# SUBSETS OF GROUPS WITH CONTEXT-FREE PREIMAGES

ALEX LEVINE

ABSTRACT. We study subsets E of finitely generated groups where the set of all words over a given finite generating set that lie in E forms a context-free language. We call these sets *recognisably context-free*. They are invariant of the choice of generating set and a theorem of Muller and Schupp fully classifies when the set  $\{1\}$  can be recognisably context-free. We show that every conjugacy class of a group G is recognisably context-free if and only if G is virtually free. We also show that a coset whose Schreier coset graph is quasi-transitive is recognisably context-free if and only if the Schreier coset graph is quasi-isometric to a tree.

#### 1. INTRODUCTION

For each finitely generated group, it is possible to define a wide variety of natural formal languages arising from different aspects of the group. One of the most widely studied is the *word problem* of a group, which is the language of all words over a given finite generating set that represent the identity. Anisimov first introduced the word problem and showed that the word problem of a group G is a regular language if and only if G is finite [1]. The class of groups with contextfree word problem was shown to be the class of virtually free groups by Muller and Schupp [24] along with a result of Dunwoody [8]. Herbst also showed that a group has a one-counter word problem if and only if it is virtually cyclic, and Holt, Owens and Thomas showed that a group is virtually abelian of rank k if and only if its word problem is the intersection of k one-counter languages [14]. Various attempts have also been made to classify groups with word problems that are poly-context-free languages [4], multiple context-free languages [19, 27] and the languages of blind k-counter automata [9].

A subset E of a finitely generated group is called *recognisably context-free* if the language of all words representing elements of E is context-free. Recognisably context-free sets were introduced by Herbst [11], although Muller and Schupp had already studied them in the guise of context-free word problems. Asking if the set  $\{1\}$  is recognisably context-free is equivalent to asking if a word problem is context-free, and thus the Muller-Schupp Theorem fully classifies in which groups  $\{1\}$  is recognisably context-free. Whilst a group must be finitely generated in order to define recognisably context-free subsets, the choice of generating set does not matter.

The complement of the word problem, called the *coword problem* has also been widely studied, and asking if the coword problem is context-free is equivalent to asking if the set  $G \setminus \{1\}$  is recognisably context-free in G. Many examples exist of non-virtually free groups with a context-free (but not deterministic) coword problem, including virtually abelian groups [16], Higman-Thompson groups and Houghton groups [21]. There is a conjecture that a finitely generated group has a context-free coword problem if and only if it embeds into Thompson's group V [3,22].

 $<sup>2020\</sup> Mathematics\ Subject\ Classification.\ 03D05,\ 20F10,\ 20F65,\ 68Q45.$ 

Key words and phrases. context-free languages, virtually free groups, recognisably context-free sets.

Herbst's study of recognisably context-free sets showed that if a group G has the property that a subset  $R \subseteq G$  is rational if and only if R is recognisably context-free, then G is virtually cyclic [11]. Various other lemmas were shown, including that the recognisably context-free sets are not affected by changing generating set, and are stable under multiplication by rational sets. A corollary to this is that in virtually free groups, rational sets are recognisably context-free. Herbst also studied the case when a finite set is recognisably (deterministic) context-free, showing that admitting a finite recognisably context-free subset, admitting a finite deterministic recognisably context-free subset and being virtually free are all equivalent [12].

Carvalho studied recognisably context-free subsets, showing that a group is virtually free if and only if for all finitely generated subgroups H of G and all subsets  $K \subseteq H$ , K is recognisably context-free in G if and only if K is recognisably context-free in H [5].

Ceccherini-Silberstein and Woess studied when subgroups can be recognisably context-free (albeit using different nomenclature), and showed that a subgroup is recognisably context-free if and only if the corresponding Schreier coset graph is a context-free graph [6]; a condition dependant on the structure of the ends of the graph.

We first consider conjugacy classes. Whilst we do not fully classify all cases when a conjugacy class can be recognisably context-free, we do are able to classify the class of groups where every conjugacy class is recognisably context-free.

**Theorem 4.10.** Let G be a finitely generated group. Then every conjugacy class of G is recognisably context-free if and only if G is virtually free.

Our final section considers subgroups and cosets where the corresponding Schreier coset graph is quasi-transitive. It is not difficult to use the Muller-Schupp Theorem to answer the question for normal subgroups, however arbitrary subgroups require more work. Using a version of Stallings' Theorem for quasi-transitive graphs [10], we show the following:

**Theorem 5.4.** Let G be a finitely generated group,  $H \leq G$  and  $g \in G$  be such that the Schreier coset graph of (G, H) is quasi-transitive. Then Hg is recognisably context-free if and only if the Schreier coset graph of (G, H) is a quasi-tree.

Since completing this paper, the author has been made aware of a result of Rodaro, released a few months earlier that proves the same result [26], when taken together with the result of Ceccherini-Silberstein and Woess [6] that classifies when a Schreier coset graph is a context-free graph. Rodaro's method uses the context-free graphs introduced by Muller and Schupp [25], whereas our method is proved using a recent generalisation of Stallings' Theorem [10], avoiding context-free graphs entirely.

We begin with the preliminary knowledge required for later sections in Section 2. Section 3 gives a collection of basic properties of recognisably free subsets. We then discuss conjugacy classes in Section 4 and conclude with our results on subgroups and cosets in Section 5.

# 2. Preliminaries

We introduce concepts that will be used later. Please note that functions will always be written to the right of their arguments.



FIGURE 1. Finite state automaton for for  $\{a^m b c^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$ , with start state  $q_0$  and accept state  $q_0$ .

2.1. Formal languages. A language over an alphabet (a finite set)  $\Sigma$  is a subset of the free monoid  $\Sigma^*$ ; the set of finite sequences of elements of  $\Sigma$ , denoted  $a_1 \cdots a_n$ , rather than  $(a_1, \ldots, a_n)$ . Words over  $\Sigma$  are elements of  $\Sigma^*$ . We will use  $\varepsilon$  to denote the empty word. Since group elements can be represented as words over a finite monoid generating set, to avoid confusion between group elements and abstract words, when writing the length of a word w we use |w|; when writing the length of a group element, we write ||g||. To avoid similar confusion between equivalence as words, and as group elements, we write  $u =_G v$  if u and v are words representing the same element of a group G and  $u \equiv v$  if u and v are equivalent as words.

2.2. **Regular languages.** We give a very brief introduction to regular languages. We refer the reader to [15, Section 2.5] or [17, Chapters 2-4] for more information.

**Definition 2.1.** Let  $\Sigma$  be an alphabet (a finite set) and let  $\Gamma$  be a  $(\Sigma \cup \{\varepsilon\})$ -edge-labelled graph. A word  $w \in \Sigma^*$  traces a path in  $\Gamma$  from a vertex  $u \in V(\Gamma)$  to  $v \in V(\Gamma)$  if there is a path  $\gamma$  in  $\Gamma$  from u to v such that concatenating the labels of the edges in  $\gamma$  (in order) yields w.

**Definition 2.2.** A finite-state automaton is a tuple  $\mathcal{A} = (\Sigma, \Gamma, q_0, F)$ , where

- (1)  $\Sigma$  is an alphabet;
- (2)  $\Gamma$  is a finite edge-labelled directed graph with labels from  $\Sigma \cup \{\varepsilon\}$ ;
- (3)  $q_0 \in V(\Gamma)$  is called the *start state*;
- (4)  $F \subseteq V(\Gamma)$  is called the set of *accept states*.

We call vertices in  $\Gamma$  states. A word  $w \in \Sigma^*$  is accepted by  $\mathcal{A}$  if there is a path in  $\Gamma$  from  $q_0$  to a state in F, where w is the word obtained by concatenating the labels of the edges in the path. The language accepted by  $\mathcal{A}$  is the set of all words accepted by  $\mathcal{A}$ . A language is called regular if it accepted by a finite-state automaton.

**Example 2.3.** We will show that the language  $L = \{a^m b c^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$  is regular over  $\{a, b, c\}$ . The finite-state automaton defined in Figure 1 accepts a language that is contained in L, as reading any word in the automaton results in reading any number of as, followed by one b, followed by any number of cs. Moreover, if  $w = a^m b c^n \in L$ , then we can use this automaton to accept w by traversing the labelled by a at  $q_0 m$  times, then reading one b to transfer to  $q_1$ , then traversing the c edge n times, before being accepted. Thus this automaton accepts L, and L is a regular language.

2.3. Context-free languages. We define context-free languages. We give a very brief introduction to this class, but the reader can find more information in [15, Section 2.6] or [17, Chapters 5-7].

**Definition 2.4.** A context-free grammar is a tuple  $\mathcal{G} = (\Sigma, V, \mathcal{P}, \mathbf{S})$ , where

- (1)  $\Sigma$  is a finite alphabet;
- (2) V is a finite alphabet, disjoint from  $\Sigma$ , called the set of *non-terminals*;

- (3)  $\mathcal{P}$  is a finite subset of  $V \times (\Sigma \cup V)^*$ , called the set of *productions*. The production  $(\mathbf{A}, \omega)$  is usually denoted  $\mathbf{A} \to \omega$ .
- (4)  $\mathbf{S} \in V$  is called the *start symbol*.

An application of a production  $\mathbf{A} \to \omega$  to a word  $\nu \in (\Sigma \cup V)^*$  that contains  $\mathbf{A}$  is the action that replaces an occurrence of  $\mathbf{A}$  in  $\omega$ . A word  $w \in \Sigma^*$  is generated by  $\mathcal{G}$ , if w can be obtained from  $\mathbf{S}$ , by a finite sequence of applications of productions. The language generated by  $\mathcal{G}$ , denoted  $L(\mathcal{G})$ , is the set of all words generated by  $\mathcal{G}$ . A language that is generated by a context-free grammar is called *context-free*.

A derivation in  $\mathcal{G}$  is a finite sequence of applications of productions. We write  $\mathbf{A} \Rightarrow^* \omega$ , for  $\mathbf{A} \in V$ and  $\omega \in (\Sigma \cup V)^*$ , if there is a derivation that takes  $\mathbf{A}$  to  $\omega$ . A non-terminal  $\mathbf{A}$  in  $\mathcal{G}$  is called useless is there is no derivation in  $\mathcal{G}$  taking the start symbol  $\mathbf{S}$  to a word in the terminals via a word (in any combination of terminals and non-terminals) containing  $\mathbf{A}$ .

# Example 2.5. The language

 $L = \{w \in \{a, a^{-1}\}^* \mid w \text{ contains the same number of occurrences } a \text{ as } a^{-1}\}$ 

is context-free. We give an example of a context-free grammar for L. Let  $\mathcal{G} = (\{a, a^{-1}\}, \{\mathbf{S}\}, \mathcal{P}, \mathbf{S})$  be a context-free grammar, where  $\mathcal{P}$  contains the productions:

 $\mathbf{S} \to \mathbf{S}a\mathbf{S}a^{-1}\mathbf{S}, \quad \mathbf{S} \to \mathbf{S}a^{-1}\mathbf{S}a\mathbf{S}, \quad \mathbf{S} \to \varepsilon.$ 

We claim that  $\mathcal{G}$  generates L. Firstly, note that every word in L can be obtained from the empty word  $\varepsilon$  by a finite sequence of free expansions; that is, by iteratively inserting a subword of the form  $aa^{-1}$  or  $a^{-1}a$ . By using the first two productions, we can therefore start with  $\mathbf{S}$  and end with every word in  $w \in L$  with a number of occurrences of  $\mathbf{S}$  'mixed in'. We can use the third production to remove all occurrences of  $\mathbf{S}$ , to end up with w. Conversely, any word that  $\mathcal{G}$  generates must be obtainable from  $\varepsilon$  by a finite sequence of free expansions, from the construction of  $\mathcal{G}$ , and so  $\mathcal{G}$  only generates words in L. Thus  $\mathcal{G}$  generates L, as required.

