# Bounds on Average Effects in Discrete Choice Panel Data Models<sup>\*</sup>

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#### Abstract

In discrete choice panel data, the estimation of average effects is crucial for quantifying the effect of covariates, and for policy evaluation and counterfactual analysis. This task is challenging in short panels with individual-specific effects due to partial identification and the incidental parameter problem. In particular, estimation of the sharp identified set is practically infeasible at realistic sample sizes whenever the number of support points of the observed covariates is large, such as when the covariates are continuous. In this paper, we therefore propose estimating outer bounds on the identified set of average effects. Our bounds are easy to construct, converge at the parametric rate, and are computationally simple to obtain even in moderately large samples, independent of whether the covariates are discrete or continuous. We also provide asymptotically valid confidence intervals on the identified set.

**Keywords:** Panel data, discrete choice, average effects, set identification, outer bounds, female labor force participation.

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### 1 Introduction

Panel data models with individual-specific effects make it possible to control for unobserved heterogeneity and confounding due to omitted variables that are constant over time. Nonlinear models are required to correctly describe discrete outcomes, and the main complication in such nonlinear panel models is the unknown distribution of unobserved heterogeneity, which constitutes an infinite-dimensional parameter. The fixed effects approach leaves this distribution unspecified, eliminating misspecification concerns (as opposed to the correlated random effects approach which models this distribution parametrically). However, lack of sufficient time-series variation in short panels means that this unknown distribution remains set-identified. An important consequence of this is a general lack of point-identification of average effects. While it is theoretically possible to recover the sharp identified set for average effects, in empirically relevant panel dimensions this often becomes an infeasible task due to a curse of dimensionality. This is a serious issue because average effects are typically the ultimate object of interest, especially from the policy perspective. In this paper, considering a general semiparametric setting, we propose alternative outer bounds which are simple to obtain and remain free of the curse of dimensionality in empirically relevant settings.

Formally, let  $Y_i = (Y_{i1}, \ldots, Y_{iT})$  be the vector of observed outcomes for individual  $i = 1, \ldots, n$ , where T is the number of time periods and n is the number of cross-sectional units. Throughout, we assume that  $n \to \infty$  but T remains fixed. The semiparametric panel models we consider in this paper describe the distribution of  $Y_i$  conditional on a vector of observed conditioning variables  $Z_i$  as

$$f_{Y|Z}(y_i|z_i) = \int_{\mathcal{A}} f(y_i | z_i, a_i; \beta) \pi(a_i | z_i) \, da_i.$$
(1)

Here,  $f(y_i | z_i, a_i; \beta)$  is the distribution of  $Y_i$  conditional on  $Z_i$  and the (vector of) unobserved individual effects  $A_i$ , and it is assumed to be known up to the finite dimensional parameter  $\beta$ . The distribution of  $A_i$  conditional on  $Z_i$ , given by  $\pi(a_i | z_i)$ , is left unrestricted. Both  $\beta$  and  $\pi = \pi(a_i | z_i)$  are unknown. The vector of conditioning variables usually consists of observed covariates  $(X_{i1}, \ldots, X_{iT})$  and/or initial conditions  $(Y_{i0}, Y_{i,-1}, \ldots)$ . Given the true distribution of  $Y_i$  conditional on  $Z_i$ , the identified set for the model parameters consists of all pairs  $(\beta, \pi)$  that satisfy (1).

In empirical research, the ultimate object of interest is generally an average effect of

the form

$$\overline{m} := \mathbb{E} m(Z_i, A_i, \beta), \tag{2}$$

where  $m(Z_i, A_i, \beta)$  is some function of interest. The exact choice of  $m(\cdot, \cdot, \cdot)$  may vary from application to application, leading to different definitions of  $\overline{m}$ ; see, among others, Chamberlain (1984), Blundell and Powell (2003, 2004), Altonji and Matzkin (2005), Wooldridge (2005a,b), Bester and Hansen (2009), Graham and Powell (2012), Hoderlein and White (2012).<sup>1</sup> Abrevaya and Hsu (2021) provide a detailed discussion of different average effects used in the literature.

The average effect in (2) can be rewritten as

$$\overline{m} = \int_{\mathcal{Z}} \int_{\mathcal{A}} m\left(z_i, a_i, \beta\right) \, \pi(a_i | z_i) \, f_Z(z_i) \, da_i dz_i$$

which clearly depends on  $(\beta, \pi)$ . In discrete choice models those model parameters (and in particular  $\pi$ ) are usually only partially-identified, implying that  $\overline{m}$  is also typically only partially-identified.

Honoré and Tamer (2006) and Chernozhukov, Fernández-Val, Hahn and Newey (2013) provide methods for obtaining the identified set when covariates are discrete. More recently, there has been an increased interest in the identification and estimation of average effects in various settings; see, e.g., Aguirregabiria and Carro (2021), Davezies, D'Haultfœuille and Laage (2021), Dobronyi, Gu and Kim (2021), Liu, Poirier and Shiu (2021), Botosaru and Muris (2022) and Botosaru, Muris and Sokullu (2022).<sup>2</sup>

Unfortunately, obtaining the sharp identified set is often practically infeasible for sample sizes typically encountered in applications, due to a curse of dimensionality. This is because obtaining the sharp identified set for  $\overline{m}$  typically requires estimates of the conditional probabilities  $f_{Y|Z}(y_i|z_i)$ . Since  $Z_i$  usually contains a time-vector of (multiple) covariates, the curse of dimensionality is obvious for continuous covariates. However, even with discrete covariates the number of conditional probabilities that would

 $<sup>^{1}</sup>$ A different quantity of interest, which we will not consider, is the quantile structural function of Imbens and Newey (2009).

<sup>&</sup>lt;sup>2</sup>Lack of point-identification of  $\pi(a_i|z_i)$  does not invariably lead to set-identification of  $\overline{m}$ . An interesting contribution in this vein is by Aguirregabiria and Carro (2021) who obtain point-identification of the average effect with respect to the lagged dependent variable in a dynamic logit model. However, such case-specific results usually remain an exception. A different route is to obtain point-identification of average effects under additional restrictions on the data generating process as in Liu, Poirier and Shiu (2021). In contrast to these approaches, our aim is to provide a method which applies to an arbitrary function  $m(Z_i, A_i, \beta)$  in a generic semiparametric framework.

need to be estimated is usually large. Suppose, for example,  $Y_{it}, X_{it} \in \{0, 1\}$ , and that  $Z_i = (X_{i1}, \ldots, X_{iT})$ . This implies  $2^{2T}$  different conditional probabilities  $f_{Y|Z}(y_i|z_i)$ ; for, say, T = 5 this yields 1,024 conditional probabilities. Estimation of objects of such numbers would require a much larger cross-sectional sample size than available in the majority of applications.

Motivated by this issue, we propose alternative bounds on the average effect  $\overline{m}$  which can be feasibly obtained in realistic data settings. Our proposal is based on finding appropriate functions  $L(Z_i, Y_i, \beta)$  and  $U(Z_i, Y_i, \beta)$  such that

$$\mathbb{E}L(Z_i, Y_i, \beta) \le \mathbb{E}m(Z_i, A_i, \beta) \le \mathbb{E}U(Z_i, Y_i, \beta).$$

We show that suitable functions  $L(\cdot, \cdot, \cdot)$  and  $U(\cdot, \cdot, \cdot)$  can be obtained by solving an appropriate linear program for each realized value of  $Z_i$ . Asymptotically valid lower and upper bounds are then given by

$$\frac{1}{n}\sum_{i=1}^{n}L(Z_i, Y_i, \widehat{\beta}) \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}U(Z_i, Y_i, \widehat{\beta}),$$

respectively, for some appropriate estimator  $\hat{\beta}$ , assuming that  $\beta$  is point-identified. We prove the validity of the proposed bounds and provide asymptotically valid inference methods on  $\overline{m}$ . Our approach allows for discrete, as well as continuous covariates. We also provide computationally feasible methods for obtaining the suggested bounds. Importantly, these do not require searching over the space of possible distributions for  $\pi(a_i|z_i)$ , but only over the domain of  $A_i$  itself. Consequently, implementation of our method is computationally straightforward and fast.

Our proposal differs from the existing literature in several ways. Firstly, we do not propose a different approach to obtaining the sharp identified set for  $\overline{m}$ ; rather, we obtain outer bounds on this set. This has the virtue of avoiding the curse of dimensionality associated with the conditioning variable  $Z_i$ . Indeed, our outer bounds can be feasibly obtained at standard sample sizes even if the vector of conditioning variables  $Z_i$  is continuous, or high-dimensional, or takes on many different values within the sample. Secondly, given our general semiparametric setting, the proposed method can easily be applied to different models (and functions  $m(Z_i, A_i, \beta)$ ) of interest, such as the static logit, dynamic logit or the more complicated random coefficient logit models.

Davezies, D'Haultfœuille and Laage (2021) propose an alternative method to achieve

inference on  $\overline{m}$ . Their paper initially focuses on inference on the sharp identified set, but they also consider "outer bounds" (different from ours) that avoid non-parametric estimation of intermediate objects, similar in spirit to our results here. However, their approach currently only applies to static logit and ordered logit models (and for several choices of average effects), while in this paper we consider general models of the form (1) (and more general average effects of the form (2)).

Throughout, we consider the case where  $\beta$  is point-identified. In principle, our approach can be extended to the case where  $\beta$  is partially identified, but we do not investigate this extension here. The reason for this is that methods for point-estimation of  $\beta$  are well-established in the literature, and these methods are regularly used by applied researchers. Indeed, for essentially every type of discrete outcome variable (e.g. binary, count data, ordered choice, multinomial choice, ...) there exist appropriate specifications for  $f(y_i | z_i, a_i; \beta)$  that allow point identification and  $\sqrt{n}$ -consistent estimation of  $\beta$  by the conditional likelihood method. In static models, this approach relies on the availability of a sufficient statistic for  $A_i$  (conditional on  $Z_i$ ), which is satisfied in exponential-family models.<sup>3</sup> In dynamic panel models, one can similarly find specifications for  $f(y_i | z_i, a_i; \beta)$  such that estimation of  $\beta$  via the generalized method of moments is possible.<sup>4</sup> More generally, the functional differencing method of Bonhomme (2012) can be viewed as a unifying framework for point-estimation of  $\beta$  in both static and dynamic panel models of the form (1).

Notice also that there are interesting models that do not require estimation of any common parameters  $\beta$ . A prominent example is the binary choice random coefficient model, which has been used in Browning and Carro (2007, 2010, 2014) to incorporate richer forms of heterogeneity; see Example 2 below, and our discussion in Section 4.3.

The rest of the paper is organized as follows: The main idea of our approach is introduced in Section 2. Section 3 provides some illustrations of the main idea, including some comparison of our outer bounds to the sharp identified set. The actual construction of our bounds in general models is explained in Section 4. Section 5 provides general inference methods and asymptotic theory. Sections 6 and 7 provide a simulation study

<sup>&</sup>lt;sup>3</sup>To provide a non-exhaustive list of examples, see, e.g., Rasch (1961), Andersen (1970), Chamberlain (1980), Chamberlain (1985) for binary choice logit; Lancaster (2000), Blundell, Griffith and Windmeijer (2002) for count data Poisson; and Das and van Soest (1999), Baetschmann, Staub and Winkelmann (2015), Muris (2017) for ordered choice logit models (using binarization).

<sup>&</sup>lt;sup>4</sup>See, for example, Honoré and Weidner (2020), Kitazawa (2022) for dynamic binary choice logit; Blundell, Griffith and Windmeijer (2002) for dynamic count data Poisson; and Honoré, Muris and Weidner (2021) for dynamic ordered choice. Honoré and Kyriazidou (2000) and Bartolucci and Nigro (2010) also consider estimation of  $\beta$  in dynamic binary choice panel models.

and present our empirical illustration, respectively. Section 8 concludes. All proofs are in the Appendix.

# 2 Bounds on average effects

We observe discrete outcomes  $Y_i \in \mathcal{Y}$ , and conditioning variables  $Z_i \in \mathcal{Z}$  for a crosssectional sample of units  $i = 1, \ldots, n$ . Unobserved heterogeneity is modeled through an unobserved latent variable  $A_i \in \mathcal{A}$ . The probability of observing  $Y_i = y$  conditional on  $Z_i = z$  and  $A_i = a$  is given by  $f(y | z, a; \beta_0)$  where  $\beta_0 \in \mathcal{B} \subset \mathbb{R}^{\dim \beta}$  and  $f: \mathcal{Y} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$  is a known function. The joint distribution of the conditioning variables  $Z_i$ and  $A_i$  is left unspecified. We focus on panel data models, where  $Y_i = (Y_{i1}, \ldots, Y_{iT})$  is a vector of outcomes  $Y_{it} \in \mathcal{Y}_t$ . The vector of conditioning variables  $Z_i$  can, for example, be equal to  $X_i = (X_{i1}, \ldots, X_{iT})$  in static models, or to  $Z_i = (X_i, Y_{i0})$  in dynamic models where  $Y_{i0}$  is the initial condition from time period t = 0. We assume that  $X_i$  is strictly exogenous, and in dynamic models (i.e. models with lagged dependent variables), we assume that  $Y_{i0}$  is observed. Our goal is to provide inference methods on average effects of the form

$$\overline{m} := \mathbb{E}\left[m\left(Z_i, A_i, \beta_0\right)\right],\tag{3}$$

where  $m: \mathcal{Z} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}$  is a known function.

To focus on the main features and intuition behind our proposed approach, in this section we abstract away from estimation of  $\beta_0$  and assume that it is known. In Section 5 we will consider the case where  $\beta_0$  is unknown but point-identified. The random coefficient model in Example 2 below provides an interesting case where no estimation of  $\beta_0$  is necessary, because the model does not feature any such common parameter. In that case, the results in this section are already fully sufficient for inference on  $\overline{m}$ .

While our approach is general enough to accommodate different panel models of interest (including dynamic ones), for illustration purposes we focus on two running examples.

**Example 1** In a static binary choice model, outcomes are generated as  $Y_{it} = 1\{X_{it}\beta_0 + A_i \geq \varepsilon_{it}\}$ , where  $X_{it}$  is a  $1 \times K$  vector of covariates, dim  $\beta_0 = K$ , and  $\varepsilon_{it}$  is a logistic or standard normal random variable. Letting  $X_{k,it}$  be the k-th covariate and  $X_{-k,it}$  be a

row matrix containing the remaining covariates, typical examples of  $m(Z_i, A_i, \beta_0)$  are

$$\frac{1}{T} \sum_{t=1}^{T} \left[ P(Y_{it} = 1 | X_{k,it} = x_1, X_{-k,it}, A_i, \beta_0) - P(Y_{it} = 1 | X_{k,it} = x_2, X_{-k,it}, A_i, \beta_0) \right], \quad (4)$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial P(Y_{it}=1|X_{it}=x_{it},A_i,\beta_0)}{\partial x_{k,it}},$$
(5)

for discrete and continuous  $X_{k,it}$ , respectively, where  $x_1, x_2 \in \mathbb{R}$ . For binary and multinomial variables, examples are  $(x_1 = 1, x_2 = 0)$  and  $(x_1 = x + 1, x_2 = x)$ , for some x. In (5),  $x_{it}$  could be equal to the observed value of  $X_{it}$  or its time average, or some other value of interest.

