$
$$f_r : (y : \mathbb{S}^2) \to P(\mathsf{base}\,,y)$$
$$f_{lr} : f_l\,\mathsf{base} \equiv f_r\,\mathsf{base}$$

Before we discuss the formalisation of Theorem 4.3 stated with the low dimensional James construction, we first construct the following function. The goal is to define a family of equivalences $f_x : \|\,\mathsf{J}_2\,\mathbb{S}^2\,\|_3 \simeq \|\,\mathsf{J}_2\,\mathbb{S}^2\,\|_3$ over $x : \mathbb{S}^2$. We do this by truncation elimination and pattern matching on x, starting with the base case:

$$\begin{aligned} f_{\mathsf{base}}\,|\,\mathsf{inl}\,(x,y)\,| &= |\,\mathsf{inl}\,(x\,,y)\,| \\ f_{\mathsf{base}}\,|\,\mathsf{inr}\,z\,| &= |\,\mathsf{inl}\,(\mathsf{base}\,,z)\,| \\ f_{\mathsf{base}}\,|\,\mathsf{push}\,(\mathsf{inl}\,x)\,i\,| &= |\,(\mathsf{push}\,(\mathsf{inl}\,x)\cdot\mathsf{push}\,(\mathsf{inr}\,x)^{-1})\,i\,| \\ f_{\mathsf{base}}\,|\,\mathsf{push}\,(\mathsf{inr}\,y)\,i\,| &= |\,\mathsf{inl}\,(\mathsf{base}\,,y)\,| \\ f_{\mathsf{base}}\,|\,\mathsf{push}\,(\mathsf{push}\,y\,j)\,i\,| &= \ldots \end{aligned}$$

where the omitted step consists of a proof that $\mathsf{push}\,(\mathsf{inl}\,\mathsf{base})\cdot\mathsf{push}\,(\mathsf{inr}\,\mathsf{base})^{-1} \equiv \mathsf{refl}$. It is an easy lemma that f_{base} is equal to the identity on $\|\,\mathsf{J}_2\,\mathbb{S}^2\,\|_3$. To complete the definition of

f_x, we need to consider the case when $x = \mathsf{surf}\, i\, j$. This amounts to providing a dependent function:

$$f_{\mathsf{surf}} : (x : \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3) \to \Omega^2\, (\|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3\, ,\, f_{\mathsf{base}}\, x)$$

To do this, we will, in particular, need to provide a family of fillers

$$Q_{(x\,,\,y)} : \mathsf{refl}_{|\,\mathsf{inl}\,(x\,,\,y)\,|} \equiv \mathsf{refl}_{|\,\mathsf{inl}\,(x\,,\,y)\,|}$$

This is a 1-type, and thus Lemma 4.5 applies. We define:

$$Q_{(\mathsf{base}\,,\,y)}\, i\, j = |\, \mathsf{inl}\, (\mathsf{surf}\, i\, j\,,\, y)\, |$$
$$Q_{(x\,,\,\mathsf{base})}\, i\, j = |\, \mathsf{inl}\, (x\,,\, \mathsf{surf}\, i\, j)\, |$$

The fact that these two constructions agree when both x and y are base is a technical but relatively straightforward lemma. Thereby, $Q_{(x\,,\,y)}$ is defined. We may now define f_{surf}:

$$f_{\mathsf{surf}}\, |\, \mathsf{inl}\, (x\,,\, y)\, | = Q_{(x\,,\,y)}$$
$$f_{\mathsf{surf}}\, |\, \mathsf{inr}\, z\, | = Q_{(\mathsf{base}\,,\,z)}$$

The higher cases are easy due to the fact that the goal becomes 0-truncated, making it sufficient to define them for $\mathsf{base} : \mathbb{S}^2$. Thus, f_x is defined for all $x : \mathbb{S}^2$.

Lemma 4.6. *For $x : \mathbb{S}^2$, f_x is an automorphism on $\|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$.*

Proof. To make coming proofs easier, this is proved by explicitly constructing the inverse analogously to f_x.

$$f^{-1}_{\mathsf{base}}\, x = f_{\mathsf{base}}\, x$$
$$f^{-1}_{\mathsf{surf}}\, x = f_{\mathsf{surf}^{-1}}\, x$$

Proving that these cancel is technical, but direct. □

We are now ready to prove the following statement, which is a rephrasing of Theorem 4.3.

Proposition 4.7. $\Omega\, \|\, \mathbb{S}^3 \,\|_4 \simeq \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$

Proof. We take $\mathbb{S}^3 = \mathsf{Susp}\, \mathbb{S}^2$, where $\mathbb{S}^2$ is defined using base/surf as above. We employ the encode-decode method and define a family of 3-types over $\|\, \mathbb{S}^3 \,\|_4$. Since the universe of 3-types is a 4-type, we may do so by truncation elimination:

$$\mathsf{Code} : \|\, \mathbb{S}^3 \,\|_4 \to 3\text{-}\mathsf{Type}$$
$$\mathsf{Code}\, |\, \mathsf{north}\, | = \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$$
$$\mathsf{Code}\, |\, \mathsf{south}\, | = \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$$
$$\mathsf{Code}\, |\, \mathsf{merid}\, x\, i\, | = \mathsf{ua}\, f_x\, i$$

We now need to define two families of functions

$$\mathsf{encode}_x : |\, \mathsf{north}\, | \equiv x \to \mathsf{Code}\, x$$
$$\mathsf{decode}_x : \mathsf{Code}\, x \to |\, \mathsf{north}\, | \equiv x$$

over $x : \|\, \mathbb{S}^3 \,\|_4$. We define encode_x by path induction, sending refl to the basepoint in $\|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$. We define decode_x by truncation elimination and pattern matching on x. The crucial step is defining $\mathsf{decode}_{|\,\mathsf{north}\,|} : \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3 \to \Omega \,\|\, \mathbb{S}^3 \,\|_4$. On point constructors, it is given by

$$\mathsf{decode}_{|\,\mathsf{north}\,|}\,(\mathsf{inl}\,(x\,,y)) = \sigma\, x \cdot \sigma\, y$$
$$\mathsf{decode}_{|\,\mathsf{north}\,|}\,(\mathsf{inr}\, z) = \sigma\, z$$

which is easily verified to be coherent with the higher constructors. The case $\mathsf{decode}_{|\,\mathsf{south}\,|}$ is immediately induced by $\mathsf{decode}_{|\,\mathsf{north}\,|}$, since $\mathsf{north} \equiv \mathsf{south}$ via $\mathsf{merid}\ \mathsf{base}$. The case $\mathsf{decode}_{|\,\mathsf{merid}\ a\, i\,|}\ y$ amounts to showing that

$$\mathsf{decode}_{|\,\mathsf{north}\,|}(f_a^{-1}\, y) \equiv \mathsf{decode}_{|\,\mathsf{north}\,|}\, y \cdot (\sigma \mid a \mid)^{-1}$$

The proof is technical but is greatly aided by Lemma 4.5. The fact that $\mathsf{decode}_x(\mathsf{encode}_x\, p) \equiv p$ for each $p : \mathsf{north} \equiv x$ holds by path induction. Finally, the fact that $\mathsf{encode}_{\mathsf{north}}(\mathsf{decode}_{\mathsf{north}}\, y) \equiv y$ holds for each $y : \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$ holds by some technical but simple path algebra. Hence $\mathsf{decode}_{|\,\mathsf{north}\,|} : \Omega \,\|\, \mathbb{S}^3 \,\|_4 \to \|\, \mathsf{J}_2\, \mathbb{S}^2 \,\|_3$ is an equivalence. □

We get Theorem 4.3 as an immediate corollary of Proposition 4.7 via the same sequence of isomorphisms as in the proof of Theorem 4.3.

4.3. Definition of the Brunerie number.

Brunerie's goal is now to analyse $\pi_3(\mathsf{J}_2\, \mathbb{S}^2)$. The first result needed is the following:

Definition 4.8 (Whitehead map)**.** Given two pointed types A and B, there is a map:

$\mathsf{W} : A * B \to \mathsf{Susp}\ A \vee \mathsf{Susp}\ B$
$\mathsf{W}\ (\mathsf{inl}\ a) = \mathsf{inr}\ \mathsf{north}$
$\mathsf{W}\ (\mathsf{inr}\ b) = \mathsf{inl}\ \mathsf{north}$
$\mathsf{W}\ (\mathsf{push}\ (a\ ,\ b)\ i) = (\mathsf{cong}\ \mathsf{inr}\ (\sigma\ b) \cdot \mathsf{push}\ \star_1{}^{-1} \cdot \mathsf{cong}\ \mathsf{inl}\ (\sigma\ a))\ i$

For our purposes, we only need the case when $A = B = \mathbb{S}^1$ (although all of the following results appear in full generality in Brunerie's thesis). We get a composite map:

$$\mathsf{e} : \mathbb{S}^3 \xrightarrow{\simeq} \mathbb{S}^1 * \mathbb{S}^1 \xrightarrow{\mathsf{W}} \mathbb{S}^2 \vee \mathbb{S}^2$$

This induces, via pre-composition, a *Whitehead product*:

$$\pi_2(\mathbb{S}^2) \times \pi_2(\mathbb{S}^2) \xrightarrow{[-,-]} \pi_3(\mathbb{S}^2)$$

by

$$[|\, f\,|, |\, g\,|] := |\, \nabla \circ (f \vee g) \circ \mathsf{e}\,|$$

Recall that we denote by i_2 the generator of $\pi_2(\mathbb{S}^2)$. Brunerie shows, in particular, the following about its relation to the Whitehead product (see [Bru16a, Proposition 3.4.4.] for the full statement).

Theorem 4.9. *The kernel of the suspension map* $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ *is generated by* $[i_2, i_2]$.

The key technical component in the proof is the *Blakers-Massey Theorem*, first formalised in HoTT by Favonia, Finster, Licata & Lumsdaine in [HFLL16]:

Theorem 4.10 (Blakers-Massey)**.** *Consider the diagram*

$$\begin{tikzcd}
A \arrow[dr, dashed, "f \sqcup g"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d, "\mathsf{inr}"] \\
& B \arrow[r, "\mathsf{inl}"'] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{Pushout}\, f\, g
\end{tikzcd}$$

where P is the pullback along inl *and* inr*, i.e.* $P = \Sigma_{(b,c):B \times C}(\mathsf{inl}\, b \equiv \mathsf{inr}\, c)$*, and $f \sqcup g$ is defined by*

$$(f \sqcup g)\, a = (f\, a\, , g\, a\, , \mathsf{push}\, a)$$

If f and g are n- respectively m-connected, then $f \sqcup g$ is $(n+m)$-connected.

