# Pauli-Villars regularization of Kaluza-Klein Casimir energy with Lorentz symmetry

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ABSTRACT: The Pauli-Villars regularization is appropriate to discuss the UV sensitivity of low-energy observables because it mimics how the contributions of new particles at high energies cancel large quantum corrections from the light particles in the effective field theory. We discuss the UV sensitivity of the Casimir energy density and pressure in an extra-dimensional model in this regularization scheme, and clarify the condition on the regulator fields to preserve the Lorentz symmetry of the vacuum state. Some of the conditions are automatically satisfied in spontaneouslybroken supersymmetric models, but supersymmetry is not enough to ensure the Lorentz symmetry. We show that the necessary regulators can be introduced as bulk fields. We also evaluate the Casimir energy density with such regulators, and its deviation from the result obtained in the analytic regularization.

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## 1 Introduction

The Casimir effect is a macroscopic quantum effect that has been observed in various experiments and the observed values are in good agreement with theoretical predictions [1–5]. The Casimir energy is defined as the energy difference between the vacuum energy in a compact space, such as a space enclosed by conducting plates, and that in a non-compact space. The vacuum energy in quantum field theory (QFT) is generally divergent and must be regularized, such as the cutoff regularization, in which the cutoff scale  $\Lambda_{cut}$  is set for the momenta of virtual particles in the loops. It is well-known that the Casimir energy remains finite even in the limit of  $\Lambda_{cut} \rightarrow \infty$ . The scale  $\Lambda_{cut}$  is regarded as a scale at which the theory under consideration breaks down and is replaced with a more fundamental theory. The Casimir effect also plays an important role in extra-dimensional models. The quantum correction for the extra-dimensional models is Kaluza-Klein (KK) Casimir energy, which depends on the compactification scale  $m_{\rm KK}$  and determines the physical properties of the extra-dimensional models [6–9]. Since the extra-dimensional models are nonrenormalizable, they should be regarded as effective theories of more fundamental ones, such as a string theory or quantum gravity (QG). Hence  $\Lambda_{\rm cut}$  can not be infinite, and may be close to  $m_{\rm KK}$ . In the latter case, the unknown ultraviolet (UV) physics can affect the KK Casimir energy. This indicates that the Casimir energy in the extra-dimensional models have regularization dependence [10–15], in contrast to the case of renormalizable theories, in which we can safely take the limit  $\Lambda_{\rm cut} \rightarrow \infty$ . In particular, one of the authors suggests the Casimir energy receives a large correction from the UV physics when  $\Lambda_{\rm cut}$  is not far from  $m_{\rm KK}$  in the cutoff regularization scheme [15].

In 3+1-dimensional QFT, there is a significant discussion regarding the Lorentz symmetry violation in the regularization of vacuum energy. Indeed, when utilizing the cutoff regularization, the UV divergences break the Lorentz symmetry [16–18]. If this Lorentz symmetry violation is considered as an actual physical phenomena, they could lead to significant cosmological issues [19]. When we consider the Friedmann-Lematre-Robertson-Walker (FLRW) universe with the metric  $ds^2 = dt^2 - a^2(t) \,\delta_{ij} dx^i dx^j$ , the semiclassical Friedmann equations for a flat universe with a vacuum state are given by

$$H^{2} = \frac{1}{3} \left( \Lambda_{cc} + \langle 0 | \hat{\rho} | 0 \rangle \right), \qquad (1.1)$$

$$2\dot{H} + 3H^2 = \Lambda_{\rm cc} - \langle 0 | \hat{p} | 0 \rangle, \qquad (1.2)$$

where  $\Lambda_{cc}$  is the cosmological constant, the hat denotes an operator,  $\hat{\rho}$  is the energy density,  $\hat{p}$  is the pressure, the dot denotes the time derivative, and  $H \equiv \dot{a}/a$  is the Hubble parameter. A combination of these equations leads to

$$\dot{H} = -\langle 0 | \hat{\rho} + \hat{p} | 0 \rangle.$$
(1.3)

In the cutoff regularization, we have

$$\langle 0 | \hat{\rho} + \hat{p} | 0 \rangle = (-1)^{\delta_{if}} \int_{0}^{\Lambda_{\text{cut}}} \frac{d^{3}k}{2(2\pi)^{3}} \left\{ \sqrt{k^{2} + m^{2}} + \frac{k^{2}}{3\sqrt{k^{2} + m^{2}}} \right\}$$

$$= (-1)^{\delta_{if}} \frac{\Lambda_{\text{cut}}^{3} \sqrt{\Lambda_{\text{cut}}^{2} + m^{2}}}{12\pi^{2}} ,$$

$$(1.4)$$

where  $\delta_{if}$  is the Kronecker delta with i = b, f for bosons and fermions respectively, and m is the mass. This shows that the UV divergences directly contribute to the dynamics of the universe. <sup>1</sup> On the other hand, we should note that (1.4) for the fermionic contribution clearly violates the null energy condition (NEC). The NEC is

<sup>&</sup>lt;sup>1</sup>We briefly mention the observational constraints on  $\langle 0 | \hat{\rho} + \hat{p} | 0 \rangle$ . These are derived from the current measurements of the dark energy and the constraints on its equation of state  $w_{\text{dark}}$ . The

known as a necessary condition to eliminate any pathological spacetime or unphysical geometry [21, 22] and it states  $T_{\mu\nu}n^{\mu}n^{\nu} \ge 0$ , for any null light-like vector  $n^{\mu}$ . This is summarized as  $\rho + p \ge 0$  for the FLRW metric. In the context of the vacuum energy of the quantum fields and its regularization, there exist issues related to the breaking of Lorentz symmetry and the violation of the NEC.

In this paper, we explore the KK Casimir energy density and pressure from compact dimension. We particularly study the UV sensitivity of the KK Casimir energy. As we will show in the next section, the analytic regularization inherently omits the UV contributions, and the cutoff regularization violates the Lorentz symmetry in the vacuum state. Therefore, we adopt the Pauli-Villars regularization, which effectively demonstrates the cancellation of large quantum corrections by the contributions of high-energy virtual particles in the effective field theory. We further specify the necessary conditions on the regulator fields to preserve Lorentz symmetry.<sup>2</sup> Although spontaneously-broken supersymmetric (SUSY) models satisfy some of these conditions, SUSY is not enough to ensure the Lorentz invariance of the vacuum state.

The rest of this paper is organized as follows. In Section 2, we review the analytic and cutoff regularizations of the KK Casimir energy density and pressure. We point out that these regularizations are not adequate to evaluate the UV sensitivity of the Casimir energy preserving the Lorentz symmetry. In Section 3, we consider the Pauli-Villars regularization to regularize the Casimir energy density, and provide the necessary conditions for regulator fields to preserve Lorentz symmetry. In Section 4, we numerically calculate the Casimir energy density and pressure in the Pauli-Villars regularization, and evaluate their dependence on the UV regulator mass scale. In Section 5, we conclude our work.