**Example 2** Our second example is the random coefficient binary choice model, given by  $Y_{it} = 1\{X_{it} A_{2,i} + A_{1,i} \ge \varepsilon_{it}\}$ , where  $A_{1,i} \in \mathbb{R}$ ,  $A_{2,i} \in \mathbb{R}^{\dim X_{it}}$ , and  $\varepsilon_{it}$  can have the logistic or standard normal distribution. This allows for richer types of heterogeneity which cannot be captured by the classical fixed effects model (see, for example, Browning and Carro 2007, 2010, 2014). For simplicity, we consider the static setting, but our approach remains valid if lagged dependent variables are included as regressors. Defining  $A_i = (A_{1,i}, A_{2,i})$ , examples for  $m(Z_i, A_i)$  in this case can be generated analogous to (4) and (5). We will later consider the case of a single discrete covariate  $X_{it}$  and focus on

$$\frac{1}{T}\sum_{t=1}^{T} \left[ P(Y_{it} = 1 | X_{it} = 1, A_i) - P(Y_{it} = 1 | X_{it} = 0, A_i) \right].$$
(6)

Our proposal for inference on  $\overline{m}$  is based on the simple idea that suitable non-random functions  $L, U : \mathcal{Z} \times \mathcal{Y} \times \mathcal{B} \rightarrow [b_{\min}, b_{\max}]$  which satisfy,

$$\sum_{y \in \mathcal{Y}} L(z, y, \beta) f(y \mid z, a; \beta) \le m(z, a, \beta) \le \sum_{y \in \mathcal{Y}} U(z, y, \beta) f(y \mid z, a; \beta),$$
(7)

can be used to obtain asymptotically valid bounds on  $\overline{m}$ . To see how, notice that when evaluated at  $\beta_0$ , the condition in (7) is equivalent to

$$\mathbb{E}\left[L(Z_i, Y_i, \beta_0) \mid Z_i = z, A_i = a\right] \le m(z, a, \beta_0) \le \mathbb{E}\left[U(Z_i, Y_i, \beta_0) \mid Z_i = z, A_i = a\right],$$

which, by the Law of Iterated Expectations, implies that

$$\mathbb{E}[L(Z_i, Y_i, \beta_0)] \le \overline{m} \le \mathbb{E}[U(Z_i, Y_i, \beta_0)].$$
(8)

This suggests that asymptotically valid bounds on  $\overline{m}$  are given by

$$\widehat{L} := \frac{1}{n} \sum_{i=1}^{n} L(Z_i, Y_i, \beta_0), \qquad \qquad \widehat{U} := \frac{1}{n} \sum_{i=1}^{n} U(Z_i, Y_i, \beta_0). \tag{9}$$

To formally show this, we impose the following regularity conditions.

#### Assumption 1

- (i)  $(Y_i, Z_i, A_i)$  are independent and identically distributed across i = 1, ..., n.
- (ii) The conditional distribution of outcomes  $Y_i$  satisfies

$$P(Y_i = y | Z_i = z, A_i = a) = f(y | z, a; \beta_0).$$

(iii) There are known bounds  $b_{\min}, b_{\max} \in \mathbb{R}$  such that  $b_{\min} \leq m(z, a, \beta) \leq b_{\max}$ , for all  $z \in \mathcal{Z}, a \in \mathcal{A}$  and  $\beta \in \mathcal{B}$ .

Assumption 1(i) demands cross-sectional sampling. Assumption 1(ii) imposes correct specification of our parametric model for  $Y_i$  conditional on  $Z_i$  and  $A_i$ . Assumption1(iii) requires uniform bounds on the functions  $m(z, a, \beta)$  that define the average effect of interest  $\overline{m}$ . This holds for typical choices for  $\overline{m}$  such as those in Examples 1 and 2, and it can easily be confirmed for any given  $m(z, a, \beta)$ .<sup>5</sup> Importantly, we do not put any restriction on the joint distribution of  $Z_i$  and  $A_i$ . In particular,  $Z_i$  can be discrete or continuous, and  $Z_i$  and  $A_i$  can be arbitrarily related.

**Theorem 1** Let Assumption 1 hold, and let  $L, U : \mathcal{Z} \times \mathcal{Y} \times \mathcal{B} \rightarrow [b_{\min}, b_{\max}]$  satisfy equation (7) for  $\beta = \beta_0$  and for all  $z \in \mathcal{Z}$ ,  $a \in \mathcal{A}$ . Let  $\overline{m}$ ,  $\widehat{L}$ ,  $\widehat{U}$  be as defined in (3) and

<sup>&</sup>lt;sup>5</sup>In principle a weaker condition such as  $b_{\min} \leq \mathbb{E}[m(Z_i, A_i, \beta_0) | A_i = a] \leq b_{\max}$  might also be used here, or bounds on second or higher-order moments of  $m(Z_i, A_i, \beta_0)$  are also conceivable, but in all the applications we consider in the paper the original Assumption 1(iii) holds, and we find it attractive that this assumption can be verified without knowing anything about the data generating process of  $Z_i$ and  $A_i$ . More generally, Assumption1(iii) could be replaced by any assumption that guarantees that  $\operatorname{Var}[L(Z_i, Y_i, \beta_0)]$ , and  $\operatorname{Var}[U(Z_i, Y_i, \beta_0)]$  are finite in Theorem 1.

(9). Then,

$$\widehat{L} + O_p(n^{-1/2}) \leq \overline{m} \leq \widehat{U} + O_p(n^{-1/2}) \quad \text{as } n \to \infty.$$

Furthermore, assume that  $\operatorname{Var}\left[L(Z_i, Y_i, \beta_0)\right] > 0$ , and  $\operatorname{Var}\left[U(Z_i, Y_i, \beta_0)\right] > 0$ , and define  $\widehat{\sigma}_L^2 := \frac{1}{n} \sum_{i=1}^n [L(Z_i, Y_i, \beta_0) - \widehat{L}]^2$  and  $\widehat{\sigma}_U^2 := \frac{1}{n} \sum_{i=1}^n [U(Z_i, Y_i, \beta_0) - \widehat{U}]^2$ . Then, for  $\alpha \in [0, 1]$ , we have

$$\lim_{n \to \infty} P\left(\widehat{L} - \frac{c_{\alpha/2}\,\widehat{\sigma}_L}{\sqrt{n}} \le \overline{m} \le \widehat{U} + \frac{c_{\alpha/2}\,\widehat{\sigma}_U}{\sqrt{n}}\right) \ge 1 - \alpha, \quad where \quad c_{\alpha/2} = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right).$$

# **3** Some further discussion of the bounds

Before moving to the general construction of our bounds in Section 4, we find it useful to give a concrete example, and to also to compare the outer bounds to the sharp identified set.

### 3.1 An illustrative example

The following example simply corresponds to the nonparametric bounds in Chernozhukov, Fernández-Val, Hahn and Newey (2013). It is therefore not representative of how we obtain the bounds in this paper in general, but we still find the example instructive, since it provides analytical expressions for bounds satisfying (7).

We consider the static binary choice model of Example 1 for the case where  $X_{it} \in \{0, 1\}$  is the only covariate and the error term  $\varepsilon_{it}$  is stationary over time t. The average effect is given by (4) with  $x_1 = 1$  and  $x_2 = 0$ , that is,

$$m(A_{i},\beta_{0}) = \frac{1}{T} \sum_{t=1}^{T} \left[ P\left(Y_{it} = 1 \mid X_{it} = 1, A_{i}, \beta_{0}\right) - P\left(Y_{it} = 1 \mid X_{it} = 0, A_{i}, \beta_{0}\right) \right]$$
  
=  $\mathbb{E} \left[ Y_{it} \mid X_{it} = 1, A_{i}, \beta_{0} \right] - \mathbb{E} \left[ Y_{it} \mid X_{it} = 0, A_{i}, \beta_{0} \right],$  (10)

where the time averaging is not needed due to stationarity.<sup>6</sup> For  $d \in \{0, 1\}$ , let

$$v(X_i, d) := \begin{cases} 0 & \text{if } X_{it} = 1 - d \text{ for all } t \in \{1, \dots, T\}, \\ 1 & \text{if } X_{it} = d \text{ for some } t \in \{1, \dots, T\}. \end{cases}$$

For  $v(X_i, d) = 1$  we define  $\overline{Y}(Y_i, X_i, d) := \sum_{t \in \mathcal{T}(X_i, d)} Y_{it} / |\mathcal{T}(X_i, d)|$  to be the average of  $Y_{it}$  over those time periods  $\mathcal{T}(X_i, d) = \{t : X_{it} = d\}$  where  $X_{it}$  equals d. For  $v(X_i, d) = 0$  we simply let  $\overline{Y}(Y_i, X_i, d) := 0.^7$  Valid outer bound functions are then given by

$$L(X_i, Y_i) = v(X_i, 1) \overline{Y}(X_i, Y_i, 1) - v(X_i, 0) \overline{Y}(X_i, Y_i, 0) - [1 - v(X_i, 0)], \quad (11)$$

$$U(X_i, Y_i) = v(X_i, 1) \overline{Y}(X_i, Y_i, 1) - v(X_i, 0) \overline{Y}(X_i, Y_i, 0) + [1 - v(X_i, 1)].$$
(12)

The stationarity assumption then guarantees that

$$\mathbb{E}\left[L(X_i, Y_i) \mid X_i, A_i\right] \le m(A_i, \beta_0) \le \mathbb{E}\left[U(X_i, Y_i) \mid X_i, A_i\right],\tag{13}$$

which is exactly the condition (7) that our bound functions are supposed to satisfy.<sup>8</sup>

Again, we want to point out that this example is not characteristic of our bounds more generally. In particular, here  $L(X_i, Y_i)$  and  $U(X_i, Y_i)$  do not depend on  $\beta_0$ , and neither the single-index structure  $X_{it}\beta_0 + A_i + \varepsilon_{it}$  nor the parametric assumption on the error distribution are utilized to show validity of the bounds — the bounds here are valid for any model  $Y_{it} = g(X_{it}, A_i, \varepsilon_{it})$ , as long as the function  $g(\cdot, \cdot, \cdot)$  is constant over t, and the conditional distribution of the shocks  $\varepsilon_{it}$  is stationary over t.

From the corresponding discussion in Chernozhukov, Fernández-Val, Hahn and Newey (2013) we also know that, as  $T \to \infty$ , the width of these bounds,  $\mathbb{E}[U(X_i, Y_i) - L(X_i, Y_i)]$ , shrinks proportionally to the probability of  $X_{it}$  being constant over t. Under appropriate distributional assumptions on  $X_{it}$  (e.g.  $X_{it}$  independent across t and random), this implies that the width of the bounds shrinks exponentially fast in T. We suspect that

<sup>8</sup>Due do stationarity we have

$$\mathbb{E}\left[L(X_i, Y_i) \mid X_i, v(X_i, 0) = v(X_i, 1) = 1, A_i\right] = m(A_i, \beta_0) = \mathbb{E}\left[U(X_i, Y_i) \mid X_i, v(X_i, 0) = v(X_i, 1) = 1, A_i\right]$$

while for  $v(X_i, d) = 0$ , the above bounds  $L(X_i, Y_i)$  and  $U(X_i, Y_i)$  simply revert to the appropriate worst-case bounds (zero or one) that are possible for the unidentified expectations

<sup>&</sup>lt;sup>6</sup>In this case,  $Z_i = X_i = (X_{i1}, \ldots, X_{iT})$ . Notice that, contrary to the general case, here  $m(A_i, \beta_0)$  does not depend on  $X_i$ . This is because the average effect is calculated with respect to specific values of  $X_{it}$  and there are no other covariates.

<sup>&</sup>lt;sup>7</sup>Essentially,  $\overline{Y}(Y_i, X_i, d)$  can be defined as any real number, given that its contribution to the bounds will be equal to zero whenever  $v(X_i, d) = 0$ .

similar results hold more generally for the bounds in this paper, but we do not actually explore large T results here.

#### **3.2** Comparison to the identified set

The key difference between our outer bounds,  $\mathbb{E}U(X_i, Y_i)$  and  $\mathbb{E}L(X_i, Y_i)$ , and the identified set for  $\overline{m}$  is how they depend on the conditional choice probabilities,  $P(Y_i|X_i)$ . In particular, while our outer bounds are a linear function of choice probabilities, the upper and lower boundaries of the identified set are complicated nonlinear functions of  $P(Y_i|X_i)$ . The goal of this subsection is to briefly explain this difference and its consequences for inference on the average effects.

For simplicity, we stick to the static binary choice example with single binary covariate discussed in the last subsection, and we assume that  $\varepsilon_{it}$  has standard logistic distribution. Let  $f(y|x, a; \beta_0)$  be the corresponding conditional distribution of  $Y_i|X_i, A_i$ . As long as we have some variation on the covariates across time,  $\beta_0$  is point-identified in this model (see e.g. Chamberlain 1985, 2010).

For  $x \in \{0,1\}^T$ , let  $p(x) := [P(Y_i = y | X_i = x) : y \in \{0,1\}^T]$  be the  $2^T$ -vector of choice probabilities conditional on  $X_i = x$ , and define  $\overline{m}(x) := \mathbb{E}[m(A_i, \beta_0) | X_i = x]$ . Next, let  $\Pi(x, p(x))$  be the set of conditional distributions  $A_i | X_i$  that are compatible with the choice probabilities p(x): that is, we have  $\pi(\cdot | x) \in \Pi(x, p(x))$  if and only if  $P(Y_i = y | X_i = x) = \int_{\mathbb{R}} f(y | x, a; \beta_0) \pi(a | x) da$ . Since  $\beta_0$  and p(x) are point-identified, the only ambiguity in the identification of  $\overline{m}(x)$  is due to the unknown distribution of  $A_i | X_i$ . Then, defining

$$L_{\mathrm{id}}(x, p(x)) := \inf_{\pi(\cdot|x)\in\Pi(x, p(x))} \int_{\mathbb{R}} m(a, \beta_0) \, \pi(a|x) da,$$
$$U_{\mathrm{id}}(x, p(x)) := \sup_{\pi(\cdot|x)\in\Pi(x, p(x))} \int_{\mathbb{R}} m(a, \beta_0) \, \pi(a|x) da,$$

the identified set for  $\overline{m} = \mathbb{E}[\overline{m}(X_i)]$  is given by  $[\mathbb{E}L_{id}(X_i, p(X_i)), \mathbb{E}U_{id}(X_i, p(X_i))]$ . All this is of course well-known. What we want to highlight here is that the above construction inevitably yields a complicated nonlinear dependence of the boundaries of the identified set on the observable choice probabilities p(x) through  $\Pi(x, p(x))$ . In contrast, our bounds

$$\mathbb{E} L(x, Y_i) = \sum_{y \in \{0,1\}^2} L(x, y) P(Y_i = y | X_i = x),$$
$$\mathbb{E} U(x, Y_i) = \sum_{y \in \{0,1\}^2} U(x, y) P(Y_i = y | X_i = x),$$

are by construction linear functions of the vector of conditional choice probabilities p(x).

This distinction between non-linearity (for the identified set) vs linearity (for our outer bounds) in p(x) has a fundamental effect on inference: the sample analogs of our bounds,  $\frac{1}{n} \sum_{i=1}^{n} L(X_i, Y_i)$  and  $\frac{1}{n} \sum_{i=1}^{n} U(X_i, Y_i)$ , avoid estimating p(x) naturally. In contrast, we are not aware of any inference procedure on the sharp identified set that would avoid consistent estimation of p(x).<sup>9</sup> Especially when p(x) is hard to estimate, the nonlinear dependence of the identified set on p(x) can cause significant issues in inference. Hence, as already mentioned in the introduction, reliable inference on the number of possible values for  $(X_i, Y_i)$ . Our outer bounds are by design immune to this.

To illustrate the points made here, we consider a brief simulation exercise. Let

$$Y_{it} = 1\{X_{it}\beta + A_i \ge \varepsilon_{it}\}, \quad A_i \sim N\left(\frac{1}{T}\sum_{t=1}^T X_{it} - \frac{1}{2}, 1\right), \quad X_{it} = x_{it}/(|\mathcal{X}| - 1),$$

where  $\varepsilon_{it} \sim \text{Logit}(0, 1)$  and  $x_{it}$  is discrete uniform with support  $[0, \mathcal{X} - 1]$ . Then,  $X_{it}$  can take on one of  $|\mathcal{X}|$  equidistant values between 0 and 1. We consider  $|\mathcal{X}| \in \{6, 12\}$ . The analysis for either case is based on 1000 replications of panels with T = 2, N = 200. The average effect of interest is as in (10). For each replication, we obtain the estimated sharp identified set and our outer bounds based on the construction in Section 4. Then we report the 2.5% and 97.5% sample quantiles of these quantities across all replications.<sup>10</sup> Doing so enables us to compare the limits of the estimated confidence intervals, without estimating the confidence bands directly. Results are presented in Figure 1. When  $|\mathcal{X}| = 6$ , the lower and upper 2.5% percentiles of the estimated bounds of the identified

<sup>&</sup>lt;sup>9</sup>Davezies, D'Haultfœuille and Laage (2021) present two different inference procedures for average effects in static panel logit models, one that relies on consistent estimation of p(x), and one that does not. In the latter case, they also obtain certain outer bounds on the identified set, in line with our discussion here.