The following lemma collects the standard closure properties of context-free languages.

**Lemma 2.6** ([15, Propositions 2.6.26, 2.6.32 and 2.6.34]). The class of context-free languages is closed under finite union, intersection with a regular languages, concatenation, Kleene star closure, image under free monoid homomorphism and preimage under free monoid homomorphism.

It is useful to be able to assume some context-free grammars we use are in Chomsky normal form. We give the definition below.

**Definition 2.7.** A context-free grammar  $(\Sigma, V, P, \mathbf{S})$  is in *Chomsky normal form* if every production is of the form  $\mathbf{A} \to \mathbf{BC}$  or  $\mathbf{A} \to \alpha$ , where  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$  and  $\alpha \in \Sigma$ .

**Lemma 2.8** ([15, Theorem 2.6.14]). Every context-free language is accepted by a context-free grammar in Chomsky normal form with no useless non-terminals.

We will also need the fact the context-free languages are closed under substitutions of context-free languages.

**Definition 2.9.** Let L and M be languages over an alphabet  $\Sigma$  and let  $a \in \Sigma$ . The substitution of a in L for M is the language of all words obtained from a word in L by replacing each occurrence of a with a word in M. That is,

 $\{u_0v_1u_1v_2\cdots v_nu_n \mid u_0au_1a\cdots au_n \in L, u_0, \ldots, u_n \in (\Sigma \setminus \{a\})^*, v_1, \ldots, v_n \in M\}.$ 

**Lemma 2.10** ([17, Theorem 7.23]). Let L and M be context-free languages over an alphabet  $\Sigma$  and let  $a \in \Sigma$ . Then the substitution of a in L for M is context-free.

2.4. **Pushdown automata.** An alternative definition for the class of context-free languages is the class of languages accepted by a pushdown automaton. We give the definition below. We refer the reader to [17, Chapter 6] for a more detailed introduction.

We informally describe a pushdown automaton before we give the definition. The idea is much the same as a finite-state automaton, with the exception that there is some memory - in the form of a finite word called the stack. When transitioning from one state to another, instead of just looking at what state one is currently in along with the letter (or word) being read, the top of the stack is also considered. When transitioning, one can remove ('pop') a (possibly empty) word from the top of the stack; that is remove a suffix. If the correct suffix does not exist in the stack, then the transition in question cannot be used. After popping a word, a new (again, possibly empty) word can be added ('pushed') to the end of the stack. All transitions take this form, and again the set of words that trace a path in the set of states, starting with the stack empty (we take empty stacks to contain precisely one letter, the bottom of stack symbol) and ending in an accept state (with any stack).

There are multiple (equivalent) definitions of pushdown automata. Some have bottom of stack symbols, whilst others do not. Most standard definitions only allow one letter (or  $\varepsilon$ ) to be read at a time. We allow any word to be read at a time, and thus this is what some authors call a *generalised pushdown automaton*.

**Definition 2.11.** A pushdown automaton is a 7-tuple  $\mathcal{A} = (Q, \Sigma, \chi, \bot, \delta, q_0, F)$ , where

- (1) Q is a finite set, called the set of *states*;
- (2)  $\Sigma$  is a (finite) alphabet;
- (3)  $\chi$  is a finite alphabet, disjoint from  $\Sigma$ , called the *stack alphabet*;
- (4)  $\perp \in \chi$  is called the *bottom of stack symbol*;
- (5)  $\delta \subseteq (Q \times \Sigma^* \times \chi^*) \times (Q \times \chi^*)$  is a finite set called the *transition relation*. We must have that pairs in  $\delta$  can only have at most one occurrence of  $\perp$  in each tuple in the pair, and if it occurs in one pair, it must occur in the other. This is to ensure that the bottom of stack symbol always tells us when the stack (the 'memory') is empty, and can never be removed. Transitions can be thought of as (not well-defined) functions, from  $Q \times \Sigma^* \times \chi^*$ to  $Q \times \chi^*$ ; they are not (necessarily) well-defined as each point in the 'domain' can have multiple 'images'.
- (6)  $q_0 \in Q$  is called the start state;
- (7)  $F \subseteq Q$  is called the set of *accept states*.

We say that Q is *deterministic* if for all stack words  $\nu \in \chi^*$ , all states  $q \in Q$  and all  $w \in \Sigma^*$ , there is a unique transition (or sequence of transitions) from q reading w for this given stack word  $\nu$ . The language *accepted* by  $\mathcal{A}$  is the language of all words w over  $\Sigma^*$  such that there is a finite sequence of transitions taking  $(q_0, \bot)$  to  $(q_f, \nu)$  whilst reading w, such that  $q_f \in F$  and  $\nu \in \chi^*$ is any stack. We write  $Q(\mathcal{A})$  and  $\chi(\mathcal{A})$  for the set of states and stack alphabet, respectively of a pushdown automaton  $\mathcal{A}$ .

**Example 2.12.** We saw in Example 2.5 that the language

 $L = \{w \in \{a, a^{-1}\}^* \mid w \text{ contains the same number of occurrences } a \text{ as } a^{-1}\}$ 

is context-free. We now define a pushdown automaton that accepts L. This idea of the pushdown automaton is to use the stack to track the freely reduced form of the word read so far, and then



FIGURE 2. Pushdown automaton defined in Example 2.12 that accepts  $L = \{w \in \{a, a^{-1}\}^* \mid w \text{ contains the same number of occurrences } a \text{ as } a^{-1}\}$ . The start state is  $q_0$  and the accept state is  $q_1$ . Each transition from a state to a state is written in the form  $(b, \alpha)/\beta$ , where b is the (terminal) letter read,  $\alpha$  is the stack word popped from the top of the stack and  $\beta$  is the stack word pushed to the top of the stack.

only accept when the stack is empty. We formally define the automaton, but Figure 2 contains a graphical representation. Our set of states will be  $\{q_0, q_1\}$ , where  $q_0$  is the start state and  $q_1$  is the (unique) accept state. The stack alphabet will be  $\chi = \{\perp, x, x^{-1}\}$ , with  $\perp$  the bottom of stack symbol. We then have six transitions from  $q_0$  to  $q_0$  and one transition from  $q_0$  to  $q_1$ :

 $\begin{array}{ll} (1) & (q_0, a, \bot) \to (q_0, x \perp); \\ (2) & (q_0, a, x) \to (q_0, xx); \\ (3) & (q_0, a, x^{-1}) \to (q_0, \varepsilon); \\ (4) & (q_0, a^{-1}, \bot) \to (q_0, x^{-1} \perp); \\ (5) & (q_0, a^{-1}, x) \to (q_0, \varepsilon); \\ (6) & (q_0, a^{-1}, x^{-1}) \to (q_0, x^{-1} x^{-1}); \\ (7) & (q_0, \varepsilon, \bot) \to (q_1, \bot); \end{array}$ 

The first six transitions simply track the freely reduced form of the word read so far (except using x rather than a) and the last transition confirms that the stack is empty; that is, that the word indeed equals the identity of the group  $\langle a | \rangle$  (that is, it lies in L) before moving to the accept state. If we move to  $q_1$  before finishing reading our word, we can never accept, as there are no transitions out of  $q_1$  that allow the rest of the word to be read.

**Lemma 2.13** ([15, Theorem 2.6.10]). A language is context-free if and only if it is accepted by a pushdown automaton.

**Definition 2.14.** A language is called *deterministic context-free* if it is accepted by a deterministic pushdown automaton.

**Lemma 2.15** ([15, Propositions 2.6.30 and 2.6.34]). The class of deterministic context-free languages is closed under complement and preimage under free monoid homomorphism. We will need the following lemma when classifying recognisably context-free cosets.

**Lemma 2.16.** Let L be a context-free language. Then L is accepted by a pushdown automaton  $\mathcal{A}$ , such that whenever an accept state in  $\mathcal{A}$  is reached, the stack is always empty.

Proof. As L is context-free, there is a pushdown automaton  $\mathcal{A}$  accepting L, with set of accept states F, stack alphabet  $\chi$  and bottom of stack symbol  $\bot$ . We modify  $\mathcal{A}$  to obtain a new pushdown automaton  $\mathcal{B}$  as follows. We start by adding two new states  $q_1$  and  $q_2$  to  $\mathcal{A}$ , and redefine the set of accept states to be  $\{q_2\}$ . We then add an  $\varepsilon$ -transition from each  $q \in F$  to  $q_1$  that does not alter the stack. For each  $\alpha \in \chi \setminus \{\bot\}$  we add an  $\varepsilon$ -transition from  $q_1$  to  $q_1$  that pops  $\alpha$  from the stack. We then add a  $\varepsilon$ -transition from  $q_1$  to  $q_2$  that pops  $\bot$  from the stack and then pushes  $\bot$  back onto the stack. By construction,  $\mathcal{B}$  can only accept when the stack is empty and  $\mathcal{B}$  must accept the same language as  $\mathcal{A}$ .

2.5. **Recognisable and rational sets.** Before we formally define recognisably context-free sets, we first cover their regular analogues: recognisable sets. We give the definition below.

**Definition 2.17.** Let G be a group with a finite generating set  $\Sigma$  and let  $\pi: \Sigma^* \to G$  be the natural homomorphism. A subset  $E \subseteq G$  is called *recognisable* with respect to  $\Sigma$  if  $E\pi^{-1}$  is a regular language.

Using the same argument as the proof of Lemma 2.21, we have that changing finite generating sets in a finitely does not affect whether a given subset is recognisable. Herbst and Thomas completely characterised recognisable subsets of groups in the following result.

**Proposition 2.18** ([13, Proposition 6.3]). A subset E of a finitely generated group G is recognisable if and only if E is a finite union of cosets of some finite-index subgroup of G.

A dual notion to recognisable sets is the concept of rational sets. Rather than having a regular full preimage, these are the image of a regular language under the natural map  $\pi$ , or equivalently.

**Definition 2.19.** Let G be a group with a finite generating set  $\Sigma$ , and let  $\pi: \Sigma^* \to G$  be the natural map. A subset  $E \subseteq G$  is called *rational* of  $E = L\pi$  for some regular language  $L \subseteq \Sigma^*$ .

As with recognisable sets, a similar argument to that in the proof of Lemma 2.21 shows that the class of rational sets is invariant under changing finite generating set.

2.6. Recognisably context-free sets. The earliest reference we can find to recognisably context-free sets is in a paper of Herbst [11]. While they are not given a name, the set of all recognisably context-free subsets of a group is denoted CF(G). Carvalho uses the term *context-free* instead of recognisably context-free [5]; we avoid this to maintain a clear distinction between recognisably context-free subsets of groups and context-free languages. We instead use the term recognisably context-free from [7]. We begin with the definition.

**Definition 2.20.** Let G be a finitely generated group,  $\Sigma$  be a finite monoid generating set, and  $\pi: \Sigma^* \to G$  be the natural homomorphism. A subset  $E \subseteq G$  is called *recognisably (deterministic)* context-free with respect to  $\Sigma$  if the full preimage  $E\pi^{-1}$  is (deterministic) context-free.