### 2 Regularizations

We take the following semiclassical treatment [24], which approximately combines QFT and general relativity (GR), and is expected to be reliable under conditions where QG is not important. We treat spacetime classically and use the expected value of the quantized stress-energy tensor in Einstein's equations. Hence, the quantum effect of matter fields on spacetime geometry can be approximately described by the

$$\rho_{\rm dark} + p_{\rm dark} = (1 + w_{\rm dark}) \,\rho_{\rm dark} \sim (1 + w_{\rm dark}) \,(10^{-3} {\rm eV})^4 \,. \tag{1.5}$$

Lorentz violation by dark energy can be formalized by the following expression:

Although some results suggest a slight phantom-like equation of state,  $w_{\text{dark}} \simeq -1.03$ , several independent observations are broadly consistent with the cosmological constant value of  $w_{\text{dark}} = -1.013^{+0.038}_{-0.043}$  [20]. Thus, the vacuum must preserve the Lorentz symmetry with the accuracy,  $\rho_{\text{dark}} + p_{\text{dark}} \lesssim \mathcal{O}(10^{-2})(10^{-3} \text{eV})^4$ .

 $<sup>^{2}</sup>$ See also Ref. [23], which discusses related issues.

semiclassical equations,<sup>3</sup>

$$G_{\mu\nu} + \Lambda_{\rm cc} g_{\mu\nu} = \langle T_{\mu\nu} \rangle \,, \tag{2.1}$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $\Lambda_{cc}$  is the cosmological constant and  $\langle T_{\mu\nu} \rangle$  is the expected value of the quantum stress-energy tensor. Phenomenologically, such treatment will suffice.<sup>4</sup>

It is known that the (quantum) vacuum is Lorentz invariant to a high accuracy from the observation [38, 39]. Therefore, the vacuum energy density  $\hat{\rho}$  must give rise to an energy-momentum tensor in the 4D Minkowski spacetime of the form,

$$\langle T_{\mu\nu}^{\mathrm{vac}} \rangle = \langle 0 | \hat{\rho} | 0 \rangle \eta_{\mu\nu},$$
(2.2)

where  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  is the Minkowski metric, and thus the quantum correction to the vacuum energy density is renormalized by the cosmological constant  $\Lambda_{cc}$ . Note that (2.2) indicates that

$$\langle 0 \mid \hat{\rho} + \hat{p} \mid 0 \rangle = 0, \tag{2.3}$$

where  $\hat{p} \equiv T_{11}^{\text{vac}} = T_{22}^{\text{vac}} = T_{33}^{\text{vac}}$  is the vacuum pressure. Therefore, the LHS of (2.3) measures the violation of the Lorentz symmetry.

#### 2.1 Formal expressions for energy density and pressure

To simplify the discussion, we consider a real scalar theory in a flat 5-dimensional spacetime, and one of the spatial dimensions is compactified on  $S^1/Z_2$ .

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\Phi\partial_{\mu}\Phi - \frac{1}{2}M_{\text{bulk}}^{2}\Phi^{2}, \qquad (2.4)$$

where  $\mu = 0, 1, \dots, 4$ , and  $M_{\text{bulk}}$  is a bulk mass parameter. The coordinate of the compact dimension is denoted as  $y \equiv x^4$ . The fundamental region of  $S^1/Z_2$  is chosen as  $0 \leq y \leq \pi R$ , where R is the radius of  $S^1$ . The real scalar field  $\Phi$  is assumed to be  $Z_2$  odd. Then the KK masses are given by

$$m_n = \sqrt{M_{\text{bulk}}^2 + \frac{n^2}{R^2}}.$$
  $(n = 1, 2, \cdots)$  (2.5)

<sup>&</sup>lt;sup>3</sup>Here we have taken the unit of the gravitational constant, i.e.,  $8\pi G_N = 1$ .

<sup>&</sup>lt;sup>4</sup>This approach has challenges. Specifically, the quantized stress-energy tensor in curved spacetime introduces higher-derivative corrections, leading to non-unitary massive ghosts and potential instability in spacetime and its perturbations, as referenced in various studies [25–35]. These quantum effects could contradict current observations if they significantly influence the universe [35]. Thus, the semiclassical gravity may not hold up under higher-perturbative calculations and may require specialized analysis methods within the effective field theory [36, 37]. In this paper, we do not consider such higher-order calculations.

The vacuum energy density and the vacuum pressure in the 4D effective theory are formally expressed as

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{\infty} \int \frac{d^3 k}{2(2\pi)^3} \sqrt{k^2 + m_n^2},$$

$$\langle 0 | \hat{p} | 0 \rangle = \frac{1}{3} \sum_{n=1}^{\infty} \int \frac{d^3 k}{2(2\pi)^3} \frac{k^2}{\sqrt{k^2 + m_n^2}}.$$

$$(2.6)$$

These obviously diverge, and we need to regularize them. In the following, we review the analytic and the momentum-cutoff regularizations, and mention unsatisfactory points for our purpose. To make the relation between them clear, we introduce the cutoff for the KK mode number  $N_{\text{cut}}$ , the momentum cutoff  $\Lambda_{\text{cut}}$  and the complexified dimension d. Then, (2.6) is regularized as

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{N_{\text{cut}}} \int_{0}^{\Lambda_{\text{cut}}} \frac{d^{d}k}{2(2\pi)^{d}\mu^{d-3}} \sqrt{k^{2} + m_{n}^{2}},$$

$$\langle 0 | \hat{p} | 0 \rangle = \frac{1}{d} \sum_{n=1}^{N_{\text{cut}}} \int_{0}^{\Lambda_{\text{cut}}} \frac{d^{d}k}{2(2\pi)^{d}\mu^{d-3}} \frac{k^{2}}{\sqrt{k^{2} + m_{n}^{2}}},$$

$$(2.7)$$

where  $\mu$  is some scale to adjust the mass dimension. Naively, the cutoff scales for the 3D momentum and the fifth one are expected to be common. Thus we assume that  $m_{N_{\text{cut}}} \simeq \Lambda_{\text{cut}}$ , or more specifically

$$N_{\rm cut} = \text{floor}\left(R\sqrt{\Lambda_{\rm cut}^2 - M_{\rm bulk}^2}\right).$$
(2.8)

Performing the  $\vec{k}$ -integral, we obtain

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{N_{\text{cut}}} \frac{m_n^{d+1}}{2(4\pi)^{d/2} \mu^{d-3} \Gamma(\frac{d}{2})} B_{1-\epsilon_n} \left(\frac{d}{2}, -\frac{d+1}{2}\right),$$

$$\langle 0 | \hat{p} | 0 \rangle = \sum_{n=1}^{N_{\text{cut}}} \frac{m_n^{d+1}}{2d(4\pi)^{d/2} \mu^{d-3} \Gamma(\frac{d}{2})} B_{1-\epsilon_n} \left(\frac{d+2}{2}, -\frac{d+1}{2}\right),$$

$$(2.9)$$

where  $\Gamma(\alpha)$  is the Euler gamma function,  $B_z(\alpha, \beta)$  is the incomplete beta function, and

$$\epsilon_n \equiv \frac{m_n^2}{\Lambda_{\rm cut}^2 + m_n^2}.$$
(2.10)

#### 2.2 Analytic regularization

#### 2.2.1 Review of conventional derivation

The most popular regularization scheme for the calculation of the Casimir energy is the combination of the dimensional regularization and the zeta-function regularization, which we call analytic regularization in this paper. Let us take the limit  $\Lambda_{\text{cut}} \to \infty$ , i.e.,  $\epsilon_n \to 0$ , keeping d-3 nonzero, in (2.9). Then, the incomplete beta function reduces the complete beta function, and becomes n-independent.