Theorem 4.9 is proved by considering the following diagram

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[dr] \arrow[drr, bend left, "\nabla \circ W"] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{S}^2 \arrow[d] \\
& \mathbb{1} \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & J_2\mathbb{S}^2
\end{tikzcd}$$

Verifying that the outer square is a pushout square is technical and we refer to Brunerie's proof for the details. Above, P is simply the fibre of $\mathsf{inr} : \mathbb{S}^2 \to \mathsf{J}_2\, \mathbb{S}^2$. The leftmost map is 2-connected since $\mathbb{S}^3$ is 2-connected and the top map is 0-connected since $\mathbb{S}^3$ and $\mathbb{S}^2$ are both 1-connected. Consequently, by Theorem 4.10, we get that the map $\mathbb{S}^3 \to P$ is 2-connected and thus induces a surjection after application of π_3. This gives the diagram:

$$\begin{tikzcd}
\pi_3(P) \arrow[r] & \pi_3(\mathbb{S}^2) \arrow[r] \arrow[dr, "\sigma_*"'] & \pi_3(\mathsf{J}_2\, \mathbb{S}^2) \arrow[d, "\cong"] \\
\pi_3(\mathbb{S}^3) \arrow[u, two heads] \arrow[ur, dashed] & & \pi_4(\mathbb{S}^3)
\end{tikzcd}$$

where the sequence on the top comes from the long exact sequence of homotopy groups associated to P. The dashed map sends the generator $i_3 : \pi_3(\mathbb{S}^3)$ to $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ by definition.

Theorem 4.9 motivates the following definition. Recall that we denote by ψ the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

Definition 4.11 (Brunerie number)**.** We define the Brunerie number $\beta : \mathbb{Z}$ by $\beta = \psi[i_2, i_2]$.

We may now prove the main result of [Bru16a, Chapter 3].

Corollary 4.12. *$\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$.*

Proof. We have a homomorphism $\sigma_* \circ \psi^{-1} : \mathbb{Z} \to \pi_4(\mathbb{S}^3)$. This composition is surjective since ψ is an isomorphism and $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective by Theorem 3.9. Since, by Theorem 4.9, the kernel of σ_* is generated by $[i_2, i_2]$, the kernel of $\sigma_* \circ \psi^{-1}$ is generated by $\psi[i_2, i_2]$, i.e. by β. The statement then follows from the first isomorphism theorem. $\square$

4.4. **Formalisation of the definition of the Brunerie number.** The formalisation of this part was straightforward. Arguably the most technical result, the Blakers-Massey theorem, was already available in the library thanks to Kang [Kan22b]. Most of the remaining results were essentially just diagram chases which, in a proof assistant, can be somewhat technical. Most work went into verifying that $\mathsf{J}_2\,\mathbb{S}^2$ is the cofibre of $\nabla \circ \mathsf{W}$, the proof of which followed Brunerie's closely.

In this section we found the only obvious mistake in Brunerie's thesis. On page 82, in his definition of the push-case for W, the path component in the middle was not inverted, making the term ill-typed. Naturally, this was of no mathematical significance and something Brunerie immediately would have noticed if he would have attempted to provide a computer formalisation of this construction.

5. Brunerie's proof that $|\beta| \equiv 2$

This section concerns the final three Chapters (4–6) of Brunerie's thesis. The main goal here is proving that $|\beta| \equiv 2$.

We will not discuss Chapter 4 in much detail. Chapter 4 is devoted to smash products and, in particular, their symmetric monoidal structure. Brunerie used this in subsequent chapters to define and prove properties about the *cup product*, a graded multiplicative operation on cohomology groups which will be used to show that $|\beta| \equiv 2$. This chapter has turned out to be incredibly difficult to formalise due to the large number of higher coherences involved in the proofs [Bru18]. In fact, the results of this chapter were proved in detail and fully formalised only 8 years after the publication of Brunerie's thesis by Ljungström [Lju24]. We remark that despite the fact that these results have now been made available to us, they are not needed. While, with these results, Brunerie's construction of the cup product appears correct, his use of smash products still leads to some rather cumbersome diagram chases (with many coherences which still need verification.)

Luckily, it turns out that Chapter 4 can be avoided altogether and that this in fact makes some difficult proofs later on very direct. For this reason, the results in Chapter 4 were omitted completely from our formalisation. The reason for this is that all results regarding smash products in Brunerie's thesis concern, in some way, pointed maps out of smash products. In this case, we may exploit the adjunction of maps out of smash products and bi-pointed maps:

$$(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$$

Here, $B \to_\star C$ is taken to be pointed by the constant map. As shown by Brunerie et al. [BLM22], it is arguably easier to define the cup product on the right-hand side of the adjunction, which effectively means that we never have to work with smash products when formalising cohomology theory. The usefulness of the approach by Brunerie et al. [BLM22] is not only witnessed by our work—it has been used by Lamiaux et al. [LLM23] and Ljungström & Mörtberg [LM24] in the development of cohomology rings and is used to describe the cup product as an instance of the delooping machinery introduced by Wärn [Wär23, Section 4.3]. We remark that the same techniques (although independent from [BLM22]) can be found in the work of Christensen & Scoccola [CS20, Section 2.4] where it is utilised in a discussion of the magma structure on loop spaces.

5.1. **Cohomology and the Hopf invariant.** [Bru16a, Chapter 5] introduces integral cohomology groups and rings, and gives a construction of the Mayer-Vietoris sequence. In more detail, Brunerie defines the integral Eilenberg-MacLane spaces by $\mathsf{K}_0 = \mathbb{Z}$ and $\mathsf{K}_n = \| \mathbb{S}^n \|_n$ for $n \geq 1$. This allows for a definition of the (integral) cohomology of X:

$$\mathsf{H}^n(X) = \| X \to \mathsf{K}_n \|_0$$

The fact that $\Omega\, \mathsf{K}_{n+1} \simeq \mathsf{K}_n$ follows by a proof completely analogous to that of Corollary 3.10. Brunerie uses this equivalence to carry over the (commutative) H-space structure on $\Omega\, \mathsf{K}_{n+1}$ to that of K_n. This provides a notion of addition $+_k : \mathsf{K}_n \times \mathsf{K}_n \to \mathsf{K}_n$ which lifts to $\mathsf{H}^n(X)$ by post-composition, thereby endowing $\mathsf{H}^n(X)$ with a group structure.

Similarly, Brunerie gives a definition of a cup product $\smile_k : \mathsf{K}_n \wedge \mathsf{K}_m \to \mathsf{K}_{n+m}$ which lifts to the usual cup product $\smile : \mathsf{H}^n(X) \times \mathsf{H}^m(X) \to \mathsf{H}^{n+m}(X)$. This is shown to induce a graded commutative ring structure on $\mathsf{H}^*(X)$ using results from Chapter 4.

The synthetic construction of the Mayer-Vietoris sequence concerns the long exact sequence

$$\begin{tikzcd} \mathsf{H}^0(D) \arrow[r] & \mathsf{H}^0(B) \times \mathsf{H}^0(C) \arrow[r] & \mathsf{H}^0(A) \arrow[dll, out=-170, in=10] \\ \mathsf{H}^1(D) \arrow[r] & \ldots \end{tikzcd}$$

where D denotes the pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$. A direct application gives us, for $n \geq 1$, that $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{Z}$ if $n = m$ and $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{1}$ otherwise. This gives, by another application of the sequence, the following result:

Lemma 5.1. *For any $f : \mathbb{S}^3 \to \mathbb{S}^2$ we have*

$$\mathsf{H}^n(\mathsf{cofib}\, f) \;\cong\; \begin{cases} \mathbb{Z} & n \in \{0, 2, 4\} \\ \mathbb{1} & \text{otherwise} \end{cases}$$

Let us briefly fix $f : \mathbb{S}^3 \to \mathbb{S}^2$. Denote by γ_2 and γ_4 the generators of $\mathsf{H}^2(\mathsf{cofib}\, f)$ and $\mathsf{H}^4(\mathsf{cofib}\, f)$ respectively given by the image of $1 : \mathbb{Z}$ under the isomorphism in Lemma 5.1. These generators may be used to define an invariant on $\mathbb{S}^3 \to \mathbb{S}^2$ called the *Hopf invariant*. This is done as follows:

Definition 5.2 (Hopf invariant)**.** The *Hopf invariant* of f is the unique integer $\mathsf{HI}\, f : \mathbb{Z}$ such that $\gamma_2 \smile \gamma_2 \equiv \mathsf{HI}\, f \cdot \gamma_4$.

We remark that the above definition is given for the more general class of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ in Brunerie's thesis. For our purposes, the above special case suffices. In particular, we may see HI as a function $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$. The following turns out to be true:

Proposition 5.3. HI *is a homomorphism* $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$*.*

Proof sketch. We first rephrase $f + g : \pi_3(\mathbb{S}^2)$ as a composition

$$\mathbb{S}^3 \to \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{f \vee g} \mathbb{S}^2 \vee \mathbb{S}^2 \xrightarrow{\nabla} \mathbb{S}^2$$

By analysing the cohomology of $\mathsf{cofib}\,(\nabla \circ (f \vee g))$ and the action on generators of the obvious maps from $\mathsf{cofib}\,(\nabla \circ (f \vee g))$, $\mathsf{cofib}\, f$ and $\mathsf{cofib}\, g$ into $\mathsf{cofib}\,(f + g)$, one arrives at the result with some elementary algebra. □

Finally, the Hopf invariant of our element of interest $[i_2, i_2]$ is computed (up to a sign), using an argument similar to that of the proof of Proposition 5.3.

Proposition 5.4. $|\mathsf{HI}\,[i_2, i_2]| \equiv 2$

We are now almost done: if there is an element $f : \pi_3(\mathbb{S}^2)$ such that $\mathsf{HI}\,f \equiv 1$, then HI is an isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Since isomorphisms of this type are unique up to a sign, Proposition 5.4 tells us that also for the standard isomorphism $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, we must have $|\psi[i_2, i_2]| \equiv 2$, i.e. $|\beta| \equiv 2$. Hence, we have so far shown the following:

Lemma 5.5. *If* $\mathsf{HI}\,f \equiv 1$ *for some* $f : \pi_3(\mathbb{S}^2)$*, then* $|\beta| \equiv 2$*.*

The final chapter of Brunerie's thesis is devoted to proving the antecedent of Lemma 5.5.