The following lemma is well-known (see, for example [11, Lemma 2.1]). We include a short proof for completeness.

**Lemma 2.21.** Let G be a finitely generated group. If  $E \subseteq G$  is recognisably (deterministic) contextfree with respect to one finite monoid generating set of G, then E is recognisably (deterministic) context-free with respect to all finite monoid generating sets of G.

*Proof.* Let  $\Sigma$  and  $\Delta$  be finite monoid generating sets for G, and  $\pi_{\Sigma} \colon \Sigma^* \to G$  and  $\pi_{\Delta} \colon \Delta^* \to G$ be the natural homomorphisms. Suppose that E is (deterministic) recognisably context-free with respect to  $\Sigma$ . For all  $a \in \Delta$  there exists  $\omega_a \in \Sigma^*$ , such that  $\omega_a =_G a$ . Define the free monoid homomorphism:

$$\phi \colon \Delta^* \to \Sigma^*$$
$$a \mapsto \omega_a$$

Then  $E\pi_{\Delta}^{-1} = (E\pi_{\Sigma}^{-1})\phi^{-1}$ , which is context-free as the class of (deterministic) context-free languages is closed under preimages of free monoid homomorphisms (Lemma 2.6 and Lemma 2.15).

As Lemma 2.21 shows, whether or not a subset of a group is recognisably context-free is not affected by the choice of generating set. Thus we can say that a subset of a group is recognisably context-free, omitting the generating set. As the above proof only relies on the fact that the class of languages is closed under preimages of free monoid homomorphisms, it holds for recognisable (regular) sets as well, so we will also omit the generating set when referring to such sets.

2.7. Quasi-isometries, trees and triangulations. Quasi-isometries between metric spaces are a central notion to geometric group theory. In the later sections we will show certain groups are virtually free is by showing that their Cayley graphs are quasi-isometric to trees. We give a brief definition along with a characterisation of graphs that are quasi-isometric to trees. We refer the reader to [23, Section 11] for an in-depth introduction to quasi-isometries from the viewpoint of geometric group theory.

**Definition 2.22.** Let X and Y be metric spaces. A function  $f: X \to Y$  is a quasi-isometry if there exist constants  $\lambda \geq 1$  and  $\mu \geq 0$ , such that:

- (1) For all  $x_1, x_2 \in X$ ,  $\frac{1}{\lambda}d(x_1, x_2) \mu \leq d((x_1)f, (x_2)f) \leq \lambda d(x_1, x_2) + \mu$ ; (2) For all  $y \in Y$  there exists  $x \in X$ , such that  $d((x)f, y) \leq \mu$ .

If a quasi-isometry from X to Y exists, we say X and Y are quasi-isometric.

**Remark 2.23.** The property of being quasi-isometric is symmetric, reflexive and transitive.

A particular class of graphs we will be using frequently is the class of graphs that are quasi-isometric to trees.

**Definition 2.24.** A graph is called a *quasi-tree* if it is quasi-isometric to a tree.

As most of the graphs we deal with will be locally finite, we define this concept as well.

**Definition 2.25.** A graph is called *locally finite* if the degree of every vertex is finite.

We now define a triangulation. The definition of a triangulation is the one used in [2]. Triangulations are a key part in the proof of the result of Muller and Schupp. Showing that there exists  $m \in \mathbb{Z}_{>0}$ such that every circuit in a given graph is *m*-triangulable is sufficient to show that this graph is a quasi-tree. This is often easier than explicitly constructing a quasi-isometry.



FIGURE 3. Triangulations

**Definition 2.26.** Let  $\Gamma$  be a graph. Let  $m, n \in \mathbb{Z}_{>0}$ . An *m*-sequence of length n in  $\Gamma$  is a sequence  $(v_0, \ldots, v_n)$  of elements of  $V(\Gamma)$ , such that  $v_0 = v_n$ , and  $d_{\Gamma}(v_i, v_{i+1}) \leq m$  for all i. An *m*-sequence is called *m*-reducible, if there exists  $i \in \{1, \ldots, n-1\}$ , such that  $d_{\Gamma}(v_{i-1}, v_{i+1}) \leq m$ . In such a case, an *m*-reduction of this *m*-sequence at i, is the operation that outputs the *m*-sequence  $(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ .

An *m*-triangulation of an *m*-sequence is a finite sequence of *m*-reductions that results in an *m*-sequence of length at most 4 (with 3 distinct points, as the first and last are equal), called the *core* of the triangulation. An *m*-sequence that admits an *m*-triangulation is called *m*-triangulable. Note that as 1-sequences are *m*-sequences, we can say that a 1-sequence is *m*-triangulable.

**Remark 2.27.** The *m*-reductions in an *m*-triangulations can be depicted by drawing a line. This lets us depict the entire triangulation as a number of lines added to our circuit (see Fig. 3).

Part of the proof of the Muller-Schupp Theorem involves showing that the Cayley graphs in groups with a context-free word problem are *m*-triangulable for some  $m \in \mathbb{Z}_{>0}$ . It is well-known that this property is equivalent to being quasi-isometric to a tree:

**Theorem 2.28** ([2, Theorem 4.7]). A graph is m-triangulable for some  $m \in \mathbb{Z}_{>0}$  if and only if it is a quasi-tree.

After this, Stalling's Theorem, together with Dunwoody's accessibility result [8] can be used to show that groups whose Cayley graphs are quasi-isometric to trees are virtually free.

2.8. Cayley graphs and Schreier coset graphs. We briefly recall the definitions of Cayley graphs and Schreier coset graphs.

**Definition 2.29.** Let G be a group with a finite inverse closed generating set  $\Sigma$ . The *(right) Cayley* graph of G with respect to  $\Sigma$  is the directed  $\Sigma$ -edge-labelled graph whose vertices are the elements of G, and with an edge labelled a from g to ga for all  $g \in G$  and  $a \in \Sigma$ .

**Theorem 2.30** ([2, Theorem 4.7]). A finitely generated group is virtually free if and only if it has a Cayley graph that is a quasi-tree.

**Definition 2.31.** Let G be a group with a finite inverse closed generating set  $\Sigma$  and let  $H \leq G$ . The *(right) Schreier coset graph* of (G, H) with respect to  $\Sigma$  is the directed  $\Sigma$ -edge-labelled graph whose vertices are the right cosets of H in G, and with an edge labelled a from each coset Hg to Hga for all  $a \in \Sigma$ .

**Remark 2.32.** As with Cayley graphs, Schreier coset graphs are dependent on the choice of generating set, however different Schreier coset graphs for the same pair (G, H), where G is finitely generated will be quasi-isometric. We will thus refer to the Schreier coset graph of (G, H) when talking about properties of graphs that are invariant under quasi-isometries.

**Definition 2.33.** A graph  $\Gamma$  is called *(vertex) transitive* if for all  $u, v \in V(\Gamma)$  there exists  $\phi \in \operatorname{Aut}(\Gamma)$  with  $(u)\phi = v$ ; that is,  $\Gamma$  has a unique automorphic orbit. We say  $\Gamma$  is *quasi-transitive* if it has finitely many automorphic orbits.

2.9. Tree amalgamations. The Muller-Schupp Theorem was proved by first showing that if a finitely generated group has a context-free word problem, then its Cayley graph is quasi-isometric to a tree. Then Stallings' Theorem together with Dunwoody's (later) accessibility result showed that a finitely generated group that is quasi-isometric to a tree is virtually free. At this point, the proof can be completed by showing that virtually free groups have (deterministic) context-free word problems.

In Section 5, we extend this result to show that a coset of a subgroup of a finitely generated group whose Schreier coset graph is quasi-transitive, is recognisably context-free if and only if the Schreier coset graph is a quasi-tree. For this, we a generalisation of Stallings' Theorem to quasi-transitive graphs. Thus gives us an alternative characterisation of quasi-trees which can be used to show that the 'language' of a transitive quasi-tree is context-free.

The definitions in the subsection are from Section 5 of [10]. We begin with the definition of a tree amalgamation. All of the graphs used here are considered to be simple graphs (that is, no multiple edges, loops or directions). Since we only use tree amalgamations to show graphs are quasi-isometric, and since forgetting directions of Cayley and Schreier graphs of groups does not affect the metric, this is sufficient for our purposes. Since we have at most one edge between two vertices, we can define edges to be subsets of the set of vertices of size 2; the two vertices in each edge being its endpoints.

**Definition 2.34.** Let  $\Gamma_1$  and  $\Gamma_2$  be graphs. Let  $(S_k^i)_{i \in I_i}$  be a collection of subsets of  $V(\Gamma_i)$ , for  $i \in \{1, 2\}$ , where each  $I_i$  is an index set, such that all  $S_k^i$  have the same cardinality and  $I_1 \cap I_2 = \emptyset$ . For each  $k \in I_1$  and  $l \in I_2$  let  $\phi_{kl} \colon S_k^1 \to S_l^2$  be a bijection, and let  $\phi_{lk} = \phi_{kl}^{-1}$ .

Let T be the  $(|I_1|, |I_2|)$ -semiregular tree; that is the bipartite tree whose vertices are partitioned into  $V(T) = V_1 \cup V_2$  such that all vertices in  $V_i$  have degree  $|I_i|$ . Let D(T) be the set of directed edges obtained from T by taking each edge  $\{u, v\}$  and taking its two directed versions (u, v) and (v, u). We also attach a labelling  $f: D(T) \to I_1 \cup I_2$  such that for all  $t \in V_i$  the set of labels of the incident is precisely the set  $I_i$  and each label occurs on precisely one incident edge.

For each  $t \in V_i$  let  $\Gamma_t$  be an isomorphic copy of  $\Gamma_i$ . Let  $S_k^t$  denote the copy of  $S_k^i$  within  $\Gamma_t$ . Let  $\Lambda$  be the disjoint union the graphs  $\Gamma_t$  for all  $t \in V(T)$ . We now quotient  $\Lambda$  as follows. For each edge  $(s,t) \in D(T)$  with (s,t) f = k and (t,s)f = l, identify all vertices  $v \in S_k^s$  with the vertex  $(v)\phi_{kl}$  in  $S_l^t$ . The quotient graph obtained is called the *tree amalgamation* of  $\Gamma_1$  with  $\Gamma_2$  over the *connecting tree* T, and denoted  $\Gamma_1 * \Gamma_2$ .

These functions  $\phi_{kl}$  called the *bonding maps* of the tree amalgamation and the sets  $S_k^i$  are called the *adhesion sets*. If all adhesion sets within a tree amalgamation are finite, the tree amalgamation is said to have *finite adhesion*. The *identification size* of a vertex  $v \in V(\Gamma_1 * \Gamma_2)$  is number of vertices in  $V(\Lambda)$  that are identified when quotienting to obtain  $\Gamma_1 * \Gamma_2$ . The tree amalgamation has *finite identification* if all identification sizes are finite. We refer the reader to [10, Examples 5.2-5.6] for a variety of examples of tree amalgamations.

As with groups we also require an accessibility result; it is not enough to say graphs can successively be expressed as tree amalgamations, this process must terminate, and the resultant graphs must be sufficiently understood. We combine these concepts in the following definition.

**Definition 2.35.** A graph  $\Gamma$  is said to admit a *terminal factorisation of finite graphs* if there exists a finite collection of finite graphs from which  $\Gamma$  can be built by a finite sequence of successive tree amalgamations with finite adhesion and finite identification.

We state two results which together give a classification of connected quasi-transitive locally finite quasi-trees. These theorems mention the property of having only thin ends. When combined together, this property is not present, so we do not define it here.

