# More Fermionic Supersymmetric Wilson Loops in Four Dimensions 

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#### Abstract

We construct supersymmetric fermionic Wilson loops along general curves in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory and along general planar curves in $\mathcal{N}=2$ superconformal $S U(N) \times S U(N)$ quiver theory. These loops are generalizations of the Zarembo loops and are cohomologically equivalent to them. In $\mathcal{N}=4$ super Yang-Mills theory, we compute their expectation values and verify the cohomological equivalence relation up to the order $g^{4}$ in perturbation theory.


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## 1 Introduction

Bogomolǹyi-Prasad-Sommerfield (BPS) Wilson loops (WLs) [1,2] in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory (SYM) play an important role in the precise checks of the AdS/CFT correspondence [3-5] since the early days. One of the precise checks is about the vacuum expectation value (vev) of a circular half-BPS WL in the fundamental representation in $\mathcal{N}=4$ SYM with gauge group $S U(N)$. Based on one-loop computations, it was conjectured [6] that, the planar limit of this vev can be obtained from a resummation of the ladder planar diagrams in the Feynman gauge. This conjecture leads to the result that this vev can be computed using a Gaussian matrix model in its own planar limit [6]. This vev is a nontrivial function of the 't Hooft coupling constant $\lambda$ and $N$. The large $N$, large $\lambda$ limit of this vev matches precisely with the prediction from the dual string theory [7, 8 using certain half-BPS F-string solutions in the $A d S_{5} \times S^{5}$ background. The conjecture about the reduction to the Gaussian matrix model was later proved by Pestun [9] using supersymmetric localization. This precise matching between the strong coupling result in the field theory side and the weakly coupled string theory result is among the earliest non-trivial validations of the AdS/CFT correspondence, extending beyond the checks about correlation functions of BPS local operators related to various non-renormalization theorems [10-12].

Many BPS WLs with fewer supersymmetries were constructed in $\mathcal{N}=4$ SYM. In Zarembo's construction [13], the loops inside a $\mathbb{R}^{n}$ subspace of $\mathbb{R}^{4}$ Euclidean space preserve $1 / 2^{n}$ of the Poincaré supercharges. We will refer to such loops as $1 / 2^{n}$ Poincaré BPS or just $1 / 2^{n}$-BPS. By direct perturbative computations, Zarembo found that the leading and next-to-leading corrections to the vev of $1 / 4$-BPS Zarembo loop vanishes in the large $N$ limit. Subsequent arguments were presented to support the result that the vev of any Zarembo loop equals unity exactly even at finite $N$ [14-16]. The holographic description of Zarembo loops using calibrated surfaces [16] also supports this result.

Another class of BPS WLs was found by Drukker, Giombi, Ricci and Trancanelli (DGRT) [17, [18. For an arbitrary curve in $S^{3}$ they found the suitable scalar coupling to the WL such that the WL preserves at least two linear combinations of Poincaré and conformal supercharges.

Different from the case of Zarembo loops, the generic DGRT loop has nontrivial vev. It was found [18, 19] that, when the DGRT loop is restricted to an $S^{2}$ submanifold, its vev can be obtained from the vev of certain ordinary WL in two-dimensional (non-supersymmetric) Yang-Mills theory on $S^{2}$ restricted to the zero-instanton sector [20, 21] 1 . A similar relation was also obtained for correlation functions of such DGRT loops and certain local operators on the same $S^{2}$ [25]. Certain classification of BPS WLs in $\mathcal{N}=4 \mathrm{SYM}$ was performed in [26].

The above BPS WLs in $\mathcal{N}=4$ SYM usually involve suitable coupling of scalars to the WLs in the construction. In the study of BPS WLs in three-dimensional super-Chern-Simons theories, fermionic BPS WLs were also constructed. In such construction [27], the WL couples to the fermions in the theory as well, besides the gauge fields and scalars. The introduction of the fermionic BPS WLs was to resolve a puzzle about the duality between BPS WLs in ABJM theory [28] and the probe F-string theory in the dual type IIA string theory on $A d S_{4} \times \mathbb{C P}^{3}$ background. 1/6-BPS bosonic WLs in ABJM theory was constructed in 29 31]. However there are probe F-string solutions [29,31] in $A d S_{4} \times \mathbb{C P}^{3}$ which are half-BPS and quite reasonably dual to WLs. But no such half-BPS WLs are found among the above $1 / 6$-BPS bosonic WLs. The construction of half-BPS fermionic WLs by Drukker and Trancanelli [27] satisfactorily resolved this problem. Later $1 / 6$-BPS fermionic WLs in ABJM theory were constructed in 32,33 . For special choice of the parameters in the constructions, such WLs will go back to the previously found $1 / 6$-BPS bosonic WLs or half-BPS WLs. It was proposed in 34 that a generic $1 / 6$-BPS fermionic WL is dual to an F -string with supersymmetric mixed boundary conditions.

One naturally wonders whether similar BPS fermionic WLs exist in four-dimensional superconformal gauge theories. In [35] we provided a positive answer to this question by explicitly constructing in four-dimensional $\mathcal{N}=2$ superconformal $S U(N) \times S U(N)$ quiver theory and $\mathcal{N}=4 \mathrm{SYM}$. In each theory, we constructed two types of fermionic BPS WLs that preserve some supersymmetries. The first type consists of WLs along an infinite timelike straight line in Lorentzian signature, which preserve one or more Poincaré supercharges. The second type consists of WLs along a circular contour in Euclidean signature, which preserve one or more linear combinations of Poincaré and conformal supercharges. Every fermionic WL belongs to the same $Q$-cohomology class as a bosonic half-BPS WL that shares the same supercharge $Q$. If we assume that this cohomological equivalence also holds true at the quantum level, we can predict that the fermionic BPS WL has the same expectation value as the bosonic one.

The aim of the present work is to investigate fermionic BPS WLs along more general contours by employing Zarembo's construction. One notable feature of the fermionic Zarembo loops is that the connections are supersymmetric invariant, whereas the supersymmetric variations of the connections of typical fermionic BPS WLs constructed previously are covariant total derivatives 2 . The number of preserved supersymmetries by the WL depends on the choice of the contour and parameters in the connection. Similar to previously constructed WLs, there exists a cohomological equivalence relation between the fermionic and bosonic Zarembo loops. In $\mathcal{N}=4 \mathrm{SYM}$, we verify the cohomological equivalence relation up to order $g^{4}$ in the perturbation theory. Our results provide new insights into the structure and

[^1]properties of BPS WLs in four-dimensional superconformal gauge theories.
The paper is organized as follows. In section 2 we review Zarembo's construction. Then we present our construction of fermionic Zarembo loops in $\mathcal{N}=4 \mathrm{SYM}$ and compute their expectation values to order $g^{4}$ at finite $N$. In section 3 we construct fermionic Zarembo loops in $\mathcal{N}=2$ superconformal $S U(N) \times S U(N)$ quiver theory and discuss their supersymmetry properties. Section 4 concludes with some remarks. Appendix A contains some conventions and useful formulas.