5.2. **Formalisation of cohomology and the Hopf invariant.** This section was largely covered by Brunerie, Ljungström and Mörtberg in [BLM22] and thus also available in `agda/cubical`. We briefly summarise:

- $+_k : \mathsf{K}_n \times \mathsf{K}_n \to \mathsf{K}_n$ was defined explicitly using a direct construction of the *Wedge Connectivity* for spheres—a generalisation of Lemma 4.5. This construction is of great convenience to our formalisation due to the fact that e.g. $\star_{\mathsf{K}_n} +_k |\,x\,| \equiv |\,x\,|$ holds definitionally. In fact, all of the basic laws governing $+_k$ are (trivially) provably path equal to refl at $\star_{\mathsf{K}_n}$, which simplifies a lot of path algebra.
- The cup product is defined via the following lift

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[d, "|\_|"'] & (\mathsf{K}_m \to_\star \mathsf{K}_{n+m}) \\ \mathsf{K}_n \arrow[ur, dashed, "\smile"'] & \end{tikzcd}$$

  for $n \geq 1$, where the top map may be thought of as being inductively defined via the equivalence

$$\begin{aligned} &(\mathbb{S}^{n+1} \to_\star (\mathsf{K}_m \to_\star \mathsf{K}_{(n+1)+m})) \\ \simeq\, &(\mathbb{S}^{n} \to_\star (\mathsf{K}_m \to_\star \Omega\,\mathsf{K}_{(n+1)+m})) \\ \simeq\, &(\mathbb{S}^{n} \to_\star (\mathsf{K}_m \to_\star \mathsf{K}_{n+m})) \end{aligned}$$

  The lift exists because the type of pointed functions $\mathsf{K}_m \to_\star \mathsf{K}_{n+m}$ is an n-type. This construction gives an inductively defined cup product which is remarkably easy to work with, as showcased in [LLM23] to compute cohomology rings of various classical spaces.
- The Mayer-Vietoris sequence was formalised by directly translating Brunerie's original proof.

Hence, what remained to be formalised in Chapter 5 was the Hopf invariant, Proposition 5.3 and Proposition 5.4. The formalisation of these propositions was straightforward and we were able to translate Brunerie's proofs in a direct manner. This is not surprising as the proofs are very algebraic.

For simplicity, we only formalised these propositions as they stand here and not their generalisations to higher spheres (i.e. as in [Bru16a, Proposition 5.4.3 & 5.4.4]). We remark, however, that the formalised proofs easily should be rephrasable for the general Hopf invariant of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$.

5.3. **The Gysin sequence.** This section corresponds to [Bru16a, Chapter 6]. In order to be able to apply Lemma 5.5, this chapter is devoted to proving that $|\mathsf{HI}\ \mathsf{hopf}| \equiv 1$, where, recall, $\mathsf{hopf} : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map—the generator of $\pi_3(\mathbb{S}^2)$ from Definition 3.5. This amounts to analysing the cup product on the cohomology of $\mathsf{cofib}\ \mathsf{hopf}$. It is well-known that $\mathsf{cofib}\ \mathsf{hopf}$ is a model of the complex projective plane $\mathbb{C}P^2$ (see e.g. [Hat02, Example 4.45]), so let us simply write $\mathbb{C}P^2$ from now on. We hence have $\mathbb{C}P^2$ defined as the following pushout:

$$\begin{tikzcd} \mathbb{S}^3 \arrow[r, "\mathsf{hopf}"] \arrow[d] & \mathbb{S}^2 \arrow[d] \\ \mathbb{1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{C}P^2 \end{tikzcd}$$

In order to show that $|\mathsf{HI}\ \mathsf{hopf}| \equiv 1$, it suffices to show that $- \smile \gamma_2 : \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^4(\mathbb{C}P^2)$ is an isomorphism for $\gamma_2 : \mathsf{H}^2(\mathbb{C}P^2)$ a generator. Brunerie does this by constructing the Gysin sequence.

Proposition 5.6 (The Gysin sequence)**.** *Let B be a pointed and 0-connected type and $P : B \to \mathsf{Type}$ be a fibration with $P \star_B \simeq \mathbb{S}^{n-1}$. Let $E = \Sigma_{b:B}(P\,b)$ be the total space of P. If there is a family of maps $c : (b : B) \to (\mathsf{Susp}\,(P\,b) \to_\star \mathsf{K}_n)$ with $c_{\star_B}$ a generator of $\mathsf{H}^n(\mathbb{S}^n)$, then there is an element $e_n : \mathsf{H}^n(B)$ and a long exact sequence*

$$\begin{tikzcd} & \dots \arrow[r] & \mathsf{H}^{i-1}(B) \arrow[dll, bend left] \\ \mathsf{H}^{i-1}(E) \arrow[r] & \mathsf{H}^{i-n}(B) \arrow[r, "- \smile e_n"] & \mathsf{H}^i(B) \arrow[dll, bend left] \\ \mathsf{H}^i(E) \arrow[r] & \dots & \end{tikzcd}$$

Moreover, c (and also e_n) exists when B is 1-connected.

In order to make use of this, we need the following result.

Proposition 5.7. *There is a fibration $P : \mathbb{C}P^2 \to \mathsf{Type}$ with $P \star_{\mathbb{C}P^2} \simeq \mathbb{S}^1$ and total space $\mathbb{S}^5$.*

Proposition 5.7 is a special case of the following result.

Proposition 5.8 (Iterated Hopf construction)**.** *Given an associative H-space A, let $h_A : A{*}A \to \mathsf{Susp}\,A$ denote the associated Hopf map. There is a fibration $\mathsf{cofib}\,h_A \to \mathsf{Type}$* with fibre A and total space $A * A * A$.

We consider the particular case when $A = \mathbb{S}^1$ in Proposition 5.8. In this case, the map $h_{\mathbb{S}^1} : \mathbb{S}^1{*}\mathbb{S}^1 \to \mathbb{S}^2$ corresponds to the usual Hopf map under the equivalence $\mathbb{S}^1{*}\mathbb{S}^1 \simeq \mathbb{S}^3$ and hence $\mathsf{cofib}\,h_{\mathbb{S}^1} \simeq \mathbb{C}P^2$. The total space of this is $\mathbb{S}^1{*}\mathbb{S}^1{*}\mathbb{S}^1$ which is equivalent to $\mathbb{S}^5$ by Proposition 3.4 and thus we have proved Proposition 5.7. The associated Gysin sequence gives us the main result of this section:

Proposition 5.9 (Hopf invariant of the Hopf map)**.** $|\mathsf{HI}\,\mathsf{hopf}| \equiv 1$

Proof. Since $\mathbb{C}P^2$ is 1-connected, Proposition 5.6 combined with Proposition 5.7 gives us an element $e_2 : \mathsf{H}^2(\mathbb{C}P^2)$ and a sequence

$$\mathsf{H}^{i-1}(\mathbb{S}^5) \to \mathsf{H}^{i-2}(\mathbb{C}P^2) \xrightarrow{- \smile e_2} \mathsf{H}^i(\mathbb{C}P^2) \to \mathsf{H}^i(\mathbb{S}^5)$$

When $1 \leq i \leq 4$, $\mathsf{H}^i(\mathbb{S}^5)$ vanishes. Setting $i = 2$, we get that e_2 must be a generator of $\mathsf{H}^2(\mathbb{C}P^2)$, and thus equal to the generator $\gamma_2 : \mathsf{H}^2(\mathbb{C}P^2)$ up to a sign. Setting $i = 4$, we get that $- \smile e_2$ must be an isomorphism of groups $\mathsf{H}^2(\mathbb{C}P^2) \cong \mathsf{H}^4(\mathbb{C}P^2)$ and hence $e_2 \smile e_2$ is a generator. Consequently, so is $\gamma_2 \smile \gamma_2$, and thus $|\mathsf{HI\ hopf}| \equiv 1$. □

Proposition 5.9 combined with Lemma 5.5 gives the desired path: $|\beta| \equiv 2$. This completes Brunerie's proof and Corollary 4.12 gives us the main result:

Theorem 5.10. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

5.4. **Formalisation of the Gysin sequence.** Formalising the results from Chapter 6 was more challenging, but was greatly aided by the alternative construction of the cup product discussed above. The first technical lemma, which is crucial for the construction of the Gysin sequence is:

Lemma 5.11. *Given $x : \mathsf{K}_n$ and $y : \mathsf{K}_m$, we have*

$$\mathsf{cong}\ (\lambda\, a \to a \smile_k y)\ (\sigma_n\, x) \equiv \sigma_{n+m}(x \smile_k y)$$

In Brunerie's thesis, this lemma relies on a result which in turn requires the symmetric monoidal structure of the smash product (in particular, it uses the *pentagon identity*). With the alternative construction of the cup product, however, this result follows immediately from the definition of the cup product.

Lemma 5.11 is used to show that the map

$$g^i : \mathsf{K}_i \to (\mathbb{S}^n \to_\star \mathsf{K}_{i+n})$$
$$g^i\, x = \lambda y \to x \smile_k \iota y$$

is an equivalence, which is crucially used in the construction of the Gysin sequence. Above, $\iota : \mathbb{S}^n \to \mathsf{K}_n$ is a generator of $\mathsf{H}^n(\mathbb{S}^n)$. For reference, g^i is the map $g^i_{\star_B}$ in the proof of [Bru16a, Proposition 6.1.2]. Brunerie's proof proceeds by induction on i: the fact that g^0 is an equivalence is easy; for the inductive step, it suffices to show that $\Omega\, g^{i+1} : \Omega\, \mathsf{K}_{i+1} \to \Omega\,(\mathbb{S}^n \to_\star \mathsf{K}_{(i+1)+n})$ is an equivalence for reasons of connectedness. This is done by showing that the following diagram commutes

$$\begin{tikzcd}
\Omega\, \mathsf{K}_{i+1} \arrow[r, "\Omega\, g^{i+1}"] \arrow[d, "\simeq"'] & \Omega\,(\mathbb{S}^n \to_\star \mathsf{K}_{(i+1)+n}) \arrow[d, "\simeq"] \\
\mathsf{K}_i \arrow[r, "g^i"'] & (\mathbb{S}^n \to_\star \mathsf{K}_{i+n})
\end{tikzcd}$$

hence getting that $\Omega\, g^{i+1}$ is an equivalence from the induction hypothesis.