**Theorem 2.36** ([18, Theorem 5.5]). A connected quasi-transitive locally finite graph has only thin ends if and only if it is a quasi-tree.

**Theorem 2.37** ([10, Theorem 7.5]). A connected quasi-transitive locally finite graph has only thin ends if and only if it admits a terminal factorisation of finite graphs.

Combining Theorem 2.36 with Theorem 2.37 gives the following.

**Theorem 2.38.** A connected quasi-transitive locally finite graph is a quasi-tree if and only if it admits a terminal factorisation of finite graphs.

#### 3. Basic properties

This section covers various basic (closure) properties of recognisably context-free sets. We begin with a result of Herbst.

**Proposition 3.1** ([11], Lemma 4.1). Let G be a finitely generated group, let  $A \subseteq G$  be recognisably context-free and let  $R \subseteq G$  be rational. Then AR and RA are recognisably context-free.

We rarely use Proposition 3.1 in its full generality. It is mostly used when R is a singleton (which is always rational as the image of a one-word language). We thus state it in this restricted form to make it clear which rational subset we are using.

**Corollary 3.2.** Let G be a finitely generated group, and let  $A \subseteq G$  be recognisably context-free. For all  $g \in G$ , Ag and gA are recognisably context-free.

An interesting corollary to Proposition 3.1 is that rational subsets of virtually free groups are recognisably context-free. The converse is not true - recognisably context-free subsets of virtually free groups are not always rational. Conjugacy classes provide one such counter-example (see for example Theorem 4.10). In fact, Herbst showed that the class of groups such that a subset is rational if and only if it is recognisably context-free is precisely the class of virtually cyclic groups [11, Theorem 3.1].

**Theorem 3.3** ([11], Lemma 4.2 and Theorem 3.1). In a finitely generated virtually free group, every rational subset is recognisably context-free. A finitely generated group G has the property that the classes of rational and recognisably context-free sets coincide if and only if G is virtually cyclic.

Using the fact that context-free languages are stable under intersections with regular languages, we can make the following observation.

**Lemma 3.4.** The intersection of a recognisably context-free set with a recognisable set is recognisably context-free.

Proof. Let G be a finitely generated group,  $C \subseteq G$  be recognisable context-free, and  $R \subseteq G$  be recognisable. Fix a finite monoid generating set  $\Sigma$  for G, and let  $\pi: \Sigma^* \to G$  be the natural homomorphism. Then  $C\pi^{-1} \cap R\pi^{-1} = (C \cap R)\pi^{-1}$ . As the intersection of a context-free language with a regular language, this language is context-free (Lemma 2.6). Thus  $C \cap R$  is recognisably context-free.

Using Lemma 3.4 along with the fact that cosets of finite-index subgroups are always recognisable, we can classify recognisably context-free subsets of a group in terms of recognisably context-free subsets of any of its finite-index subgroups.

**Proposition 3.5** ([5, Proposition 3.6]). Let G be a finitely generated group, and H be a finite-index subgroup. Let T be a (finite) right transversal for H in G. Suppose C is a recognisably context-free subset of G. Then for each  $t \in T$  there exists a recognisably context-free  $C_t$  of H, such that

$$C = \bigcup_{t \in T} C_t t$$

It is well-known that the class of context-free languages is stable under preimages of free monoid homomorphisms. We prove an analogous statement holds for recognisably context-free subsets of a given group.

**Proposition 3.6.** Let G and H be a finitely generated groups and  $\phi: G \to H$  be a epimorphism. If  $A \subseteq H$  is a recognisably context-free subset of H, then  $A\phi^{-1}$  is a recognisably context-free subset of G.

Proof. Fix a finite monoid generating set  $\Sigma$  for G. Then  $\Sigma\phi$  is a finite monoid generating set for H. Let  $\pi_G \colon \Sigma^* \to G$  and  $\pi_H \colon (\Sigma\phi)^* \to H$  be the natural homomorphisms. Let  $\bar{\phi} \colon \Sigma^* \to (\Sigma\phi)^*$  be the homomorphism that extends  $a \mapsto a\phi$  for all  $a \in \Sigma$ . We will show that  $A\phi^{-1}\pi_G^{-1} = A\pi_H^{-1}\bar{\phi}^{-1}$ . To show this, we need to show that  $w\bar{\phi}\pi_H \in A$  if and only if  $w\pi_G\phi \in A$ . But this is true, as the element of H that  $w\bar{\phi}$  represents is  $w\pi_G\phi$ .

#### 4. Conjugacy classes

The aim of this section is to classify all finitely generated groups where every conjugacy class is recognisably context-free, which ends up being the class of virtually free groups. We do not provide a full classification of when conjugacy classes are recognisably context-free, although we briefly discuss the case when a group admits a recognisably context-free conjugacy class.

Most of the work in this section is therefore to prove that conjugacy classes in finitely generated virtually free groups are recognisably context-free. Since virtually free groups always admit finite-index normal free subgroups, we can define the multiplication in a finitely generated virtually free group using a finite-index free normal subgroup, a (finite) right transversal, and the action (by automorphisms) of the transversal on the normal subgroup. We begin with the definition of a  $\phi$ -cyclic permutation, where  $\phi$  is an automorphism of a free group; a generalisation of cyclic permutations.

Thus before we can show that conjugacy classes in virtually free groups are recognisably contextfree, we must first show that  $\phi$ -twisted conjugacy classes are recognisably context-free in free groups, where  $\phi$  is a virtually inner automorphism. We start by defining virtually inner automorphisms.

**Definition 4.1.** Let F be a finite rank free group. We say an automorphism  $\phi \in Aut(F)$  is *virtually inner* if there exists  $k \in \mathbb{Z}_{\geq 0}$  such that  $\phi^k$  is an inner automorphism.

**Definition 4.2.** Let F be a finite rank free group with basis  $\Sigma$ , and  $\phi \in \operatorname{Aut}(F)$ . Define  $\sim_{\phi}$  on the set of freely reduced words in  $(\Sigma \cup \Sigma^{-1})^*$  to be the transitive closure of the binary relation

 $\{(uv, v(u\phi)) \mid u, v \in \Sigma^* \text{ freely reduced}\} \cup \{(v(u\phi)), uv) \mid u, v \in \Sigma^* \text{ freely reduced}\},\$ 

where uv is the freely reduced word obtained by concatenating and freely reducing, and  $v(u\phi)$  is the freely reduced word obtained by applying  $\phi$ , concatenating and freely reducing.

As  $\sim_{\phi}$  is defined on the set of freely reduced words, we can therefore define  $\sim_{\phi}$  on F as well. We say g is a  $\phi$ -cyclic permutation if  $g \sim_{\phi} h$ . We say g is a  $\phi$ -twisted conjugate of h if there exists  $x \in F$  such that  $x^{-1}g(x\phi) = h$ . A  $\phi$ -twisted conjugacy class is an equivalence class of the equivalence relation of being  $\phi$ -twisted conjugate.

Virtually inner automorphisms and twisted conjugacy classes have been used before to study variants of the conjugacy problem in virtually free groups; for example in [20] to study the generalised conjugacy problem.

We start by studying the set of  $\phi$ -cyclic permutations of a given element.

**Lemma 4.3.** Let F be a finite rank free group and  $\phi \in Aut(F)$  be virtually inner. Let  $g \in F$ . Then there is a finite set  $X \subseteq F$  and an element  $h \in F$  such that the set of  $\phi$ -cyclic permutations of g is

$$\{h^{-n}ph^n \mid n \in \mathbb{Z}, p \in X\}$$

*Proof.* Since  $\phi$  is virtually inner, there exists  $k \in \mathbb{Z}_{>0}$  such that  $\phi^k = \psi$  for some inner automorphism  $\psi$ . Then every  $\phi$ -cyclic permutation of  $g = x_1 \cdots x_m$ , where each  $x_i$  is a generator of F, has the form

$$v(x_{i+1}\phi^n)(x_{i+2}\phi^n)\cdots(x_m\phi^n)(x_1\phi^{n+1})\cdots(x_{i-1}\phi^{n+1})(u\phi),$$

for some  $n \in \mathbb{Z}$ , where  $u, v \in \Sigma^*$  are such that  $uv = (x_i)\phi^n$ . In addition, as  $\phi^k = \psi$ , we can rewrite this as

$$v(x_{i+1}\phi^{(n \bmod k)}\psi^{\lfloor \frac{n}{k}\rfloor})\cdots(x_{m}\phi^{(n \bmod k)}\psi^{\lfloor \frac{n}{k}\rfloor})(x_{1}\phi^{(n \bmod k)+1}\psi^{\lfloor \frac{n}{k}\rfloor})\cdots(x_{i-1}\phi^{(n \bmod k)+1}\psi^{\lfloor \frac{n}{k}\rfloor})(u\phi),$$

where  $uv = (x_i)\phi^{(n \mod k)}\psi^{\lfloor \frac{n}{k} \rfloor}$ . Since  $\psi$  is inner, it is defined by conjugation by some element  $h \in F$ . Let  $r = \lfloor \frac{n}{k} \rfloor$ . Then the above expression becomes

(1) 
$$vh^{-r}(x_{i+1})\phi^{(n \mod k)} \cdots (x_m \phi^{(n \mod k)})(x_1 \phi^{(n \mod k)+1}) \cdots (x_{i-1} \phi^{(n \mod k)+1})h^r(u\phi)$$

where  $uv = h^{-r}(x_i)\phi^{(n \mod k)}h^r$ . We have  $u = h^{-r}u'h^r$  and  $v = h^{-r}v'h^r$ , where u' and v' are freely reduced, and  $h^{-r}u'v'h^r = uv = h^{-r}(x_i)\phi^{n \mod k}h^r$ , and so  $u'v' = (x_i)\phi^{n \mod k}$ . Since  $\psi$  is a power of  $\phi$ , they commute, and so  $u\phi = (h^{-r}u'h^r)\phi = h^{-r}(u'\phi)h^r$ . We can therefore rewrite (1) as

$$h^{-r}v'(x_{i+1})\phi^{(n \mod k)}\cdots(x_m\phi^{(n \mod k)})(x_1\phi^{(n \mod k)+1})\cdots(x_{i-1}\phi^{(n \mod k)+1})(u'\phi)h^r,$$

Thus every  $\phi$ -cyclic permutation of g is an  $h^r$ -conjugate of an expression of the form

$$v'(x_{i+1})\phi^{(n \mod k)} \cdots (x_m \phi^{(n \mod k)})(x_1 \phi^{(n \mod k)+1}) \cdots (x_{i-1} \phi^{(n \mod k)+1})(u'\phi).$$

Since there are finitely many possibilities for i, u', v' and  $n \mod k$ , these expressions define finitely many elements of F. Moreover, all such expressions represent  $\phi$ -cyclic permutations of g, and so the set of  $\phi$ -cyclic permutations of g is in the stated form.

Before we can show that the set of  $\phi$ -cyclic permutations of a given element forms a recognisably context-free set, we need the following lemma.

**Lemma 4.4** ([11, Lemma 4.6]). Let F be a free group with free basis  $\Sigma$  and natural homomorphism  $\pi: (\Sigma \cup \Sigma^{-1})^* \to F$ . Let  $E \subseteq F$  be such that there is a context-free language  $L \subseteq E\pi^{-1}$  that contains the freely reduced forms of every element of E. Then E is recognisably context-free.