 $<sup>^{10}</sup>$ The outer bounds presented here, based on the construction in Section 4, provide much narrower bounds than the simple analytical expressions in (11)-(12). This is not surprising, given that our bounds utilize stronger model assumptions.



Figure 1: Sample quantiles of estimates of the outer bounds and the identified set. The DGP is  $Y_{it} = 1\{X_{it}\beta + A_i \geq \varepsilon_{it}\}$  where  $A_i \sim N(T^{-1}\sum_{t=1}^T X_{it} - 1/2, 1), X_{it} = x_{it}/(|\mathcal{X}| - 1), x_{it}$ is discrete uniform with support  $[0, \mathcal{X} - 1]$ , and  $\varepsilon_{it} \sim \text{Logit}(0, 1)$ . The average effect under consideration is  $\mathbb{E}[Y_{it} | X_{it} = 1, A_i, \beta_0] - \mathbb{E}[Y_{it} | X_{it} = 0, A_i, \beta_0]$ . For each  $\beta_0 \in [-2, 2]$ , the quantiles are calculated across 1000 replications of panels with T = 2 and N = 200. The left panel contains the results for  $|\mathcal{X}| = 6$  whereas the results for  $|\mathcal{X}| = 12$  are presented in the right panel.

set provide valid coverage. However, when  $|\mathcal{X}|$  increases to 12, the same percentiles do not even include the average effect itself almost all the time. This reflects an underlying bias in the estimation of the sharp identified set. The outer bounds are immune to this issue, and still provide valid coverage. This example illustrates that, although the outer bounds are not sharp, they can be more reliable in inference compared to estimators of the sharp identified set itself. The results suggest, as expected, that issues arise as the cardinality of the support of the covariate increases. Therefore, the case with continuous  $X_{it}$  will be subject to more pronounced issues.

### 4 Construction of the bounds

We now introduce our general construction of the bound functions  $L(z, y, \beta)$  and  $U(z, y, \beta)$ . To concentrate solely on bound construction, in this section we still consider the case with known  $\beta_0$ . A full theory with estimated  $\beta_0$  is provided in Section 5. In terms of implementation, the construction methods remain the same for given  $\beta$ , independent of whether it is  $\beta_0$  or its estimate.

The example in Section 3.1 illustrates that in a particular model and for a particular average effect of interest, it might be possible to obtain analytic expressions for those

bound functions. But for the class of semi-parametric panel models and average effects introduced in Section 2, it does not appear likely that analytic expressions for the bounds can be obtained in general. We therefore introduce a computational method for obtaining  $L(z, y, \beta)$  and  $U(z, y, \beta)$  based on solving linear programs.

The distinction between analytic expressions for the bound functions and a computational method is analogous to the distinction between the functional differencing method in Bonhomme (2012) and the analytical moment functions in Honoré and Weidner (2020) for the purpose of inference on  $\beta$ . The former paper applies to the same class of semi-parametric panel models that we consider here, but it only provides a computational method to find valid moment functions. The latter paper only applies to specific models, but allows for the explicit analytical constructions of valid moment functions for  $\beta$ . From the perspective of this comparison, our current paper is analogous to Bonhomme (2012).

In obtaining asymptotically valid bounds, the key requirement on the functions  $L(z, y, \beta)$  and  $U(z, y, \beta)$  is that they satisfy (7) and that they are bounded. Of course, one wants the estimated bounds on  $\overline{m}$  to be informative, in the sense that the interval in (7) is as narrow as possible. At the same time, importantly, for given z and  $\beta$ ,  $L(z, y, \beta)$  and  $U(z, y, \beta)$  have to be chosen such that (7) holds for all  $a \in \mathcal{A}$ . This can be reformulated as a standard optimisation problem. Namely, for any given  $z \in \mathcal{Z}$  and  $\beta \in \mathcal{B}$  we can choose  $L(z, y, \beta) = \ell(y)$  and  $U(z, y, \beta) = u(y)$  as solutions to the following optimization problem with some appropriate objective function  $Q(\ell(\cdot), u(\cdot), z, \beta)$ ,<sup>11</sup>

$$\min_{\ell,u:\mathcal{Y}\to\mathbb{R}} Q(\ell(\cdot), u(\cdot), z, \beta)$$
subject to
$$\forall y \in \mathcal{Y}: \ b_{\min} \leq \ell(y) \leq u(y) \leq b_{\max}$$
and
$$\forall a \in \mathcal{A}: \ \sum_{y \in \mathcal{Y}} \ell(y) \ f(y \mid z, a; \beta) \leq m(z, a, \beta) \leq \sum_{y \in \mathcal{Y}} u(y) f(y \mid z, a; \beta).$$
(14)

In the current setting where we assume that  $\beta_0$  is known, (14) will be solved at  $\beta = \beta_0$ . When  $\beta_0$  is estimated, (14) will be solved at some estimate  $\beta = \hat{\beta}$ . When no common parameter is estimated (as in Example 2), the objective function and the constraints will be free of  $\beta$ .

<sup>&</sup>lt;sup>11</sup>The solutions  $L(z, y, \beta) = \ell(y)$  and  $U(z, y, \beta) = u(y)$  may not be unique. But in a practical implementation some concrete solution will still be obtained by the specific linear solver used for implementation, and Theorem 1 is still valid, since it only depends on the constraints being satisfied.

The restrictions of the program (14) guarantee the conditions of Theorem 1 and also impose that  $\ell(y) \leq u(y)$ . Consequently, any choice of the objective function  $Q(\ell(\cdot), u(\cdot), z, \beta)$  yields valid bounds with  $\hat{L} \leq \hat{U}$ . It is important to stress that in order to construct the bounds  $\hat{L}$  and  $\hat{U}$  we only need to solve the program in (14) once for every  $i \in \{1, ..., n\}$  at  $z = Z_i$ . Contrary to the sharp identified set, construction of our bounds does not involve conditional choice probabilities, and therefore remains free of the curse of dimensionality.

Display (14) states our approach to obtaining bounds in its most general form, in the sense that the econometrician can choose any objective function  $Q(\ell(\cdot), u(\cdot), z, \beta)$ that she sees fit. It is computationally attractive to consider objective functions which turn the optimization problem into a linear program, and we now discuss two intuitive choices of objective functions that are indeed linear in  $\ell(\cdot)$  and  $u(\cdot)$ .

### 4.1 Choice of objective function for linear program

#### 4.1.1 Baseline linear program

A linear program can be implemented by using the objective function

$$Q(\ell(\cdot), u(\cdot), z, \beta) = \int_{\mathcal{A}} \sum_{y \in \mathcal{Y}} \left[ u(y) - \ell(y) \right] f(y \mid z, a; \beta) \, p(a \mid z) \, da, \tag{15}$$

where p(a|z) is some (potentially non-proper) "prior distribution". Our bounds are valid for any choice of "prior", but if p(a|z) happens to be equal to the true distribution of  $A_i|Z_i$ , then this objective function yields the narrowest expected bounds that satisfy the constraints of (14). Essentially, the function p(a|z) allows for considering a *weighted average* over  $a \in \mathcal{A}$ . In the absence of any additional information on a one can simply use p(a|z) = 1, which is indeed what we use in all our applications below.

#### 4.1.2 Uniform linear program

If we are unwilling to specify a prior p(a|z), then we can choose the objective function

$$Q(\ell(\cdot), u(\cdot), z, \beta) = \max_{a \in \mathcal{A}} \left[ \sum_{y \in \mathcal{Y}} \left[ u(y) - \ell(y) \right] f(y \mid z, a; \beta) \right],$$
(16)

where instead of integrating over  $a \in \mathcal{A}$  with a prior distribution we choose the worstcase value of  $a \in \mathcal{A}$  that maximizes the expected bounds  $\sum_{y \in \mathcal{Y}} [u(y) - \ell(y)] f(y | z, a; \beta)$ . Hence, we call the ensuing approach the *uniform linear program*. To be precise, this objective function cannot be used directly to yield a linear program since it is not linear in u(y) and  $\ell(y)$ ; however, an equivalent representation of this problem as a linear program is obtained as follows:

$$\min_{\{s \in \mathbb{R}, \ell, u : \mathcal{Y} \to \mathbb{R}\}} s$$
subject to
$$\forall y \in \mathcal{Y} : b_{\min} \leq \ell(y) \leq u(y) \leq b_{\max}$$

$$\forall a \in \mathcal{A} : \sum_{y \in \mathcal{Y}} [u(y) - \ell(y)] f(y \mid z, a; \beta) \leq s$$
and
$$\forall a \in \mathcal{A} : \sum_{y \in \mathcal{Y}} \ell(y) f(y \mid z, a; \beta) \leq m(z, a, \beta) \leq \sum_{y \in \mathcal{Y}} u(y) f(y \mid z, a; \beta).$$
(17)

In this linear program, the variable set is extended by  $s \in \mathbb{R}$ . When profiling out  $s \in \mathbb{R}$ from this program one finds that for given  $\ell, u : \mathcal{Y} \to \mathbb{R}$  the optimal s is given by

$$s = \max_{a \in \mathcal{A}} \left[ \sum_{y \in \mathcal{Y}} \left[ u(y) - \ell(y) \right] f(y \mid z, a; \beta) \right],$$
(18)

which is identical to the objective function in (16). Thus, solving the linear program in (17) gives the desired bound functions  $L(z, y, \beta) = \ell(y)$  and  $U(z, y, \beta) = u(y)$  that correspond to choosing the objective function (16) in our general program (14).

#### 4.2 Implementational details

In practice, we usually cannot solve the linear programs (14) and (17) exactly. This is because the functions (15) and (18) require evaluation over  $\mathcal{A}$  which typically has infinite cardinality. Instead, we approximate these objects by choosing a subset of grid points  $\mathcal{A}_{g} \subset \mathcal{A}$  and imposing the constraints only at  $a \in \mathcal{A}_{g}$ . This yields

$$Q(\ell(\cdot), u(\cdot), z, \beta) = \sum_{a \in \mathcal{A}_g} \sum_{y \in \mathcal{Y}} \left[ u(y) - \ell(y) \right] f(y \mid z, a; \beta) p(a \mid z) ,$$

and

$$s = \max_{a \in \mathcal{A}_g} \left[ \sum_{y \in \mathcal{Y}} \left[ u(y) - \ell(y) \right] f(y \mid z, a; \beta) \right].$$

The size of the grid  $\mathcal{A}_g$  directly controls the number of restrictions in (14) and so, especially in complicated applications, computational concerns may put a limit on how fine the grid  $\mathcal{A}_g$  can be. However, even then, our approach provides an easy way to obtain solutions that work on a much finer grid. To illustrate how, let  $\mathcal{A}_g, \mathcal{A}_G \subset \mathcal{A}$  be two grids where the cardinality of  $\mathcal{A}_G$  is (much) larger than that of  $\mathcal{A}_g$ . Let  $(L(z, y, \beta), U(z, y, \beta))$ be the solution to (14) on  $\mathcal{A}_g$ . It is computationally almost cost-free to check whether this solution satisfies the restriction (7) on  $\mathcal{A}_G$ . If violations occur, one can adjust the original solution  $(L(z, y, \beta), U(z, y, \beta))$  to fit the restriction (7) on  $\mathcal{A}_G$ , thereby obtaining a valid solution to the constraints on the much finer grid  $\mathcal{A}_G$ . A simple way to do this is to replace the original solution by  $(\dot{L}(z, y, \beta), \dot{U}(z, y, \beta))$  where

$$\begin{split} \dot{L}(z,y,\beta) &:= L(z,y,\beta) + \min\left\{0, \min_{a \in \mathcal{A}_G} \left[m(z,a,\beta) - \sum_{y \in \mathcal{Y}} L(z,y,\beta) f(y|z,a;\beta)\right]\right\},\\ \dot{U}(z,y,\beta) &:= U(z,y,\beta) + \max\left\{0, \max_{a \in \mathcal{A}_G} \left[m(z,a,\beta) - \sum_{y \in \mathcal{Y}} U(z,y,\beta) f(y|z,a;\beta)\right]\right\}. \end{split}$$

Here, we simply add the maximum deviation across all grid points to the original solution, automatically yielding a solution that satisfies (7) on  $\mathcal{A}_G$ . The grid  $\mathcal{A}_G$  can be made very fine, making the difference between  $\mathcal{A}_G$  and  $\mathcal{A}$  negligible. Of course, depending on the case at hand, one can devise options that yield less "conservative" solutions compared to  $(\dot{L}(z, y, \beta), \dot{U}(z, y, \beta))$ . We suggest comparing results from different selections of  $(\mathcal{A}_g, \mathcal{A}_G)$  in order to find some  $\mathcal{A}_g$  which is computationally feasible and yet yields reliable bounds.

Computational properties will also depend on the cardinality of  $\mathcal{Y}$ , which determines the number of variables in the program (14). Although the cardinality of  $\mathcal{Y}$  is finite in a fixed-T setting, the size of the support can still be large enough to cause computational difficulties. In logit-based applications it is relatively straightforward to mitigate this problem, by re-writing the restriction (7) in terms of the conditional density of the sufficient statistic. We next illustrate this for Examples 1 and 2. **Example 1 (continued)** Fix z and  $\beta$ . Let  $y = (y_1, ..., y_T)'$  and define

$$\overline{P}(k|z,a,\beta) := \sum_{\{y: \sum_t y_t = k\}} P(y|z,a,\beta),$$

the conditional density of the sufficient statistic  $\sum_{t=1}^{T} y_t$ . Then, there is some  $\overline{u}(k)$  such that

$$m(z, a, \beta) \le \sum_{y \in \mathcal{Y}} u(y) P(y|z, a, \beta) = \sum_{k=0}^{T} \overline{u}(k) \overline{P}(k|z, a, \beta), \quad \forall a \in \mathbb{R}$$

As such, one can solve the problem for  $\overline{u}(k)$ , k = 0, ..., T, and then use, for instance,  $u(y) = \overline{u}(k)$  for all y with  $\sum_{t=1}^{T} y_t = k$ . This effectively decreases the number of variables from  $2^T$  to T + 1. An analogous argument applies to  $\ell(y)$ .

**Example 2 (continued)** Suppose that  $z_t$  is binary. In this case there are two unobserved effects, and so the argument will be based on the sufficient statistics  $\sum_{t=1}^{T} y_t$  and  $\sum_{t=1}^{T} y_t z_t$ . Fix  $z = (z_1, ..., z_T)'$ . Define  $\overline{P}(k_1, k_2 | z, a_1, a_2)$ , the conditional density of  $k_1 = \sum_{t=1}^{T} y_t$  and  $k_2 = \sum_{t=1}^{T} y_t z_t$ . Then, similar to Example 1, there is some  $\overline{u}(k_1, k_2)$  such that

$$m(z, a_1, a_2) \leq \sum_{y \in \mathcal{Y}} u(y) P(y|z, a_1, a_2)$$
  
= 
$$\sum_{k_1=0}^T \sum_{k_2=0}^T \overline{u}(k_1, k_2) \overline{P}(k_1, k_2|z, a_1, a_2), \quad \forall a_1 \in \mathbb{R} \text{ and } a_2 \in \mathbb{R},$$

implying that it is sufficient to solve the linear program for  $\overline{u}(k_1, k_2)$ . At first sight, it appears that the number of variables in this problem is  $(T + 1)^2$ . However, notice that one cannot have  $k_2 > k_1$ . Moreover, for a given z some combinations of  $(k_1, k_2)$  will have zero probability. Consequently, the actual number of variables will usually be less than  $(T + 1)^2$ . We note this method will not work with continuous covariates, since in that case z has infinite support and so does  $k_2$ .