$$\lim_{\epsilon_n \to 0} B_{1-\epsilon_n} \left( \frac{d}{2}, -\frac{d+1}{2} \right) = B \left( \frac{d}{2}, -\frac{d+1}{2} \right) = -\frac{\Gamma(\frac{d}{2})\Gamma(-\frac{d+1}{2})}{2\sqrt{\pi}},$$
$$\lim_{\epsilon_n \to 0} B_{1-\epsilon_n} \left( \frac{d+2}{2}, -\frac{d+1}{2} \right) = B \left( \frac{d+2}{2}, -\frac{d+1}{2} \right) = \frac{d\Gamma(\frac{d}{2})\Gamma(-\frac{d+1}{2})}{2\sqrt{\pi}}.$$
 (2.11)

Thus, (2.9) become

$$\langle 0 | \hat{\rho} | 0 \rangle = \frac{\mu^4}{2(4\pi)^{d/2}\Gamma(\frac{d}{2})} B\left(\frac{d}{2}, -\frac{d+1}{2}\right) \sum_{n=1}^{\infty} \left(\frac{m_n}{\mu}\right)^{d+1}$$

$$= -\frac{\mu^4 \Gamma(-\frac{d+1}{2})}{2(4\pi)^{\frac{d+1}{2}}} \sum_{n=1}^{\infty} \left(\frac{m_n}{\mu}\right)^{d+1},$$

$$\langle 0 | \hat{p} | 0 \rangle = \frac{\mu^4}{2d(4\pi)^{d/2}\Gamma(\frac{d}{2})} B\left(\frac{d+2}{2}, -\frac{d+1}{2}\right) \sum_{n=1}^{\infty} \left(\frac{m_n}{\mu}\right)^{d+1}$$

$$= \frac{\mu^4 \Gamma(-\frac{d+1}{2})}{2(4\pi)^{\frac{d+1}{2}}} \sum_{n=1}^{\infty} \left(\frac{m_n}{\mu}\right)^{d+1}.$$

$$(2.12)$$

The infinite sum over the KK modes is evaluated by the zeta-function regularization technique [8, 40, 41]. Using the formula (B.10) with (B.11) in Appendix, the energy density is expressed as

$$\langle 0 | \hat{\rho} | 0 \rangle = -\frac{\mu^{3-d} \Gamma(-\frac{d+1}{2})}{2(4\pi)^{\frac{d+1}{2}} R^{d+1}} \sum_{n=1}^{\infty} \left( \bar{M}_{\text{bulk}}^2 + n^2 \right)^{\frac{d+1}{2}}$$

$$= \frac{\mu^{3-d} M_{\text{bulk}}^{d+1} \Gamma(-\frac{d+1}{2})}{4(4\pi)^{(d+1)/2}} - \frac{\mu^{3-d} \Gamma(-\frac{d+2}{2})}{8(4\pi)^{d/2}} R M_{\text{bulk}}^{d+2}$$

$$- \frac{\mu^{3-d} M_{\text{bulk}}^{\frac{d+2}{2}}}{(2\pi)^{d+1} R^{d/2}} \sum_{n=1}^{\infty} n^{-\frac{d+2}{2}} K_{\frac{d+2}{2}} \left( 2\pi n \bar{M}_{\text{bulk}} \right),$$

$$(2.13)$$

where  $\overline{M}_{\text{bulk}} \equiv RM_{\text{bulk}}$ , and  $K_{\alpha}(z)$  is the modified Bessel function of the second kind. The first term diverges as  $d \to 3$ , but it does not depend on R and is irrelevant to the stabilization of the extra dimension. Thus we simply neglect it. We require that the vacuum energy density in the decompactified limit  $R \to \infty$  vanishes [42]. Thus the Casimir energy density, which is a function of R, is defined as

$$\frac{\langle 0 | \hat{\rho} | 0 \rangle_{\text{Casimir}}(R)}{\pi R} \equiv \frac{\langle 0 | \hat{\rho} | 0 \rangle(R)}{\pi R} - \lim_{R \to \infty} \frac{\langle 0 | \hat{\rho} | 0 \rangle(R)}{\pi R}.$$
 (2.14)

Note that the subtraction should be performed for the 5D energy density since the second term is the quantity in the decompactified limit. Then, the second term in

(2.13) is cancelled, and we obtain

$$\langle 0 | \hat{\rho} | 0 \rangle_{\text{Casimir}} = -\frac{\mu^{3-d} M_{\text{bulk}}^{\frac{d+2}{2}}}{(2\pi)^{d+1} R^{d/2}} \sum_{n=1}^{\infty} n^{-\frac{d+2}{2}} K_{\frac{d+2}{2}} \left( 2\pi n R M_{\text{bulk}} \right)$$
  
$$\rightarrow -\frac{M_{\text{bulk}}^{\frac{5}{2}}}{16\pi^4 R^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^{-\frac{5}{2}} K_{\frac{5}{2}} (2\pi n R M_{\text{bulk}}).$$
(2.15)

We have taken the limit  $d \to 3$  at the last step. Similarly, the vacuum pressure is calculated as

$$\langle 0 | \hat{p} | 0 \rangle_{\text{Casimir}} \equiv \langle 0 | \hat{p} | 0 \rangle - R \lim_{R \to \infty} \frac{\langle 0 | \hat{p} | 0 \rangle}{R}$$

$$= \frac{\mu^{3-d} M_{\text{bulk}}^{\frac{d+2}{2}}}{(2\pi)^{d+1} R^{d/2}} \sum_{n=1}^{\infty} n^{-\frac{d+2}{2}} K_{\frac{d+2}{2}} \left(2\pi n R M_{\text{bulk}}\right)$$

$$\rightarrow \frac{M_{\text{bulk}}^{\frac{5}{2}}}{16\pi^4 R^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^{-\frac{5}{2}} K_{\frac{5}{2}} (2\pi n R M_{\text{bulk}}).$$

$$(2.16)$$

In the massless case  $M_{\text{bulk}} = 0$ , (2.15) reduces to the well-known form,

$$\langle 0 \,|\, \hat{\rho} \,|\, 0 \rangle_{\text{Casimir}} = -\frac{\mu^{3-d} \Gamma(\frac{d+2}{2}) \zeta(d+2)}{2^{d+2} \pi^{\frac{3}{2}d+2} R^{d+1}} \to -\frac{3\zeta(5)}{128\pi^6 R^4},\tag{2.17}$$

where  $\zeta(s)$  is the Riemann zeta function.