## 2 Fermionic supersymetric WLs in $\mathcal{N}=4 \mathbf{S Y M}$

### 2.1 Zarembo loop

Let us begin by briefly reviewing Zarembo's construction [13]. The Euclidean action of $\mathcal{N}=4$ SYM with gauge group $S U(N)$ is

$$
\begin{equation*}
S_{\mathcal{N}=4}=\int_{\mathbf{R}^{4}} d^{4} x\left(\frac{1}{2} \operatorname{Tr}\left(F_{M N} F^{M N}\right)+i \operatorname{Tr}\left(\Psi^{c} \Gamma^{M} D_{M} \Psi\right)\right) . \tag{1}
\end{equation*}
$$

The $\Gamma^{M}$ 's are ten-dimensional gamma matrices. We use the index conventions $M, N=0, \ldots, 9$ and $I, J=4, \ldots, 9 . \Psi$ satisfies the chiral condition $\Gamma^{0123456789} \Psi=\Psi$ and $\Psi^{c}=\Psi^{T} C$ is the charge conjugation of $\Psi$. For more detailed conventions, please refer to Appendix A. The action is invariant under the superconformal transformations:

$$
\begin{align*}
& \delta A_{M}=-i \xi^{c} \Gamma_{M} \Psi, \\
& \delta \Psi=\frac{1}{2} F_{M N} \Gamma^{M N} \xi-2 \Gamma^{I} A_{I} \vartheta . \tag{2}
\end{align*}
$$

where $\xi=\theta+x^{\mu} \Gamma_{\mu} \vartheta$ with $\mu=0, \ldots, 3$. $\xi$ satisfies the chiral condition $\Gamma^{0123456789} \xi=\xi$. The constant spinors $\theta$ and $\vartheta$ generate Poincaré supersymmetry transformations and conformal supersymmetry transformations respectively. In $\mathcal{N}=4 \mathrm{SYM}$, a natural generalization of the ordinary WL is the Maldacena-Wilson loop:

$$
\begin{equation*}
W=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i g \int_{C} d \tau\left(A_{\mu} \dot{x}^{\mu}(\tau)+i|\dot{x}| \Theta^{I}(\tau) A_{I}\right)\right) . \tag{3}
\end{equation*}
$$

Local supersymmetry requires the norm of $\Theta^{I}$ to be one. One simple example of a globally supersymmetric WL is the one with $C$ a straight line and $\Theta^{I}$, s being constants. A remarkable generalization has been proposed by Zarembo [13]. The Zarembo loops are defined by

$$
\begin{equation*}
\Theta^{I}(\tau)=M_{\mu}^{I} \frac{\dot{x}^{\mu}}{|\dot{x}|} . \tag{4}
\end{equation*}
$$

where $M_{\mu}^{I}$ is a constant projection matrix:

$$
\begin{equation*}
M_{\mu}^{I} M_{\nu}^{J} \delta_{I J}=\delta_{\mu \nu} \tag{5}
\end{equation*}
$$

Considering Poincaré supersymmetry variation of the loop, some supersymmetries will be preserved if the equation

$$
\begin{equation*}
\dot{x}^{\mu}\left(\Gamma_{\mu}+i \Gamma_{I} M_{\mu}^{I}\right) \theta=0, \tag{6}
\end{equation*}
$$

has nontrivial solutions. When the contour is a generic curve in $\mathbb{R}^{4}, \theta$ satisfies four constraints

$$
\begin{equation*}
\left(\Gamma_{\mu}+i \Gamma_{I} M_{\mu}^{I}\right) \theta=0, \quad \mu=0,1,2,3 \tag{7}
\end{equation*}
$$

and the WL is $1 / 16$ Poincaré BPS. When the contour of the loop is inside a subspace $\mathbb{R}^{n}$, there are $n$ independent constraints and the loop is $1 / 2^{n} \operatorname{BPS} 3$. Especially, if the operator lies on a straight line there is only one constraint equation on $\theta$ so it is $1 / 2$ BPS. Zarembo loop operators on non-straight curves can only be constructed in Euclidean signature. The reason is that if there is more than one constraint equation, at least one of them corresponds to a space direction and contradicts the real condition of the spinor in Lorentz signature [36]. Therefore in this work, we focus on WL operators in Euclidean space.

### 2.2 Fermionic loop

We now generalize Zarembo's construction by including coupling to the fermionic fields in $\mathcal{N}=4$ SYM. The connection contains both bosonic and fermionic components. BPS fermionic WLs along a straight line were constructed in [35]. The fermionic component can be obtained by acting on a specific linear combination of the scalars with a supersymmetry generator $Q_{s}$ that preserves the loop. The fermionic supersymmetry generator $Q_{s}$ is defined as $\delta_{\theta}=\chi Q_{s}$, where the bosonic spinor $s$ is related to $\theta$ as $\theta=\chi s$ and $\chi$ is a real Grassmann variable.

To construct a BPS WL on a non-straight curve, we start with a bosonic connection

$$
\begin{equation*}
L_{\mathrm{bos}}=g \dot{x}^{\mu}\left(A_{\mu}+i M_{\mu}^{I} A_{I}\right), \tag{8}
\end{equation*}
$$

which is $Q_{s}$-invariant. When $s$ is constrained by at least two projection equations, i.e.

$$
\begin{equation*}
\left(\Gamma_{\mu_{1}}+i \Gamma_{I} M_{\mu_{1}}^{I}\right) s=\left(\Gamma_{\mu_{2}}+i \Gamma_{I} M_{\mu_{2}}^{I}\right) s=0 \tag{9}
\end{equation*}
$$

one can prove that

$$
\begin{equation*}
s^{c} \Gamma_{M} s=0, \tag{10}
\end{equation*}
$$

by using the $S O(4) \times S O(6)$ symmetry (and parity invariance if needed) to transform $M_{\mu}^{I}$ to a simple form $\delta_{I-4}^{\mu}$. Therefore we find

$$
\begin{equation*}
Q_{s}^{2} A_{M}=Q_{s}\left(-i s^{c} \Gamma_{M} \Psi\right)=0 \tag{11}
\end{equation*}
$$

A supersymmetric fermionic loop can be constructed as

$$
\begin{equation*}
W_{\mathrm{fer}}=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} L d \tau\right), \tag{12}
\end{equation*}
$$

with a $Q_{s}$-invariant connection

$$
\begin{equation*}
L=L_{\mathrm{bos}}+g|\dot{x}| Q_{s}\left(m^{I}(\tau) A_{I}\right)=L_{\mathrm{bos}}-i g|\dot{x}| m^{I}(\tau) s^{c} \Gamma_{I} \Psi . \tag{13}
\end{equation*}
$$

For the BPS WL constructed in [35], its connection transforms under supersymmetry as a covariant derivative. However, for the BPS WL we constructed here, its connection itself

[^2]is supersymmetric invariant. Therefore, we can directly use the trace to construct the WL, without the need to construct supermatrices and take the supertrace as in the case of [35].