An analogous idea for non-logit applications (where the outlined approach does not necessarily exist) is to reduce the dimension of the problem by partitioning the support  $\mathcal{Y}$  using some meaningful criterion. One can, for example, partition  $\mathcal{Y}$  such that  $\arg \max_{a \in \mathcal{A}_g} P(y|z, a, \beta)$  is the same for all  $y \in \mathcal{Y}$  in the same subset. Generation of the partition can also be based on extra information specific to the application or data at hand. Another way to decrease the computational complexities is to solve the linear program separately for the upper and lower bounds. Note that (14) puts the restriction  $\ell(\cdot) \leq u(\cdot)$  to avoid any crossover between the upper and lower bounds. Solving the bound problem separately would drop this convenient additional condition. However, for moderately large T the probability of such a crossover between the bounds is expected to be quite low, and a potential solution of the crossover problem is, for example, given in Stoye (2020).

#### 4.3 Comparison to the identified set (cont'd)

Continuing the discussion in Section 3.2, we now investigate how the outer bounds compare to the sharp identified set, in the specific cases of typical logit-based binary choice models. These are the static logit and random coefficient logit models, and dynamic variants thereof. In all cases (except for the random coefficient dynamic logit model) we use the linear program in (17). The analysis in this subsection is at the population level, that is, we compare our outer bounds to the population sharp identified set, estimation of which is challenging whenever the support of the conditioning variables  $(Z_1, \ldots, Z_T)$  is not small relative to the sample size.<sup>12</sup>

For static logit we consider both the discrete and continuous covariate cases, with the data generating processes (DGPs) given by

$$Y_{it} = 1 \{ X_{it}\beta + A_i \ge \varepsilon_{it} \}, \quad A_i \sim N(0, 1), \quad X_{it} = 1 \{ A_i \ge \eta_{it} \}, \quad \eta_{it} \sim N(0, 1), \quad (19)$$

and

$$Y_{it} = 1 \{ X_{it}\beta + A_i \ge \varepsilon_{it} \}, \quad A_i \sim N(0, 1), \quad X_{it} \sim N(A_i, 1),$$
(20)

respectively. In both cases,  $\varepsilon_{it} \sim \text{Logit}(0, 1)$ . For the discrete covariate case we consider the average effect based on (4) with  $(x_1, x_2) = (1, 0)$ . The analysis for the continuous covariate case focuses on the average effect based on (5). To focus solely on the difference between the bounds and the identified set, we set  $\beta = \beta_0$  (a simulation analysis for obtaining bounds when  $\beta$  is estimated will be provided in Section 6).

Results are presented in Figure 2, where the reported outer bounds are the aver-

<sup>&</sup>lt;sup>12</sup>Similar to Chernozhukov, Fernández-Val, Hahn and Newey (2013) we obtain the sharp identified set by solving an appropriate linear program.



Figure 2: Comparison of the outer bounds and the identified set for the static logit model  $Y_{it} = 1 \{X_{it}\beta + A_i \ge \varepsilon_{it}\}$ , where  $\varepsilon_{it} \sim \text{Logit}(0, 1)$  and  $A_i \sim N(0, 1)$ . Results for each  $\beta_0 \in [-2, 2]$  are based on 1000 replications of panels with cross-section size n = 1000. Reported outer bounds are the cross-replication averages. Left panel: single discrete covariate  $X_{it} = 1 \{A_i \ge \eta_{it}\}$  where  $\eta_{it} \sim N(0, 1)$ . The average effect of interest is based on (4) with  $(x_1, x_2) = (1, 0)$ . Right panel: single continuous covariate,  $X_{it} \sim N(A_i, 1)$ . The average effect of interest is based on (5).

ages of the estimated bounds across 1000 replications of panels with n = 1000.<sup>13</sup> The identified set and the outer bounds are obtained for  $\beta_0 \in [-2, 2]$ . The support of  $A_i$ is approximated by a grid of 100 equidistant points between -5 and 5. Several observations are in order. First, in all cases, the outer bounds mimic the behaviour of the identified set. In particular, both the identified set and our bounds shrink to a point when  $\beta_0 = 0$  but become wider as  $|\beta_0|$  increases. At T = 5 both the bounds and the identified set become almost a point for the majority of  $\beta_0$  we consider. Also, both types of bounds yield the correct sign for the average effect. Second, the difference between the identified set and the outer bounds vanishes almost completely at T = 5. This is an important result: as mentioned previously, obtaining the identified set in applications with moderate T is practically infeasible due to the large number of conditional probabilities P(Y = y | Z = z) one has to estimate, even when Z is discrete. Our results show that the method proposed here stands out as a viable and computationally feasible alternative in such cases.

The random coefficient example is based on the DGP

$$Y_{it} = 1 \{ X_{it} A_{2,i} + A_{1,i} \ge \varepsilon_{it} \}, \quad A_{1,i} \sim N(0, 1/\sqrt{2}), \quad A_{2,i} \sim N(A_2, 1/\sqrt{2}), \quad (21)$$

$$X_{it} = 1 \{ A_{1,i} \ge \eta_{it} \}, \quad \eta_{it} \sim N(0,1),$$
(22)

where  $\varepsilon_{it} \sim \text{Logit}(0, 1)$ . Our interest is in identifying the average effect based on (6). We note that Theorem 1 fully applies here, as there are no structural parameters to be estimated.

Results are based on 1000 replications, and are presented in Figure 3. We consider n = 1000 and  $T \in \{3, 5, 8, 10\}$  with  $A_2 \in [-2, 2]$ . To approximate the supports of  $A_{i,1}$  and  $A_{2,i}$  we use grids of 50 equidistant points between -5/5 and -7/7, respectively. Not surprisingly, the presence of a random coefficient renders the average effect more difficult to identify. Indeed, for small T even the sign of the average effect remains inconclusive for values of  $A_2$  close to zero. More importantly, although the identified set becomes narrower as T increases, it does not shrink to a point even when T is 8 or 10. For reasons discussed before, obtaining the identified set for such large T will in practice be infeasible. Simulation results confirm that our proposed method provides a reliable alternative. Indeed, the outer bounds are quite close to the identified set at T = 8, 10.

<sup>&</sup>lt;sup>13</sup>Since we are averaging over a large number of replications, the bounds reported in Figure 2 are essentially equal to the population outer bounds  $\mathbb{E}[L(Z_i, Y_i, \beta_0)]$  and  $\mathbb{E}[U(Z_i, Y_i, \beta_0)]$ , which justifies the comparison to the identified set. The same comment applies to the comparisons made in Figures 3-5.



Figure 3: Comparison of the outer bounds and the identified set for the random coefficient logit model  $Y_{it} = 1 \{X_{it}A_{2,i} + A_{1,i} \ge \varepsilon_{it}\}$ , where  $\varepsilon_{it} \sim \text{Logit}(0,1)$ ,  $A_{1,i} \sim N(0,1/\sqrt{2})$ ,  $A_{2,i} \sim N(A_2, 1/\sqrt{2})$ ,  $X_{it} = 1 \{A_{1,i} \ge \eta_{it}\}$  and  $\eta_{it} \sim N(0,1)$ . The average effect of interest is based on (6). Results for each  $A_2 \in [-2,2]$  are based on 1000 replications of panels with cross-section size n = 1000. Reported outer bounds are the cross-replication averages.

We next focus on the dynamic logit model with a continuous covariate. The DGP is

$$Y_{it} = 1 \{ Y_{i,t-1}\gamma + X_{it}\beta + A_i \ge \varepsilon_{it} \} \quad \text{for } t = 1, \dots, T,$$
$$Y_{i0} = 1 \{ X_{i0}\beta + A_i \ge \varepsilon_{i0} \}, \quad X_{it} \sim N(A_i, 1), \quad A_i \sim N(0, 1), \quad \varepsilon_{it} \sim \text{Logit}(0, 1),$$

and we consider the average effect

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[m(X_{it}, A_i, \gamma, \beta)|X_{it}\right]\right],\tag{23}$$



Figure 4: Comparison of the outer bounds and the identified set for the dynamic logit model  $Y_{it} = 1 \{Y_{i,t-1}\gamma + X_{it}\beta + A_i \ge \varepsilon_{it}\}$ , where  $\varepsilon_{it} \sim \text{Logit}(0,1)$ ,  $A_i \sim N(0,1)$ ,  $X_{it} \sim N(A_i,1)$ , and  $Y_{i0} = 1\{X_{i0}\beta + A_i \ge \varepsilon_{i0}\}$ . Results for each  $\beta_0 \in [-2,2]$  are based on 1000 replications of panels with cross-section size n = 1000. Reported outer bounds are the cross-replication averages. The average effect of interest is based on (23).

where

$$m(X_{it}, A_i, \gamma, \beta) = P(Y_{it} = 1 | Y_{i,t-1} = 1, X_{it}, A_i, \gamma, \beta)$$
$$- P(Y_{it} = 1 | Y_{i,t-1} = 0, X_{it}, A_i, \gamma, \beta).$$

This is an interesting case, since Aguirregabiria and Carro (2021) have shown that in a dynamic logit model with a single covariate, the average effect in (23) will under certain conditions be point-identified. The comparison in this part is then that between the outer bounds and the *point-identified* average effect. We investigate the behaviour of the outer bounds in this case in panels of size n = 1000 and  $T \in \{4, 6, 8\}$  with  $\beta = 1$  and  $\gamma \in [-2, 2]$ . The support of  $A_i$  is approximated by a grid of 50 equidistant points between -5 and 5. The results, presented in Figure 4, are based on 1000 replications and confirm that the outer bounds nearly point-identify the average effect, unless when  $\gamma$  is large; however this issue tends to disappear as T increases. This is not surprising since under a large  $\gamma$ , the term  $\gamma Y_{i,t-1}$  will act similar to a fixed effect.

Finally, we consider the random coefficient dynamic logit model given by

$$Y_{it} = 1 \{ Y_{i,t-1}A_{2,i} + A_{1,i} \ge \varepsilon_{it} \} \text{ for } t = 1, \dots, T,$$
  
$$Y_{i0} = 1 \{ A_{1,i} \ge \varepsilon_{i0} \}, \quad A_{1,i} \sim N(0, 1/\sqrt{2}) \quad A_{2,i} \sim N(A_2, 1/\sqrt{2}), \quad \varepsilon_{it} \sim \text{Logit}(0, 1).$$



Figure 5: Comparison of the outer bounds and the identified set for the random coefficient dynamic logit model  $Y_{it} = 1 \{Y_{i,t-1}A_{2,i} + A_{1,i} \ge \varepsilon_{it}\}$ , where  $\varepsilon_{it} \sim \text{Logit}(0,1)$ ,  $A_{1,i} \sim N(0,1/\sqrt{2})$ ,  $A_{2,i} \sim N(A_2, 1/\sqrt{2})$ , and  $Y_{i0} = 1 \{A_{1,i} \ge \varepsilon_{i0}\}$ . The average effect of interest is based on (24). Results for each  $A_2 \in [-2, 2]$  are based on 1000 replications of panels with cross-section size n = 1000. Reported outer bounds are the cross-replication averages.

For this exercise, we focus on the average effect

$$\mathbb{E}\left[P(Y_{it}=1|Y_{i,t-1}=1,A_{1,i},A_{2,i})-P(Y_{it}=1|Y_{i,t-1}=0,A_{1,i},A_{2,i})\right].$$
(24)

We consider 1000 replications where n = 1000 and  $T \in \{4, 6, 8, 10\}$ , and vary  $A_2$  between -2 and 2. As in the static logit variant of this model, the supports of  $A_{i,1}$  and  $A_{2,i}$  are approximated by grids of 50 equidistant points between -5/5 and -7/7, respectively. Figure 5 reveals that the identified set can be quite wide. This is in line with the earlier observations for the random coefficient static logit model. However, the

identified set becomes wider as  $A_2$  increases. This is similar to the asymmetry observed in the dynamic logit case. When  $A_2$  is large,  $Y_i$  is more likely to be a vector of 1s. Hence, again, the effect of  $A_{i,2}Y_{i,t-1}$  is hard to distinguish from that of  $A_{i,1}$ . In results not reported here, we observed that the outer bounds do not perform well when the uniform linear program (17) is used. This highlights that in certain cases the uniform linear program can yield too conservative bounds. We therefore used the baseline linear program which utilizes (15) in obtaining the bounds reported in Figure 5. The resulting outer bounds perform well in tracking the identified set as T increases.

# 5 Accounting for estimated common parameters

We now consider the case where the common parameter vector  $\beta_0$  has to be estimated. Our construction of the bound functions  $L(z, y, \beta)$  and  $U(z, y, \beta)$  remains essentially unchanged in that case, but they are now evaluated at some consistent estimator of  $\beta_0$ , rather than the true  $\beta_0$ . The goal of this section is to provide asymptotic results that account for the noise in the estimation of  $\beta_0$ .

If the bound functions  $L(z, y, \beta)$  and  $U(z, y, \beta)$  were differentiable in  $\beta$ , then accounting for the estimation of  $\beta_0$  when providing one-sided confidence intervals on the bounds  $\mathbb{E}[L(Z_i, Y_i, \beta_0)]$  and  $\mathbb{E}[U(Z_i, Y_i, \beta_0)]$  would be a straightforward application of the delta method. Unfortunately, because we obtain  $L(z, y, \beta)$  and  $U(z, y, \beta)$  as the solution to a linear program, it is generally not possible to verify any smoothness (or even uniqueness or continuity) of those functions in  $\beta$ . The convergence rate and inference results in this section therefore make no assumption whatsoever on the continuity or smoothness of the bound functions.<sup>14</sup>

### 5.1 Consistency and convergence rate of the estimated bounds

Before discussing inference on  $\overline{m}$ , our first goal is to show that the population bounds  $\mathbb{E}[L(Z_i, Y_i, \beta_0)]$  and  $\mathbb{E}[U(Z_i, Y_i, \beta_0)]$  can be estimated at  $\sqrt{n}$  rate, even if  $\beta_0$  is estimated. For that purpose, we split the set of observations  $\{1, \ldots, n\}$  into the disjoint subsets  $\mathcal{I}_1 = \{1, \ldots, \lfloor n/2 \rfloor\}$  and  $\mathcal{I}_2 = \{\lfloor n/2 \rfloor + 1, \ldots, n\}$ . For any subset of observed units

<sup>&</sup>lt;sup>14</sup>One could, alternatively, construct  $L(z, y, \beta)$  and  $U(z, y, \beta)$  such that they still satisfy the assumptions of Theorem 1, but are also smooth in  $\beta$  (e.g. in a particular model for a particular average effect of interest, one may simply find explicit analytic expressions for the bound functions). We leave the exploration of such possibilities to future work.

 $\mathcal{I} \subset \{1, \ldots, n\}$  we denote by  $Y_{(\mathcal{I})}$  and  $Z_{(\mathcal{I})}$  the collection of all observations  $Y_i$  and  $Z_i$  with  $i \in \mathcal{I}$ . Furthermore, we define the function  $\bar{s} : \{1, \ldots, n\} \to \{1, 2\}$  by

$$\bar{s}(i) := \begin{cases} 2 & \text{if } i \in \mathcal{I}_1, \\ 1 & \text{if } i \in \mathcal{I}_2. \end{cases}$$

For each  $s \in \{1, 2\}$  we have an estimator  $\widehat{\beta}_s = \widehat{\beta}_s(Y_{(\mathcal{I}_s)}, Z_{(\mathcal{I}_s)})$  that only depends on the observed data  $(Y_i, Z_i)$  for  $i \in \mathcal{I}_s$ . In other words,  $\widehat{\beta}_1$  and  $\widehat{\beta}_2$  are estimators of  $\beta$  obtained using the first and second half-sample, respectively. Our estimates for the upper and lower bounds in (9) then generalize to

$$\widehat{L}_S := \frac{1}{n} \sum_{i=1}^n L\left(Z_i, Y_i, \widehat{\beta}_{\overline{s}(i)}\right), \qquad \qquad \widehat{U}_S := \frac{1}{n} \sum_{i=1}^n U\left(Z_i, Y_i, \widehat{\beta}_{\overline{s}(i)}\right).$$
(25)

Notice that the "cross-fitting" construction in (25) ensures that for any i,  $(Z_i, Y_i)$  and  $\widehat{\beta}_{\bar{s}(i)}$  are always from two different half-samples, and therefore independent of each other. Consequently, conditional on the half-sample  $\mathcal{I}_{\bar{s}(i)}$ ,  $L(Z_i, Y_i, \widehat{\beta}_{\bar{s}(i)})$  and  $U(Z_i, Y_i, \widehat{\beta}_{\bar{s}(i)})$  are independently distributed over i. In contrast, if the bound estimators were based on  $\widehat{\beta}$  obtained from the full-sample,  $L(Z_i, Y_i, \widehat{\beta})$  and  $U(Z_i, Y_i, \widehat{\beta})$  would be arbitrarily dependent over i, ruling out a standard Law of Large Numbers. Along with reasonable assumptions on the behavior of  $\widehat{\beta}_{\bar{s}(i)}$ , as well as smoothness conditions on the functions  $f(y|z, a; \beta)$  and  $m(z, a, \beta)$  in  $\beta$ , the conditional independence is sufficient for proving the consistency of the bounds in (25) for  $\mathbb{E}[L(Z_i, Y_i, \beta_0)]$  and  $\mathbb{E}[U(Z_i, Y_i, \beta_0)]$ .