From (2.15) and (2.16), we can see that the sum of the KK Casimir energy density and pressure are exactly zero,

$$\left\langle 0 \left| \hat{\rho} + \hat{p} \right| 0 \right\rangle_{\text{Casimir}} = 0.$$
(2.18)

Thus, the Lorentz symmetry and NEC are both preserved in this regularization.<sup>5</sup>

#### 2.2.2 Cutoff sensitivity in analytic regularization

Although the formula (2.15) or (2.16) is useful because of its rapid convergent property, the analytic continuation processes make it difficult to see how the divergent terms are removed. It is well-known that this regularization only captures the logarithmic divergences, and is insensitive to the power-law divergences of  $\Lambda_{\rm cut}$ . To see the situation, let us review the procedure we have performed in (2.11) in more detail. As long as  $\Lambda_{\rm cut}$  is kept finite, the incomplete beta functions in (2.9) are well-defined for any values of the dimension d. Before taking the limit  $\Lambda_{\rm cut} \to \infty$ , let us consider a case that d < -1. Then, using (A.6) in the Appendix, the incomplete beta functions

<sup>&</sup>lt;sup>5</sup>We can already see this in the formal expressions in (2.12).

are expanded as

$$B_{1-\epsilon_n}\left(\frac{d}{2}, -\frac{d+1}{2}\right) = B\left(\frac{d}{2}, -\frac{d+1}{2}\right) + \frac{2}{d+1}\epsilon_n^{-\frac{d+1}{2}} - \frac{d-2}{d-1}\epsilon_n^{\frac{1-d}{2}} + \frac{(d-2)(d-4)}{d-3}\epsilon_n^{\frac{3-d}{2}} + \mathcal{O}\left(\epsilon_n^{\frac{5-d}{2}}\right),$$

$$B_{1-\epsilon_n}\left(\frac{d+2}{2}, -\frac{d+1}{2}\right) = B\left(\frac{d+2}{2}, -\frac{d+1}{2}\right) + \frac{2}{d+1}\epsilon_n^{-\frac{d+1}{2}} - \frac{d}{d-1}\epsilon_n^{\frac{1-d}{2}} + \frac{d(d-2)}{d-3}\epsilon_n^{\frac{3-d}{2}} + \mathcal{O}\left(\epsilon_n^{\frac{5-d}{2}}\right).$$
(2.19)

Since all the powers in RHS are positive for d < -1, we can safely take the limit  $\Lambda_{\text{cut}} \rightarrow \infty$  (i.e.,  $\epsilon_n \rightarrow 0$ ), and drop all  $\epsilon_n$ -dependent terms. After dropping them, we can move d to a value close to 3. This is what we have done in (2.11). However, if we keep the  $\epsilon_n$ -dependent terms when we move d to a value close to 3, the second and the third terms in RHS of (2.19) have negative powers, and correspond to the quartic and quadratic divergences, respectively.<sup>6</sup> Therefore, what we have done in (2.11) is just dropping the quartic and quadratic divergent terms by hand.

A similar prescription has been performed when we apply the zeta-function regularization for the infinite sum over the KK modes. If we keep the cutoff  $\Lambda_{\text{cut}}$  finite, the incomplete functions in (2.9) depend on the KK level n, and cannot be factored out from the summation over n. Therefore, it is not easy to perform the exact calculation of (2.9). Hence we investigate the following expression instead.

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{\infty} \frac{m_n^{d+1}}{2(4\pi)^{d/2} \mu^{d-3} \Gamma(\frac{d}{2})} B_{1-\epsilon_n} \left(\frac{d}{2}, -\frac{d+1}{2}\right) e^{-a^2 n^2},$$
(2.20)

where  $a \equiv 1/N_{\text{cut}}$  is a tiny positive constant. Instead of the sharp cutoff at  $n = N_{\text{cut}}$ , we introduce the damping factor  $e^{-a^2n^2}$ , which suppresses the contribution of heavy KK modes with  $m_n > \Lambda_{\text{cut}}$ .<sup>7</sup> Then, (2.20) is rewritten as

$$\langle 0 | \hat{\rho} | 0 \rangle = \frac{1}{2(4\pi)^{d/2} \mu^{d-3} R^{d+1} \Gamma(\frac{d}{2})} U\left(\frac{d}{2}, -\frac{d+1}{2}; \bar{M}_{\text{bulk}}^2\right),$$
(2.21)

where  $U(\alpha, \beta; M^2)$  is defined in (B.1) in Appendix. According to the expression (B.3)

 $^6\mathrm{Besides},$  the fourth terms also diverge as  $d\to3$  and contain logarithmic divergent terms.

<sup>&</sup>lt;sup>7</sup>To simplify the discussion, we approximate a as  $a = (\Lambda_{\text{cut}} R)^{-1}$ .

with (B.4) and (B.12), this has the following terms.

$$U\left(\frac{d}{2}, -\frac{d+1}{2}; \bar{M}_{\text{bulk}}^{2}\right) = \frac{(\Lambda_{\text{cut}}R)^{d+1}}{d(d+1)} - \frac{\bar{M}_{\text{bulk}}^{2}(\Lambda_{\text{cut}}R)^{d-1}}{2d(d-1)} + \frac{3\bar{M}_{\text{bulk}}^{4}(\Lambda_{\text{cut}}R)^{d-3}}{8d(d-3)} - \frac{C_{1}(-\frac{d+1}{2}; \bar{M}_{\text{bulk}}^{2})}{d\sqrt{\pi}} (\Lambda_{\text{cut}}R)^{d+2} - \frac{C_{2}(-\frac{d+1}{2}; \bar{M}_{\text{bulk}}^{2})}{d\sqrt{\pi}} (\Lambda_{\text{cut}}R)^{d} - \frac{C_{3}(-\frac{d+1}{2}; \bar{M}_{\text{bulk}}^{2})}{d\sqrt{\pi}} (\Lambda_{\text{cut}}R)^{d-2} + \tilde{U}_{2}(d; \bar{M}_{\text{bulk}}^{2}) + \cdots,$$

$$(2.22)$$

where  $C_i(\beta; M^2)$  (i = 1, 2, 3) are defined in (B.13), and

$$\tilde{U}_{2}(d; \bar{M}_{\text{bulk}}^{2}) \equiv \frac{2(\Lambda_{\text{cut}}R)^{d}}{d} \sum_{n=1}^{\infty} \sqrt{(\Lambda_{\text{cut}}R)^{2} + \bar{M}_{\text{bulk}}^{2} + n^{2}} \exp\left(-\frac{n^{2}}{(\Lambda_{\text{cut}}R)^{2}}\right), \quad (2.23)$$

and the ellipsis denotes terms that appeared in (2.13) and irrelevant terms that will vanish in the limit of  $\Lambda_{\rm cut} \to \infty$  when d = 3. In the limit of  $d \to 3$  keeping  $\Lambda_{\rm cut}$ finite, the terms shown in (2.22) represent power-law divergent terms up to quintic in  $\Lambda_{\rm cut}$ . This is expected because we are considering the 5D theory. In the derivation of (2.13), we have taken the limit of  $\Lambda_{\rm cut} \to \infty$  for d < -2, where all terms shown in (2.22) vanish. However, this treatment is equivalent to just dropping those terms by hand. Therefore, the analytic regularization is inappropriate for studying the UV sensitivity of the Casimir energy density or pressure.