We use Lemma 4.3 and Lemma 4.4 to show the following.

**Lemma 4.5.** Let F be a finite rank free group and  $\phi \in \operatorname{Aut}(F)$  be virtually inner. Let  $\Sigma$  be a free basis for F and let  $w \in (\Sigma \cup \Sigma^{-1})^*$  be freely reduced. Let  $E \subseteq F$  denote the set of all elements that can be written as  $\phi$ -cyclic permutations of w. Then E is recognisably context-free.

Proof. Lemma 4.3 tells us that there is a finite set  $X \subseteq F$  and an element  $h \in F$  such that  $E = \{h^{-n}ph^n \mid n \in \mathbb{Z}, p \in X\}$ . Since finite unions of context-free languages are context-free (Lemma 2.6), it suffices to show that if  $p \in F$ , then  $E_p = \{h^{-n}ph^n \mid n \in \mathbb{Z}\}$  is recognisably context-free. Using Lemma 4.4, it suffices to show that there exists a context-free language  $L \subseteq E_p \pi^{-1}$  that contains all freely reduced words in  $E_p \pi^{-1}$ . We construct a context-free grammar for L.

Let u denote the freely reduced form h. Note that there is a finite set Y of words such that the freely reduced words in  $E_p \pi^{-1}$  are the freely reduced words in the set

$$\{u^{-n}vu^n \mid n \in \mathbb{Z}, v \in Y\}.$$

Thus if we take  $L = \{u^{-n}vu^n \mid n \in \mathbb{Z}, u \in Y\}$ , then  $L \subseteq E_p\pi^{-1}$  and L contains all freely reduced words in  $E_p\pi^{-1}$ , and so it suffices to show that L is context-free. Again, since finite unions of context-free languages are context-free (Lemma 2.6), it suffices to show that for any freely reduced  $v \in Y$ ,  $L_v = \{u^{-n}vu^n \mid n \in \mathbb{Z}\}$  is context-free.

Fix  $v \in Y$ . We define a context-free grammar for  $L_v$ . Our set of non-terminals will be  $\{\mathbf{S}, \mathbf{T}, \mathbf{U}\}$ , with **S** the start symbol. The set of productions  $\mathcal{P}$  is defined by

$$\mathcal{P} = \{ \mathbf{S} \to \mathbf{T}, \mathbf{S} \to \mathbf{U}, \mathbf{T} \to u^{-1}\mathbf{T}u, \mathbf{T} \to v, \mathbf{U} \to u\mathbf{U}u^{-1}, \mathbf{U} \to v \}.$$

Any derivation using these productions and starting at **S**, either goes straight to **T** or to **U**. When in **T**, we can add  $u^{-1}$  at the beginning and u at the end, or replace the **T** with v. Thus the set of words derived with first production  $\mathbf{S} \to \mathbf{T}$  is  $\{u^{-n}vu^n \mid n \in \mathbb{Z}_{\geq 0}\}$ . By symmetry, those derived through **U** are  $\{u^{-n}vu^n \mid n \in \mathbb{Z}_{\leq 0}\}$ , and so the grammar  $(\Sigma \cup \Sigma^{-1}, \{\mathbf{S}, \mathbf{T}, \mathbf{U}\}, \mathcal{P}, \mathbf{S})$  generates  $L_v$ , as required.

We now use the fact that the set of  $\phi$ -cyclic permutations of a given element is recognisably contextfree to prove that the  $\phi$ -twisted conjugacy classes in free groups are recognisably context-free in finite extensions corresponding to the automorphism  $\phi$ .

The conjugacy problem in free groups can be solved using two facts: every conjugacy class has only finitely many cyclically reduced words, and every freely reduced word representing an element of a conjugacy class can be expressed in the form  $uxu^{-1}$ , with u a freely reduced word and x a cyclically reduced word. The first fact we replace with Lemma 4.7, and the following result is an analogue of the latter for  $\phi$ -twisted conjugates.

**Proposition 4.6.** Let F be a finite rank free group with basis  $\Sigma$ , and let  $\phi \in Aut(F)$ . Let C be a  $\phi$ -twisted conjugacy class of F. Let X be the set of freely reduced words representing the minimal

length elements in C and their  $\phi$ -cyclic permutations. Let  $C_{red}$  be the set of freely reduced words over  $\Sigma \cup \Sigma^{-1}$  representing elements of C. Then

(2) 
$$C_{red} \subseteq \{v_1 u v_2 \mid u \in X, v_1, v_2 \text{ are freely reduced representatives for } g^{-1}, g\phi, \text{ for some } g \in F\}$$

*Proof.* Note that in the expression (2), we mean equality as words; there is no free reduction involved. Let  $w \in C_{\text{red}}$ . Then w is  $\phi$ -twisted conjugate to some element  $h \in F$  such that the freely reduced representative for h lies in X. Thus w can be obtained from  $v_1uv_2$  by freely reducing, where  $u \in X$  is of minimal length, and  $v_1$  and  $v_2$  are the freely reduced representatives for  $g^{-1}$  and  $g\phi$ , respectively for some  $g \in F$ . If there is no free reduction between  $v_1$  and u, and u and  $v_2$ , we will have shown (2). So we will modify the expression to show that such a form will always exist.

We have  $u \equiv xu'y$ ,  $v_1 \equiv v'_1x^{-1}$ ,  $v_2 \equiv y^{-1}v'_2$  and  $w \equiv v'_1uv'_2$ , for some freely reduced  $x, y, u', v'_1, v'_2 \in (\Sigma \cup \Sigma^{-1})^*$ . If the word  $w \equiv v'_1uv'_2$  is of the form  $z^{-1}p(z\phi)$ , for freely reduced words  $p, z, z\phi$ , and  $z \neq \varepsilon$  then it suffices to show that p lies in the form stated in (2). Since |p| < |w| and  $p \in C_{\text{red}}$ , we can use induction to conclude that p is of the form stated in (2).

Thus we can assume that w is not of the form  $z^{-1}p(z\phi)$ . We have that (after freely reducing  $v_1^{-1}\phi$ ), that  $v_1^{-1}\phi \equiv v_2$ , and so  $(v_1')^{-1}\phi$  (after being freely reduced) and  $v_2'$  are both suffixes of  $v_2$ . However, since we are assuming that w is not of the form  $z^{-1}p(z\phi)$ , for freely reduced words p, z,  $z\phi$ , we have that  $v_2'$  and (the freely reduced form of)  $(v_1')^{-1}\phi$  must not have a common suffix. Since they are both suffixes of the same word, we conclude that one must be empty. If  $(v_1')^{-1}\phi$  is empty, then  $v_1'$  is empty, since  $\phi$  is an automorphism of F. We can therefore split into the cases when  $v_1'$  or  $v_2'$  are empty.

Case 1:  $v'_1 = \varepsilon$ .

Then  $w \equiv u'v'_2$ , and  $v_1uv_2 \equiv x^{-1}xu'yy^{-1}v'_2$ . Since  $x\phi \equiv v_1^{-1}\phi =_F v_2 \equiv y^{-1}v'_2$ , we can  $\phi$ -cyclically permute  $u \equiv xu'y$  to  $u'yy^{-1}v'_2 =_F u'v'_2 \equiv w$ . Thus  $w\pi$  is a  $\phi$ -cyclic permutation of  $u\pi$ , and  $u \in C_{\text{red}}$ , as required.

Case 2:  $v'_2 = \varepsilon$ .

Then  $w \equiv v'_1 u'$  and  $v_1 u v_2 \equiv v'_1 x^{-1} x u' y y^{-1}$ . Since  $v_1^{-1} \phi =_F v_2 \equiv y^{-1}$ , we have that  $y \phi^{-1} =_F v_1 \equiv v'_1 x^{-1}$ . Thus  $u \equiv x u' y$  is a  $\phi$ -cyclic permutation (using the relation 'backwards') of  $(y \phi^{-1}) x u' =_F v'_1 x^{-1} x u' =_F v'_1 u' \equiv w$ . Thus  $w \pi$  is a  $\phi$ -cyclic permutation of  $u \pi$ , and  $u \in C_{\text{red}}$ , as required.  $\Box$ 

**Lemma 4.7.** Let G be a finitely generated virtually free group. Let F be a finite-index normal free subgroup of G, and T be a right transversal for F in G. Let  $\Sigma$  be a basis for F and let  $\# \notin \Sigma \cup \Sigma^{-1} \cup T$  be a new letter. Let  $\phi \in \operatorname{Aut}(F)$ . Then the language

$$\{u \# v \mid u, v \in (\Sigma \cup \Sigma^{-1} \cup T)^*, u^{-1}\phi = v\}$$

is context-free.

Proof. We construct a pushdown automaton accepting the language  $L_{\phi}$  stated in the lemma. We start by constructing a pushdown automaton  $\mathcal{A}$  that accepts  $g\pi^{-1}$  for some  $g \in G$ . We first describe the automaton  $\mathcal{A}$ . Our description is based on the proof of the Muller-Schupp Theorem in [15], and generalises Example 2.12. The idea is given a word, we track the normal form ht of the prefix 'read so far', by storing the freely reduced form on the stack, and having a state  $q_t$  for each  $t \in T$ . Let  $t_0$  be the unique element of  $T \cap F$ . Thus our stack alphabet will be  $\{\bot\} \cup \Sigma \cup \Sigma^{-1}$ , where  $\bot$  is the bottom of stack symbol. We add an additional state p to be our unique accept state, and have a transition from  $q_{t_h}$  with stack w to p, where  $t_g \in T$  is the unique coset representative such that

 $g = h_g t_g$  for some (unique)  $h_g \in F$ , and w is the freely reduced form of  $h_g$ . Our start state will simply be  $q_{t_0}$ .

It remains to add transitions between the states  $q_t$  for each  $t \in T$ . We will be using  $\Sigma \cup \Sigma^{-1} \cup \{\chi\}$  as our stack alphabet, with  $\chi$  the bottom of stack symbol. For each  $a \in \Sigma \cup \Sigma^{-1} \cup T$  and each  $t \in T$ , we have that ta = ht' for some  $h \in F$  and  $t' \in T$ . Thus when in state  $q_t$  and reading a, we want to transition to state  $q_{t'}$ , then add the freely reduced form  $\mu_h$  of h to the stack, and then freely reduce. Since we cannot simply freely reduce the stack, we have multiple transitions from  $q_t$  to  $q_{t'}$  when reading a; one transition for each pair  $(\omega, x)$ , where  $\omega$  is a suffix of the freely reduced form of  $h^{-1}$  (that is,  $\omega^{-1}$  is a prefix of  $\mu_h$ ) and  $x \in \Sigma \cup \Sigma^{-1}$  is such that  $x\omega$  is not a suffix of  $\mu_h^{-1}$ . This transition then pops  $x\omega$  from the stack, and pushes x followed by the 'remainder' of  $\mu_h$ ; that is the (unique) freely reduced word  $\nu$ , such that  $\mu_h \equiv \omega^{-1}\nu$ . To deal with the case when the stack is empty (that is, it contains only the symbol  $\bot$ , we add the transition from  $q_t$  to  $q_{t'}$  when reading a that pops  $\bot$  and pushes  $\bot \mu_h$ .