#### Assumption 2

- (i) For  $s \in \{1, 2\}$  the estimator  $\widehat{\beta}_s = \widehat{\beta}_s(Y_{(\mathcal{I}_s)}, Z_{(\mathcal{I}_s)})$  satisfies  $\widehat{\beta}_s = \beta_0 + O_p(n^{-1/2})$ .
- (ii) There exists  $\epsilon > 0$  such that for an  $\epsilon$ -ball  $B_{\epsilon}(\beta_0)$  around  $\beta_0$  we have

$$\sup_{\beta \in B_{\epsilon}(\beta_0)} \sum_{y \in \mathcal{Y}} \mathbb{E} \left\| \frac{\partial f\left(y \mid Z_i, A_i; \beta\right)}{\partial \beta} \right\| < \infty, \qquad \sup_{\beta \in B_{\epsilon}(\beta_0)} \mathbb{E} \left\| \frac{\partial m\left(Z_i, A_i, \beta\right)}{\partial \beta} \right\| < \infty.$$

**Theorem 2** Let Assumptions 1 and 2 hold, and let  $L, U : \mathbb{Z} \times \mathcal{Y} \times \mathcal{B} \rightarrow [b_{\min}, b_{\max}]$  be two non-random functions that satisfy (7) for all  $z \in \mathbb{Z}$ ,  $a \in \mathcal{A}$  and  $\beta \in \mathcal{B}$ . Let, finally,  $\overline{m}$  be as defined in (3), and  $\hat{L}_S$  and  $\hat{U}_S$  be as defined in (25). Then, as  $n \to \infty$ , we have

$$\widehat{L}_S + O_p(n^{-1/2}) \leq \overline{m} \leq \widehat{U}_S + O_p(n^{-1/2}).$$

This theorem generalizes consistency of the outer bounds to the case of estimated  $\beta_0$ . The proof is straightforward and provided in the appendix. By contrast, obtaining inference results under estimated  $\beta_0$  is more complicated due to the linear program yielding potentially non-smooth bound functions. This non-smoothness is not specific to our case. The bottom line is that one cannot simply deploy the delta method to account for randomness introduced by the estimation of  $\beta_0$ , and so a different approach is needed. In the remainder of this section, we introduce two inference methods.

#### 5.2 First inference method

Our first inference method is inspired by the handling of common parameters in the "perturbed bootstrap" approach of Chernozhukov, Fernández-Val, Hahn and Newey (2013). The idea is to simply take the union of our "known  $\beta_0$ " confidence intervals in Section 2 over a confidence set of the unknown  $\beta_0$ . For that purpose, define

$$\widehat{L}(\beta) := \frac{1}{n} \sum_{i=1}^{n} L(Z_i, Y_i, \beta), \qquad \qquad \widehat{U}(\beta) := \frac{1}{n} \sum_{i=1}^{n} U(Z_i, Y_i, \beta),$$
$$\widehat{\sigma}_L^2(\beta) := \frac{1}{n} \sum_{i=1}^{n} \left[ L(Z_i, Y_i, \beta) - \widehat{L}(\beta) \right]^2, \qquad \widehat{\sigma}_U^2(\beta) := \frac{1}{n} \sum_{i=1}^{n} \left[ U(Z_i, Y_i, \beta) - \widehat{U}(\beta) \right]^2.$$

We then have the following theorem.

**Theorem 3** Let, for some  $0 < \gamma < 1$ ,  $\mathcal{B}_{1-\gamma}$  be such that  $\lim_{n\to\infty} P(\beta_0 \in \mathcal{B}_{1-\gamma}) \ge 1-\gamma$ . Then,

$$\lim_{n \to \infty} P\left\{ \inf_{\beta \in \mathcal{B}_{1-\gamma}} \left[ \widehat{L}(\beta) - \frac{c_{\alpha/2} \,\widehat{\sigma}_L(\beta)}{\sqrt{n}} \right] \le \overline{m} \le \sup_{\beta \in \mathcal{B}_{1-\gamma}} \left[ \widehat{U}(\beta) + \frac{c_{\alpha/2} \,\widehat{\sigma}_U(\beta)}{\sqrt{n}} \right] \right\} \ge 1 - \alpha - \gamma,$$
  
where  $c_{\alpha/2} = \Phi^{-1} \left( 1 - \frac{\alpha}{2} \right).$ 

Theorem 3 provides a straightforward albeit potentially conservative way of obtaining confidence bands that incorporate the uncertainty due to estimation of  $\beta_0$ . This uncertainty is captured by  $\gamma$  whereas  $\alpha$  parameterizes the uncertainty due to estimation of the population outer bounds by sample averages. For a desired level of confidence 1 - c, one can trade off between these two sources of uncertainty by choosing  $\alpha$  and  $\gamma$ as desired. Another option is to find the narrowest confidence interval across all  $(\alpha, \gamma)$ such that  $c = \alpha + \gamma$ . Notice that the infimum and supremum cannot be calculated exactly, so one has to do a grid search across a sufficiently large selection of  $\beta \in \mathcal{B}_{1-\gamma}$ . Especially when  $\beta$  contains several parameters, this method can be demanding. Nevertheless, the attraction of Theorem 3 is that as long as a valid confidence interval for  $\beta_0$  can be constructed, inference on  $\overline{m}$  requires only a straightforward application of the methods described in Section 2. Fortunately, there is a large literature on obtaining valid confidence intervals on the common parameters  $\beta_0$  in the type of panel data model with fixed effects that we consider here; see, for example, Arellano (2003) and Arellano and Bonhomme (2011) for reviews, as well as our discussion in the introduction.

Interestingly, the confidence set for  $\beta_0$  in Theorem 3 can in principle also accommodate cases where  $\beta_0$  is not point-identified, as long as a valid confidence set  $\mathcal{B}_{1-\gamma}$  can be constructed. We leave the exploration of this idea to future work.

#### 5.3 Second inference method

As mentioned before, evaluating the infimum and supremum over  $\beta \in \mathcal{B}_{1-\gamma}$  in Theorem 3 can be challenging. As an alternative inference method, we therefore suggest to modify the linear program that is used to calculate the upper and lower bounds for  $\overline{m}$  such that the uncertainty about  $\beta_0$  is accounted for within the constraints of the linear program.

In Section 2, the crucial requirement on our bound functions  $L(z, y, \beta)$  and  $U(z, y, \beta)$ was that they satisfy the inequalities in (7) for a fixed value  $\beta$ . To account for the fact that the true  $\beta_0$  is unknown, we now slightly generalize this idea. Given a *finite* set  $\mathcal{B}_{sub} \subset \mathcal{B}$  of possible values for  $\beta$ , we demand that the bound functions  $L(z, y, \mathcal{B}_{sub})$  and  $U(z, y, \mathcal{B}_{sub})$  satisfy the inequalities in (7) for each value  $\beta \in \mathcal{B}_{sub}$ , that is, we demand

$$\forall \beta \in \mathcal{B}_{\text{sub}} : \sum_{y \in \mathcal{Y}} L(z, y, \mathcal{B}_{\text{sub}}) f(y \mid z, a; \beta) \le m(z, a, \beta) \le \sum_{y \in \mathcal{Y}} U(z, y, \mathcal{B}_{\text{sub}}) f(y \mid z, a; \beta).$$
(26)

As in (7), we want the inequality in (26) to hold for all  $z \in \mathbb{Z}$  and  $a \in \mathcal{A}$ .<sup>15</sup>

Next, for each half-sample  $s \in \{1, 2\}$ , let  $\widehat{\mathcal{B}}_s$  be a set of points estimated only from observations  $i \in \mathcal{I}_s$ , such that the convex hull of  $\widehat{\mathcal{B}}_s$ ,  $\operatorname{Conv}(\widehat{\mathcal{B}}_s)$ , provides a  $1 - \gamma/2$ confidence set for  $\beta_0$ . For example, for a one-dimensional parameter  $\beta$ , we choose  $\widehat{\mathcal{B}}_s =$  $\{\widehat{\beta}_{\mathrm{low},s}, \widehat{\beta}_{\mathrm{up},s}\}$  to consist of the lower and upper bounds of a confidence interval for  $\beta_0$ . Then,  $\operatorname{Conv}(\widehat{\mathcal{B}}_s) = [\widehat{\beta}_{\mathrm{low},s}, \widehat{\beta}_{\mathrm{up},s}]$  is just a standard confidence interval in that case. More

<sup>&</sup>lt;sup>15</sup>For consistency of notation, our previous bounds  $L(z, y, \beta)$  and  $U(z, y, \beta)$  could have been written as  $L(z, y, \{\beta\})$  and  $U(z, y, \{\beta\})$  to agree with (26), but this is a minor mismatch of notation.

generally, we have to find a confidence set that can be generated as a convex hull of a finite number of points.<sup>16</sup> Let also diam ( $\mathcal{B}_{sub}$ ) be the diameter of the set  $\mathcal{B}_{sub}$ . Finally, we define

$$\widehat{L}_C := \frac{1}{n} \sum_{i=1}^n L\left(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}\right), \qquad \qquad \widehat{U}_C := \frac{1}{n} \sum_{i=1}^n U\left(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}\right).$$
(27)

We require the following additional assumptions for this inference method, which strengthen Assumption 2(ii) and also formalize the requirement that  $\text{Conv}(\widehat{\mathcal{B}}_s)$  is a confidence interval.

#### Assumption 3

- (i)  $\mathbb{E}[\operatorname{diam}(\widehat{\mathcal{B}}_s)]^2 = o(n^{-1/2})$  and  $\lim_{n \to \infty} P\{\beta_0 \in \operatorname{Conv}(\widehat{\mathcal{B}}_s)\} \ge 1 \gamma/2$  where  $s \in \{1, 2\}$ .
- (ii) There exists  $\epsilon > 0$  such that for an  $\epsilon$ -ball  $B_{\epsilon}(\beta_0)$  around  $\beta_0$  we have

$$\sup_{\beta \in B_{\epsilon}(\beta_0)} \sum_{y \in \mathcal{Y}} \mathbb{E} \left\| \frac{\partial^2 f\left(y \mid Z_i, A_i; \beta\right)}{\partial \beta^2} \right\| < \infty, \quad \sup_{\beta \in B_{\epsilon}(\beta_0)} \mathbb{E} \left\| \frac{\partial^2 m\left(Z_i, A_i, \beta\right)}{\partial \beta^2} \right\| < \infty.$$

Then, the following lemma shows that conditional on  $\beta_0 \in \widehat{\mathcal{B}}_{\bar{s}}$ ,  $L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}})$  and  $U(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}})$  provide valid bounds on  $\overline{m}$  in expectation.

**Lemma 1** Let Assumptions 1 and 3 hold. Let  $L(\cdot, \cdot, \cdot)$  and  $U(\cdot, \cdot, \cdot)$  satisfy (26) for all  $z \in \mathbb{Z}$  and  $a \in \mathcal{A}$ , and let  $\widehat{\mathcal{B}}_{\overline{s}}$  be as defined after display (26). Let  $\mathcal{B}_{sub} \subset \mathcal{B}$  be such that  $\beta_0 \in \text{Conv}(\mathcal{B}_{sub})$ . Then, for sufficiently large n, we have

$$\mathbb{E}\left[L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}}) \mid \widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{sub}\right] + O\left([\operatorname{diam}(\mathcal{B}_{sub})]^2\right) \leq \overline{m}$$
$$\leq \mathbb{E}\left[U(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}}) \mid \widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{sub}\right] + O\left([\operatorname{diam}(\mathcal{B}_{sub})]^2\right).$$

Once Lemma 1 is obtained, then all that is left to do is to account for the sampling uncertainty when replacing the expected value over  $L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}})$  and  $U(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}})$  by the sample averages in (27), analogously to Theorem 1.

<sup>&</sup>lt;sup>16</sup>If  $\beta$  is higher-dimensional, then one simple choice for  $\hat{\mathcal{B}}_s$  would be the Cartesian product of onedimensional confidence bounds for each component of  $\beta$ , using a Bonferroni correction to maintain the correct confidence level  $1 - \gamma/2$ . Less conservative (though possibly more complicated) constructions that might allow for a lower cardinality of  $\hat{\mathcal{B}}_s$  are also possible.

**Theorem 4** Let Assumptions 1 and 3 hold. For  $s \in \{1, 2\}$  and  $\overline{s} = s - 3$ , let  $L(\cdot, \cdot, \cdot)$ and  $U(\cdot, \cdot, \cdot)$  satisfy (26) for all  $z \in \mathbb{Z}$  and  $a \in \mathcal{A}$ , and be such that  $b_{\min} \leq L(z, y, \widehat{\mathcal{B}}_{\overline{s}}) \leq b_{\max}$  and  $b_{\min} \leq U(z, y, \widehat{\mathcal{B}}_{\overline{s}}) \leq b_{\max}$ . Assume further that  $\operatorname{Var} [L(Z_i, Y_i, \beta)] > 0$  and  $\operatorname{Var} [U(Z_i, Y_i, \beta)] > 0$  for all  $\beta$  in some neighborhood around  $\beta_0$ . Let  $\overline{m}$ ,  $\widehat{L}_C$ ,  $\widehat{U}_C$  be as defined in (3) and (27), let  $\widehat{\sigma}_{L,s}$  and  $\widehat{\sigma}_{U,s}$  be the sample standard deviations over  $i \in \mathcal{I}_s$ of  $L(Z_i, Y_i, \widehat{\mathcal{B}}_{\overline{s}})$  and  $U(Z_i, Y_i, \widehat{\mathcal{B}}_{\overline{s}})$ , respectively.<sup>17</sup> Let  $\alpha \in [0, 1]$ . Then, as  $n \to \infty$  we have:

$$\lim_{n \to \infty} P\left(\widehat{L}_C - \frac{c_{\alpha/4}\left(\widehat{\sigma}_{L,1} + \widehat{\sigma}_{L,2}\right)/2}{\sqrt{n/2}} \le \overline{m} \le \widehat{U}_C + \frac{c_{\alpha/4}\left(\widehat{\sigma}_{U,1} + \widehat{\sigma}_{U,2}\right)/2}{\sqrt{n/2}}\right) \ge 1 - \alpha - \gamma,$$
  
with  $c_{\alpha/4} = \Phi^{-1}\left(1 - \frac{\alpha}{4}\right).$ 

Theorem 1 demands that equation (7) holds, but does not specify any explicit construction of the bound functions. Analogously, Theorem 4 requires that equation (26) hold, but again does not specify any explicit construction of the bounds. In order to actually construct the bounds we use the methods described earlier, but we replace the constraint (7) by (26). Specifically, the program in display (14) then gets modifies as follows: For any given  $z \in \mathbb{Z}$  and any finite set  $\mathcal{B}_{sub} \subset \mathcal{B}$  with  $\overline{\beta} = |\mathcal{B}_{sub}|^{-1} \sum_{\beta \in \mathcal{B}_{sub}} \beta$ we can choose  $L(z, y, \mathcal{B}_{sub}) = \ell(y)$  and  $U(z, y, \mathcal{B}_{sub}) = u(y)$  as solutions to the following optimization problem:

$$\min_{\ell,u:\mathcal{Y}\to\mathbb{R}} Q(\ell(\cdot), u(\cdot), z, \overline{\beta})$$
subject to
$$\forall y \in \mathcal{Y}: \ b_{\min} \leq \ell(y) \leq u(y) \leq b_{\max}$$
and
$$\forall \beta \in \mathcal{B}_{\text{sub}}: \ \forall a \in \mathcal{A}: \ \sum_{y \in \mathcal{Y}} \ell(y) \ f(y \mid z, a; \beta) \leq m(z, a, \beta) \leq \sum_{y \in \mathcal{Y}} u(y) f(y \mid z, a; \beta).$$
(28)

Useful choices for the objective function  $Q(\ell(\cdot), u(\cdot), z, \overline{\beta})$  are already described in Section 4.1. For example, by choosing  $Q(\ell(\cdot), u(\cdot), z, \overline{\beta})$  as in (15) we again have to solve a linear program to obtain the bounds.