#### 2.3 Cutoff regularization

Next, we consider the cutoff regularization. Take the limit  $d \to 3$ , keeping  $\Lambda_{\text{cut}}$  finite, in (2.9). Then we obtain

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{N_{\text{cut}}} \frac{m_n^4}{8\pi^2} B_{1-\epsilon_n} \left( \frac{3}{2}, -2 \right)$$

$$= \sum_{n=1}^{N_{\text{cut}}} \left\{ \frac{\Lambda_{\text{cut}} \sqrt{\Lambda_{\text{cut}}^2 + m_n^2} (2\Lambda_{\text{cut}}^2 + m_n^2)}{32\pi^2} - \frac{m_n^4}{32\pi^2} \ln \frac{\Lambda_{\text{cut}} + \sqrt{\Lambda_{\text{cut}}^2 + m_n^2}}{m_n} \right\},$$

$$\langle 0 | \hat{p} | 0 \rangle = \sum_{n=1}^{N_{\text{cut}}} \frac{m_n^4}{24\pi^2} B_{1-\epsilon_n} \left( \frac{5}{2}, -2 \right)$$

$$= \sum_{n=1}^{N_{\text{cut}}} \left\{ \frac{\Lambda_{\text{cut}} \sqrt{\Lambda_{\text{cut}}^2 + m_n^2} (2\Lambda_{\text{cut}}^2 - 3m_n^2)}{96\pi^2} + \frac{m_n^4}{32\pi^2} \ln \frac{\Lambda_{\text{cut}} + \sqrt{\Lambda_{\text{cut}}^2 + m_n^2}}{m_n} \right\}.$$

$$(2.24)$$

The sum of the vacuum energy density and pressure is

$$\langle 0 \,|\, \hat{\rho} + \hat{p} \,|\, 0 \rangle = \sum_{n=1}^{N_{\rm cut}} \frac{\Lambda_{\rm cut}^3 \sqrt{\Lambda_{\rm cut}^2 + m_n^2}}{12\pi^2},\tag{2.25}$$

where the logarithmic terms exactly cancel but the cut-off divergences remain. Namely, the Lorentz symmetry is violated in this regularization. If  $\Phi$  is replaced with a 5D fermion, an overall minus sign appears in the above expressions. Hence NEC is also violated in that case.

For a light mode with  $m_n \ll \Lambda_{\text{cut}}$ , its contribution to the vacuum energy and the vacuum pressure can be expanded as

$$\tilde{\rho}(m_n) \equiv \frac{m_n^4}{8\pi^2} B_{1-\epsilon_n} \left(\frac{3}{2}, -2\right) = \frac{1}{16\pi^2} \left(\Lambda_{\text{cut}}^4 + \Lambda_{\text{cut}}^2 m_n^2 + \frac{m_n^4}{8}\right) - \frac{m_n^4}{32\pi^2} \ln \frac{2\Lambda_{\text{cut}}}{m_n} + \mathcal{O}\left(\frac{m_n^6}{\Lambda_{\text{cut}}^2}\right), \tilde{\rho}(m_n) \equiv \frac{m_n^4}{24\pi^2} B_{1-\epsilon_n} \left(\frac{5}{2}, -2\right) = \frac{1}{48\pi^2} \left(\Lambda_{\text{cut}}^4 - \Lambda_{\text{cut}}^2 m_n^2 - \frac{7m_n^4}{8}\right) + \frac{m_n^4}{32\pi^2} \ln \frac{2\Lambda_{\text{cut}}}{m_n} + \mathcal{O}\left(\frac{m_n^6}{\Lambda_{\text{cut}}^2}\right).$$
(2.26)

After summing over the KK modes, the leading terms of  $\langle 0 | \hat{\rho} | 0 \rangle$  and  $\langle 0 | \hat{\rho} | 0 \rangle$  are

$$\langle 0 | \hat{\rho} | 0 \rangle_{\text{leading}} \sim \frac{N_{\text{cut}} \Lambda_{\text{cut}}^4}{16\pi^2} \sim \frac{R \Lambda_{\text{cut}}^5 \sqrt{1 - M_{\text{bulk}}^2 / \Lambda_{\text{cut}}^2}}{16\pi^2},$$

$$\langle 0 | \hat{p} | 0 \rangle_{\text{leading}} \sim \frac{N_{\text{cut}} \Lambda_{\text{cut}}^4}{48\pi^2} \sim \frac{R \Lambda_{\text{cut}}^5 \sqrt{1 - M_{\text{bulk}}^2 / \Lambda_{\text{cut}}^2}}{48\pi^2},$$

$$(2.27)$$

where  $N_{\text{cut}}$  is defined in (2.8). Since these are proportional to R, they are canceled in the Casimir energy density and pressure defined in (2.14) and (2.16), respectively. However, the other terms remain and violate the Lorentz symmetry. Thus the cutoff regularization is considered to be problematic for the calculation of the Casimir energy [16–18].

From the physical point of view, contributions of massive KK modes near the cutoff scale  $\Lambda_{cut}$  should be suppressed by UV physics. In the previous work [15], we introduced a damping function, such as

$$g_{\rm damp}(n) = \exp\left(-\frac{n^2}{2N_{\rm cut}^2}\right),\tag{2.28}$$

or

$$g_{\rm damp}(n) = \frac{1}{2} \left[ 1 + \tanh\left\{ A\left(1 - \frac{n}{N_{\rm cut}}\right) \right\} \right], \qquad (2.29)$$

where  $A \gtrsim 10$  is a positive constant that controls the steepness around the cutoff scale, and inserted it into the expression (2.9) as

$$\langle 0 | \hat{\rho} | 0 \rangle = \sum_{n=1}^{\infty} \frac{m_n^{d+1}}{2(4\pi)^{d/2} \Gamma(\frac{d}{2})} B_{1-\epsilon_n} \left(\frac{d}{2}, -\frac{d+1}{2}\right) g_{\text{damp}}(n).$$
(2.30)

Then we obtain a finite value for the Casimir energy density, which agrees with the value obtained by (2.16).<sup>8</sup> The cutoff regularization considered in this subsection corresponds to the limit of  $A \to \infty$  in (2.29). It is known that the regularization with such a sharp cutoff provides a divergent Casimir energy density, and should not be applied to the calculations for the Casimir energy density and pressure [15].

#### 3 Pauli-Villars regularization

As mentioned in Sec. 2.3, the contributions of massive KK modes near  $\Lambda_{\rm cut}$  should be suppressed by the UV physics, such as contributions of new particles with masses of  $\mathcal{O}(\Lambda_{\rm cut})$ . Such contributions can be mimicked by the Pauli-Villars regulators. However, a single regulator that has opposite statistics and a large mass  $M_{\rm reg}$  is not enough to suppress contributions of the KK modes heavier than  $M_{\rm reg}$ .<sup>9</sup> Hence, for each KK mode with mass  $m_n$ , we introduce k species of regulators. Then, its contributions to the Casimir energy density and pressure are modified as

$$\rho_n \equiv \tilde{\rho}(m_n) - \sum_{i=1}^k c_i \tilde{\rho}(M_i),$$
  
$$p_n \equiv \tilde{p}(m_n) - \sum_{i=1}^k c_i \tilde{p}(M_i),$$
(3.1)

where  $\tilde{\rho}$  and  $\tilde{p}$  are defined in (2.26),  $M_i$  and an integer  $c_i$  denote the mass and the degree of freedom for the *i*-th regulator, respectively. We assume that all  $M_i$  $(i = 1, 2, \dots, k)$  are of  $\mathcal{O}(M_{\text{reg}})$ . Note that we introduce both bosonic  $(c_i < 0)$  and fermionic  $(c_i > 0)$  regulators. Then, using the expanded expressions in (2.26), we have

$$\rho_{n} = \frac{1}{16\pi^{2}} \left\{ \left( 1 - \sum_{i} c_{i} \right) \Lambda_{\text{cut}}^{4} + \Lambda_{\text{cut}}^{2} \left( m_{n}^{2} - \sum_{i} c_{i} M_{i}^{2} \right) + \frac{1}{8} \left( m_{n}^{4} - \sum_{i} c_{i} M_{i}^{4} \right) \right\} - \frac{m_{n}^{4}}{32\pi^{2}} \ln \frac{2\Lambda_{\text{cut}}}{m_{n}} + \sum_{i} c_{i} \frac{M_{i}^{4}}{32\pi^{2}} \ln \frac{2\Lambda_{\text{cut}}}{M_{i}} + \mathcal{O}\left(\frac{M_{\text{reg}}^{6}}{\Lambda_{\text{cut}}^{2}}\right),$$