We now construct a pushdown automaton accepting  $L_{\phi}$ . We start by taking two (disjoint) copies of the automaton  $\mathcal{A}$ :  $\mathcal{A}_1$  and  $\mathcal{A}_2$ . The automaton  $\mathcal{A}_1$  will be for  $g\pi^{-1}$  for some fixed  $g \in G$  with its accept state considered not an accept state (and so the choice of g does not matter), and  $\mathcal{A}_2$ will be for  $1\pi^{-1}$ . We modify the transitions of  $\mathcal{A}_1$ , so that whenever we read  $a \in \Sigma \cup \Sigma^{-1} \cup T$ , we instead use the transition for  $a^{-1}$ . That is, if we were in state t with stack  $\mu_h$ , we would end up in the state and stack corresponding to  $\mu_h ta^{-1}$ . Similarly, we modify  $\mathcal{A}_2$  so that whenever we read awe act as if we read  $a\phi$ . Again, this means that in state t with stack  $\mu_h$ , we move to the state-stack pair corresponding to  $\mu_h t(a\phi)$ . Our start state will be the start state of  $\mathcal{A}_1$ , and our accept state will be the accept state of  $\mathcal{A}_2$ .

We add an additional transition between every state in  $\mathcal{A}_1$  to the corresponding state in  $\mathcal{A}_2$  that does not alter the stack (that is, it pushes and pops  $\varepsilon$  from the stack) when reading #.

We can now show that  $\phi$ -twisted conjugacy classes of free groups are recognisably context-free in finite-index overgroups.

**Proposition 4.8.** Let F be a finite rank free group and  $\phi \in Aut(F)$  be virtually inner. Let G be such that F is a finite-index normal subgroup of G. Then every  $\phi$ -twisted conjugacy class of F is recognisably context-free in G.

Proof. Let  $\Sigma$  be a free basis for F and  $\pi: (\Sigma \cup \Sigma^{-1})* \to F$  be the natural map. Let  $C_{\phi}$  be a  $\phi$ -twisted conjugacy class. Using Lemma 4.4, in order to show that  $C_{\phi}$  is recognisably context-free, it suffices to construct a context-free language  $L \subseteq C_{\phi}\pi^{-1}$  that contains every freely reduced word in  $C_{\phi}\pi^{-1}$ . Proposition 4.6, shows us that all such words are of the form  $v_1uv_2$ , where  $v_1$  and  $v_2$  are freely reduced representative for  $g^{-1}$  and  $g\phi$ , for some  $g \in F$ , and u is a  $\phi$ -cyclic permutation of a minimal length word in  $C_{\phi}$ .

It therefore suffices to show that the language of all words of the form  $v_1uv_2$  where  $v_1$  and  $v_2$  are (not necessarily freely reduced) representatives for  $g^{-1}$  and  $g\phi$ , for some  $g \in F$ , and u is a  $\phi$ -cyclic permutation of a minimal length word in  $C_{\phi}$ , is a context-free language. Lemma 4.7 tells us that

$$\{u \# v \mid u, v \in (\Sigma \cup \Sigma^{-1} \cup T)^*, u^{-1}\phi = v\}$$

is context-free. In addition, if  $E \subseteq F$  denotes the set of elements that can be expressed as  $\phi$ -cyclic permutations of a minimal length word in  $C_{\phi}$ , then Lemma 4.5, together with the fact that

finite unions of context-free languages are context-free (Lemma 2.6) shows that  $E\pi^{-1}$  is contextfree. Since context-free languages are closed under substitutions by other context-free languages (Lemma 2.10), the language

$$L = \{ uxv \mid u, v \in (\Sigma \cup \Sigma^{-1} \cup T)^*, u^{-1}\phi = v, x \in E\pi^{-1} \}$$

is context-free. As  $L \subseteq C_g \pi^{-1}$ , and by construction contains every freely reduced word in  $C_g \pi^{-1}$ , Lemma 4.4 tells us that  $C_g$  is recognisably context-free.

We can now show that conjugacy classes in finitely generated virtually free groups are recognisably context-free.

**Proposition 4.9.** Let G be a finitely generated virtually free group. Then every conjugacy class of G is recognisably context-free.

Proof. Let F be a finite-index normal free subgroup of G. Fix a finite right transversal T for F in G. We have that every element of G can be written in the form ht where  $h \in F$  and  $t \in T$ . Since F is finitely generated free, we can write every element of F uniquely as a freely reduced word with respect to a (finite) basis  $\Sigma$ . We will use  $\Sigma \cup \Sigma^{-1} \cup T$  as our (monoid) generating set. Since F is normal and finite-index, elements of G (in particular elements of T) act on F by automorphisms of finite order. For each  $t \in T$  we write  $\phi_t$  to denote the automorphism of F defined by  $h \mapsto tht^{-1}$ . Note that for each  $t \in T$  there exists  $k \in \mathbb{Z}_{>0}$  such that  $t^k \in F$ , and so  $\phi_t^k \colon x \mapsto t^{-k}xt^k$ , and we have shown that  $\phi_t^k$  is an inner automorphism of F. Thus all automorphisms  $\phi_t$  are virtually inner.

Fix a conjugacy class C of G, and a representative  $h_0t_0 \in C$ . Let  $ht \in G$ . We have that  $ht \in C$  if and only if there exists  $x \in G$  such that  $xh_0t_0x^{-1} = ht$ . We can write any such x = ys, where  $y \in F$  and  $s \in T$ . So

(3) 
$$ht = ysht_0s^{-1}y^{-1} = y(h_0\phi_s)(y^{-1}\phi_s^{-1}\phi_{t_0}\phi_s)st_0s^{-1}$$

Thus  $ht \in C$  if and only if there exists  $y \in F$  and  $s \in T$  such that (3) is satisfied. Moreover, if we fix  $s \in T$ , then (3) becomes a twisted conjugacy class of F, using the (fixed) automorphism  $\phi_s$ , multiplied by a fixed element of F and then a fixed element of T; that is, the normal form for the fixed element  $st_0s^{-1}$ . Since finite unions of context-free languages are context-free (Lemma 2.6), it is sufficient to show that the set of elements that lie in a set of the form  $C_{\phi}hs$  is recognisably context-free, where  $C_{\phi}$  is a  $\phi$ -twisted conjugacy class of F,  $h \in F$  and  $s \in T$ . This follows from Corollary 3.2 together with the fact that  $C_{\phi}$  is recognisably context-free (Proposition 4.8).

Combining Proposition 4.9 with the fact that  $\{1\}$  is a conjugacy class that is recognisably contextfree in a group G if and only if G is virtually free (by the Muller-Schupp Theorem), we have the following:

**Theorem 4.10.** Let G be a finitely generated group. Then every conjugacy class of G is recognisably context-free if and only if G is virtually free.

The following example, due to Corentin Bodart, shows that there exist non-virtually free groups that admit recognisably deterministic context-free conjugacy classes (they are in fact recognisable).

**Example 4.11.** Let H be a finitely generated abelian group and let

$$G = \langle H \cup \{t\} \mid \{tht = h^{-1} \mid h \in H\} \cup \{t^2 = 1\} \rangle$$

We look at the conjugacy class of t in G. Since  $th = h^{-1}t$  for all  $h \in H$ , we have for all  $h \in H$ that  $h^{-1}th = h^{-2}t$ . In particular, the conjugacy class of t contains the coset  $H^2t$ . Conversely,  $ifh^{2}t \in H^{2}t$ , then  $h^{2}t = hth^{-1}$ , and so  $h^{2}t$  is conjugate to t. We can thus conclude that the conjugacy class of t is equal to the coset  $H^{2}t$ .

In addition, H is index 2 in G and  $H^2$  is finite index in H, and so  $H^2$  is finite-index in G. By Proposition 2.18,  $H^2t$  is recognisable in G, and so the conjugacy class of t is recognisable in G.

# 5. Subgroups and cosets

We conclude by giving a classification of when subgroups and cosets with quasi-transitive Schreier coset graphs of finitely generated groups are recognisably context-free. Ceccherini-Silberstein and Woess provided a full classification by showing that a subgroup (and hence coset, using Corollary 3.2) is recognisably context-free if and only if the Schreier coset graph is what is called a context-free graph [6], a term introduced by Muller and Schupp [25] which depends on the ends of a graph. Woess continued the study of these graphs in [28].

As mentioned earlier, since the release of this work, the author has been made aware of a result of Rodaro, released a few months earlier that proves the main result of this section [26], when taken together with the result of Ceccherini-Silberstein and Woess [6] that classifies when a Schreier coset graph is a context-free graph. Rodaro's method uses the context-free graphs introduced by Muller and Schupp [25]. We prove this using a recent generalisation of Stallings' Theorem [10], avoiding the notion of a context-free graph.

We consider the case when a Schreier coset graph is quasi-transitive; that is, it has finitely many automorphic orbits, and show that a coset with quasi-transitive coset graph is recognisably context-free if and only if the corresponding Schreier coset graph is a quasi-tree. If the subgroup in question were normal, we could use the Muller-Schupp Theorem to show that the Cayley graph of the quotient group must be a quasi-tree. Since this is always isomorphic to the Schreier coset graph, this proves the result. Stated in terms of properties of the quotient rather than Schreier coset graphs this is a coset Hg within a group G is recognisably context-free if and only if G/H is virtually free.

Extending this to the non-normal case requires more work. In light of Corollary 3.2, it is sufficient to answer the question for subgroups. The fact that a subgroup H being recognisably contextfree implies that the Schreier coset graph of H is a quasi-tree is not too difficult to show using the same argument as the Muller-Schupp Theorem. The converse of this is much more difficult. The main stumbling block arises from the fact that the Muller-Schupp proof shows that groups with context-free word problem have Cayley graphs quasi-isometric to trees, then uses Stallings' Theorem and Dunwoody's accessibility result to show that these groups must be virtually free, and then shows that virtually free groups have context-free word problem. The difficulty we have here is replacing Stallings' Theorem and Dunwoody's accessibility result, as we are working with Schreier coset graphs rather than groups.

A recent result of Hamann, Lehner, Miraftab and Rühmann does prove a version of Stallings' Theorem for connected quasi-transitive graphs [10]. In addition, they show that such quasi-trees will be 'accessible' in their sense. We state these results in Subsection 2.9.

We begin with the more straightforward direction.

**Proposition 5.1.** Let G be a finitely generated group and  $H \leq G$  be such that the Schreier coset graph of H is quasi-transitive. If H is recognisably context-free then the Schreier coset graph of H in G is a quasi-tree.

19

Proof. Fix a finite monoid generating set  $\Sigma$  for G and let  $\pi: \Sigma^* \to G$  be the natural homomorphism. As  $H\pi^{-1}$  is context-free, Lemma 2.8 tells us there is a context-free grammar  $\mathcal{G} = (V, \Sigma, \mathcal{P}, \mathbf{S})$  that is in Chomsky normal form and has no useless non-terminals, such that the language of  $\mathcal{G}$  is  $H\pi^{-1}$ . Let  $\mathbf{A} \in V$ , and suppose  $w_1, w_2 \in \Sigma^*$  are such that  $\mathbf{A} \Rightarrow^* w_1$  and  $\mathbf{A} \Rightarrow^* w_2$ . Then there exist  $\sigma, \tau \in \Sigma^*$  such that  $(\sigma w_1 \tau)\pi, (\sigma w_2 \tau)\pi \in H$ . So  $(\sigma w_1 w_2^{-1} \sigma^{-1})\pi = (\sigma w_1 \tau \tau^{-1} w_2^{-1} \sigma^{-1})\pi \in H$ . We have thus shown that  $(w_1 w_2^{-1})\pi$  is conjugate to an element of H. Moreover, a conjugating element is  $\sigma\pi$ .