While the general idea of Brunerie's proof of this statement is correct, it was difficult to formalise directly. The primary reason for this is that Brunerie does not pay much attention to the fact that the objects of interest are not just functions, but *pointed* functions. In particular, his argument for the commutativity of the diagram above treats $g^i\, x$ as a plain function rather than a pointed function. Fortunately for us, the whole proof is very direct with the alternative definition of the cup product. Formalising Brunerie's proof with pointedness of functions respected would have been hard, especially without machinery external to [Bru16a] (e.g. [BLM22, Lemma 14.]).

After these subtleties were dealt with, the formalisation of the Gysin sequence could proceed following Brunerie's proof closely. In our initial formalisation, we made a slight adjustment to the indexing of the Gysin sequence. This removed some bureaucracy but happened at the cost of generality.[3] This made verifying that Proposition 5.9 slightly less direct, because we no longer had access to the case

$$\mathsf{H}^1(\mathbb{S}^5) \longrightarrow \mathsf{H}^0(\mathbb{C}P^2) \xrightarrow{- \smile e_2} \mathsf{H}^2(\mathbb{C}P^2) \longrightarrow \mathsf{H}^2(\mathbb{S}^5)$$

which is used by Brunerie to show that the element $e_2 : \mathsf{H}^2(\mathbb{C}P^2)$, for which $- \smile e_2 : \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^4(\mathbb{C}P^2)$ is an isomorphism, is indeed a generator. However, in practice, this is not a big problem. In fact, it provides a nice example of a proof by computation. It is very direct to manually show that the map $i : \mathbb{C}P^2 \to \mathsf{K}_2$ induced by $i(\mathsf{inl}\, x) = |\, x\, |$ is equal to the underlying map of e_2. The fact that i generates $\mathsf{H}^2(\mathbb{C}P^2)$ can then be verified by computation: applying the isomorphism $\mathsf{H}^2(\mathbb{C}P^2) \cong \mathbb{Z}$ to $|\, i\, |$ returns 1 by normalisation in `Cubical Agda`. We stress, for those skeptical of this method, that it also is very direct to provide a "manual" formalisation of this fact.

The final step of the formalisation was Proposition 5.8, i.e. the iterated Hopf construction. Although technical, the formalisation could be carried out following Brunerie closely.

6. The simplified proof and normalisation of a Brunerie number

It turns out that not only Chapter 4, but also Chapters 5–6 can be avoided. As conjectured by Brunerie, it would be possible to do this by simply normalising the Brunerie number. While we still cannot normalise his original definition of it, we can at least provide a computation of a substantially simplified Brunerie number. This is defined via a more tractable description of the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ as a composition of simpler isomorphisms, relying on an alternative definition of π_3 in terms of $\mathbb{S}^1 * \mathbb{S}^1$. The idea is then to trace $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ step by step through these isomorphisms. This gives a sequence of new Brunerie numbers and one of these normalises to -2 in `Cubical Agda` in a matter of seconds.

The trick to give a more tractable definition of $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ is to redefine the third homotopy group of a type A as $\pi_3^*(A) = \|\, \mathbb{S}^1 * \mathbb{S}^1 \to_\star A \,\|_0$. This reformulation of π_3 can be given an explicit group structure, such that pre-composition by $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$ induces an isomorphism $\pi_3(A) \cong \pi_3^*(A)$. Let us first set up machinery we need (and a bit more).

6.1. Interlude: joins and smash products of spheres.

We have seen that the equivalence $\mathbb{S}^3 \simeq \mathbb{S}^1 * \mathbb{S}^1$ played a crucial role in Brunerie's original proof. What is less clear, however, is what this equivalence actually looks like. It turns out that it is closely related to the multiplication $\mathbb{S}^1 \wedge \mathbb{S}^1 \to \mathbb{S}^2$ and, as such, has a rather direct and algebraic description. Let us therefore briefly study this multiplication and describe its relation to the decomposition of spheres into joins. Although we only need low-dimensional special cases of these facts, we take the opportunity to tell the general story.

Remark. *In this subsection only, we will use the definition $\mathbb{S}^n := \mathsf{Susp}^n\, \mathsf{Bool}$. In particular, this means that we redefine $\mathbb{S}^1 := \mathsf{Susp}\, \mathsf{Bool}$ instead of using the $\mathsf{base}/\mathsf{loop}$ construction. This is only done for ease of presentation and is not used in the formalisation.*

[3] A more general form of the Gysin sequence using Brunerie's indexing has later been added to `agda/cubical`.

Let us use the following (explicit) definition of the smash product $A \wedge B$.

$$
\begin{aligned}
&\mathsf{data}\ \_\wedge\_\ (A\ B : \mathsf{Pointed}) : \mathsf{Type}\ \mathsf{where}\\
&\quad \star_\wedge : A \wedge B\\
&\quad \langle \_ \rangle : A \times B \to A \wedge B\\
&\quad \mathsf{push}_l : (a : A) \to \langle\, a\,,\, \star_B \,\rangle \equiv \star_\wedge\\
&\quad \mathsf{push}_r : (b : B) \to \langle\, \star_A\,,\, b \,\rangle \equiv \star_\wedge\\
&\quad \mathsf{push}_{lr} : \mathsf{push}_l\ \star_A \equiv \mathsf{push}_r\ \star_B
\end{aligned}
$$

This construction is well-known and can easily be seen to be (bi-)functorial (see e.g. [Lju24, Definition 6]), i.e. given pointed maps $f : A \to_\star C$ and $g : B \to_\star D$, there is a map $f \wedge g : A \wedge B \to C \wedge D$ with $(f \wedge g)\langle x\,, y\rangle := \langle f\,x\,, g\,y\rangle$.

The first goal is to define a multiplication $\mathbb{S}^n \wedge \mathbb{S}^m \to \mathbb{S}^{n+m}$. To facilitate future proofs, we first introduce the following construction which lifts maps $A \times B \to C$ to maps $(\mathsf{Susp}\,A) \times B \to \mathsf{Susp}\,B$:

$$
\begin{aligned}
&\hat{}\,\_ : (A \times B \to C) \to ((\mathsf{Susp}\ A) \times B \to \mathsf{Susp}\ C)\\
&(\hat{}\ f)\ (\mathsf{north}\ ,\ b) = \mathsf{north}\\
&(\hat{}\ f)\ (\mathsf{south}\ ,\ b) = \mathsf{north}\\
&(\hat{}\ f)\ (\mathsf{merid}\ a\ i\ ,\ b) = \sigma\ (f\ (a\ ,\ b))\ i
\end{aligned}
$$

The function $\widehat{\ }f$ is pointed in the left-argument by construction. It is pointed also in the right-argument if this is the case for f. Hence, given any function $g : A \wedge B \to C$, we also get (with some abuse of notation) $\widehat{\ }g : (\mathsf{Susp}\,A) \wedge B \to \mathsf{Susp}\,C$.

Lemma 6.1. *For any pointed types A and B, the map $\widehat{\ }\mathsf{id}_{A \wedge B} : (\mathsf{Susp}\,A) \wedge B \to \mathsf{Susp}\,(A \wedge B)$ is an equivalence.*

Proof. The inverse of $\widehat{\ }\mathsf{id}_{A \wedge B}$ is induced by the map $A \times B \to \Omega\,((\mathsf{Susp}\,A) \wedge B)$ defined by mapping $(a, b) : A \times B$ to the composite loop given by:

$$\star_\wedge \xrightarrow{\mathsf{push_r}^{-1}} \langle \mathsf{north}, b\rangle \xrightarrow{\mathsf{cong}\ \langle -\,, b\rangle\ (\sigma\,a)} \langle \mathsf{north}\,, b\rangle \xrightarrow{\mathsf{push_r}} \star_\wedge$$

The fact that these maps cancel follows by some technical but elementary path algebra. For the details, we refer to the formalisation. □

Lemma 6.2. *If $f : A \wedge B \to C$ is an equivalence, then so is $\widehat{\ }f : \mathsf{Susp}\,A \wedge B \to \mathsf{Susp}\ A$*

Proof. Using equivalence induction (see e.g. [Uni13, Corollary 5.8.5]), it is enough to prove the lemma for $C := A \wedge B$ and $f := \mathsf{id}_{A \wedge B}$. In this case, the statement is precisely that of Lemma 6.1. □

These lemmas allow us to define an equivalence $\wedge_{n,m} : \mathbb{S}^n \wedge \mathbb{S}^m \simeq \mathbb{S}^{n+m}$ (we borrow this notation from [Bru16a, Proposition 4.2.2]). We will write $\smile\, : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^{n+m}$ for the underlying function, i.e. $x \smile y := \wedge_{n,m}\langle x\,, y\rangle$. The name $\smile$ is suggestive: modulo the quotient maps $\mathbb{S}^\bullet \to \mathsf{K}_\bullet$, it is precisely the cup product; this justifies overloading the symbol. We define it by induction on n. In the case $n = 0$, we define it on canonical points $\langle x\,, y\rangle$ by case distinction on x:

$$
\begin{aligned}
\mathsf{false} \smile y &= y\\
\mathsf{true} \smile y &= \star_{\mathbb{S}^m}
\end{aligned}
$$

This map induces a map on the full smash product $\mathbb{S}^0 \wedge \mathbb{S}^m$ [Bru16a, Section 4.1]. In fact, it is an equivalence, and thereby $\wedge_{0,m}$ is defined. For $n > 0$ we use the fact that $\mathbb{S}^n := \mathsf{Susp}\,\mathbb{S}^{n-1}$ and simply define define $\wedge_{n,m} := \widehat{\wedge}_{n-1,m}$. By Lemma 6.1, this is an equivalence.