$$p_{n} = \frac{1}{48\pi^{2}} \left\{ \left( 1 - \sum_{i} c_{i} \right) \Lambda_{\text{cut}}^{4} - \Lambda_{\text{cut}}^{2} \left( m_{n}^{2} - \sum_{i} c_{i} M_{i}^{2} \right) - \frac{7}{8} \left( m_{n}^{4} - \sum_{i} c_{i} M_{i}^{4} \right) \right\} + \frac{m_{n}^{2}}{32\pi^{2}} \ln \frac{2\Lambda_{\text{cut}}}{m_{n}} - \sum_{i} c_{i} \frac{M_{i}^{4}}{32\pi^{2}} \ln \frac{2\Lambda_{\text{cut}}}{M_{i}} + \mathcal{O}\left(\frac{M_{\text{reg}}^{6}}{\Lambda_{\text{cut}}^{2}}\right).$$

$$(3.2)$$

<sup>&</sup>lt;sup>8</sup>With the damping function in (2.29), the parameter A has to be chosen to a value in the appropriate region to obtain a consistent value with (2.16).

<sup>&</sup>lt;sup>9</sup>In the Pauli-Villars regularization,  $M_{\rm reg}$  plays a role of  $\Lambda_{\rm cut}$  in the cutoff regularization.

If we require the integers  $c_i$   $(i = 1, 2, \dots, k)$  to satisfy [17, 23]<sup>10</sup>

$$\sum_{i=1}^{k} c_i = 1, \qquad \sum_{i=1}^{k} c_i M_i^2 = m_n^2, \qquad \sum_{i=1}^{k} c_i M_i^4 = m_n^4, \tag{3.3}$$

the Lorentz-violating terms are canceled, and obtain

$$\rho_n = -\frac{m_n^4}{32\pi^2} \ln \frac{M_{\text{reg}}}{m_n} + \sum_{i=1}^k c_i \frac{M_i^4}{32\pi^2} \ln \frac{M_{\text{reg}}}{M_i},$$
$$p_n = \frac{m_n^4}{32\pi^2} \ln \frac{M_{\text{reg}}}{m_n} - \sum_{i=1}^k c_i \frac{M_i^4}{32\pi^2} \ln \frac{M_{\text{reg}}}{M_i},$$
(3.4)

in the limit of  $\Lambda_{\rm cut} \to \infty$ . Hence we have

$$\rho_n + p_n = 0. \tag{3.5}$$

The first condition in (3.3) is the requirement of the balance between the bosonic and fermionic degrees of freedom. The second one has the same form as the supertrace mass formula in a model that has spontaneously broken supersymmetry (SUSY) [43]. Namely, the first two conditions in (3.3) are automatically satisfied in such a model. To preserve the Lorentz symmetry, however, the third condition is also necessary. It is intriguing to discuss the possibility of constructing a SUSY model in which all conditions in (3.3) are satisfied [23].

To suppress the contributions of the massive KK modes heavier than  $M_{\rm reg}$ , we should also require that

$$\lim_{n \to \infty} \rho_n = \lim_{n \to \infty} p_n = 0. \tag{3.6}$$

This is rewritten as

$$\lim_{n \to \infty} \left( m_n^4 \ln \frac{m_n^2}{M_{\text{reg}}^2} - \sum_{i=1}^k c_i M_i^4 \ln \frac{M_i^2}{M_{\text{reg}}^2} \right) = 0.$$
(3.7)

In the case of

$$k = 3, \quad c_1 = 1, \quad c_2 = -c_3,$$
 (3.8)

we can solve (3.3), and obtain

$$M_2^2 = \frac{c_2 - 1}{2c_2} M_1^2 + \frac{c_2 + 1}{2c_2} m_n^2,$$
  

$$M_3^2 = \frac{c_2 + 1}{2c_2} M_1^2 + \frac{c_2 - 1}{2c_2} m_n^2.$$
(3.9)

<sup>&</sup>lt;sup>10</sup>Wolfgang Pauli found these constraints (3.3).

For  $c_2 = 3$ , we have 4 bosonic and 4 fermionic degrees of freedom in total and can be embedded into a chiral multiplet in a (spontaneously broken) SUSY model. In the following, we consider the case of (3.8) with  $c_2 = 3$  as a specific example.

We assume that  $M_i^2$  (i = 1, 2, 3) are functions of  $m_n^2$  and  $M_{\text{reg}}^2$ . In solving (3.7), we are interested in the KK modes with  $m_n \gg M_{\text{reg}}$ . Thus, we expand  $M_1^2$  as

$$M_1^2 = \alpha m_n^2 \left( 1 + \beta_1 \delta + \beta_2 \delta^2 + \cdots \right), \qquad (3.10)$$

where  $\delta \equiv M_{\rm reg}^2/m_n^2$ . Using this expression and (3.9), we can expand the LHS in (3.7) as

$$m_n^4 \ln \frac{m_n^2}{M_{\text{reg}}^2} - \sum_{i=1}^3 c_i M_i^4 \ln \frac{M_i^2}{M_{\text{reg}}^2}$$
  
=  $\left( \mathcal{C}_1 m_n^4 + \mathcal{C}_2 m_n^2 M_{\text{reg}}^2 + \mathcal{C}_3 M_{\text{reg}} \right) \ln \frac{m_n^2}{M_{\text{reg}}^2}$   
+  $\mathcal{C}_4 m_n^4 + \mathcal{C}_5 m_n^2 M_{\text{reg}}^2 + \mathcal{C}_6 M_{\text{reg}}^4 + \mathcal{O}\left(\frac{M_{\text{reg}}^6}{m_n^2}\right),$  (3.11)

where the coefficients  $C_i$   $(i = 1, 2, \dots, 6)$  are functions of  $\alpha$ ,  $\beta_1$  and  $\beta_2$ . The requirement (3.7) indicates that all  $C_i$   $(i = 1, 2, \dots, 6)$  vanish. We find that  $C_1$ ,  $C_2$ and  $C_3$  automatically vanish, and do not give any constraints on  $\alpha$ ,  $\beta_1$  and  $\beta_2$ . The coefficient  $C_4$  is a function of only  $\alpha$ ,

$$C_4 = -\alpha^2 \ln \alpha - \frac{(\alpha+2)^2}{3} \ln \frac{\alpha+2}{3} + \frac{(2\alpha+1)^2}{3} \ln \frac{2\alpha+1}{3}, \qquad (3.12)$$

and the solution of  $C_4 = 0$  is  $\alpha = 1$ . Under the condition  $\alpha = 1$ , we can easily see that both  $C_5$  and  $C_6$  vanish identically. Therefore, we can take  $\beta_2 = 0$ , and assume that

$$M_1^2 = m_n^2 + \beta_1 M_{\rm reg}^2, \tag{3.13}$$

as a solution to (3.7). If we rescale  $M_{\text{reg}}^2$ , we can always set  $\beta_1 = 1$ . As a result, we can choose a solution of (3.7) (and (3.3)) as

$$M_1^2(m_n^2, M_{\text{reg}}^2) = M_{\text{reg}}^2 + m_n^2,$$
  

$$M_2^2(m_n^2, M_{\text{reg}}^2) = \frac{1}{3}M_{\text{reg}}^2 + m_n^2,$$
  

$$M_3^2(m_n^2, M_{\text{reg}}^2) = \frac{2}{3}M_{\text{reg}}^2 + m_n^2.$$
(3.14)

This result indicates that the regulators can be regarded as the KK modes for 5D bulk fields. In fact, if we introduce one fermionic 5D field with the (squared) bulk mass  $M_{\text{bulk}}^2 + M_{\text{reg}}^2$ , three fermionic 5D fields with  $M_{\text{bulk}}^2 + \frac{1}{3}M_{\text{reg}}^2$ , and three bosonic

5D fields with  $M_{\text{bulk}}^2 + \frac{2}{3}M_{\text{reg}}^2$ , the conditions (3.3) and (3.6) are satisfied for each KK mode.