For each  $\mathbf{A} \in V$  choose a word  $w_{\mathbf{A}} \in \Sigma^*$  such that  $\mathbf{A} \Rightarrow^* w_{\mathbf{A}}$  (such a derivation always exists as  $\mathbf{A}$  is not useless). Let  $M = \max\{|w_{\mathbf{A}}| \mid \mathbf{A} \in V\}$ . Suppose  $u \in H\pi^{-1}$ . Then w labels a circuit in the Schreier coset graph  $\Gamma$  of H, with basepoint H. Suppose in a derivation of u in  $\mathcal{G}$  we have  $\mathbf{S} \Rightarrow^* \sigma \mathbf{A}\tau \Rightarrow^* \sigma v\tau \equiv w$ . Note that  $(v^{-1}w_{\mathbf{A}})\pi \in H^{\sigma\pi}$ . In particular,  $(v^{-1}w_{\mathbf{A}})\pi$  lies in the stabiliser of  $H(\sigma\pi)$ . So replacing  $\sigma vv^{-1}w_{\mathbf{A}}\tau$  also traces a circuit in  $\Gamma$  with basepoint H, and so  $\sigma w_{\mathbf{A}}\tau$  does as well. Since  $|w_{\mathbf{A}}| \leq M$ , this will be an M-reduction. We can therefore apply the Muller-Schupp method to M-triangulate every circuit in  $\Gamma$  with basepoint H by replacing subwords derived from each non-terminal  $\mathbf{A}$  with  $w_{\mathbf{A}}$ .

To achieve this, we go through the derivation of a word u in  $H\pi^{-1}$ , (ignoring productions of the form  $\mathbf{A} \to a$ ), and for each production of the form  $\mathbf{A} \to \mathbf{BC}$ , we have an *M*-reduction from the start of the subword of u derived from **BC** to the end (using the label  $w_{\mathbf{A}}$ ).

To show that every circuit in  $\Gamma$  is triangulable, not just those with a basepoint in H, it is sufficient to show that for all automorphic orbits of the Schreier coset graph there is a basepoint Ht such that every circuit with a basepoint Ht is triangulable. Fix a set T of representatives for these automorphic orbits. Since  $\Gamma$  is quasi-transitive, we can choose T to be finite.

Let  $Ht \in T$  and let  $u \in \Sigma^*$  trace a path in  $\Gamma$  from Ht to Ht. Fix a word  $w_t \in \Sigma^*$  representing t. Then  $w_t u w_t^{-1}$  labels a path in  $\Gamma$  from H to itself, and so this circuit is M-triangulable. Thus the circuit traced by u with basepoint Ht is  $(M + |w_t|)$ -triangulable. Let  $K = \max_{Ht \in T} |w_t|$ . We can conclude that the Schreier graph of H in G is (M + K)-triangulable, and so by Theorem 2.28, it is a quasi-tree.

We now prove that if a Schreier coset graph of a subgroup of a finitely generated group is a quasitree, then then the subgroup is recognisably context-free. We first need some definitions, based on definitions in [25].

**Definition 5.2.** A *finitely generated graph* is a  $\Sigma$ -labelled graph, where  $\Sigma$  is an alphabet, such that

- (1)  $\Gamma$  is connected;
- (2)  $\Gamma$  has uniformly bounded degree (that is, there exists d > 0 such that the degree of every vertex is at most d);
- (3)  $\Sigma$  is finite.

Let  $\Gamma$  be a finitely generated graph, with edges labelled using an alphabet  $\Sigma$ . Fix a vertex  $v_0$  and a finite set F of vertices of  $\Gamma$ . The *language* of  $\Gamma$  with respect to the *origin*  $v_0$  and *accepting states* F is the set of all words that trace a path in  $\Gamma$  from  $v_0$  to a vertex in F.

We now show that taking a 'nice' tree amalgamation of finitely generated graphs that both have context-free languages yields a graph with a context-free language.

**Lemma 5.3.** Let  $\Gamma_1$  and  $\Gamma_2$  be finitely generated quasi-transitive graphs with edges labelled from an alphabet  $\Sigma$ , whose languages are context-free with respect to any origin and any finite set of accepting states. Then every tree amalgamation of  $\Gamma_1$  and  $\Gamma_2$  with finite adhesion and finite identification also has a context-free language with respect to any origin and any finite set of accepting states.

Proof. Let  $(S_k^i)_{k \in I_i}$  be the adhesion sets of the tree amalgamation in  $\Gamma_i$ , for each  $i \in \{1, 2\}$ , and assume  $I_1$  and  $I_2$  are disjoint. Let T be the connecting tree. Note that as finite unions of contextfree languages are context-free (Lemma 2.6), it suffices to show that the language of  $\Gamma_1 * \Gamma_2$  is context-free with respect to any origin and any singleton set of accepting states.

Fix an origin vertex  $u_0$  in  $\Gamma_1 * \Gamma_2$ . This lies in a copy of  $\Gamma_1$  or  $\Gamma_2$ ; without loss of generality assume it lies in a copy of  $\Gamma_1$ , corresponding to a pair of vertices  $(v_0, t_0) \in V(\Gamma_1) \times V(T)$ . Since  $\Gamma_1$  has a context-free language with respect to any origin and any finite set of accepting states, for each  $u \in \bigcup_{k \in I_1} S_k^1$ , there is a pushdown automaton that accepts the language of  $\Gamma_1$  with respect to the origin  $v_0$  and the accepting state u.

We can then take the finite union of these pushdown automata across all  $u \in \bigcup_{k \in I_1} S_k^1$ , to obtain a pushdown automaton  $\mathcal{A}_0$  that accepts the language of all words that trace a path in  $\Gamma_1$  from  $v_0$ to a vertex in  $\bigcup_{k \in I_1} S_k^1$ . Moreover, as we constructed this as a (disjoint) finite union of pushdown automata, we can assume that the set of accept states is partitioned into the vertices lying in  $\bigcup_{k \in I_1} S_k^1$ .

Now let  $i \in \{1, 2\}$ . Since the language of  $\Gamma_i$  is context-free with respect to any origin and any set of accepting states, for each  $u, v \in \bigcup_{k \in I_i} S_k^i$ , we can construct a pushdown automaton  $\mathcal{B}_{i,u,v}$  accepting the language of all words that trace a path in  $\Gamma_i$  from u to v.

Fix an accepting state  $q \in V(\Gamma_1 * \Gamma_2)$ . Recall that the directed edge 'version' D(T) of T admits an edge-labelling using  $I_1 \sqcup I_2$ . Since D(T) is the directed 'version' of a tree, there is a unique minimal path in D(T) from  $t_0$  to each vertex  $t \in V(T)$ . This path traces a word  $w_t \in (I_1 \sqcup I_2)^*$ , and thus we can uniquely describe each vertex in T using a word over  $I_1 \sqcup I_2$ . Let  $t_q \in V(T)$  and  $v_q \in \Gamma_i$  be a (not necessarily unique) pair, such that q is the image of  $v_q$  under the canonical map from the copy of  $\Gamma_1$  or  $\Gamma_2$  corresponding to  $t_q$  to  $\Gamma_1 * \Gamma_2$ . Fix  $j_q \in \{1, 2\}$  such that  $\Gamma_{j_q}$  is the corresponding graph.

Similar to the pushdown automata  $\mathcal{B}_{i,u,v}$  for each vertex  $u \in \bigcup_{k \in I_{j_q}} S_k^i$ , we construct a pushdown automaton  $\mathcal{C}_u$  to be the pushdown automata that accept the language of all words that trace a path in  $\Gamma_{j_q}$  from u to  $v_q$ .

We can assume that all of the pushdown automata we have defined have pairwise disjoint sets of states and stack alphabets. We also assume the stack alphabets are all pairwise disjoint from  $I_1$  and  $I_2$ . By Lemma 2.16, we can assume that whenever a word is accepted by any of these pushdown automata, the stack is empty; that is, the only symbol on the stack is the bottom of stack symbol.

We now use the pushdown automata  $\mathcal{B}_{i,u,v}$ ,  $\mathcal{C}_x$  and  $\mathcal{A}_0$  to construct a (non-deterministic) pushdown automaton  $\mathcal{D}$  accepting the language of  $\Gamma_1 * \Gamma_2$  as follows:

(1) Our set of states will be

$$Q(\mathcal{A}_0) \sqcup \bigsqcup_{i \in \{1,2\}} \bigsqcup_{u,v \in \bigcup_{k \in I_i} S_k^i} Q(\mathcal{B}_{i,u,v}) \sqcup \bigsqcup_{u \in \bigcup_{k \in I_{j_a}} S_k^{j_a}} Q(\mathcal{C}_u) \sqcup \{p\}$$

where p is a new state.

- (2) Our alphabet will be  $\Sigma$ .
- (3) Our start state will be the start state  $q_0$  of  $\mathcal{A}_0$ .
- (4) Our accept state will be p.
- (5) Our stack alphabet will be

$$\chi(\mathcal{A}_0) \sqcup \bigsqcup_{i \in \{1,2\}} \bigsqcup_{u,v \in \bigcup_{k \in I_i} S_k^i} \chi(\mathcal{B}_{i,u,v}) \sqcup \bigsqcup_{u \in \bigcup_{k \in I_{i_a}} S_k^{j_q}} \chi(\mathcal{C}_u) \sqcup I_1 \sqcup I_2.$$

- (6) Our bottom of stack symbol will be the bottom of stack symbol  $\perp_0$  of  $\mathcal{A}_0$ .
- (7) Our transitions will be all of those of the following forms:
  - (a) All transitions entirely within  $\mathcal{A}_0$  or some  $\mathcal{B}_{i,u,v}$  or  $\mathcal{C}_u$ ;
    - (b) For each  $i \in \{1, 2\}$ , each  $u, v \in \bigcup_{k \in I_i} S_k^i$ , and each bonding map  $\phi$  such that  $v \in \text{dom } \phi$ , there is an  $\varepsilon$ -transition from each accept state of  $\mathcal{B}_{i,u,v}$  to the start state of  $\mathcal{B}_{j,v\phi,x}$ , where  $j \in \{1, 2\} \setminus \{i\}$  and for every  $x \in \bigcup_{k \in I_j} S_k^j$ . Since all of the automata  $\mathcal{A}_0$  and  $\mathcal{B}_{i,u,v}$  have empty stacks when arriving in an accept state, when making this transition, the stack will have the form  $\perp_0 w \perp_{i,u,v}$ , where  $w \in (I_1 \sqcup I_2)^*$  and  $\perp_{i,u,v}$  is the start symbol of  $\mathcal{B}_{i,u,v}$ . We pop  $\perp_{i,u,v}$  from the stack, along with the topmost symbol k in w. If dom  $\phi \neq S_k^i$ , we push k back onto the stack, followed by the (unique)  $l \in I_i$  such that im  $\phi = S_l^j$ , and then the bottom of stack symbol for  $\mathcal{B}_{j,v\phi,x}$ . If dom  $\phi = S_k^i$ , then we don't push k back onto the stack; we only push the bottom of stack symbol for  $\mathcal{B}_{j,v\phi,x}$ .
    - (c) There are transitions analogous to those in (b), except starting in  $\mathcal{A}_0$  and ending in some  $\mathcal{B}_{i,u,v}$ . To avoid any ambiguity, we formally state these as well. As mentioned earlier, the accept states of  $\mathcal{A}_0$  are partitioned into parts corresponding to the vertices  $u \in \bigcup_{k \in I_1} S_k^1$ . Fix such a vertex u. For each bonding map  $\phi$  such that  $u \in \text{dom } \phi$ , there is an  $\varepsilon$ -transition from each accept state of  $\mathcal{A}_0$  that lies in the part of the partition corresponding to u to the start state of  $\mathcal{B}_{2,v\phi,x}$  for all  $x \in \bigcup_{l \in I_2} S_k^l$ . In such a case the stack will be of the form  $\perp_0$ , and we pop  $\perp_0$  from the stack then push  $\perp_0 l$ , where  $l \in I_2$  is unique such that im  $\phi = S_l^2$ .
    - (d) There are transitions analogous to those in (b), except starting in some  $\mathcal{B}_{i,u,v}$  and ending in some  $\mathcal{C}_x$ . Naturally, these only start in automata  $\mathcal{B}_{i,u,v}$  where  $i \neq j_q$ , as  $\mathcal{C}_x$ corresponds to  $\Gamma_{j_q}$ .
    - (e) If  $j_q \neq 1$ , then there are transitions analogous to those in (c), starting in  $\mathcal{A}_0$  and ending in some  $\mathcal{C}_u$ .
    - (f) From each accept state of each  $C_u$ , there is an  $\varepsilon$ -transition to p, that pops  $\perp_0 w_{t_q} \$_u$ , where  $\$_u$  is the bottom of stack symbol of  $C_u$ . We then push  $\perp_0$  back onto the stack.