When the contour is a generic curve in $\mathbb{R}^{4}$, the WL is $1 / 16 \mathrm{BPS}$. When the loop lies in a subspace, we would like to find all the $u$ such that $Q_{u} L=0$. To be concrete, we take $M_{\mu}^{I}=\delta_{\mu}^{I-4}$. When the WL is inside the 01 plane, $s$ is constrained by two projection equations:

$$
\begin{equation*}
\left(1+i \Gamma_{04}\right) s=\left(1+i \Gamma_{15}\right) s=0, \tag{14}
\end{equation*}
$$

and $u$ satisfies the same constraints because $Q_{u} L_{\mathrm{bos}}=0$. When $m^{4}=m^{5}=0, Q_{u} L=$ $Q_{u} Q_{s}\left(m^{I} A_{I}\right)=0$ requires

$$
\begin{equation*}
m^{I} s^{c} \Gamma_{45 I} u=0, \tag{15}
\end{equation*}
$$

so the solution space of $u$ is three-dimensional and the WL is $3 / 16 \mathrm{BPS}$. Otherwise, the solution is $u \propto s$, which can be derived from the vanishing of the $F_{0 P}$ and $F_{1 P}$ terms, and the WL is $1 / 16$ BPS.

When WLs lie along the 012 subspace, $s$ and $u$ are constrained by three projection equations. Therefore $s^{c} \Gamma_{I J K} u \neq 0$ only if $\{I, J, K\}=\{4,5,6\}$ and $s^{c} \Gamma_{456} u=0$ only if $u \propto s$. So when $m^{4}=m^{5}=m^{6}=0$, the WL is $1 / 8 \mathrm{BPS}$. Otherwise, the WL is $1 / 16$ BPS.

### 2.3 Expectation values in perturbation theory

The fermionic Zarembo loop is classically $Q_{s}$-cohomological equivalent to the bosonic one:

$$
\begin{equation*}
\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} L d \tau\right)-\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} L_{\mathrm{bos}} d \tau\right)=Q_{s} V, \tag{16}
\end{equation*}
$$

where $V$ can be constructed as

$$
\begin{align*}
V & =\frac{1}{N} \sum_{n=1}^{\infty} \operatorname{Tr} \mathcal{P}\left(e^{i \int L_{\mathrm{bos}} d \tau} \int_{\tau_{1}>\tau_{2}>\ldots>\tau_{n}} d \tau_{1} d \tau_{2} \ldots d \tau_{n} \Lambda\left(\tau_{1}\right) F\left(\tau_{2}\right) \ldots F\left(\tau_{n}\right)\right),  \tag{17}\\
\Lambda & =m^{I} A_{I}, \quad F=Q_{s} \Lambda .
\end{align*}
$$

If the $Q_{s}$-cohomological equivalence holds at the quantum level, the expectation values of the bosonic and fermionic loops should be equal. The expectation values of the bosonic Zarembo loops are known to be exactly one [14-16]. In this subsection, we compute the expectation value of the fermionic Zarembo loop to order $g^{4}$ in perturbation theory to test the $Q_{s}$-cohomological equivalence. We will use regularization by dimensional reduction [37] as in [6. We do not take the planar limit in this computation.

Let us first review the calculation of the vacuum expectation value of the bosonic loop in [13]. In the Feynman gauge, the tree-level propagators take the form

$$
\begin{align*}
\left\langle A(x)_{M}^{a} A(y)_{N}^{b}\right\rangle_{0} & =\delta^{a b} \delta_{M N} D(x-y),  \tag{18}\\
\left\langle\Psi(x)^{a} \bar{\Psi}(y)^{b}\right\rangle_{0} & =i \delta^{a b} \Gamma^{\mu} \partial_{\mu} D(x-y) . \tag{19}
\end{align*}
$$

Although the explicit forms of the propagators will not be necessary for our discussion below, we give the tree level and one-loop corrected propagators in regularization by dimensional reduction for completeness in Appendix $\triangle$ and the explicit form of $D(x)$ in $2 \omega$ dimensions is

$$
\begin{equation*}
D(x)=\frac{\Gamma(\omega-1)}{4 \pi^{\omega}} \frac{1}{\left(x^{2}\right)^{\omega-1}} . \tag{20}
\end{equation*}
$$



Figure 1: Feynman diagram for the bosonic loop at order $g^{2}$.

At order $g^{2}$, the Feynman diagram depicted in figure 1 vanishes because

$$
\begin{equation*}
\operatorname{Tr}\left\langle L_{\mathrm{bos}}\left(x_{1}\right) L_{\mathrm{bos}}\left(x_{2}\right)\right\rangle_{0} \propto \dot{x}_{1}^{\mu} \dot{x}_{2}^{\nu}\left(\delta_{\mu \nu}-\delta_{I J} M_{\mu}^{I} M_{\nu}^{J}\right) D\left(x_{1}-x_{2}\right)=0 \tag{21}
\end{equation*}
$$