Before ending this section, we comment on the relation to analytic regularization. In that regularization,  $\rho_n$  and  $p_n$  are read off from (2.12) as

$$\rho_n = -p_n = -\frac{\mu^4 \Gamma(-\frac{d+1}{2})}{2(4\pi)^{\frac{d+1}{2}}} \left(\frac{m_n}{\mu}\right)^{d+1} = -\frac{m_n^4}{32\pi^2} \left(\frac{m_n^2}{4\pi\mu^2}\right)^{\frac{d-3}{2}} \Gamma\left(\frac{3-d}{2}-2\right)$$
$$= -\frac{m_n^4}{32\pi^2} \left\{\frac{1}{3-d} - \frac{1}{2} \left(\ln\frac{m_n^2}{4\pi\mu^2} - \frac{3}{2} + \gamma_{\rm E}\right) + \mathcal{O}\left(\frac{d-3}{2}\right)\right\},\tag{3.15}$$

where  $\gamma_{\rm E}$  is the Euler-Mascheroni constant. We have used (A.3) at the last equality. After the minimal subtraction, we have

$$\rho_n = -p_n = \frac{m_n^4}{64\pi^2} \ln \frac{m_n^2}{\mu^2}.$$
(3.16)

To match (3.2) with this result, a further additional condition has to be imposed [17].

$$\sum_{i=1}^{k} c_i M_i^4 \ln \frac{M_i^2}{\mu^2} = 0.$$
(3.17)

Therefore, the number of the regulator species has to be chosen as  $k \ge 4$ . Then, (3.2) agrees with (3.16) in the limit of  $\Lambda_{\text{cut}} \to \infty$ . However, we do not have a simple solution of (3.3) when  $k \ge 4$ . For example, if we assume that k = 4 and  $c_3 = -c_4$ , we obtain from (3.3)

$$M_3^2 = -\mathcal{A} + \mathcal{B},$$
  

$$M_4^2 = \mathcal{A} + \mathcal{B},$$
(3.18)

where

$$\mathcal{A} \equiv \frac{c_1 M_1^2 + (1 - c_1) M_2^2 - m_n^2}{2c_3},$$
$$\mathcal{B} \equiv \frac{c_1 M_1^4 + (1 - c_1) M_2^4}{4c_3 \mathcal{A}}.$$
(3.19)

Plugging this into (3.17) and solving it, we can express  $M_2^2$  in terms of  $M_1^2$  in principle. As a result,  $M_i^2$  (i = 2, 3, 4) can be expressed as functions of  $M_1^2$  and  $m_n^2$ . However, we do not have analytic expressions for them in general.

As we will see in the next section, even if the condition (3.17) is not imposed, the result well agrees with the one obtained in the analytic regularization (2.15) as long as min  $(M_1^2, M_2^2, M_3^2) > m_{\rm KK} \equiv R^{-1}$ ,

#### 4 Regulator-mass dependence of Casimir energy

In this section, we will numerically calculate the Casimir energy density and pressure in the Pauli-Villars regularization, and evaluate their dependence on the regulator mass scale  $M_{\text{reg}}$ . As a specific example, we choose the regulator masses as (3.14). In this case, the energy density and pressure for the vacuum are expressed as

$$\langle 0 | \hat{\rho} | 0 \rangle^{\text{PV}} = - \langle 0 | \hat{p} | 0 \rangle^{\text{PV}}$$

$$= \frac{M_{\text{reg}}^4}{64\pi^2} \sum_{n=1}^{\infty} \left( \hat{m}_n^4 \ln \hat{m}_n^2 - \sum_{i=1}^3 c_i \hat{M}_i^4 \ln \hat{M}_i^2 \right)$$

$$= \frac{M_{\text{reg}}^4}{64\pi^2} \sum_{n=1}^{\infty} F(an),$$

$$(4.1)$$

where  $a \equiv (M_{\text{reg}}R)^{-1}$ , and

$$\begin{split} \hat{m}_{n}^{2} &\equiv \frac{m_{n}^{2}}{M_{\text{reg}}^{2}} = \hat{M}_{\text{bulk}}^{2} + a^{2}n^{2}, \qquad \hat{M}_{\text{bulk}} \equiv \frac{M_{\text{bulk}}}{M_{\text{reg}}}, \\ \hat{M}_{1}^{2} &\equiv \frac{M_{1}^{2}}{M_{\text{reg}}^{2}} = 1 + \frac{m_{n}^{2}}{M_{\text{reg}}^{2}} = 1 + \left(\hat{M}_{\text{bulk}}^{2} + a^{2}n^{2}\right), \\ \hat{M}_{2}^{2} &\equiv \frac{M_{2}^{2}}{M_{\text{reg}}^{2}} = \frac{1}{3} + \frac{m_{n}^{2}}{M_{\text{reg}}^{2}} = \frac{1}{3} + \left(\hat{M}_{\text{bulk}}^{2} + a^{2}n^{2}\right), \\ \hat{M}_{3}^{2} &\equiv \frac{M_{3}^{2}}{M_{\text{reg}}^{2}} = \frac{2}{3} + \frac{m_{n}^{2}}{M_{\text{reg}}^{2}} = \frac{2}{3} + \left(\hat{M}_{\text{bulk}}^{2} + a^{2}n^{2}\right), \\ F(x) &\equiv \left(\hat{M}_{\text{bulk}}^{2} + x^{2}\right)^{2}\ln\left(\hat{M}_{\text{bulk}}^{2} + x^{2}\right) \\ &- \left(1 + \hat{M}_{\text{bulk}}^{2} + x^{2}\right)^{2}\ln\left(1 + \hat{M}_{\text{bulk}}^{2} + x^{2}\right) \\ &- 3\left(\frac{1}{3} + \hat{M}_{\text{bulk}}^{2} + x^{2}\right)^{2}\ln\left(\frac{1}{3} + \hat{M}_{\text{bulk}}^{2} + x^{2}\right) \\ &+ 3\left(\frac{2}{3} + \hat{M}_{\text{bulk}}^{2} + x^{2}\right)^{2}\ln\left(\frac{2}{3} + \hat{M}_{\text{bulk}}^{2} + x^{2}\right). \end{split}$$
(4.2)

Fig. 1 shows the profile of the function F(x) for various values of  $\hat{M}_{\text{bulk}}$ . We can see that the contribution of the KK modes damps around x = 1, which corresponds to the regulator mass scale  $M_{\text{reg}}$ .