 $<sup>\</sup>frac{1}{1^{7} \text{Formally, letting } \widehat{L}_{C,s} = (2/n) \sum_{i \in \mathcal{I}_{s}} L(Z_{i}, Y_{i}, \widehat{\mathcal{B}}_{\bar{s}(i)}) \text{ and } \widehat{U}_{C,s} = (2/n) \sum_{i \in \mathcal{I}_{s}} U(Z_{i}, Y_{i}, \widehat{\mathcal{B}}_{\bar{s}(i)}) \text{ we have } \widehat{\sigma}_{L,s}^{2} := (2/n) \sum_{i \in \mathcal{I}_{s}} [L(Z_{i}, Y_{i}, \widehat{\mathcal{B}}_{\bar{s}(i)}) - \widehat{L}_{C,s}]^{2} \text{ and } \widehat{\sigma}_{U,s}^{2} := (2/n) \sum_{i \in \mathcal{I}_{s}} [U(Z_{i}, Y_{i}, \widehat{\mathcal{B}}_{\bar{s}(i)}) - \widehat{U}_{C,s}]^{2}.$ 

### 6 Simulation evidence

In this part we investigate the small sample behavior of the proposed bounds and confidence bands. We focus on the static logit and random coefficient logit models. The setting largely follows Section 4.3. In particular, we use the DGPs in (19) and (21)-(22) with a single discrete covariate, and focus on the same average effects. The main difference is that we now estimate  $\beta_0$  in the static logit model, and also provide confidence bands. The results, presented in Figures 6-8, provide the population average effect, and the cross-replication averages of estimated bounds and 95% confidence bands.

We first consider the static logit model.  $\beta_0$  is estimated using the conditional likelihood method. For inference we use the two inference methods proposed in Sections 5.2 and 5.3. In either case, we consider 1000 replications of panels of size n = 5000 and  $T \in \{3, 5, 8\}$ . For  $\mathcal{A}_q$ , we use a grid of 100 equidistant points between -5 and 5.

The results using the inference method of Section 5.2 are based on  $\gamma = 0.0001$  and  $\mathcal{B}_{1-\gamma}$  is approximated by a grid of 5000 equidistant points on  $\mathcal{B}_{1-\gamma}$ . Outer bounds for this case are obtained by the uniform linear program of Section 4.1.2. Results are presented in Figure 6. For moderate T, which is the main focus of this study, both the bounds and the confidence bands are quite tight. Interestingly, this is despite the fact that the bounds are based on a uniform linear program. In all cases, the confidence bands are, not surprisingly, conservative, with some improvement as T increases. However, this is acceptable in view of the bands being quite tight. While we have not tried this option, coverage rates may be improved by using the method of Imbens and Manski (2004).

We next consider the inference approach of Section 5.3, the results of which are presented in Figure 7. Confidence bands are based on  $\alpha = \frac{2}{3} \times 0.05$  and  $\gamma = \frac{1}{3} \times 0.05$ .<sup>18</sup> Both the confidence bands and the outer bounds are based on the linear program defined in (28). Relative to the inference method of Section 5.2, there are two differences: first, the confidence bands are overall quite tight around the estimated bounds, across all T. This is not surprising given that the inference method of Section 5.2 is based on the infemum/supremum bands. Second, while the outer bounds improve with T, they are not as tight as the bounds produced by the linear program in (14). This probably results from (28) incorporating the uncertainty due to  $\hat{\beta}$  in outer bound estimation (as opposed

<sup>&</sup>lt;sup>18</sup>The choice of  $\alpha = 2\gamma$  is not crucial and was only imposed to compensate for the fact that the confidence interval  $\text{Conv}(\widehat{\mathcal{B}}_s)$  is subject to one Bonferroni split, whereas the interval for estimated outer bounds is subject to two.



Figure 6: Simulation results for the static logit model with a single discrete covariate:  $Y_{it} = 1 \{X_{it}\beta + A_i \geq \varepsilon_{it}\}$  where  $\varepsilon_{it} \sim \text{Logit}(0, 1), A_i \sim N(0, 1), X_{it} = 1 \{A_i \geq \eta_{it}\}$  and  $\eta_{it} \sim N(0, 1)$ . Average effects are based on (4) with  $(x_1, x_2) = (1, 0)$ . Results for each  $\beta_0 \in [-2, 2]$  are based on 1000 replications of panels with cross-section size n = 5000. For each replication,  $\hat{L}$  and  $\hat{U}$  are obtained by the linear program of Section 4.1.2, using the conditional likelihood estimator  $\hat{\beta}$  of  $\beta_0$ . Confidence intervals are based on the inference method of Section 5.2, using  $\gamma = 0.0001$ .  $\mathcal{B}_{1-\gamma}$  is approximated by a grid of 5000 equidistant points. Reported confidence intervals and  $(\hat{L}, \hat{U})$  are cross-replication averages. The lower right panel presents the coverage rates.

to (14) which incorporates the same in the inference stage). This is also possibly why the coverage rate of this second approach is almost always 100%.

We move to the random coefficient static logit example. Figure 8 presents results based on 1000 replications of panels of size n = 1000 and  $T \in \{3, 5, 10\}$ . We construct



Figure 7: Simulation results for the static logit model with a single discrete covariate:  $Y_{it} = 1 \{X_{it}\beta + A_i \geq \varepsilon_{it}\}$  where  $\varepsilon_{it} \sim \text{Logit}(0, 1), A_i \sim N(0, 1), X_{it} = 1 \{A_i \geq \eta_{it}\}$  and  $\eta_{it} \sim N(0, 1)$ . Average effects are based on (4) with  $(x_1, x_2) = (1, 0)$ . Results for each  $\beta_0 \in [-2, 2]$  are based on 1000 replications of panels with cross-section size n = 5000. For each replication, outer bounds and confidence bands are obtained by the methods outlined in Section 5.3, using the conditional likelihood estimator  $\hat{\beta}$  of  $\beta_0$ . Confidence intervals are based on  $\alpha = \frac{2}{3} \times 0.05$  and  $\gamma = \frac{1}{3} \times 0.05$ . Reported confidence intervals and  $(\hat{L}, \hat{U})$  are cross-replication averages. The lower right panel presents the coverage rates.



Figure 8: Simulation results for the random coefficient logit model with a single discrete covariate:  $Y_{it} = 1 \{X_{it}A_{2,i} + A_{1,i} \ge \varepsilon_{it}\}$ , where  $\varepsilon_{it} \sim \text{Logit}(0,1)$ ,  $A_{1,i} \sim N(0,1/\sqrt{2})$ ,  $A_{2,i} \sim N(A_2, 1/\sqrt{2})$ ,  $X_{it} = 1 \{A_{1,i} \ge \eta_{it}\}$  and  $\eta_{it} \sim N(0,1)$ . Average effects are based on (6). Results for each  $A_2 \in [-2,2]$  are based on 1000 replications of panels with cross-section size n = 1000. For each replication,  $\hat{L}$  and  $\hat{U}$  are obtained by the linear program in (17) Confidence intervals are based on Theorem 1. Reported confidence intervals and  $(\hat{L}, \hat{U})$  are cross-replication averages. The lower right panel presents the coverage rates.

 $\mathcal{A}_g$  using 50 equidistant grid points between -5 and 5 for  $A_{1,i}$ , and between -7 and 7 for  $A_{2,i}$ , leading to 2,500 grid points in total. We note that the average effects and outer bounds are the same as in Section 4.3, since no parameter estimation is involved in this setting. The new result is the confidence bands, which are based on Theorem 1. On average the confidence bands are reasonably tight around the outer bounds.

A general observation across the three simulation exercises is that while the confidence bands are generally tight around the bounds, the coverage rates are conservative. We however note that conservative coverage rates would not be surprising for confidence bands around the sharp identified set either (e.g., see Imbens and Manski 2004 and Stoye 2020).

### 7 Empirical Analysis

We consider an empirical analysis of female labor force participation, using the National Longitudinal Survey of Youth (NLSY) 1979 dataset. Our sample consists of data on women who were married throughout the sample and who were not in active forces or going to school.<sup>19</sup> Also, we only include individuals who were observed at all periods under consideration.

First, we consider a random coefficient logit specification:

$$LFP_{it} = 1\left\{\alpha_i + \beta_i \, kids3_{it} \ge \varepsilon_{it}\right\},\tag{29}$$

where, for individual *i* and at time *t*,  $LFP_{it}$  is the labor force participation indicator whereas  $kids_{it}$  is a binary variable which equals one if the individual has at least one child below the age of three. This is almost identical to the example considered by Chernozhukov, Fernández-Val, Hahn and Newey (2013), except that they assume a homogeneous coefficient  $\beta$  for all individuals. Our objective is to obtain a confidence interval on the average effect

$$\mathbb{E}\left[P(LFP_{it}=1|kids_{it}=1,\alpha_i,\beta_i)-P(LFP_{it}=1|kids_{it}=0,\alpha_i,\beta_i)\right].$$

Our sample period for this analysis covers all even years from 1986 to 1998, which

<sup>&</sup>lt;sup>19</sup>An individual is classified as "in the labor force" if her status was recorded as *working*, *with job not* at work or *unemployed*. Individuals are considered as not in the labor force if their recorded status was keeping house, unable to work or other.

yields data on 929 individuals over seven years. For comparison, we also report the average effects based on the fixed effects logit (FE logit) and probit (FE probit) models, as well as the linear fixed effects model. We note that all these alternatives impose homogeneity of  $\beta_i$ , and calculate the average effects using estimated ( $\alpha_i, \beta$ ). Hence, they provide a point-estimate for the average effect. We also use the bias-corrected logit (BC logit) and probit (BC probit) methods, which analytically correct  $\hat{\beta}$  for the incidental parameter bias. We note that none of these alternative methods are designed for short-T samples where average effects are not necessarily point-identified. For all methods under consideration, we provide the 95% confidence intervals. For the outer bounds this is obtained by using the normal approximation of Theorem 1.

Results for this first illustration are reported in the top panel of Table 1. All methods agree that having at least one child younger than three has a negative impact on labor force participation. This is also in line with the results obtained by Chernozhukov, Fernández-Val, Hahn and Newey (2013) who consider a shorter sample, covered by our dataset (see their Table III). The confidence intervals for the outer bounds are wider than the rest, but this is normal as it is the only method that allows for heterogeneity of  $\beta_i$ . Heterogeneity of  $\beta_i$  is quite likely, as the effect of having a child younger than three will vary depending on various conditions. For example, families with higher income will have easier (and better) access to child care. Geographical proximity of grandparents (who can, at least from time to time, provide child care) is also likely to have an effect on  $\beta_i$ . Moreover, the effect of having children younger than three may differ depending on the actual number of children. The wider confidence bands provided by our method reflect all such considerations.

In the second illustration, we consider the static logit specification with a richer set of covariates:

$$LFP_{it} = 1\left\{\alpha_i + \beta \, kids3_{it} + \gamma \, educ_{it} + \delta \, \ln(spouseinc_{it}) \ge \varepsilon_{it}\right\},\tag{30}$$

where  $educ_{it}$  is the highest completed grade (as of May 1 of the survey year) and  $spouseinc_{it}$  is the total income of the spouse from wages and salary in past calendar year. The sample for this exercise covers all even years from 1990 to 1998. We do not include individuals whose spouse had zero income at any point during this period. Average effects for the covariates  $kids_{it}$  and  $educ_{it}$  are based on (4), where we use  $(x_1, x_2) = (1, 0)$  and  $(x_1, x_2) = (educ_{it} + 1, educ_{it})$ , respectively. Average effects for log spouse income are calculated using (5) with  $x_{k,it} = \ln(spouseinc_{it})$ . The outer bounds

$LFP_{it} = 1 \{ \alpha_i + \beta_i \ kids 3_{it} \ge \varepsilon_{it} \},  n = 929,  T = 7$						
	heterogeneous $\beta_i$	$\beta_i = \beta$				
kids3	$(\widehat{L}, \widehat{U})$ 245;067 [272,044]	FE logit 122 [142,102]	BC logit 133 [153,113]	FE probit 122 [142,102]	BC probit 132 [151,112]	Linear model 124 [144,104]
$LFP_{it} = 1 \{ \alpha_i + \beta \ kids_{it} + \gamma \ educ_{it} + \delta \ \ln(spouseinc_{it}) \ge \varepsilon_{it} \},  n = 993,  T = 5$						
	$(\widehat{L},\widehat{U})$	FE logit	BC logit	FE probit	BC probit	Linear model
kids3	101;098 $_{[169,045]}$	097 $[118,076]$	120 [141,099]	096 [116,076]	116 [137,096]	095 [117,072]
educ	.018;.020 [036,.078]	.018 $[024,.060]$	.022 [020,.063]	.016 [023,.056]	$.020 \\ \left[020  , .059 \right]$	$.011 \\ \left[017  , .039 \right]$
$\ln(spouseinc)$	101;040	$057$ $_{[165,.051]}$	069 $[177,.039]$	$047$ $_{[137,.042]}$	$057$ $_{[146,.033]}$	042 [065,020]

Table 1: Empirical analysis results. For the average effects of interest in each case, see the discussion in Section 7.  $(\hat{L}, \hat{U})$  are the outer bounds. FE logit and FE probit are the fixed effects panel logit and panel probit models. BC logit and BC probit are the bias-corrected versions, which analytically correct for the incidental parameter bias in estimating  $\beta_0$ . Linear model is the linear panel fixed effects model. Numbers in brackets are the 95% confidence bands. All methods other than the outer bounds provide point estimates of the average effects. In addition, on the top panel these alternative methods impose homogeneity of  $\beta_i$ .

are obtained using the uniform linear program of Section 4.1.2 whereas the inference approach of Section 5.3 is used to generate the confidence bands.<sup>20</sup>

Results are reported in the bottom panel of Table 1. All methods agree that the average effect of *kids*3 is negative. For *educ*, all confidence bands are ambiguous about the size of the effect. However, for all methods these bands are mostly on the positive side. In addition, estimated average effects and outer bounds all point to a positive effect of *educ* on labor force participation. Finally, for log of spouse income, confidence bands by all alternatives (other than the linear model) are inconclusive about the sign of the average effect, though they mostly lie on the negative side. Interestingly, confidence bands for all methods other than the linear probability model lie partially outside the confidence intervals for the outer bounds. This is not necessarily surprising, given that none of the alternative methods considered here are designed to work in short samples.

# 8 Conclusion

In this paper, we have introduced a new method for estimating bounds on average effects in discrete choice panel data models with fixed effects, including two approaches for obtaining asymptotically valid confidence intervals on the average effects. For realistic models and sample sizes, inference based on our outer bounds is easier and more robust than inference based on the sharp identified set. A key strength of our approach is its broad applicability: it is suitable for models with both discrete and continuous covariates, and it can be adapted for a variety of static and dynamic panel models.

We have focused here on the case where the common model parameters  $\beta_0$  are pointidentified and can be estimated at the parametric rate, but our approach can, in principle, be extended to cases where structural parameters are only partially-identified.