Because of the same reason, diagram (b) in figure 2 is zero. The one-loop propagators are

$$
\begin{align*}
& \left\langle A(x)_{\mu}^{a} A(y)_{\nu}^{b}\right\rangle_{1}=g^{2} N \delta^{a b} \frac{\Gamma(\omega-1) \Gamma(\omega-2)}{32 \pi^{2 \omega}(2 \omega-3)}\left(\frac{\delta_{\mu \nu}}{\left((x-y)^{2}\right)^{2 \omega-3}}-\frac{\partial_{\mu} \partial_{\nu}\left(\left((x-y)^{2}\right)^{4-2 \omega}\right)}{8(\omega-3)(\omega-2)}\right),  \tag{22}\\
& \left\langle A(x)_{I}^{a} A(y)_{J}^{b}\right\rangle_{1}=g^{2} N \delta^{a b} \delta_{I J} \frac{\Gamma(\omega-1) \Gamma(\omega-2)}{32 \pi^{2 \omega}(2 \omega-3)} \frac{1}{\left((x-y)^{2}\right)^{2 \omega-3}},  \tag{23}\\
& \left\langle\Psi(x)^{a} \bar{\Psi}(y)^{b}\right\rangle_{1}=-i g^{2} N \delta^{a b} \frac{\Gamma(\omega-1) \Gamma(\omega-2)}{8 \pi^{2 \omega}} \frac{\left(x^{\mu}-y^{\mu}\right) \Gamma_{\mu}}{\left((x-y)^{2}\right)^{2 \omega-2}} . \tag{24}
\end{align*}
$$

Because the one-loop scalar and vector propagators are equal up to a total derivative, diagram (a) in figure 2 vanishes. To compute diagram (c), we use

$$
\begin{equation*}
\left\langle\operatorname{Tr}\left(L_{\mathrm{bos}}\left(x_{1}\right) L_{\mathrm{bos}}\left(x_{2}\right) L_{\mathrm{bos}}\left(x_{3}\right)\right) \operatorname{Tr}\left(\partial_{M} A_{N}(x)\left[A^{M}(x), A^{N}(x)\right]\right)\right\rangle_{0}=0, \tag{25}
\end{equation*}
$$

where the convention $\partial_{I}=0$ is used. Here when one $L_{\mathrm{bos}}$ is contracted with $\partial_{M} A_{N}$ and another $L_{\mathrm{bos}}$ with $A^{N}$, one can find that the result is proportional to $\delta_{\mu}^{N} \delta_{\nu N}-M_{\mu}^{I} M_{\nu}^{J} \delta_{I J}=0$ and thus diagram (c) vanishes. Therefore the vev of the bosonic loop equals unity up to order $g^{4}$.

For the fermionic loop, we need to consider diagrams with fermion insertions. We assume that the parameters $m^{I}$ 's are independent of $g$. At order $g^{2}$, the Feynman diagram depicted in figure 3 vanishes:

$$
\begin{align*}
& \operatorname{Tr}\left\langle\int d \tau_{1>2} g m^{I}\left(\tau_{1}\right) \bar{s} \Gamma_{I} \Psi\left(\tau_{1}\right) g m^{J}\left(\tau_{2}\right) \bar{\Psi}\left(\tau_{2}\right) \Gamma_{J} s\right\rangle_{0} \\
= & i g^{2} N \int d \tau_{1>2} m^{I}\left(\tau_{1}\right) m^{J}\left(\tau_{2}\right) \bar{s} \Gamma_{I} \Gamma^{\mu} \Gamma_{J} s \partial_{\mu} D\left(x_{1}-x_{2}\right)  \tag{26}\\
= & 0,
\end{align*}
$$

where we have used $s^{c} \Gamma_{M} s=0$ and $s^{c} \Gamma_{M_{1} M_{2} M_{3}} s=0$.

a

b

c

Figure 2: Feynman diagrams for the bosonic loop at order $g^{4}$. Diagram (b) represents all distinct types of contractions including non-planar contributions.


Figure 3: Feynman diagram involving $F$ at order $g^{2}$.


Figure 4: Feynman diagrams involving $F$ at order $g^{4}$. Diagrams (b) and (c) represent all distinct types of contractions including non-planar contributions.

The order $g^{4}$ Feynman diagrams involving $F=-i m^{I} \bar{s} \Gamma_{I} \Psi$ are shown in figure 4. Since the one-loop fermion propagator is proportional to $x^{\mu} \Gamma_{\mu}$ as the tree-level one, diagram (a) does not contribute. Diagrams (b) and (c) vanish for the same reason. Diagram (d) vanishes because it contains the following structure:

$$
\begin{equation*}
\dot{x}^{\mu} s^{c} \Gamma^{K} \Gamma_{\rho}\left(\Gamma_{\mu}+i M_{\mu}^{I} \Gamma_{I}\right) \Gamma_{\nu} \Gamma^{J} s=0, \tag{27}
\end{equation*}
$$

where we have used $s^{c} \Gamma_{M} s=s^{c} \Gamma_{M_{1} M_{2} M_{3}} s=0$ and the anti-communication relations of the gamma matrices to move $\left(\Gamma_{\mu}+i M_{\mu}^{I} \Gamma_{I}\right)$ to the place just before $s$. Then $\dot{x}^{\mu}\left(\Gamma_{\mu}+i M_{\mu}^{I} \Gamma_{I}\right) s=0$ has been used. Therefore the expectation value of the fermionic Zarembo loop is trivial and the $Q_{s}$-cohomological equivalence is confirmed up to order $g^{4}$.

### 2.4 WL with conformal supersymmetries

The WL constructed above typically does not preserve conformal supersymmetry. However, for a circular contour, the WL may preserve conformal symmetry. In this section, we will provide an example which is a generalization of one special $1 / 4$-BPS bosonic loop studied in 38 being also a special case of Zarembo loops. Let us consider WLs on a circle $\left(x^{0}, x^{1}, x^{2}, x^{3}\right)=r(\cos \tau, \sin \tau, 0,0)$. We start with the bosonic connection:

$$
\begin{equation*}
L_{\mathrm{bos}}=g\left(\dot{x}^{\mu} A_{\mu}+i r \cos \tau A_{4}+i r \sin \tau A_{5}\right) \tag{28}
\end{equation*}
$$

Referring to the Poincaré supersymmetry generator notation, we employ $S_{v}$ with a bosonic spinor $v$ to represent the superconformal generator. The preserved supersymmetries by $L_{\mathrm{bos}}$ satisfy

$$
\begin{equation*}
\left(-\sin \tau \Gamma_{0}+\cos \tau \Gamma_{1}+i \cos \tau \Gamma_{4}+i \sin \tau \Gamma_{5}\right)\left(s+r\left(\cos \tau \Gamma_{0}+\sin \tau \Gamma_{1}\right) v\right)=0 \tag{29}
\end{equation*}
$$

The result is

$$
\begin{equation*}
\left(1+i \Gamma_{14}\right) s=\left(1-i \Gamma_{05}\right) s=\left(1+i \Gamma_{14}\right) v=\left(1-i \Gamma_{05}\right) v=0, \tag{30}
\end{equation*}
$$

so $L_{\text {bos }}$ preserves four Poincaré supersymmetries and four conformal supersymmetries. We take $F=g|\dot{x}| m^{6}(\tau) Q_{s} A_{6}=-i g r m^{6}(\tau) s^{c} \Gamma_{6} \Psi$. As discussed in subsection 2.2, the connection $L=L_{\text {bos }}+F$ preserves $Q_{u}$ with $s^{c} \Gamma_{456} u=0$.