Let us try to transfer this construction from smash products to joins. To begin with, consider the following map, defined for any two pointed types A and B:

$\mathsf{pinch} : A * B \to \mathsf{Susp}\ (A \wedge B)$
$\mathsf{pinch}\ (\mathsf{inl}\ a) = \mathsf{north}$
$\mathsf{pinch}\ (\mathsf{inr}\ b) = \mathsf{south}$
$\mathsf{pinch}\ (\mathsf{push}\ (a\ ,\ b)\ i) = \mathsf{merid}\ \langle\ a\ ,\ b\ \rangle\ i$

Proposition 6.3. *For any two pointed types A and B, the map* pinch *is an equivalence.*

Proof. While we have, in our formalisation, explicitly constructed an inverse of pinch and proved directly that the two maps cancel, a recent (independent) result by Cagne et al. [CBKB24] allows us to give a more principled proof. We proceed by noting that for any pointed type C, we have

$$(\mathsf{Susp}\,(A \wedge B) \to_\star C) \simeq (A \wedge B \to_\star \Omega\, C) \simeq (A \to_\star (B \to_\star \Omega\, C)) \simeq (A * B \to_\star C)$$

where the first equivalence comes from the adjunction between Susp and Ω and the second from the adjunction between smash products and doubly pointed maps. The third equivalence is [CBKB24, Lemma 6.1]. This shows that $\mathsf{Susp}\,(A \wedge B)$ and $A * B$ have the same elimination principle, which implies the desired statement. □

Let $\mathsf{F}_{n,m} : \mathbb{S}^n * \mathbb{S}^m \to \mathbb{S}^{n+m+1}$ denote the following composition:

$$\mathbb{S}^n * \mathbb{S}^m \xrightarrow{\mathsf{pinch}} \mathsf{Susp}\,(\mathbb{S}^n \wedge \mathbb{S}^m) \xrightarrow{\mathsf{Susp}\,(\wedge_{n,m})} \mathsf{Susp}\,\mathbb{S}^{n+m} =: \mathbb{S}^{n+m+1}$$

Unfolding the definition of $\mathsf{F}_{n,m}$ we see that it has an incredibly compact description:

$\mathsf{F}_{n,m} : \mathbb{S}^n * \mathbb{S}^m \to \mathbb{S}^{n+m+1}$
$\mathsf{F}_{n,m}\ (\mathsf{inl}\ a) = \mathsf{north}$
$\mathsf{F}_{n,m}\ (\mathsf{inr}\ b) = \mathsf{north}$
$\mathsf{F}_{n,m}\ (\mathsf{push}\ (a\ ,\ b)\ i) = \sigma\ (a \smile b)\ i$

Since $\mathsf{F}_{n,m}$ is a composition of two equivalences, we immediately arrive at the following result.

Proposition 6.4. $\mathsf{F}_{n,m}$ *is an equivalence*

We have already seen that the special case $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$ plays an important role in Brunerie's proof. Now that we have moved from the rather opaque definition of this equivalence presented in Brunerie's thesis to the definition in terms of the very explicit function $\mathsf{F}_{n,m}$, we can hope to better understand its role in the definition of the Brunerie number. Since the only non-trivial component of the construction of $\mathsf{F}_{n,m}$ is $\smile$, we may hope that it inherits some of its properties. We study these now.

Let us first analyse the interaction of $\smile$ with inversion. Recall, given a pointed type A we can define inversion on $\mathsf{Susp}\,A$ by:

$\text{-} : \mathsf{Susp}\ A \to \mathsf{Susp}\ A$
$\text{-}\ \mathsf{north} = \mathsf{north}$

- $\mathsf{south} = \mathsf{north}$
- $(\mathsf{merid}\ a\ i) = \sigma\ a\ (\sim i)$

We get sphere inversion by letting $- : \mathbb{S}^n \to \mathbb{S}^n$ be boolean negation when $n := 0$ and the suspension inversion defined above when $n > 1$.[4]

Proposition 6.5. *The multiplication $\smile$ is graded-commutative, i.e. for $x : \mathbb{S}^n$ and $y : \mathbb{S}^m$, we have $x \smile y \equiv -^{nm}(y \smile x)$.*

For the proof, we refer to [BLM22, Proposition 18] which is the corresponding statement for the cup product on $\mathsf{K}_n := \| \mathbb{S}^n \|_n$ and whose proof directly applies also in our setting. Associativity follows, just like in the proof of [BLM22, Proposition 17], by sphere induction:

Proposition 6.6. *The multiplication $\smile$ is associative.*

Proof. Let $x : \mathbb{S}^n$, $y : \mathbb{S}^m$ and $z : \mathbb{S}^k$. We show that $x \smile (y \smile z) \equiv (x \smile y) \smile z$ by induction on n and x. When $n = 0$, the two equalities

$$\mathsf{false} \smile (y \smile z) \equiv (\mathsf{false} \smile y) \smile z$$
$$\mathsf{true} \smile (y \smile z) \equiv (\mathsf{true} \smile y) \smile z$$

hold definitionally. For the inductive step, we use suspension elimination on $x : \mathbb{S}^{n+1}$. The two equalities

$$\mathsf{north} \smile (y \smile z) \equiv (\mathsf{north} \smile y) \smile z$$
$$\mathsf{south} \smile (y \smile z) \equiv (\mathsf{south} \smile y) \smile z$$

also hold definitionally. So, by inspection of the definition of $\smile$, we need to show that

$$\sigma(x \smile y) \equiv \mathsf{cong}\ (- \smile z)\ (\sigma(x \smile y))$$

Using the action of cong on path composition, we can unfold the right-hand side as follows:

$$\begin{aligned} \mathsf{cong}\ (- \smile z)\ (\sigma(x \smile y)) &\equiv \mathsf{cong}\ (- \smile z)\ (\mathsf{merid}\ (x \smile y)) \cdot \mathsf{cong}\ (- \smile z)\ (\mathsf{merid}\ \mathsf{north})^{-1} \\ &:= \sigma(x \smile y) \cdot \sigma(\mathsf{north} \smile y)^{-1} \\ &\equiv \sigma(x \smile y) \end{aligned}$$

□

6.2. **Homotopy groups in terms of joins.** As we have seen in Brunerie's construction of the Hopf map, it is often easier to describe maps of type $\mathbb{S}^n * \mathbb{S}^m \to A$ than those of type $\mathbb{S}^{n+m+1} \to A$. However, the definition of homotopy groups we have relied on so far uses the latter type. This forces us to translate back and forth whenever we want to use the definition in terms of joins. The key strategy behind our new calculation of the Brunerie number is to rephrase homotopy groups in terms of maps out of joins of spheres.

Definition 6.7. Given a pointed type A, we define $\pi^*_{n+m+1}(A) := \| \mathbb{S}^n * \mathbb{S}^m \to_\star A \|_0$.

Clearly, this is equivalent to the usual definition of $\pi_{n+m+1}(A)$ via pre-composition by $\mathsf{F}_{n,m}$. However, $\pi^*_{n+m+1}(A)$ can be endowed with an explicit group structure which $\mathsf{F}_{n,m}$ turns out to respect. In order to construct the group structure on $\pi^*_{n+m+1}(A)$, let us

[4] If we prefer to use the base/loop-construction of $\mathbb{S}^1$, we may define the inversion map simply by sending loop to loop^{-1}.

construct a map $\ell : A \times B \to \Omega\,(A * B)$ for all pointed types A and B. Recall, we take $A * B$ to be pointed by $\mathsf{inl}\,\star_A$. We define ℓ by:

$$\ell(a\,, b) := \mathsf{push}\,(\star_A\,, \star_B) \cdot \mathsf{push}\,(a\,, \star_B)^{-1} \cdot \mathsf{push}\,(a\,, b) \cdot \mathsf{push}\,(\star_A\,, b)^{-1}$$

Note that ℓ is pointed in both arguments. Let us also define explicitly (once and for all) a pointed version of cong taking a pointed functions $f : A \to_\star B$ to a pointed function $\mathsf{cong}_\star\, f : \Omega\, A \to_\star \Omega\, B$. We define it by

$$\mathsf{cong}_\star\, f\, p := \star_f^{-1} \cdot \mathsf{cong}\; f\; p \cdot \star_f$$

where, recall, $\star_f : f\ \star_A\ \equiv \star_B$. In other words, $\mathsf{cong}_\star$ is the functorial action of Ω.

We can now add two functions f and g of type $A * B \to_\star C$ by

$(f +^* g) : A * B \to C$
$(f +^* g)\ (\mathsf{inl}\ a) = \star_C$
$(f +^* g)\ (\mathsf{inr}\ b) = \star_C$
$(f +^* g)\ (\mathsf{push}\ (a\ ,\ b)\ i) = (\mathsf{cong}_\star\, f\ (\ell\ (a\ ,\ b)) \cdot \mathsf{cong}_\star\, g\ (\ell\ (a\ ,\ b)))\ i$

We take this function to be pointed by refl. Note that, since ℓ is pointed in both arguments, both $\mathsf{cong}\ (f +^* g)\ (\mathsf{push}\,(a\,, \star_B))$ and $\mathsf{cong}\ (f +^* g)\ (\mathsf{push}\,(\star_A\,, b))$ vanish. Let us compare this with the addition on the usual definition homotopy groups. In general, we may add any two functions f and g of type $\mathsf{Susp}\, A \to_\star B$ by

$(f +^{\mathsf{Susp}} g) : \mathsf{Susp}\ A \to B$
$(f +^{\mathsf{Susp}} g)\ \mathsf{north} = \star_B$
$(f +^{\mathsf{Susp}} g)\ \mathsf{south} = \star_B$
$(f +^{\mathsf{Susp}} g)\ (\mathsf{merid}\ a\ i) = (\mathsf{cong}_\star\, f\ (\sigma\ a) \cdot \mathsf{cong}_\star\, g\ (\sigma\ a))\ i$

This is precisely the construction used to define the group structure on π_n whenever $n > 0$. Note that, by construction, we have $\mathsf{cong}\ (f +^{\mathrm{Susp}} g)\ (\mathsf{merid}\ \star_A) \equiv \mathsf{refl}$.