The automaton  $\mathcal{D}$  works as follows. The automata  $\mathcal{A}_0$ ,  $\mathcal{B}_{i,u,v}$  and  $\mathcal{C}_x$  simulate the copies of  $\Gamma_1$ and  $\Gamma_2$  used to define  $\Gamma_1 * \Gamma_2$ . We use the stack (behind the bottom of stack symbol of whichever automaton we are currently in) to track the position within the connecting tree T that we are in, with  $t_0$  being used as a root. The transitions between each of the automata  $\mathcal{A}_0$ ,  $\mathcal{B}_{i,u,v}$  and  $\mathcal{C}_x$ simulate the bonding maps, as they identify vertices in  $\Gamma_1$  and  $\Gamma_2$ . We need  $\mathcal{A}_0$  to be a separate automaton to deal with the multiple accepting vertices we can start with (after that, we just pass to a different automaton  $\mathcal{B}_{i,u,v}$ ) for different accepting states v). The automaton  $\mathcal{C}_x$  is separate to make transitioning to the accept state p more straightforward. Transitioning from  $\mathcal{C}_x$  to p requires us to be in the vertex  $v_q$  corresponding to q, and the transition confirms that our stack reads  $t_q$ ; that is, we are in the correct position within T, before accepting.

Using Theorem 2.38, we can build any Schreier coset graph by iteratively taking tree amalgamations, starting with a collection of finite graphs. Lemma 5.3 tells us that each of these tree amalgamations

preserves the property of having a context-free language. We can thus use this to show that a subgroup whose Schreier coset graph is a quasi-transitive quasi-tree will be recognisably contextfree. We now formally state the characterisation of recognisably context-free cosets of subgroups with quasi-transitive Schreier coset graphs we have been working towards.

**Theorem 5.4.** Let G be a finitely generated group,  $H \leq G$  and  $g \in G$  be such that the Schreier coset graph of (G, H) is quasi-transitive. Then Hg is recognisably context-free if and only if the Schreier coset graph of (G, H) is a quasi-tree.

*Proof.* In light of Corollary 3.2, it suffices to show that H is recognisably context-free if and only if the Schreier coset graph  $\Gamma$  of (G, H) is a quasi-tree. The fact that H is recognisably context-free implies that  $\Gamma$  is a quasi-tree is Proposition 5.1. So it remains to show that if  $\Gamma$  is a quasi-tree tree, then H is recognisably context-free.

Suppose  $\Gamma$  is a quasi-tree. Since  $\Gamma$  is quasi-transitive, we can apply Theorem 2.38 to show that  $\Gamma$  can be built from a (finite) collection of finite graphs by successive tree amalgamations with finite adhesion and finite identification. Each of the finite graphs will have a context-free language with respect to any origin and any set of accepting states, as the languages of finite graphs are always regular. We can then apply Lemma 5.3 to show that each of the successive tree amalgamation preserves the properties of having a context-free language with respect to any origin and any set of accepting states. Thus  $\Gamma$  has a context-free language with respect to any origin and any set of accepting states. In particular, the language of all words that trace a path in  $\Gamma$  from H to H is context-free. Since this is precisely the set of words in  $H\pi^{-1}$ , H is recognisably context-free.  $\Box$ 

**Remark 5.5.** Recall that Herbst and Thomas proved that a subset E of a group G is recognisable if and only if E is a finite union of cosets of some finite-index subgroup of G (Proposition 2.18). In light of Theorem 5.4 (or alternatively the classification of when generic subgroups are recognisably context-free due to Ceccherini-Silberstein and Woess [6]) it is natural to ask whether an analogous statement may be true for recognisably context-free subsets, using subgroups with quasi-tree Schreier coset graphs in place of finite-index subgroups (and not necessarily use a fixed subgroup). If we consider  $\mathbb{Z}$ , then it is easy to see that such a statement cannot be true. As the coword problem of  $\mathbb{Z}$  is context-free (by the Muller-Schupp Theorem),  $\mathbb{Z} \setminus \{0\}$  is a recognisably context-free subset of  $\mathbb{Z}$ . The only subgroups of  $\mathbb{Z}$  with Schreier coset graphs that are quasi-trees are infinite ones. Their cosets are of the form  $\{ax + b \mid x \in \mathbb{Z}\}$  with  $a \in \mathbb{Z} \setminus \{0\}$  and  $b \in \mathbb{Z}$ . It is not difficult to see that any finite union of these sets that does not contain zero, must necessarily miss infinitely many elements.

#### Acknowledgements

I would like to thank Corentin Bodart, André Carvalho, Gemma Crowe, Luke Elliott, Matthias Hamann, Mark Kambites, Alan Logan, Carl-Fredrik Nyberg Brodda, Davide Perego and Nóra Szakács and Martin van Beek for answering questions, mathematical discussions, directing me to references and pointing out errors in previous versions, all of which greatly helped with this work. During this work, I was supported by the Heilbronn Institute for Mathematical Research.

### References

<sup>[1]</sup> A. V. Anisimov, Group languages, Cybernetics and Systems Analysis 7 (1971), 594-601.

Yago Antolín, On Cayley graphs of virtually free groups, Groups Complex. Cryptol. 3 (2011), no. 2, 301–327. MR2898895

- [3] Collin Bleak, Francesco Matucci, and Max Neunhöffer, Embeddings into Thompson's group V and coCF groups, J. Lond. Math. Soc. (2) 94 (2016), no. 2, 583–597. MR3556455
- [4] Tara Brough, Groups with poly-context-free word problem, Groups Complex. Cryptol. 6 (2014), no. 1, 9–29. MR3200359
- [5] André Carvalho, Algebraic and context-free subsets of subgroups, Theoret. Comput. Sci. 980 (2023), 114229. MR4652858
- [6] Tullio Ceccherini-Silberstein and Wolfgang Woess, Context-free pairs of groups I: Context-free pairs and graphs, European J. Combin. 33 (2012), no. 7, 1449–1466. MR2923462
- [7] Laura Ciobanu, Alex Evetts, and Alex Levine, Effective equation solving, constraints and growth in virtually abelian groups, arXiv e-prints (2023). arXiv:math/2309.00475.
- [8] M. J. Dunwoody, The accessibility of finitely presented groups, Invent. Math. 81 (1985), no. 3, 449–457. MR807066
- Murray Elder, Mark Kambites, and Gretchen Ostheimer, On groups and counter automata, Internat. J. Algebra Comput. 18 (2008), no. 8, 1345–1364. MR2483126
- [10] Matthias Hamann, Florian Lehner, Babak Miraftab, and Tim Rühmann, A Stallings type theorem for quasitransitive graphs, J. Combin. Theory Ser. B 157 (2022), 40–69. MR4438888
- Thomas Herbst, On a subclass of context-free groups, RAIRO Inform. Théor. Appl. 25 (1991), no. 3, 255–272. MR1119044
- [12] \_\_\_\_\_, Some remarks on a theorem of Sakarovitch, J. Comput. System Sci. 44 (1992), no. 1, 160–165. MR1161110
- [13] Thomas Herbst and Richard M. Thomas, Group presentations, formal languages and characterizations of onecounter groups, Theoret. Comput. Sci. 112 (1993), no. 2, 187–213. MR1216320
- [14] Derek F. Holt, Matthew D. Owens, and Richard M. Thomas, Groups and semigroups with a one-counter word problem, J. Aust. Math. Soc. 85 (2008), no. 2, 197–209. MR2470538
- [15] Derek F. Holt, Sarah Rees, and Claas E. Röver, Groups, languages and automata, London Mathematical Society Student Texts, vol. 88, Cambridge University Press, Cambridge, 2017. MR3616303
- [16] Derek F. Holt, Sarah Rees, Claas E. Röver, and Richard M. Thomas, Groups with context-free co-word problem, J. London Math. Soc. (2) 71 (2005), no. 3, 643–657. MR2132375
- [17] John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman, Introduction to automata theory, languages and computations, Third, Pearson Education, Inc, 2007.
- [18] Bernhard Krön and Rögnvaldur G. Möller, Quasi-isometries between graphs and trees, J. Combin. Theory Ser. B 98 (2008), no. 5, 994–1013. MR2442593
- [19] Robert P. Kropholler and Davide Spriano, Closure properties in the class of multiple context-free groups, Groups Complex. Cryptol. 11 (2019), no. 1, 1–15. MR4000593
- [20] Manuel Ladra and Pedro V. Silva, The generalized conjugacy problem for virtually free groups, Forum Math. 23 (2011), no. 3, 447–482. MR2805191
- [21] J. Lehnert and P. Schweitzer, The co-word problem for the Higman-Thompson group is context-free, Bull. Lond. Math. Soc. 39 (2007), no. 2, 235–241. MR2323454
- [22] Jörg Lenhert, Gruppen von quasi-automorphismen, Ph.D. Thesis, 2008.
- [23] John Meier, Groups, graphs and trees, London Mathematical Society Student Texts, vol. 73, Cambridge University Press, Cambridge, 2008. An introduction to the geometry of infinite groups. MR2498449
- [24] David E. Muller and Paul E. Schupp, Groups, the theory of ends, and context-free languages, J. Comput. System Sci. 26 (1983), no. 3, 295–310. MR710250
- [25] \_\_\_\_\_, The theory of ends, pushdown automata, and second-order logic, Theoret. Comput. Sci. **37** (1985), no. 1, 51–75. MR796313
- [26] Emanuele Rodaro, Generalizations of the muller-schupp theorem and tree-like inverse graphs, arXiv e-prints (2023). arXiv:math/2302.06664v3.
- [27] Sylvain Salvati, MIX is a 2-MCFL and the word problem in Z<sup>2</sup> is captured by the IO and the OI hierarchies, J. Comput. System Sci. 81 (2015), no. 7, 1252–1277. MR3354791
- [28] Wolfgang Woess, Context-free pairs of groups II—cuts, tree sets, and random walks, Discrete Math. 312 (2012), no. 1, 157–173. MR2852518

Department of Mathematics, Alan Turing Building, The University of Manchester, Manchester M13 9PL, UK

Email address: alex.levine@manchester.ac.uk