Acting $S_{v}$ on $F$ we get

$$
\begin{equation*}
S_{v} F=-\frac{i}{2} g r m^{6} s^{c} \Gamma_{6} \Gamma^{M N}\left(r \cos \tau \Gamma_{0}+r \sin \tau \Gamma_{1}\right) v F_{M N}+2 i g r m^{6} s^{c} \Gamma_{6} \Gamma^{J} v A_{J} . \tag{31}
\end{equation*}
$$

We find $F$ preserves $S_{v}$ with $v \propto \Gamma_{6} s$. Therefore the WL associated with the connection $L$ preserves three Poincaré supercharges and one conformal supercharge, so it is $1 / 8$-BPS.

## 3 Fermionic Zarembo loops in $\mathcal{N}=2$ superconformal $S U(N) \times$ $S U(N)$ quiver theory

In this section, we construct fermionic Zarembo loops in the $\mathcal{N}=2$ superconformal $S U(N) \times$ $S U(N)$ quiver theory which can be obtained via orbifolding $\mathcal{N}=4 \mathrm{SYM}$ by $\mathbb{Z}_{2}$. There are
two $\mathcal{N}=2$ vector multiplets for the two gauge group factors. The component fields can be arranged into $2 \times 2$ block matrices:

$$
\begin{align*}
& A_{m}=\left(\begin{array}{cc}
A_{m}^{(1)} & 0 \\
0 & A_{m}^{(2)}
\end{array}\right), \quad m=0, \ldots, 5 \\
& \lambda_{\alpha}=\left(\begin{array}{cc}
\lambda_{\alpha}^{(1)} & 0 \\
0 & \lambda_{\alpha}^{(2)}
\end{array}\right), \quad \alpha=1,2 \tag{32}
\end{align*}
$$

where $A_{\mu}$ with $\mu=0, \ldots, 3$ is the gauge field and $A_{4,5}$ are two real scalars. We use sixdimensional spinorial notations for the spinors. The gaugino fermions $\lambda_{1}$ and $\lambda_{2}$ are $S O(6)$ Weyl spinors of chirality -1 for $\Gamma^{012345}$. There are also two bifundamental hypermultiplets with component fields:

$$
q^{\alpha}=\left(\begin{array}{cc}
0 & q^{(1) \alpha}  \tag{33}\\
q^{(2) \alpha} & 0
\end{array}\right), \quad \psi=\left(\begin{array}{cc}
0 & \psi^{(1)} \\
\psi^{(2)} & 0
\end{array}\right)
$$

where $q^{1,2}$ are complex scalars and $\psi$ is an $S O(6)$ Weyl spinor of chirality +1 for $\Gamma^{012345}$. The Euclidean action of the $\mathcal{N}=2$ gauge theory is

$$
\begin{align*}
S_{\mathcal{N}=2}= & \int d^{4} x\left(\operatorname { T r } \left(\frac{1}{2} F_{m n} F^{m n}+i \bar{\lambda}^{\alpha} \Gamma^{m} D_{m} \lambda_{\alpha}+2 D_{m} q_{\alpha} D^{m} q^{\alpha}+2 i \bar{\psi} \Gamma^{m} D_{m} \psi\right.\right. \\
& \left.-2 \sqrt{2} g \bar{\lambda}^{\alpha A} q_{\alpha} T_{a} \psi+2 \sqrt{2} g \bar{\psi} T_{a} q^{\alpha} \lambda_{\alpha}^{A}\right)+2 g^{2} \operatorname{Tr}\left(q_{\alpha} T^{a} q^{\beta}\right) \operatorname{Tr}\left(q_{\beta} T_{a} q^{\alpha}\right) \\
& \left.-g^{2} \operatorname{Tr}\left(q_{\alpha} T_{a} q^{\alpha}\right) \operatorname{Tr}\left(q_{\beta} T^{a} q^{\beta}\right)\right) \tag{34}
\end{align*}
$$

where $T^{a}$ are the generators of the gauge group. We define $\bar{\lambda}^{\alpha}$ as $\bar{\lambda}^{\alpha}=-\epsilon^{\alpha \beta} \lambda_{\beta}^{c}$ where $\epsilon^{\alpha \beta}$ is the antisymmetric symbol with $\epsilon^{12}=1$. The fermions $\psi$ and $\bar{\psi}$ are independent. The $\mathcal{N}=2$ superconformal symmetry is still preserved when one leaves the orbifold point by independently varying the coupling constants for the two factors of the gauge group. The two coupling constants can be written as:

$$
g=\left(\begin{array}{cc}
g^{(1)} I_{N} & 0  \tag{35}\\
0 & g^{(2)} I_{N}
\end{array}\right)
$$

where $I_{N}$ denotes the $N \times N$ identity matrix. The definitions of the covariant derivatives are

$$
\begin{align*}
D_{\mu} \lambda & =\partial_{\mu} \lambda-i g\left[A_{\mu}, \lambda\right]  \tag{36}\\
D_{\mu} q^{\alpha} & =\partial_{\mu} q^{\alpha}-i g A_{\mu} q^{\alpha}  \tag{37}\\
D_{\mu} q_{\alpha} & =\partial_{\mu} q_{\alpha}+i g q_{\alpha} A_{\mu}  \tag{38}\\
D_{\mu} \Psi & =\partial_{\mu} \Psi-i g A_{\mu} \Psi \tag{39}
\end{align*}
$$