Proposition 6.8. *Given $f, g : \mathbb{S}^{n+m+1} \to_\star A$, we have*

$$(f +^{\mathrm{Susp}} g) \circ \mathsf{F}_{n,m} \equiv (f \circ \mathsf{F}_{n,m}) +^* (g \circ \mathsf{F}_{n,m})$$

Proof. The two functions agree on inl and inr by refl. Let us consider the action on $\mathsf{push}(x\,, y)$. We have

$$\begin{aligned}
\mathsf{cong}\ ((f +^{\mathrm{Susp}} g) \circ \mathsf{F}_{n,m})\ (\mathsf{push}\,(x\,, y)) &:= \mathsf{cong}\ (f +^{\mathrm{Susp}} g)\ (\sigma(x \smile y)) \\
&\equiv \mathsf{cong}\ (f +^{\mathrm{Susp}} g)\ (\mathsf{merid}\ (x \smile y)) \\
&\quad \cdot \mathsf{cong}\ (f +^{\mathrm{Susp}} g)\ (\mathsf{merid\ north})^{-1} \\
&\equiv \mathsf{cong}\ (f +^{\mathrm{Susp}} g)\ (\mathsf{merid}\ (x \smile y))
\end{aligned}$$

which, by definition, unfolds to

$$\mathsf{cong}_\star\, f\,(\sigma(x \smile y)) \cdot \mathsf{cong}_\star\, g\,(\sigma(x \smile y)) \tag{6.1}$$

On the other hand, $\mathsf{cong}\ ((f \circ \mathsf{F}_{n,m}) +^* (g \circ \mathsf{F}_{n,m}))\ (\mathsf{push}\,(x\,, y))$ unfolds to

$$\mathsf{cong}_\star\, f\,(\mathsf{cong}\ \mathsf{F}_{n,m}\ (\ell(x\,, y))) \cdot \mathsf{cong}_\star\, g\,(\mathsf{cong}\ \mathsf{F}_{n,m}\ (\ell(x\,, y))) \tag{6.2}$$

Hence, comparing (6.1) and (6.2), we see that it is enough to show that

$$\mathsf{cong}\ \mathsf{F}_{n,m}\ (\ell(x\,, y)) \equiv \sigma(x \smile y)$$

Unfolding ℓ, we get

$$\begin{aligned}\mathsf{cong}\ \mathsf{F}_{n,m}\ (\ell(x\,,y)) &\equiv \sigma(\star_{\mathbb{S}^n} \smile \star_{\mathbb{S}^m}) \cdot \sigma(x \smile \star_{\mathbb{S}^m})^{-1} \cdot \sigma(x \smile y) \cdot \sigma(\star_{\mathbb{S}^n} \smile y)^{-1} \\ &\equiv \sigma\ \mathsf{north} \cdot \sigma\ \mathsf{north}^{-1} \cdot \sigma(x \smile y) \cdot \sigma\ \mathsf{north}^{-1} \\ &\equiv \sigma(x \smile y)\end{aligned}$$ □

Proposition 6.9. *For any pointed type A, the set $\pi^*_{n+m+1}(A)$ is a group with group structure induced by $+^*$. Furthermore, pre-composition $(\mathsf{F}_{n,m})^* : \pi_{n+m+1}(A) \to \pi^*_{n+m+1}(A)$ is an isomorphism.*

Proof. We know that $(\mathsf{F}_{n,m})^*$ is an equivalence of types. By Proposition 6.8 and the Structure Identity Principle [Uni13, Section 9.8], it induces a path

$$(\pi_{n+m+1}(A), +^{\mathrm{Susp}}) \equiv (\pi^*_{n+m+1}(A), +^*)$$

of raw monoids (i.e. elements of type $\Sigma_{A:\mathsf{Type}}(A \times A \to A)$). Since the the left-hand side of this equality can be extended to form a group, so can the right-hand side. This is precisely what we set out to show. □

The following result follows in exactly the same manner.

Proposition 6.10. *π^*_{n+m+1} is functorial with its action on maps being defined by post-composition.*

6.3. **The new synthetic proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.** Let us now return to the new proof. We will use $\smile$ from above in dimensions $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$. We remark that, by Proposition 6.5, it is anti-commutative in these dimensions. In order to make the following constructions somewhat more direct, let us return to the base/loop definition of $\mathbb{S}^1$. Under the equivalence, $\mathsf{Susp}\,\mathsf{Bool} \simeq \mathbb{S}^1$, the multiplication is described by

$$\begin{aligned}&\_\smile\_ : \mathbb{S}^1 \to \mathbb{S}^1 \to \mathbb{S}^2 \\ &\mathsf{base} \smile y = \mathsf{north} \\ &(\mathsf{loop}\ i) \smile y = \sigma\ y\ i\end{aligned}$$

In addition to anti-commutativity and associativity, we have the following distributivity-like fact about $\smile$:

Lemma 6.11. *For $x, y : \mathbb{S}^1$, we have $x \smile (x + y) \equiv x \smile y$*

Proof. We proceed by $\mathbb{S}^1$-induction on x. The equality $\mathsf{base} \smile (\mathsf{base} + y) \equiv \mathsf{base} \smile y$ holds by refl, so we are left to verify the equality

$$\mathsf{cong}\ (x \mapsto x \smile (x + y))\ \mathsf{loop} \equiv \sigma\, y$$

Simplifying the left-hand side using functoriality of binary cong [LM24, Definition 1], we get

$$\begin{aligned}\mathsf{cong}\ (x \mapsto x \smile (x + y))\ \mathsf{loop} &\equiv \mathsf{cong}\ (x \mapsto \mathsf{north} \smile (x + y))\ \mathsf{loop} \\ &\quad\cdot \mathsf{cong}\ (x \mapsto x \smile (\mathsf{north} + y))\ \mathsf{loop} \\ &:= \mathsf{refl} \cdot \sigma\, y \equiv \sigma\, y\end{aligned}$$ □

We now redefine $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ via the following decomposition, primarily defined in terms of post- and pre-composition with $\mathsf{F}_{1,1} : \mathbb{S}^1 * \mathbb{S}^1 \cong \mathbb{S}^3$ and its inverse. In what follows, let us simply write $\mathsf{F} := \mathsf{F}_{1,1}$ and $\pi_3^*(A) := \pi_{1+1+1}^*(A) := \| \mathbb{S}^1 * \mathbb{S}^1 \to_\star A \|_0$. We also remind the reader of the map $\mathsf{h} : \mathbb{S}^1 * \mathbb{S}^1 \to \mathbb{S}^2$ from Definition 3.5 for which h_* is an isomorphism–this follows from Proposition 3.11.

Definition 6.12. Let $\theta : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ be defined by the following sequence of isomorphisms

$$\pi_3(\mathbb{S}^2) \xrightarrow{\mathsf{F}^*} \pi_3^*(\mathbb{S}^2) \xrightarrow{(\mathsf{h}_*)^{-1}} \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\mathsf{F}_*} \pi_3^*(\mathbb{S}^3) \xrightarrow{(\mathsf{F}^{-1})^*} \pi_3(\mathbb{S}^3) \xrightarrow{\xi} \mathbb{Z}$$

where the last map can be chosen to be any reasonable description of the isomorphism $\xi : \pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ sending i_3 to 1.

The goal is to trace the image of $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ under θ. Let us define the following three underlying functions of elements $\eta_1 : \pi_3^*(\mathbb{S}^2)$, $\eta_2 : \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1)$ and $\eta_3 : \pi_3^*(\mathbb{S}^3)$:

```
η₁-fun : S¹ * S¹ → S²
η₁-fun (inl x) = north
η₁-fun (inr y) = north
η₁-fun (push (x , y) i) = (σ y · σ x) i

η₂-fun : S¹ * S¹ → S¹ * S¹
η₂-fun (inl x) = inr (- x)
η₂-fun (inr y) = inr y
η₂-fun (push (x , y) i) = (push (y - x , - x) ⁻¹ · push (y - x , y)) i

η₃-fun : S¹ * S¹ → S³
η₃-fun (inl x) = north
η₃-fun (inr y) = north
η₃-fun (push (x , y) i) = (σ (x ⌣ y) ⁻¹ · σ (x ⌣ y) ⁻¹) i
```

The claim is now that the image of $[i_2, i_2]$ under the chain of isomorphisms can be described as follows:

$$[i_2, i_2] \xmapsto{\mathsf{F}^*} \eta_1 \xmapsto{(\mathsf{h}_*)^{-1}} \eta_2 \xrightarrow{\mathsf{F}_*} \eta_3 \xmapsto{(\mathsf{F}^{-1})^*} (-2)i_3 \xmapsto{\xi} \pm 2$$

Lemma 6.13. $\mathsf{F}^* [i_2, i_2] \equiv \eta_1$

Proof. The definition of η_1 matches that of $|\nabla \circ \mathsf{W}| : \pi_3^*(\mathbb{S}^2)$, and so the statement holds by construction of the Whitehead product. □

Lemma 6.14. $(\mathsf{h}_*)^{-1} \eta_1 \equiv \eta_2$

Proof. Applying h_* on both sides gives the equation $\eta_1 \equiv \mathsf{h}_* \eta_2$. Thus, we are done if we can show that $\eta_1\text{-fun}\, a \equiv \mathsf{h}\, (\eta_2\text{-fun}\, a)$ for $a : \mathbb{S}^1 * \mathbb{S}^1$. We do it by induction on a. When a is $\mathsf{inl}\, x$ or $\mathsf{inr}\, y$, the equality holds by refl. Thus, it remains to show that

$$\mathsf{cong}\ \eta_1\text{-fun}\ (\mathsf{push}\,(x\,,y)) \equiv \mathsf{cong}\ (\mathsf{h} \circ \eta_2\text{-fun})\ (\mathsf{push}\,(x\,,y))$$

We show the identity by unfolding the right-hand side:

$$\begin{aligned}
\mathsf{cong}\ (\mathsf{h} \circ \eta_2\text{-}\mathsf{fun})\ (\mathsf{push}\,(x\,,y)) &:= \mathsf{cong}\ \mathsf{h}\ (\mathsf{push}\,(y-x\,,-x)^{-1} \cdot \mathsf{push}\,(y-x\,,y)) \\
&\equiv \mathsf{cong}\ \mathsf{h}\ (\mathsf{push}\,(y-x\,,-x))^{-1} \cdot \mathsf{cong}\ \mathsf{h}\ (\mathsf{push}\,(y-x\,,y)) \\
&:= \sigma((-x)-(y-x))^{-1} \cdot \sigma(y-(y-x)) \\
&\equiv \sigma(-y)^{-1} \cdot \sigma\,x \\
&\equiv \sigma\,y \cdot \sigma\,x \\
&=: \mathsf{cong}\ \eta_1\text{-}\mathsf{fun}\ (\mathsf{push}\,(x\,,y))
\end{aligned}$$

□

Lemma 6.15. $\mathsf{F}_*\,\eta_2 \equiv \eta_3$

Proof. The identity follows if we can show that $\mathsf{F}(\eta_2\text{-}\mathsf{fun}\,a) \equiv \eta_3\text{-}\mathsf{fun}\,a$ for $a : \mathbb{S}^1 * \mathbb{S}^1$. Again, the identity holds by refl when a is $\mathsf{inl}\,x$ or $\mathsf{inr}\,y$. So it remains to show that

$$\mathsf{cong}\ (\mathsf{F}\ \circ\ \eta_2\text{-}\mathsf{fun})\ (\mathsf{push}\,(x\,,y)) \equiv \mathsf{cong}\ \eta_3\text{-}\mathsf{fun}\ (\mathsf{push}\,(x\,,y))$$