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<sup>&</sup>lt;sup>20</sup>We first obtain the confidence bands across a selection of  $\alpha$  and  $\gamma$  such that  $\alpha + \gamma = 0.05$ , and then report the shortest confidence interval among these.

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# A Appendix with Proofs

**Proof of Theorem 1.**  $\underline{\#}$  Part (i): Define  $U_i = U(Z_i, Y_i, \beta_0)$  and  $\overline{U} = \mathbb{E}(U_i)$ . We have  $Var(U_i) < \infty$  since  $U_i$  is, by design, uniformly bounded. Then, by Assumption 1 and Chebychev's inequality, for any  $\varepsilon > 0$  we have

$$P\left\{ |\widehat{U} - \overline{U}| \ge \varepsilon \right\} = P\left\{ \left[ \frac{1}{n} \sum_{i=1}^{n} \left( U_i - \overline{U} \right) \right]^2 \ge \varepsilon^2 \right\}$$
$$\le \frac{1}{n^2 \epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left[ \left( U_i - \overline{U} \right) \left( U_j - \overline{U} \right) \right] = \frac{\operatorname{Var}(U_i)}{n \epsilon^2} = O\left(\frac{1}{n}\right).$$

We therefore have  $\widehat{U} - \overline{U} = O_p(n^{-1/2})$ . According to (8) we have  $\overline{m} \leq \overline{U}$ , and therefore  $\overline{m} \leq \widehat{U} + O_p(n^{-1/2})$ . By analogous arguments we obtain  $\widehat{L} + O_p(n^{-1/2}) \leq \overline{m}$ .

<u># Part (ii)</u>: Define  $\sigma_U^2 = \operatorname{Var}[U(Z_i, Y_i, \beta_0)]$ . By the Weak Law of Large Numbers we have  $\widehat{\sigma}_U^2 \to_p \sigma_U^2$ . Remember that  $\sigma_U^2 > 0$ , by assumption. Then, by the Lindeberg–Lévy CLT it follows that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}U_{i}-\overline{U}\overset{d}{\to}\mathcal{N}(0,\sigma_{U}^{2}),$$

and also using the continuous mapping theorem we thus obtain

$$\lim_{n \to \infty} P\left(\overline{U} \le \widehat{U} + \frac{c_{\alpha/2}\widehat{\sigma}_U}{\sqrt{n}}\right) = \Phi(c_{\alpha/2}).$$
(31)

By analogous arguments,

$$\lim_{n \to \infty} P\left(\overline{L} \ge \widehat{L} - \frac{c_{\alpha/2}\widehat{\sigma}_L}{\sqrt{n}}\right) = \Phi(c_{\alpha/2}).$$
(32)

Next, notice that

$$P\left(\widehat{L} - \frac{c_{\alpha/2}\,\widehat{\sigma}_L}{\sqrt{n}} \le \overline{m} \le \widehat{U} + \frac{c_{\alpha/2}\,\widehat{\sigma}_U}{\sqrt{n}}\right) = P\left(\overline{m} \ge \widehat{L} - c_{\alpha/2}\frac{\widehat{\sigma}_L}{\sqrt{n}} \cap \overline{m} \le \widehat{U} + c_{\alpha/2}\frac{\widehat{\sigma}_U}{\sqrt{n}}\right)$$
$$\ge P\left(\overline{L} \ge \widehat{L} - c_{\alpha/2}\frac{\widehat{\sigma}_L}{\sqrt{n}} \cap \overline{U} \le \widehat{U} + c_{\alpha/2}\frac{\widehat{\sigma}_U}{\sqrt{n}}\right)$$
$$= 1 - P\left(\overline{L} \le \widehat{L} - c_{\alpha/2}\frac{\widehat{\sigma}_L}{\sqrt{n}} \cup \overline{U} \ge \widehat{U} + c_{\alpha/2}\frac{\widehat{\sigma}_U}{\sqrt{n}}\right)$$
$$\ge 1 - P\left(\overline{L} \le \widehat{L} - c_{\alpha/2}\frac{\widehat{\sigma}_L}{\sqrt{n}}\right) - P\left(\overline{U} \ge \widehat{U} + c_{\alpha/2}\frac{\widehat{\sigma}_U}{\sqrt{n}}\right),$$
(33)

where in the first inequality we have used  $\overline{L} \leq \overline{m} \leq \overline{U}$ . Using (31) and (32) in (33), and then taking limits, we finally obtain

$$\lim_{n \to \infty} P\left(\widehat{L} - \frac{c_{\alpha/2}\,\widehat{\sigma}_L}{\sqrt{n}} \le \overline{m} \le \widehat{U} + \frac{c_{\alpha/2}\,\widehat{\sigma}_U}{\sqrt{n}}\right) \ge 1 - \left(1 - \Phi(c_{\alpha/2})\right) - \left(1 - \Phi(c_{\alpha/2})\right) = 1 - \alpha,$$

as stated.  $\blacksquare$ 

**Proof of Theorem 2.** For  $s \in \{1, 2\}$  let  $\bar{s} = 3 - s$  and  $n_s = |\mathcal{I}_s|$ , which is either  $\lfloor n/2 \rfloor$  or  $\lceil n/2 \rceil$ . Define also

$$\widehat{L}_s = \frac{1}{n_s} \sum_{i \in \mathcal{I}_s} L(Z_i, Y_i, \widehat{\beta}_{\overline{s}}), \qquad \overline{L}(\beta) = \mathbb{E}\left[\sum_{y \in \mathcal{Y}} L(Z_i, y, \beta) f(y \mid Z_i, A_i; \beta)\right].$$

Note, importantly, that whenever  $i \in \mathcal{I}_s$ 

$$\overline{L}(\beta) = \mathbb{E}\left[\left|\sum_{y \in \mathcal{Y}} L\left(Z_i, y, \beta\right) f\left(y \mid Z_i, A_i; \beta\right)\right| Y_{(\mathcal{I}_{\bar{s}})}, Z_{(\mathcal{I}_{\bar{s}})}\right]$$

due to cross-sectional independence. Now, conditional on  $(Y_{(\mathcal{I}_{\bar{s}})}, Z_{(\mathcal{I}_{\bar{s}})})$  the terms  $L(Z_i, Y_i, \hat{\beta}_{\bar{s}})$  are independent and identically distributed across i and have a variance bounded by  $(b_{\max} - b_{\min})^2$ , which implies that

$$\operatorname{Var}(\widehat{L}_s \,|\, Y_{(\mathcal{I}_{\bar{s}})}, \, Z_{(\mathcal{I}_{\bar{s}})}) \leq \frac{(b_{\max} - b_{\min})^2}{n_s} = O(n^{-1}).$$

By an application of Markov's inequality we therefore obtain

$$\widehat{L}_s = \mathbb{E}\left[ \left| L(Z_i, Y_i, \widehat{\beta}_{\overline{s}}) \right| Y_{(\mathcal{I}_{\overline{s}})}, Z_{(\mathcal{I}_{\overline{s}})} \right] + O_p(n^{-1/2}),$$

where here and in the following  $i \in \mathcal{I}_s$ . Evaluating the expectation over  $Y_i$  gives

$$\widehat{L}_{s} = \mathbb{E} \left[ \sum_{y \in \mathcal{Y}} L(Z_{i}, y, \widehat{\beta}_{\bar{s}}) f(y \mid Z_{i}, A_{i}; \beta_{0}) \mid Y_{(\mathcal{I}_{\bar{s}})}, Z_{(\mathcal{I}_{\bar{s}})} \right] + O_{p}(n^{-1/2}) \\
= \mathbb{E} \left[ \sum_{y \in \mathcal{Y}} L(Z_{i}, y, \widehat{\beta}_{\bar{s}}) f(y \mid Z_{i}, A_{i}; \widehat{\beta}_{\bar{s}}) \mid Y_{(\mathcal{I}_{\bar{s}})}, Z_{(\mathcal{I}_{\bar{s}})} \right] \\
- \mathbb{E} \left[ \sum_{y \in \mathcal{Y}} L(Z_{i}, y, \widehat{\beta}_{\bar{s}}) \frac{\partial f(y \mid Z_{i}, A_{i}; \widetilde{\beta})}{\partial \beta'} \mid Y_{(\mathcal{I}_{\bar{s}})}, Z_{(\mathcal{I}_{\bar{s}})} \right] (\widehat{\beta}_{\bar{s}} - \beta_{0}) + O_{p}(n^{-1/2}) \\
= \overline{L}(\widehat{\beta}_{\bar{s}}) + O_{p}(n^{-1/2}),$$
(34)

where in the second step we performed a mean-value expansion of  $f(y | Z_i, A_i; \beta)$  around  $\beta_0$ , with  $\tilde{\beta}$  being some value between  $\beta_0$  and  $\hat{\beta}_{\bar{s}}$ , and in the last step we used the definition of  $\overline{L}(\beta)$  as well as  $\hat{\beta}_{\bar{s}} - \beta_0 = O_p(n^{-1/2})$  and

$$\begin{aligned} \left\| \mathbb{E} \left[ \sum_{y \in \mathcal{Y}} L(Z_i, y, \widehat{\beta}_{\overline{s}}) \frac{\partial f(y \mid Z_i, A_i; \widetilde{\beta})}{\partial \beta} \middle| Y_{(\mathcal{I}_{\overline{s}})}, Z_{(\mathcal{I}_{\overline{s}})} \right] \right\| \\ &\leq \max(|b_{\min}|, |b_{\max}|) \sup_{\beta \in B_{\epsilon}(\beta_0)} \sum_{y \in \mathcal{Y}} \mathbb{E} \left\| \frac{\partial f(y \mid Z_i, A_i; \beta)}{\partial \beta} \right\| = O(1). \end{aligned}$$

Here we also used that by the consistency of  $\widehat{\beta}_{\overline{s}}$  one has  $\widetilde{\beta} \in B_{\epsilon}(\beta_0)$ , for an  $\epsilon > 0$ , with probability approaching one. Next, we define

$$\overline{m}(\beta) = \mathbb{E}\left[m\left(Z_i, A_i, \beta\right)\right]$$

Then, by another mean-values expansion in  $\beta$  we find that

$$\overline{m} = \overline{m}(\beta_0) = \overline{m}(\widehat{\beta}_{\overline{s}}) + O_p(n^{-1/2}).$$
(35)

By condition (7),  $\overline{L}(\widehat{\beta}_{\overline{s}}) \leq \overline{m}(\widehat{\beta}_{\overline{s}})$ , and together with (34) and (35) this implies  $\widehat{L}_s + O_p(n^{-1/2}) \leq \overline{m}$ , as stated. The derivation of  $\overline{m} \leq \widehat{U}_s + O_p(n^{-1/2})$  is analogous. This completes the proof.

**Proof of Theorem 3.** Let  $\widehat{U} = \widehat{U}(\beta_0)$ ,  $\overline{U} = \mathbb{E}[U(Z_i, Y_i, \beta_0)]$  and  $\widehat{\sigma}_U^2 = \widehat{\sigma}_U^2(\beta_0)$ , and let  $\widehat{L}, \overline{L}$ , and  $\widehat{\sigma}_L^2$  be defined analogously. Remember that it was already obtained in (31) and (32) in the proof of Theorem 1 that

$$\lim_{n \to \infty} P\left(\overline{U} \le \widehat{U} + \frac{c_{\alpha/2}\widehat{\sigma}_U}{\sqrt{n}}\right) = \Phi(c_{\alpha/2}) \quad \text{and} \quad \lim_{n \to \infty} P\left(\overline{L} \ge \widehat{L} - \frac{c_{\alpha/2}\widehat{\sigma}_L}{\sqrt{n}}\right) = \Phi(c_{\alpha/2}).$$

To keep the notation simple, define (with some abuse of notation)

$$L(1-\gamma,\alpha) = \inf_{\beta \in \mathcal{B}_{1-\gamma}} \left( \widehat{L}(\beta) - c_{\alpha/2} \frac{\widehat{\sigma}_L(\beta)}{\sqrt{n}} \right), \qquad L_0(\alpha) = \left( \widehat{L} - c_{\alpha/2} \frac{\widehat{\sigma}_L}{\sqrt{n}} \right),$$
$$U(1-\gamma,\alpha) = \sup_{\beta \in \mathcal{B}_{1-\gamma}} \left( \widehat{U}(\beta) + c_{\alpha/2} \frac{\widehat{\sigma}_U(\beta)}{\sqrt{n}} \right), \qquad U_0(\alpha) = \left( \widehat{U} + c_{\alpha/2} \frac{\widehat{\sigma}_U}{\sqrt{n}} \right).$$

Now, notice that

$$P(L_{0}(\alpha) \leq \overline{m} \leq U_{0}(\alpha)) \geq P(\overline{L} \geq L_{0}(\alpha) \cap \overline{U} \leq U_{0}(\alpha))$$
  
=1 - P( $\overline{L} \leq L_{0}(\alpha) \cup \overline{U} \geq U_{0}(\alpha)$ )  
 $\geq$ 1 - P( $\overline{L} \leq L_{0}(\alpha)$ ) - P( $\overline{U} \geq U_{0}(\alpha)$ ), (36)

where in obtaining the first inequality we have used  $\overline{L} \leq \overline{m} \leq \overline{U}$ . Moreover, analogous to the arguments used in the Proof of Theorem 11 of Chernozhukov, Fernández-Val, Hahn and Newey (2013),

$$P(L_{0}(\alpha) \leq \overline{m} \leq U_{0}(\alpha)) = P\left(L_{0}(\alpha) \leq \overline{m} \leq U_{0}(\alpha) \cap \beta_{0} \in \mathcal{B}_{1-\gamma}\right) + P\left(L_{0}(\alpha) \leq \overline{m} \leq U_{0}(\alpha) \cap \beta_{0} \notin \mathcal{B}_{1-\gamma}\right) \leq P\left(L_{0}(\alpha) \leq \overline{m} \leq U_{0}(\alpha) \cap \beta_{0} \in \mathcal{B}_{1-\gamma}\right) + P\left(\beta_{0} \notin \mathcal{B}_{1-\gamma}\right) \leq P\left(L(1-\gamma,\alpha) \leq \overline{m} \leq U(1-\gamma,\alpha)\right) + \gamma.$$
(37)

Combining (36) and (37), and taking limits, it follows that

$$\lim_{n \to \infty} P\left(L(1-\gamma, \alpha) \le \overline{m} \le U(1-\gamma, \alpha)\right) \ge 1 - \alpha - \gamma,$$

as stated.  $\blacksquare$ 

# **B** Online appendix with additional Proofs

**Proof of Lemma 1.** Given  $\beta_0 \in \text{Conv}(\mathcal{B}_{\text{sub}})$ , by the definition of the convex hull we have  $\beta_0 = \sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_\beta \beta$ , for some convex weights  $\lambda_\beta \ge 0$  such that  $\sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_\beta = 1$ . To keep the notation simple, define  $\ell(y) = L(z, y, \mathcal{B}_{\text{sub}})$  and  $u(y) = U(z, y, \mathcal{B}_{\text{sub}})$ . Then, under the assumption that  $L(z, y, \widehat{\mathcal{B}}_{\bar{s}})$  and  $U(z, y, \widehat{\mathcal{B}}_{\bar{s}})$  satisfy (26) for all  $\beta \in \mathcal{B}_{\text{sub}}$  and  $a \in \mathcal{A}$ , conditional on the event  $\widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{\text{sub}}$  we have

$$\sum_{y \in \mathcal{Y}} \ell(y) f(y \mid z, a; \beta) \le m(z, a, \beta) \le \sum_{y \in \mathcal{Y}} u(y) f(y \mid z, a; \beta).$$

Multiplying this expression by  $\lambda_{\beta}$  and summing over  $\beta \in \mathcal{B}_{sub}$  then gives

$$\sum_{y \in \mathcal{Y}} \ell(y) \sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_{\beta} f(y \mid z, a; \beta) \leq \sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_{\beta} m(z, a, \beta) \leq \sum_{y \in \mathcal{Y}} u(y) \sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_{\beta} f(y \mid z, a; \beta).$$
(38)