The $\mathcal{N}=2$ superconformal transformations are:

$$
\begin{align*}
& \delta A_{m}=-i \bar{\xi}^{\alpha} \Gamma_{m} \lambda_{\alpha}=i \bar{\lambda}^{\alpha} \Gamma_{m} \xi_{\alpha}, \\
& \delta q^{\alpha}=-i \sqrt{2} \bar{\xi}^{\alpha} \psi, \\
& \delta q_{\alpha}=-i \sqrt{2} \bar{\psi} \xi_{\alpha}, \\
& \delta \lambda_{\alpha}^{A}=\frac{1}{2} F_{m n}^{A} \Gamma^{m n} \xi_{\alpha}+2 i g q_{\alpha} T^{A} q^{\beta} \xi_{\beta}-i g q_{\beta} T^{A} q^{\beta} \xi_{\alpha}-2 A_{a}^{A} \Gamma^{a} \vartheta_{\alpha},  \tag{40}\\
& \delta \bar{\lambda}^{\alpha A}=-\frac{1}{2} \bar{\xi}^{\alpha} F_{m n}^{A} \Gamma^{m n}-2 i g q_{\beta} T^{A} q^{\alpha} \bar{\xi}^{\beta}+i g q_{\beta} T^{A} q^{\beta} \bar{\xi}^{\alpha}+2 \bar{\vartheta}^{\alpha} A_{a}^{A} \Gamma^{a}, \\
& \delta \psi=-\sqrt{2} D_{m} q^{\alpha} \Gamma^{m} \xi_{\alpha}-2 \sqrt{2} q^{\alpha} \vartheta_{\alpha}, \\
& \delta \bar{\psi}=\sqrt{2} \bar{\xi}^{\alpha} \Gamma^{m} D_{m} q_{\alpha}-2 \sqrt{2} \bar{\vartheta}^{\alpha} q_{\alpha},
\end{align*}
$$

where $\xi_{\alpha}=\theta_{\alpha}+x^{\mu} \Gamma_{\mu} \vartheta_{\alpha}$ has chirality -1 for $\Gamma^{012345}$ and the index $a=4,5$. The constant spinors $\theta_{\alpha}$ and $\vartheta_{\alpha}$ are parameters associated with Poincaré supersymmetry and conformal supersymmetry respectively.

There are only two real adjoint scalars in the $\mathcal{N}=2$ theory. One can define a planar bosonic Zarembo loop:

$$
\begin{equation*}
W_{\mathrm{bos}}=\frac{1}{2 N} \operatorname{Tr} \mathcal{P} e^{i \int \mathrm{~d} \tau L_{\mathrm{bos}}(\tau)}, \quad L_{\mathrm{bos}}=g \dot{x}^{\mu}\left(A_{\mu}+i M_{\mu}^{a} A_{a}\right) . \tag{41}
\end{equation*}
$$

Without loss of generality, we choose a contour inside the 01 plane and take $M_{\mu}^{a}=\delta_{\mu+4}^{a}$. The Poincaré supersymmetries preserved by this bosonic WL satisfy

$$
\begin{equation*}
\dot{x}^{\mu}\left(\Gamma_{\mu}+i \Gamma_{a} M_{\mu}^{a}\right) \theta_{\alpha}=0 \tag{42}
\end{equation*}
$$

Parameterizing $\theta_{\alpha}$ as $\theta_{\alpha}=\chi s_{\alpha}$ with a real Grassmann number $\chi$, we find

$$
\begin{equation*}
\left(\frac{1}{2}+\frac{i}{2} \Gamma_{04}\right) s_{\alpha}=\left(\frac{1}{2}+\frac{i}{2} \Gamma_{15}\right) s_{\alpha}=0 . \tag{43}
\end{equation*}
$$

For each $\alpha$ there is only one linearly independent solution and $s_{1} \propto s_{2}$. So the WL preserves two supersymmetries.

The connection of the fermionic loop can be constructed as a supermatrix

$$
\begin{equation*}
L=L_{\mathrm{bos}}+F, \tag{44}
\end{equation*}
$$

where the fermionic part takes the form:

$$
F=g|\dot{x}|\left(\zeta^{c} \psi+\bar{\psi} \eta\right), \quad \zeta^{c}=\left(\begin{array}{cc}
\zeta^{(1) c} I_{N} & 0  \tag{45}\\
0 & \zeta^{(2) c} I_{N}
\end{array}\right), \quad \eta=\left(\begin{array}{cc}
\eta^{(2)} I_{N} & 0 \\
0 & \eta^{(1)} I_{N}
\end{array}\right)
$$

The bosonic spinors $\zeta$ and $\eta$ satisfy

$$
\begin{equation*}
\left(\frac{1}{2}+\frac{i}{2} \Gamma_{04}\right) \zeta=\left(\frac{1}{2}+\frac{i}{2} \Gamma_{15}\right) \zeta=\left(\frac{1}{2}+\frac{i}{2} \Gamma_{04}\right) \eta=\left(\frac{1}{2}+\frac{i}{2} \Gamma_{15}\right) \eta=0, \tag{46}
\end{equation*}
$$

and they can depend on $\tau$.

We define the fermionic supersymmetry generator $Q_{s}$ as $\delta_{\theta}=\sqrt{2} \chi Q_{s}$ by extracting a real Grassmann number $\chi$. For any $s_{\alpha}$ satisfying (43), we have $Q_{s} F=0$. Therefore we can define a BPS WL by using a trace or supertrace:

$$
\begin{equation*}
W_{\mathrm{tr}}=\frac{1}{2 N} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} L d \tau\right), \quad \text { or } \quad W_{\mathrm{str}}=\frac{1}{2 N} \mathrm{~s} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} L d \tau\right) . \tag{47}
\end{equation*}
$$

Both of them preserve two Poincaré supersymmetries. It is straightforward to show that $F$ can be constructed by acting $Q_{s}$ on a linear combination of $q_{\alpha}$ and $q^{\alpha}$, so the The fermionic Zarembo loops are classically $Q_{s}$-cohomological equivalent to the bosonic one.

## 4 Conclusion and discussions

In this paper, we have constructed fermionic Zarembo loops in four-dimensional $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ superconformal $S U(N) \times S U(N)$ quiver theories. These loops are generalizations of the bosonic Zarembo loops. In the construction, we strongly made use of special properties of Poincaré supercharges preserved by the bosonic Zarembo loops. We examined how the choice of contour and connection parameters affects the number of supersymmetries preserved by the fermionic Zarembo loops. We have shown that the fermionic Zarembo loops are cohomologically equivalent to the bosonic ones and computed their expectation values in perturbation theory up to order $g^{4}$ in $\mathcal{N}=4$ SYM. We have also discussed the possibility of preserving conformal supercharges.

Our results provide new examples of BPS WLs in four-dimensional superconformal gauge theories. They also raise some open questions and potential extensions of our work. It would be interesting to study the holographic duals of the fermionic Zarembo loops in IIB superstring theory on $A d S_{5} \times S^{5}$ or its orbifold background. It would be worthwhile to consider other generalizations of known fermionic BPS WLs such as possible fermionic DGRT loops.