Just like in the proof of Lemma 6.15, we show this simply by unfolding the definitions of the, in this case, left-hand side. We get:

$$\begin{aligned}
\mathsf{cong}\ (\mathsf{F} \circ \eta_2\text{-}\mathsf{fun})\ (\mathsf{push}\,(x\,,y)) &:= \mathsf{cong}\ \mathsf{F}\ (\mathsf{push}\,(y-x\,,-x)^{-1} \cdot \mathsf{push}\,(y-x\,,y)) \\
&\equiv \mathsf{cong}\ \mathsf{F}\ (\mathsf{push}\,(y-x\,,-x))^{-1} \cdot \mathsf{cong}\ \mathsf{F}\ (\mathsf{push}\,(y-x\,,y)) \\
&:= \sigma((y-x) \smile (-x))^{-1} \cdot \sigma((y-x) \smile y) \\
&\equiv \sigma((-x) \smile ((-x)+y)) \cdot \sigma(y \smile (y-x))^{-1} \\
&\equiv \sigma((-x) \smile y) \cdot \sigma(y \smile (-x))^{-1} \\
&\equiv \sigma(x \smile y)^{-1} \cdot \sigma(y \smile (-x))^{-1} \\
&\equiv \sigma(x \smile y)^{-1} \cdot \sigma(x \smile y)^{-1} \\
&=: \mathsf{cong}\ \eta_3\text{-}\mathsf{fun}\ (\mathsf{push}\,(x\,,y))
\end{aligned}$$

where the fourth and seventh equalities come from anti-commutativity and the fifth equality from Lemma 6.11. The fact that σ commutes with inversion is used throughout. □

Theorem 6.16. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

Proof. By uniqueness (up to a sign) of isomorphisms $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, it suffices, according to Corollary 4.12, to show that the image of $[i_2, i_2]$ under θ is ± 2. That is:

$$(\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1} \circ \mathsf{F}^*)[i_2, i_2] \equiv \pm 2$$

By Lemma 6.13, Lemma 6.14 and Lemma 6.15, it suffices to show that

$$(\xi \circ (\mathsf{F}^{-1})^*)\,\eta_3 \equiv \pm 2$$

One can easily show that $\mathsf{F}^{-1}\,\eta_3 \equiv (-2)\,i_3$, and hence

$$(\xi \circ (\mathsf{F}^{-1})^*)\,\eta_3 \equiv (-2)\,(\xi\,i_3) \equiv -2$$

□

In addition to providing a much shorter proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, this gives us a sequence of new Brunerie numbers, $\beta_1, \beta_2, \beta_3 : \mathbb{Z}$, of decreasing complexity:

$$\begin{aligned}
\beta_1 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1})\,\eta_1 \\
\beta_2 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_*)\,\eta_2 \\
\beta_3 &= (\xi \circ (\mathsf{F}^{-1})^*)\,\eta_3
\end{aligned}$$

This gives new hope for Brunerie's conjecture about a proof by normalisation. This may be captured as follows:

Theorem 6.17 (New Brunerie numbers)**.** *If either of* $\beta_1, \beta_2, \beta_3 : \mathbb{Z}$ *normalises to* ± 2*, then* $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$*.*

Ideally, we could normalise β_1. This, however, turns out to be difficult, as it does not bypass the main hurdle of computing the inverse of the isomorphism $\pi_3^*(\mathbb{S}^2) \cong \pi_3^*(\mathbb{S}^1 * \mathbb{S}^1)$ induced by the Hopf map, which has a rather indirect construction coming from the LES of homotopy groups associated to the Hopf fibration. This problem does not apply to β_2, for which the computation does not rely on the problematic inverse. Unfortunately, also β_2 fails to normalise in reasonable time in `Cubical Agda`. This is surprising, as the only maps playing a fundamental role here are two applications of the equivalence $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$, which is not too involved, and one application of ξ which may be compactly described via

$$\pi_3(\mathbb{S}^3) \xrightarrow{|\text{-}|_*} \mathsf{H}^3(\mathbb{S}^3) \xrightarrow{\cong} \mathbb{Z}$$

and computes relatively well if the last isomorphism is constructed as in [BLM22].[5] We have hence, at the time of writing, not been able to normalise even β_2, despite many optimisations of the functions involved. We are, however, able to normalise β_3 after some minor modifications to η_3 and the map $\pi_3^*(\mathbb{S}^3) \to \mathbb{Z}$. This optimised version of β_3, normalises to -2 in `Cubical Agda` in just under 4 seconds, thereby giving us an at least partially computer-assisted proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

We emphasise again that β_2 is a vastly simplified version of β since the isomorphism $\pi_3(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^3)$ never has to be computed. Hence, it is rather surprising that computations break down already at this stage. This tells us that `Cubical Agda` has a long way to go before any direct computation of the original β is feasible. We hope that this could be useful for benchmarking in future optimisations of `Cubical Agda` and related systems.

Finally, we address the elephant in the room: why is there a minus sign popping up? In other words, have we really chosen the, in some way, canonical isomorphism? The isomorphism $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ maps, as expected, i_3 to 1, so it can hardly be the culprit. Neither can the equivalence $\mathsf{F} : \mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$, since it is applied equally in the constructions of hopf and of $[i_2, i_2]$. We could, however, have defined the push-case for h by

$\mathsf{h}\ (\mathsf{push}\ (x\ ,\ y)\ i) = \sigma\ (x\ \text{-}\ y)\ i$

in which case θ would have sent $[i_2, i_2]$ to 2 and hopf to 1 (note that this is only possible since altering h would alter the definition of θ). The construction of h that we have given is, however, precisely the one which fell out by unfolding our formalisation Brunerie's construction of the corresponding map. If this indeed is what Brunerie intended, we may also conclude that the original Brunerie number β is equal to -2. We stress that this merely is a fun fact and of no mathematical importance to Brunerie's proof or our formalisation.

6.4. **A stand-alone proof of Brunerie's theorem?** We saw above that the new proof of $\beta \equiv \pm 2$ together with Corollary 4.12 implies Brunerie's theorem. However, what conclusions can we draw concerning the cardinality of $\pi_4(\mathbb{S}^3)$ in the absence of Corollary 4.12? In other words, how self-contained is the new proof? While the fact that $\beta \equiv \pm 2$ does not

[5] As noted in [Lju20], the Freudenthal suspension theorem should be avoided here as it has a tendency to lead to very slow computations. This is another way in which we deviate from Brunerie's β.

automatically imply that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, it does provide all ingredients necessary for a stand-alone proof of the following fact:

Theorem 6.18. *If $\pi_4(\mathbb{S}^3) \not\cong \mathbb{1}$, then $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.*

Before we prove Theorem 6.18, we need to analyse the action of suspension on Whitehead products and, in particular, on $[i_2, i_2] : \pi_3(\mathbb{S}^2)$. In what follows, let A, B and C be pointed types and let us fix two pointed functions $f : \mathsf{Susp}\, A \to_\star C$ and $g : \mathsf{Susp}\, B \to_\star C$. The Whitehead product of f and g can be understood as the composition

$$A * B \xrightarrow{\mathsf{W}} \mathsf{Susp}\, A \vee \mathsf{Susp}\, B \xrightarrow{f \vee g} C \vee C \xrightarrow{\nabla} C$$

We remark that this construction has been independently studied by Cagne et al. [CBKB24, Definition 6.3] who call it the 'generalised Whitehead product'. After a bit of massaging, this function can be given a very simple description:

$$\begin{aligned}
&(f \cdot_w g) : A * B \to C\\
&(f \cdot_w g)\ (\mathsf{inl}\ x) = \star_C\\
&(f \cdot_w g)\ (\mathsf{inr}\ y) = \star_C\\
&(f \cdot_w g)\ (\mathsf{push}\ (x\ ,\ y)\ i) = (\mathsf{cong}_\star\, g\ (\sigma\ y) \cdot \mathsf{cong}_\star\, f\ (\sigma\ x))\ i
\end{aligned}$$

We remark that this composition gives η_1-fun when $A = B = \mathbb{S}^1$, $C = \mathbb{S}^2$ and $f = g = \mathsf{id}_{\mathbb{S}^2}$.

Our aim is to show that $f \cdot_w g$ vanishes under suspension. To this end, let us consider a function very similar to $f \cdot_w g$:

$$\begin{aligned}
&\gamma : A * B \to C\\
&\gamma\ (\mathsf{inl}\ x) = \star_C\\
&\gamma\ (\mathsf{inr}\ y) = \star_C\\
&\gamma\ (\mathsf{push}\ (x\ ,\ y)\ i) = (\mathsf{cong}_\star\, f\ (\sigma\ x) \cdot \mathsf{cong}_\star\, g\ (\sigma\ y))\ i
\end{aligned}$$

Despite the similarity of $f \cdot_w g$ and γ, the latter turns out to be trivial.

Lemma 6.19. *γ is constant.*

Proof. We show that $\gamma\, a \equiv \star_C$ for all $a : \mathbb{S}^1 * \mathbb{S}^1$ by induction on a. When a is $\mathsf{inl}\, x$, the left-hand side reduces to $\star_C$, so we need to provide a path $\star_C \equiv \star_C$. Instead of choosing the obvious path refl, we provide $\mathsf{cong}_\star\, f\, (\sigma\, x) : \star_C \equiv \star_C$. When a is $\mathsf{inr}\, y$, we have the same goal. This time, we provide the path $\mathsf{cong}_\star\, g\, (\sigma\, y)^{-1}$. For the final step, i.e. the action of γ on $\mathsf{push}\, (x\, , y)$, we need to provide a filler of the following square of paths:

$$\begin{tikzcd}
\star_C \arrow[r, equal, "\mathsf{refl}"] & \star_C \\
\star_C \arrow[u, "\mathsf{cong}_\star\, f\, (\sigma\, x)"] \arrow[r, "\mathsf{cong}_\star\, f\, (\sigma\, x) \cdot \mathsf{cong}_\star\, g\, (\sigma\, y)"'] & \star_C \arrow[u, "(\mathsf{cong}_\star\, g\, (\sigma\, y))^{-1}"']
\end{tikzcd}$$

Squares of this shape always have a filler by definition of path composition, and thus the statement holds. □

Now, although $f \cdot_w g$ and γ may look similar, it it is now, in light of Lemma 6.19, clear that they are not the same. This happens because the actions of the functions on $\mathsf{push}\, (x\, , y)$ only are the same up to commutation of paths—something which is not always legal in the possibly non-commutative loop space $\Omega\, C$. Nevertheless, after suspending the function, the situation is different:

Lemma 6.20. *The pointed functions* $\mathsf{Susp}\,(f \cdot_w g), \mathsf{Susp}\,\gamma : \mathsf{Susp}\,(A * B) \to_\star \mathsf{Susp}\,C$ *are equal.*

Proof. Under the adjunction $\mathsf{Susp} \dashv \Omega$, it is enough to show that for every $a : A * B$, we have an equality of loops in $\Omega\,(\mathsf{Susp}\,C)$:

$$\sigma((f \cdot_w g)\,a) \equiv \sigma(\gamma\,a)$$

We proceed by induction on a. When a is $\mathsf{inl}\,x$ or $\mathsf{inr}\,y$, the equality holds by refl. Thus, it remains to show that

$$\mathsf{cong}\ \sigma\ (\mathsf{cong}\ (f \cdot_w g)\ (\mathsf{push}\,(x\,,y))) \equiv \mathsf{cong}\ \sigma\ (\mathsf{cong}\ \gamma\ (\mathsf{push}\,(x\,,y)))$$

As before, this is a simple exercise in unfolding the definitions of each respective function:

$$\begin{aligned}
\mathsf{cong}\ \sigma\ (\mathsf{cong}\ (f \cdot_w g)\ (\mathsf{push}\,(x\,,y))) &:= \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, g\,(\sigma\,y) \cdot \mathsf{cong}_\star\, f\,(\sigma\,x)) \\
&\equiv \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, g\,(\sigma\,y)) \cdot \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, f\,(\sigma\,x)) \\
&\equiv \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, f\,(\sigma\,x)) \cdot \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, g\,(\sigma\,y)) \qquad \text{(EH)} \\
&\equiv \mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, f\,(\sigma\,x) \cdot \mathsf{cong}_\star\, g\,(\sigma\,y)) \\
&\equiv \mathsf{cong}\ \gamma\ (\mathsf{push}\,(x\,,y))
\end{aligned}$$

where the step labelled (EH) is an application of the Eckmann-Hilton argument which says that path composition in $\Omega^2\,A$ is commutative for any pointed type A [Uni13, Theorem 2.1.6]. In particular, since we may interpret $\mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, f\,(\sigma\,x))$ and $\mathsf{cong}\ \sigma\ (\mathsf{cong}_\star\, g\,(\sigma\,y))$ as loops in $\Omega^2\,(\mathsf{Susp}\,C)$, the identity holds. ∎

Proposition 6.21. *The pointed function* $\mathsf{Susp}\,(f \cdot_w g) : \mathsf{Susp}\,(A * B) \to_\star \mathsf{Susp}\,C$ *is constant.*

Proof. Since γ is constant and constant functions are preserved by suspension, Lemma 6.19 gives us the desired equality of (pointed) functions:

$$\mathsf{Susp}\,(f \cdot_w g) \equiv \mathsf{Susp}\,\gamma \equiv \mathsf{const}_C$$ ∎

As we have seen before, setting $A = \mathbb{S}^n$ and $B = \mathbb{S}^m$ in the definition of $f \cdot_w g$, so that $f : \mathbb{S}^{n+1} \to_\star C$ and $g : \mathbb{S}^{m+1} \to_\star C$ we obtain the usual Whitehead product. $[|\,f\,|, |\,g\,|] : \pi_{n+m+1}(C)$, that is

$$[|\,f\,|, |\,g\,|] \equiv |\,(f \cdot_w g) \circ \mathsf{F}^{-1}_{n,m}\,|$$

Let us translate Proposition 6.21 to a result concerning these maps.

Proposition 6.22. *For* $f : \mathbb{S}^n \to_\star C$ *and* $g : \mathbb{S}^m \to_\star C$*, their Whitehead product* $(f \cdot_w g) \circ \mathsf{F}^{-1}_{n,m}$ *vanishes under suspension, i.e.*

$$\mathsf{Susp}\,(|\,(f \cdot_w g) \circ \mathsf{F}^{-1}_{n,m}\,|) \equiv \mathsf{const}_C$$

Proof. The result follows immediately from the fact that the action of suspension $\mathsf{Susp} : (X \to_\star Y) \to (\mathsf{Susp}\,X \to_\star \mathsf{Susp}\,Y)$ is functorial and from Proposition 6.21. ∎

We get the following classically well-known theorem as an immediate corollary:

Theorem 6.23. *For any* $x : \pi_{n+1}(C)$ *and* $y : \pi_{m+1}(C)$*, the Whitehead product* $[x, y] : \pi_{n+m+1}(C)$ *lies in the kernel of the suspension map* $\sigma_* : \pi_{n+m+1}(C) \to \pi_{n+m+2}(\mathsf{Susp}\,C)$

We now have all that we need in order to prove Theorem 6.18.

Proof of Theorem 6.18. By the Freudenthal suspension theorem, we know that $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective. Furthermore, we know that the domain of this function is isomorphic to $\mathbb{Z}$ via θ from Definition 6.12 and thus we have a surjection $\sigma_* \circ \theta^{-1} : \mathbb{Z} \to \pi_4(\mathbb{S}^3)$. We know from the new direct calculation of the Brunerie number that $\theta^{-1}(-2) \equiv [i_2, i_2]$ and thus we have

$$\sigma_*(\theta^{-1}(-2)) \equiv \sigma_*[i_2, i_2] \equiv 0_{\pi_4(\mathbb{S}^3)}$$

where the second equality comes from Theorem 6.23. Hence, we have shown that there exists a surjection from $\mathbb{Z}$ onto $\pi_4(\mathbb{S}^3)$ with -2 in its kernel. This implies the theorem. □

Now, with Theorem 6.18 in mind, we seem to be very close to having produced a remarkably short proof of Brunerie's theorem. All that remains is showing that $\pi_4(\mathbb{S}^3)$ is not trivial. This, however, turns out not to be entirely straightforward. One possible proof uses the so called *Steenrod Squares*. This is a cohomology operation which was originally defined in HoTT by Brunerie [Bru16b] and whose theory was recently made available in HoTT by Ljungström and Wärn [LW24]. Such an approach, however, can hardly be said to simplify Brunerie's original proof, as the Steenrod Squares are rather advanced constructions. A solution to this problem which would truly be impressive would be a direct construction of two elements $x, y : \pi_4(\mathbb{S}^3)$ and a proof that $x \not\equiv y$. While this appears to be difficult to do by hand, we can, since we are working constructively, reformulate this problem as a computational challenge.

Challenge. *Construct a function* $f : \pi_4(\mathbb{S}^3) \to \mathsf{Bool}$ *and an element* $e : \pi_4(\mathbb{S}^3)$ *such that*
- $f\, 0_{\pi_4(\mathbb{S}^3)}$ *computes to* true *and*
- $f\, e$ *computes to* false.

In fact, such a computation was successfully run by Jack [Jac23] in `cubicaltt` [CCHM]. Unfortunately, `Cubical Agda` has not yet been able to perform the computation.

7. Conclusion

In this paper, we have presented three formalisations of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ in the `Cubical Agda` system. For the different proofs that $|\beta| \equiv 2$, the line count is roughly as follows:

(1) Brunerie's original proof [$\sim 9,000$ LOC]
(2) A direct calculation of β [~ 600 LOC]
(3) A computer-assisted reformulation of (2) [~ 400 LOC]

As always, the number of lines of code (LOC) should be taken with a grain of salt. First, the 9,000 LOC in the first formalisation exclude over 8,000 LOC from [Kan22b, Cav20, BLM22] which we have imported as libraries. In addition, these numbers also exclude many elementary results used in the formalisation, including ~ 9000 LOC for Chapters 1–3. We also stress that the line count for formalisations (2) and (3) only concern the part of the proof discussed in section 6.

Formalisation (1), which constituted the bulk of this paper, was a formalisation of Brunerie's pen-and-paper proof, taking some convenient shortcuts when possible. The problem of formalising Brunerie's proof has been a widely discussed open problem in HoTT/UF, and we hope that our efforts here provide a satisfactory solution to it. Formalisations (2) and (3) were of a simplified calculation of the Brunerie number, β. The very similar proofs (2) and (3) differ in that (3) uses `Cubical Agda` to carry out part of the computation of the new Brunerie number automatically. Perhaps equally important, we have seen that (3) provides us with new Brunerie numbers $\beta_1, \beta_2 : \mathbb{Z}$ which are far simpler than the original one, but still do not normalise in a reasonable amount of time. Our hope is that these can prove useful in future optimisations of `Cubical Agda` and related systems, as they could help shed some light on where the normalisation of the original Brunerie number breaks down.

We remark that proofs (1) and (2) could be done in Book HoTT and do not use any cubical machinery in a fundamental way, making them interpretable in any suitably structured $(\infty, 1)$-topos [Shu19]. We hence claim that, in our formalisations, we do not crucially rely on computations using univalence and HITs to prove anything that we could not have proved by hand in Book HoTT. Nevertheless, the `Cubical Agda` system has been very helpful in the formalisation, primarily due to its native support for HITs and definitional computation rules for higher constructors. Formalisation (3), however, is only valid in a system with computational support for univalence as it crucially relies on normalisation of proof terms involving univalence. It would be interesting to run this in other cubical systems, like `cubicaltt` [CCHM], `redtt` [Redb], `cooltt` [Reda], etc.

In addition to the above, we have also taken the opportunity to include some important constructions and results concerning joins of spheres and Whitehead products. In particular, we have given a very explicit definition of the decomposition of spheres into joins of spheres, given a new construction of homotopy groups in terms of maps out of joins of spheres and shown that Whitehead products vanish under suspension. The vanishing of Whitehead products allowed us to extend (2) to a stand-alone proof of the fact that $\pi_4(\mathbb{S}^3)$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Interestingly, another direct proof using an entirely different approach of this very fact was recently announced by Baker [Bak24]. Baker's argument is concerned with showing that a certain path constructed via the Eckmann-Hilton argument generates $\pi_3(\mathbb{S}^2)$ and then concludes that two times this generator must vanish under suspension due to the so called *syllepsis* [SK22]. We leave it to future work to investigate if anything interesting can be said about the relation between Baker's proof and ours.

We also remark that our formalisation of Brunerie's proof does not cover all results of Brunerie's thesis in full generality. For instance, we have not developed his proof concerning Whitehead products in full generality. We leave this generalisation for future work. This would tie in nicely with another possible direction of future research, namely that of investigating whether the approach outlined in section 6 can be used to compute other Whitehead products. In addition, describing their graded quasi-Lie algebra structure is work in progress.

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