Using Assumption 3 we can employ a second-order mean value expansion of  $f(y | z, a; \beta)$ around  $\beta_0$ , to find for  $\beta \in \mathcal{B}_{sub}$ ,

$$\begin{aligned} f(y \mid z, a; \beta) &= f(y \mid z, a; \beta_0) + \frac{\partial f(y \mid z, a; \beta_0)}{\partial \beta'} (\beta - \beta_0) + (\beta - \beta_0)' \frac{\partial^2 f(y \mid z, a; \widetilde{\beta})}{\partial \beta \partial \beta'} (\beta - \beta_0), \\ &= f(y \mid z, a; \beta_0) + \frac{\partial f(y \mid z, a; \beta_0)}{\partial \beta'} (\beta - \beta_0) + O\left(d_{\text{sub}}^2\right), \end{aligned}$$

where  $\tilde{\beta}$  is a mean value between  $\beta$  and  $\beta_0$  and  $d_{sub} = diam(\mathcal{B}_{sub})$ . By definition  $\sum_{\beta \in \mathcal{B}_{sub}} \lambda_{\beta}(\beta - \beta_0) = 0$ . It then follows from the expansion in the previous display that

$$\sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_{\beta} f(y \mid z, a; \beta) = f(y \mid z, a; \beta_0) + O\left(d_{\text{sub}}^2\right).$$
(39)

Analogous arguments also yield

$$\sum_{\beta \in \mathcal{B}_{\text{sub}}} \lambda_{\beta} m(z, a, \beta) = m(z, a, \beta_0) + O\left(d_{\text{sub}}^2\right).$$
(40)

Then, by combining (38), (39) and (40) we obtain

$$\sum_{y \in \mathcal{Y}} \ell(y) f(y \mid z, a; \beta_0) + O\left(d_{\text{sub}}^2\right) \le m(z, a, \beta_0) \le \sum_{y \in \mathcal{Y}} u(y) f(y \mid z, a; \beta_0) + O\left(d_{\text{sub}}^2\right).$$
(41)

Taking expectations of all sides of (41) finally yields,

$$\mathbb{E}\left[L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}}) \mid \widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{\text{sub}}\right] + O\left(d_{\text{sub}}^2\right) \leq \overline{m} \leq \mathbb{E}\left[U(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}}) \mid \widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{\text{sub}}\right] + O\left(d_{\text{sub}}^2\right),$$

where the conditioning on  $\widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{sub}$  is required as the derivations leading up to (41) are based on this condition. Finally,  $\mathbb{E}[m(Z_i, A_i, \beta_0) | \widehat{\mathcal{B}}_{\bar{s}} = \mathcal{B}_{sub}] = \mathbb{E}[m(Z_i, A_i, \beta_0)]$  follows since the marginal distribution of  $(Y_i, Z_i, A_i)$  is independent across *i*. This completes the proof.

**Proof of Theorem 4.** Before moving to the proof, we make a series of definitions for notational brevity. First, define

$$\overline{L}(\widehat{\mathcal{B}}_{\overline{s}(i)}) = \mathbb{E}\left[L(Z_i, Y_i, \widehat{\mathcal{B}}_{\overline{s}(i)}) \mid \widehat{\mathcal{B}}_{\overline{s}(i)}\right],\$$

where the expectation is with respect to the joint distribution of  $(Y_i, Z_i)$  with  $i \in \mathcal{I}_s$ , conditional on  $\widehat{\mathcal{B}}_{\bar{s}(i)}$ . We next define the centered quantity

$$\widetilde{L}(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}) = L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}) - \overline{L}(\widehat{\mathcal{B}}_{\bar{s}(i)}),$$

and the half-sample averages

$$\widetilde{L}_{C,s} = \frac{2}{n} \sum_{i \in \mathcal{I}_s} \widetilde{L}(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}) \quad \text{and} \quad \widehat{L}_{C,s} = \frac{2}{n} \sum_{i \in \mathcal{I}_s} L(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)}).$$

The corresponding quantities  $\overline{U}(\widehat{\mathcal{B}}_{\bar{s}(i)})$ ,  $\widetilde{U}(Z_i, Y_i, \widehat{\mathcal{B}}_{\bar{s}(i)})$ ,  $\widetilde{U}_{C,s}$  and  $\widehat{U}_{C,s}$  are defined analogously. We further define

$$\mathcal{L}_{C,s} = \widehat{L}_{C,s} - \frac{c_{\alpha/4}\,\widehat{\sigma}_{L,s}}{\sqrt{n/2}} \quad \text{and} \quad \mathcal{U}_{C,s} = \widehat{U}_{C,s} + \frac{c_{\alpha/4}\,\widehat{\sigma}_{U,s}}{\sqrt{n/2}}.$$

Notice that  $\hat{L}_C = (\hat{L}_{C,1} + \hat{L}_{C,2})/2$  and  $\hat{U}_C = (\hat{U}_{C,1} + \hat{U}_{C,2})/2$ . Finally, let

$$\mathcal{L}_C = rac{\mathcal{L}_{C,1} + \mathcal{L}_{C,2}}{2}$$
 and  $\mathcal{U}_C = rac{\mathcal{U}_{C,1} + \mathcal{U}_{C,2}}{2}$ 

These quantities are equivalent to the lower and upper bounds in the probability statement of Theorem 4. We therefore want to prove

$$P(\mathcal{L}_C \leq \overline{m} \leq \mathcal{U}_C) \geq 1 - \alpha - \gamma + o(1), \quad \text{as } n \to \infty.$$

Lemma 1 states that conditional on  $\beta_0 \in \text{Conv}(\widehat{\mathcal{B}}_{\bar{s}(i)})$ , we have

$$\overline{L}(\widehat{\mathcal{B}}_{\bar{s}(i)}) + \delta_L(\widehat{\mathcal{B}}_{\bar{s}(i)}) \le \overline{m} \le \overline{U}(\widehat{\mathcal{B}}_{\bar{s}(i)}) + \delta_U(\widehat{\mathcal{B}}_{\bar{s}(i)}),$$
(42)

where we have introduced the notation  $\delta_L(\widehat{\mathcal{B}}_{\bar{s}(i)})$  and  $\delta_U(\widehat{\mathcal{B}}_{\bar{s}(i)})$  for the upper and lower bound  $O([\operatorname{diam}(\widehat{\mathcal{B}}_{\bar{s}})]^2)$  remainder terms in Lemma 1. In what follows, let  $\mathcal{A}_s$  denote the event that  $\beta_0 \in \operatorname{Conv}(\widehat{\mathcal{B}}_s)$ , with the complement given by  $\mathcal{A}_s^c$ . Now, observe that

$$P\left(\mathcal{L}_{C,s} \geq \overline{m} \bigcup \mathcal{U}_{C,s} \leq \overline{m}\right)$$

$$= P\left(\left\{\mathcal{L}_{C,s} \geq \overline{m} \bigcup \mathcal{U}_{C,s} \leq \overline{m}\right\} \bigcap \mathcal{A}_{\bar{s}}\right) + P\left(\left\{\mathcal{L}_{C,s} \geq \overline{m} \bigcup \mathcal{U}_{C,s} \leq \overline{m}\right\} \bigcap \mathcal{A}_{\bar{s}}^{c}\right)$$

$$\leq P\left(\mathcal{L}_{C,s} \geq \overline{m} \bigcap \mathcal{A}_{\bar{s}}\right) + P\left(\mathcal{U}_{C,s} \leq \overline{m} \bigcap \mathcal{A}_{\bar{s}}\right) + P\left(\mathcal{A}_{\bar{s}}^{c}\right)$$

$$= P\left(\mathcal{L}_{C,s} \geq \overline{m} \mid \mathcal{A}_{\bar{s}}\right) P(\mathcal{A}_{\bar{s}}) + P\left(\mathcal{U}_{C,s} \leq \overline{m} \mid \mathcal{A}_{\bar{s}}\right) P(\mathcal{A}_{\bar{s}}) + P\left(\mathcal{A}_{\bar{s}}^{c}\right)$$

$$\leq P\left(\mathcal{L}_{C,s} \geq \overline{L}(\widehat{\mathcal{B}}_{\bar{s}}) + \delta_{L,\bar{s}} \mid \mathcal{A}_{\bar{s}}\right) P(\mathcal{A}_{\bar{s}}) + P\left(\mathcal{U}_{C,s} \leq \overline{U}(\widehat{\mathcal{B}}_{\bar{s}}) + \delta_{U,\bar{s}} \mid \mathcal{A}_{\bar{s}}\right) P(\mathcal{A}_{\bar{s}}) + \frac{\gamma}{2} + o(1)$$

$$= P\left(\mathcal{L}_{C,s} \geq \overline{L}(\widehat{\mathcal{B}}_{\bar{s}}) + \delta_{L,\bar{s}}\right) + P\left(\mathcal{U}_{C,s} \leq \overline{U}(\widehat{\mathcal{B}}_{\bar{s}}) + \delta_{U,\bar{s}}\right) + \frac{\gamma}{2} + o(1), \qquad (43)$$

where in the second to last step we have used Assumption 3(i) and (42), and we defined  $\delta_{L,\bar{s}} = \delta_L(\widehat{\mathcal{B}}_{\bar{s}})$  and  $\delta_{U,\bar{s}} = \delta_U(\widehat{\mathcal{B}}_{\bar{s}})$ . Next, we obtain

$$P\left(\mathcal{L}_{C} \leq \overline{m} \leq \mathcal{U}_{C}\right) \geq P\left(\mathcal{L}_{C,1} \leq \overline{m} \leq \mathcal{U}_{C,1} \bigcap \mathcal{L}_{C,2} \leq \overline{m} \leq \mathcal{U}_{C,2}\right)$$

$$\geq 1 - P\left(\mathcal{L}_{C,1} \geq \overline{m} \bigcup \mathcal{U}_{C,1} \leq \overline{m}\right) - P\left(\mathcal{L}_{C,2} \geq \overline{m} \bigcup \mathcal{U}_{C,2} \leq \overline{m}\right)$$

$$\geq 1 - \gamma - \sum_{s=1}^{2} \left[P\left(\mathcal{L}_{C,s} \geq \overline{L}(\widehat{\mathcal{B}}_{\overline{s}}) + \delta_{L,\overline{s}}\right) + P\left(\mathcal{U}_{C,s} \leq \overline{U}(\widehat{\mathcal{B}}_{\overline{s}}) + \delta_{U,\overline{s}}\right)\right] + o(1)$$

$$\geq 1 - \gamma - \sum_{s=1}^{2} P\left(\sqrt{n/2}\frac{\widetilde{L}_{C,s}}{\widehat{\sigma}_{L,s}} \geq c_{\alpha/4} + \sqrt{n/2}\frac{\delta_{L,\overline{s}}}{\widehat{\sigma}_{L,s}}\right)$$

$$- \sum_{s=1}^{2} P\left(\sqrt{n/2}\frac{\widetilde{U}_{C,s}}{\widehat{\sigma}_{U,s}} \leq -c_{\alpha/4} + \sqrt{n/2}\frac{\delta_{U,\overline{s}}}{\widehat{\sigma}_{U,s}}\right) + o(1) \qquad (44)$$

where the third inequality follows from (43) and the last inequality applies the various definitions we made earlier.

It remains to show that the probabilities  $P(\cdot)$  that explicitly appear in the last inequality of (44) are all bounded from above by  $\alpha/4 + o(1)$ . To show this, we first note that conditional on  $\mathcal{G}_{\bar{s}} = \{(Y_j, Z_j) : j \in \mathcal{I}_{\bar{s}}\}, \widetilde{L}(Z_i, Y_i, \mathcal{B}_{\bar{s}})$  is centered and iid over  $i \in \mathcal{I}_s$ . Next, define  $M_{r,\bar{s}} = \mathbb{E}\left[|\widetilde{L}(Z_i, Y_i, \mathcal{B}_{\bar{s}})|^r \mid \mathcal{G}_{\bar{s}}\right]$ . Since  $L(z, y, \widehat{\mathcal{B}}_{\bar{s}})$  is bounded by  $b_{\min}$  and  $b_{\max}$ ,  $M_{r,\bar{s}}$  exists for any r > 0. It follows by Theorem 1.1 of Bentkus and Götze (1996) that there exists some  $k_{\bar{s}} > 0$  such that

$$\sup_{c\in\mathbb{R}} \left| P\left( \left| \sqrt{n/2} \frac{\widetilde{L}_{C,s}}{\widehat{\sigma}_{L,s}} < c \right| \mathcal{G}_{\bar{s}} \right) - \Phi(c) \right| \le \frac{1}{\sqrt{n/2}} \frac{k_{\bar{s}} M_{3,\bar{s}}}{(M_{2,\bar{s}})^{3/2}}.$$

The second part of this upper bound is finite for any  $\overline{s}$ . Hence, it equivalently holds that

$$P\left(\left.\sqrt{n/2}\frac{\widetilde{L}_{C,s}}{\widehat{\sigma}_{L,s}} < c \right| \mathcal{G}_{\bar{s}}\right) = \Phi(c) + O\left(n^{-1/2}\right),\tag{45}$$

where the rate  $O(n^{-1/2})$  holds uniformly over  $c \in \mathbb{R}$ . Choosing  $c = c_{\alpha/4} + \sqrt{n/2}\delta_{L,\bar{s}}/\hat{\sigma}_{L,s}$ , equation (45) yields

$$P\left(\left.\sqrt{n/2}\frac{\widetilde{L}_{C,s}}{\widehat{\sigma}_{L,s}} \ge c_{\alpha/4} + \sqrt{n/2}\frac{\delta_{L,\bar{s}}}{\widehat{\sigma}_{L,s}}\right| \mathcal{G}_{\bar{s}}\right) = 1 - \Phi\left(c_{\alpha/4} + \sqrt{\frac{n}{2}}\frac{\delta_{L,\bar{s}}}{\widehat{\sigma}_{L,s}}\right) + O\left(n^{-1/2}\right)$$
$$= 1 - \Phi\left(c_{\alpha/4}\right) + O\left(\left.\sqrt{\frac{n}{2}}\left|\frac{\delta_{L,\bar{s}}}{\widehat{\sigma}_{L,s}}\right| + \frac{1}{\sqrt{n}}\right) = \frac{\alpha}{4} + O\left(\left.\sqrt{\frac{n}{2}}\left|\frac{\delta_{L,\bar{s}}}{\widehat{\sigma}_{L,s}}\right| + \frac{1}{\sqrt{n}}\right)\right),$$
(46)

where the second equality expands  $\Phi$  around  $c_{\alpha/4}$ , and the final equality follows from the definition of  $c_{\alpha/4}$ . Taking expectations over  $\mathcal{G}_{\bar{s}}$  in (46) and applying the Law of Iterated Expectations yields

$$P\left(\sqrt{n/2}\frac{\widetilde{L}_{C,s}}{\widehat{\sigma}_{L,s}} \ge c_{\alpha/4} + \sqrt{n/2}\frac{\delta_{L,\bar{s}}}{\widehat{\sigma}_{L,s}}\right) = \frac{\alpha}{4} + O\left(\sqrt{\frac{n}{2}}\mathbb{E}\left|\delta_{L,\bar{s}}\right| \mathbb{E}\left|\frac{1}{\widehat{\sigma}_{L,s}}\right| + \frac{1}{\sqrt{n}}\right) = \frac{\alpha}{4} + o\left(1\right)$$

$$\tag{47}$$

where in obtaining the final o(1) rate we have used Assumption 3(i). We have also used the assumption  $\operatorname{Var}[L(Z_i, Y_i, \beta)] > 0$  which implies that  $\mathbb{E}|1/\widehat{\sigma}_{L,s}|$  is bounded for some sufficiently large n. By analogous arguments one obtains

$$P\left(\sqrt{n/2}\frac{\widetilde{U}_{C,s}}{\widehat{\sigma}_{U,s}} \le -c_{\alpha/4} + \sqrt{n/2}\frac{\delta_{U,\bar{s}}}{\widehat{\sigma}_{U,s}}\right) = \frac{\alpha}{4} + o(1).$$
(48)

Combining (44), (47) and (48) yields the stated result.  $\blacksquare$