Both $\mathcal{N}=4$ SYM and ABJM theories are integrable in the planar limit. If we insert a composite operator inside the WL, the WL provides boundary conditions/interactions for the open spin chain from the composite operator. Half-BPS WLs in both theories lead to integrable open spin chains [39-41]. The correlation function of a WL and a single trace operator which is an eigenstate of the dilatation operator is proportional to the overlap of a boundary state and a Bethe state. For half-BPS WLs in the antisymmetric representation in $\mathcal{N}=4$ theory, such boundary states are integrable in the planar limit [42]. For bosonic $1 / 6$-BPS WLs and half-BPS WLs in the fundamental representation in ABJM theory, such boundary states are integrable at least at tree level in the planar limit [43]. It is interesting to study whether the fermionic BPS WLs constructed in [35] and this paper can lead to integrable open chains and/or integrable boundary states.

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## A Conventions and useful formulas

The action of $\mathcal{N}=4 \mathrm{SYM}$ in Euclidean signature is:

$$
\begin{equation*}
S_{\mathcal{N}=4}=\int d^{4} x\left(\frac{1}{2} \operatorname{Tr}\left(F_{M N} F^{M N}\right)+i \operatorname{Tr}\left(\bar{\Psi} \Gamma^{M} D_{M} \Psi\right)\right) \tag{48}
\end{equation*}
$$

We split the ten-dimensional indices $M, N, P=0, \ldots 9$ into two groups: $\mu, \nu=0,1,2,3$ and $I, J=4, \ldots, 9$. We denote $A_{M}=\left(A_{\mu}^{a}, \Phi_{I}^{a}\right) T^{a}, \Psi=\Psi^{a} T^{a}$ and $\Gamma^{M}=\left(\Gamma^{\mu}, \Gamma^{i}\right)$. In Euclidean signature $\bar{\Psi}=\Psi^{T} C$ where $C$ is the charge conjugation matrix. The field strength is defined as

$$
\begin{equation*}
F_{M N}=\partial_{M} A_{N}-\partial_{N} A_{M}-i g\left[A_{M}, A_{N}\right] \tag{49}
\end{equation*}
$$

And the covariant derivative is defined as

$$
\begin{equation*}
D_{M} \Psi=\partial_{M} \Psi-i g\left[A_{M}, \Psi\right] \tag{50}
\end{equation*}
$$

Following the conventions in [35], we use the representation for the ten-dimensional gamma matrices:

$$
\begin{align*}
\Gamma_{(10)}^{m} & =I_{4} \otimes \Gamma_{(6)}^{m}, \quad m=0, \ldots, 5 \\
\Gamma_{(10)}^{p} & =\Gamma_{(4)}^{10-p} \otimes \Gamma_{(6)}^{012345}, \quad p=6, \ldots, 9 \tag{51}
\end{align*}
$$

The four-dimensional gamma matrices are

$$
\Gamma_{(4)}^{j}=\left(\begin{array}{cc}
0 & -i \sigma_{j}  \tag{52}\\
i \sigma_{j} & 0
\end{array}\right), \quad \Gamma_{(4)}^{4}=\left(\begin{array}{cc}
0 & I_{2} \\
I_{2} & 0
\end{array}\right)
$$

where $\sigma_{j}$ 's denote the Pauli matrices. The six-dimensional gamma matrices are

$$
\begin{align*}
& \Gamma_{(6)}^{0}=-\sigma_{2} \otimes \sigma_{3} \otimes \sigma_{3} \\
& \Gamma_{(6)}^{1}=\sigma_{1} \otimes \sigma_{3} \otimes \sigma_{3} \\
& \Gamma_{(6)}^{2}=-I_{2} \otimes \sigma_{1} \otimes \sigma_{3} \\
& \Gamma_{(6)}^{3}=-I_{2} \otimes \sigma_{2} \otimes \sigma_{3}  \tag{53}\\
& \Gamma_{(6)}^{4}=I_{2} \otimes I_{2} \otimes \sigma_{1} \\
& \Gamma_{(6)}^{5}=I_{2} \otimes I_{2} \otimes \sigma_{2}
\end{align*}
$$

which were also used in section 3, The charge conjugate matrices can be defined as

$$
\begin{align*}
C_{(4)} & =\left(\begin{array}{cc}
i \sigma_{2} & 0 \\
0 & i \sigma_{2}
\end{array}\right),  \tag{54}\\
C_{(6)} & =\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),  \tag{55}\\
C_{(10)} & =C_{(4)} \otimes C_{(6)} . \tag{56}
\end{align*}
$$

The commutation relations and normalization of the generators of the Lie algebra $s u(N)$ are,

$$
\begin{equation*}
\left[T^{a}, T^{b}\right]=i f^{a b c} T^{c}, \quad \operatorname{Tr} T^{a} T^{b}=\frac{1}{2} \delta^{a b} \tag{57}
\end{equation*}
$$

The structure constants satisfy the identity

$$
\begin{equation*}
\sum_{c, d} f^{a c d} f^{b c d}=N \delta^{a b} \tag{58}
\end{equation*}
$$

Adding terms involving the ghosts and gauge fixing terms to the Euclidean $\mathcal{N}=4$ SYM action, we get

$$
\begin{align*}
S_{\mathcal{N}=4}^{\mathrm{total}}=\int & d^{4} x \frac{1}{2}\left(\frac{1}{2}\left(F_{\mu \nu}^{a}\right)^{2}+\left(\partial_{\mu} \Phi_{i}^{a}+g f^{a b c} A_{\mu}^{b} \Phi_{i}^{c}\right)^{2}+i \bar{\Psi}^{a} \Gamma^{\mu}\left(\partial_{\mu} \Psi^{a}+g f^{a b c} A_{\mu}^{b} \Psi^{c}\right)\right. \\
& +i g f^{a b c} \bar{\Psi}^{a} \Gamma^{i} \Phi_{i}^{b} \Psi^{c}-g^{2} \sum_{i<j} f^{a b c} f^{a d e} \Phi_{i}^{b} \Phi_{j}^{c} \Phi_{i}^{d} \Phi_{j}^{e}+\partial^{\mu} \bar{c}^{a}\left(\partial_{\mu} c^{a}+g f^{a b c} A_{\mu}^{b} c^{c}\right)  \tag{59}\\
& \left.+\xi\left(\partial^{\mu} A_{\mu}^{a}\right)^{2}\right)
\end{align*}
$$

The unrenormalized gluon and scalar propagators up to one-loop order in the Feynman gauge using regularization by dimensional reduction can be found in [6]. Up to one loop order, the gluon propagator in $d=2 \omega$ in momentum space is

$$
\begin{equation*}
\Delta_{\mu \nu}^{a b}(p)=\delta^{a b} \frac{\delta_{\mu \nu}}{p^{2}}-g^{2} N \frac{\Gamma(2-\omega) \Gamma(\omega) \Gamma(\omega-1)}{(4 \pi)^{\omega} \Gamma(2 \omega)} \cdot 4(2 \omega-1) \delta^{a b} \frac{\delta_{\mu \nu}-p_{\mu} p_{\nu} / p^{2}}{\left(p^{2}\right)^{3-\omega}} \tag{60}
\end{equation*}
$$

and the scalar propagator is

$$
\begin{equation*}
D_{I J}^{a b}(p)=\delta^{a b} \frac{\delta_{I J}}{p^{2}}-g^{2} N \frac{\Gamma(2-\omega) \Gamma(\omega) \Gamma(\omega-1)}{(4 \pi)^{\omega} \Gamma(2 \omega)} \cdot 4(2 \omega-1) \frac{\delta_{i j} \delta^{a b}}{\left(p^{2}\right)^{3-\omega}} \tag{61}
\end{equation*}
$$

The fermion propagator to one loop order can be computed as

$$
\begin{align*}
S^{a b}(p) & =-\delta^{a b} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}}-g^{2} \int \frac{d^{2 \omega} l}{(2 \pi)^{2 \omega}} \frac{-p_{\mu} \Gamma^{\mu}}{p^{2}}\left(f^{a c d} \Gamma^{M}\right) \frac{-\left(p_{\nu}-l_{\nu}\right) \Gamma^{\nu}}{(p-l)^{2}}\left(f^{d c b} \Gamma_{M}\right) \frac{1}{l^{2}} \frac{-p_{\rho} \Gamma^{\rho}}{p^{2}} \\
& =-\delta^{a b} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}}-N \delta^{a b} g^{2} \int \frac{d^{2 \omega} l}{(2 \pi)^{2 \omega}} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}} \Gamma^{M} \frac{\left(p_{\nu}-l_{\nu}\right) \Gamma^{\nu}}{(p-l)^{2}} \Gamma_{M} \frac{1}{l^{2}} \frac{p_{\rho} \Gamma^{\rho}}{p^{2}}  \tag{62}\\
& =-\delta^{a b} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}}+8 N \delta^{a b} g^{2} \int \frac{d^{2 \omega} l}{(2 \pi)^{2 \omega}} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}} \frac{\left(p_{\nu}-l_{\nu}\right) \Gamma^{\nu}}{(p-l)^{2} l^{2}} \frac{p_{\rho} \Gamma^{\rho}}{p^{2}} \\
& =-\delta^{a b} \frac{p_{\mu} \Gamma^{\mu}}{p^{2}}+g^{2} N \delta^{a b} \frac{\Gamma(2-\omega) \Gamma(\omega) \Gamma(\omega-1)}{(4 \pi)^{\omega} \Gamma(2 \omega)} \cdot 8(2 \omega-1) \frac{p_{\mu} \Gamma^{\mu}}{\left(p^{2}\right)^{3-\omega}},
\end{align*}
$$

where we have used

$$
\begin{equation*}
\int \frac{d^{2 \omega} l}{(2 \pi)^{2 \omega}} \frac{1}{\left(l^{2}+2 l \cdot p+M^{2}\right)^{A}}=\frac{\Gamma(A-\omega)}{(4 \pi)^{\omega} \Gamma(A)\left(M^{2}-p^{2}\right)^{A-\omega}}, \tag{63}
\end{equation*}
$$

and the Feynman parameterization formula

$$
\begin{equation*}
\prod_{i} A_{i}^{-n_{i}}=\frac{\Gamma\left(\sum n_{i}\right)}{\prod_{i} \Gamma\left(n_{i}\right)} \int_{0}^{1} d x_{1} \cdots d x_{k} x_{1}^{n_{1}-1} \cdots x_{k}^{n_{k}-1} \frac{\delta\left(1-\sum x_{i}\right)}{\left(\sum_{i} A_{i} x_{i}\right)^{\sum n_{i}}} . \tag{64}
\end{equation*}
$$

The propagators in position space can be obtained via the Fourier transform

$$
\begin{equation*}
\int \frac{d^{2 \omega} p}{(2 \pi)^{2 \omega}} \frac{e^{i p \cdot x}}{p^{2 s}}=\frac{\Gamma(\omega-s)}{4^{s} \pi^{\omega} \Gamma(s)} \frac{1}{\left(x^{2}\right)^{\omega-s}} . \tag{65}
\end{equation*}
$$

The results are

$$
\begin{align*}
\Delta_{\mu \nu}^{a b}(x) & =\delta^{a b} \delta_{\mu \nu} \frac{\Gamma(\omega-1)}{4 \pi^{\omega}} \frac{1}{\left(x^{2}\right)^{\omega-1}} \\
& +g^{2} N \delta^{a b} \frac{\Gamma(\omega-1) \Gamma(\omega-3)}{64 \pi^{2 \omega}(2 \omega-3)} \frac{\delta_{\mu \nu} x^{2}(2 \omega-5)+x_{\mu} x_{\nu}(6-4 \omega)}{\left(x^{2}\right)^{2 \omega-2}},  \tag{66}\\
D_{I J}^{a b}(x) & =\delta^{a b} \delta_{I J} \frac{\Gamma(\omega-1)}{4 \pi^{\omega}} \frac{1}{\left(x^{2}\right)^{\omega-1}}+g^{2} N \delta^{a b} \delta_{I J} \frac{\Gamma(\omega-1) \Gamma(\omega-2)}{32 \pi^{2 \omega}(2 \omega-3)} \frac{1}{\left(x^{2}\right)^{2 \omega-3}},  \tag{67}\\
S^{a b}(x) & =-i \delta^{a b} \frac{\Gamma(\omega)}{2 \pi^{\omega}} \frac{x^{\mu} \Gamma_{\mu}}{\left(x^{2}\right)^{\omega}}-i g^{2} N \delta^{a b} \frac{\Gamma(\omega-1) \Gamma(\omega-2)}{8 \pi^{2 \omega}} \frac{x^{\mu} \Gamma_{\mu}}{\left(x^{2}\right)^{2 \omega-2}} . \tag{68}
\end{align*}
$$

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[^1]:    ${ }^{1}$ This restriction leads to the Wu-Mandelstam-Leibbrandt prescription of reguralization $22,24$.
    ${ }^{2}$ The connection in this covariant derivative is just the connection used to define the WL.

[^2]:    ${ }^{3}$ Precisely speaking, Zarembo only investigated the Poincaré supercharges preserved by these WLs. The counting of the supercharges in this and the next subsections is only for Poincaré supercharges as well.