According to (2.14), the Casimir energy and pressure are given by

$$\langle 0 | \hat{\rho} | 0 \rangle_{\text{Casimir}}^{\text{PV}} = - \langle 0 | \hat{p} | 0 \rangle_{\text{Casimir}}^{\text{PV}} = \frac{M_{\text{reg}}^4}{64\pi^2} \Delta(a), \qquad (4.3)$$



Figure 1. The profile of the function F(x) defined in (4.2). The bulk mass is chosen as  $\hat{M}_{\text{bulk}} = 0$  (solid), 0.1 (dashed), 0.3 (dotted) and 0.5 (dotdashed) from bottom to top.

where  $^{11}$ 

$$\Delta(a) \equiv \sum_{n=1}^{\infty} F(an) - \int_0^\infty dx \ F(ax) + \frac{1}{2}F(0).$$
 (4.5)

In order to evaluate  $\Delta(a)$ , the Euler-Maclaurin formula is useful [44–46]. Then we obtain

$$\Delta(a) = \lim_{N_{\rm cut} \to \infty} \left\{ \sum_{n=1}^{N_{\rm cut}} F(an) - \int_{0}^{N_{\rm cut}} dx \ F(ax) + \frac{1}{2}F(0) \right\}$$

$$= \lim_{N_{\rm cut} \to \infty} \left\{ \sum_{n=0}^{N_{\rm cut}} F(an) - \int_{0}^{N_{\rm cut}} dx \ F(ax) \right\} - \frac{1}{2}F(0)$$

$$= \lim_{N_{\rm cut} \to \infty} \left[ \frac{1}{2} \left\{ F(0) + F(aN_{\rm cut}) \right\} + \sum_{p=1}^{\text{floor}(q/2)} \frac{B_{2p}a^{2p-1}}{(2p)!} \left\{ F^{(2p-1)}(aN_{\rm cut}) - F^{(2p-1)}(0) \right\} + R_q \right] - \frac{F(0)}{2}$$

$$= -\sum_{p=1}^{\text{floor}(q/2)} \frac{B_{2p}a^{2p-1}}{(2p)!} F^{(2p-1)}(0) + \lim_{N_{\rm cut} \to \infty} R_q, \qquad (4.6)$$

where  $B_{2p}$  are the Bernoulli numbers, q is an integer greater than 1, and

$$R_q \equiv (-1)^{q-1} \int_0^{N_{\rm cut}} dx \; \frac{B_q(x - \text{floor}\,(x))}{q!} a^q F^{(q)}(ax), \tag{4.7}$$

 $^{11}$ We have used that

$$\lim_{a \to 0} \sum_{n=1}^{\infty} aF(an) = \int_0^{\infty} dx \ F(x) - \frac{a}{2}F(0) = a \int_0^{\infty} dx \ F(ax) - \frac{a}{2}F(0).$$
(4.4)

See Sec. 3.3 of Ref. [15] for details.



Figure 2. The ratio  $r_{\rm cas}$  defined in (4.11) as a function of  $m_{\rm KK}/M_{\rm reg}$ . The bulk mass is chosen as  $M_{\rm bulk}/M_{\rm reg} = 0$  (solid), 0.1 (dashed), 0.2 (dotted), and 0.3 (dotdashed), respectively.

with the Bernoulli polynomial  $B_q(x)$ . At the last step in (4.6), we have used that

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} F^{(1)}(x) = \dots = \lim_{x \to \infty} F^{(q-1)}(x) = 0.$$
(4.8)

Here we set q = 2. Then, noting that  $F^{(1)}(0) = 0$  from (C.1), (4.6) becomes

$$\Delta(a) = -\int_0^\infty dx \; \frac{B_2(x - \text{floor}(x))a^2}{2} F^{(2)}(ax)$$
$$= -\frac{a^2}{2} \sum_{l=0}^\infty \int_0^1 dx \; B_2(x) F^{(2)}(a(x+l)), \tag{4.9}$$

where the explicit form of  $F^{(2)}(x)$  is shown in (C.1) in Appendix, and

$$B_2(x) = x^2 - x + \frac{1}{6}. (4.10)$$

To see the deviation of the Casimir energy (4.3) from the one obtained in the analytic regularization (2.15), we define

$$r_{\rm cas} \equiv \frac{\langle 0 \mid \hat{\rho} \mid 0 \rangle_{\rm Casimir}^{\rm PV}}{\langle 0 \mid \hat{\rho} \mid 0 \rangle_{\rm Casimir}^{\rm anal}},\tag{4.11}$$

where  $\langle 0 | \hat{\rho} | 0 \rangle_{\text{Casimir}}^{\text{anal}}$  denotes (2.15). Fig. 2 shows the ratio  $r_{\text{cas}}$  as a function of  $a = m_{\text{KK}}/M_{\text{reg}}$ , where  $m_{KK} \equiv 1/R$  is the KK mass scale. We can see that the result obtained by the Pauli-Villars regularization well agrees with that of the analytic regularization as long as the compactification scale  $m_{\text{KK}}$  is well below the regulator mass scale  $M_{\text{reg}}$ .

Before ending the section, one comment is in order. The above results can also be expressed by using the analytic regularized formula (2.15). As mentioned below (3.14), the current choice of the Pauli-Villars regulators can be understood as 5D fields. Thus, the Casimir energy density in (4.1) is also expressed as

$$\langle 0|\hat{\rho}|0\rangle_{\text{Casimir}}^{\text{PV}} = \mathcal{E}(R, M_{\text{bulk}}) - \mathcal{E}\left(R, \sqrt{M_{\text{bulk}}^2 + M_{\text{reg}}^2}\right) - 3\mathcal{E}\left(R, \sqrt{M_{\text{bulk}}^2 + \frac{1}{3}M_{\text{reg}}^2}\right) + 3\mathcal{E}\left(R, \sqrt{M_{\text{bulk}}^2 + \frac{2}{3}M_{\text{reg}}^2}\right), \quad (4.12)$$

where

$$\mathcal{E}(R,M) \equiv -\frac{M^{\frac{5}{2}}}{16\pi^4 R^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^{-\frac{5}{2}} K_{\frac{5}{2}}(2\pi n R M).$$
(4.13)

Thus, (4.11) can be rewritten as

$$r_{\rm cas} = 1 - \Delta \left( \bar{M}_{\rm bulk}, \sqrt{\bar{M}_{\rm bulk}^2 + \bar{M}_{\rm reg}^2} \right) - 3\Delta \left( \bar{M}_{\rm bulk}, \sqrt{\bar{M}_{\rm bulk}^2 + \frac{1}{3}\bar{M}_{\rm reg}^2} \right) + 3\Delta \left( \bar{M}_{\rm bulk}, \sqrt{\bar{M}_{\rm bulk}^2 + \frac{2}{3}\bar{M}_{\rm reg}^2} \right), \quad (4.14)$$

where  $\bar{M}_{\text{bulk}} = RM_{\text{bulk}}, \ \bar{M}_{\text{reg}} = RM_{\text{reg}}, \text{ and}$ 

$$\Delta(\bar{M}_1, \bar{M}_2) \equiv \left(\frac{\bar{M}_2}{\bar{M}_1}\right)^{\frac{5}{2}} \frac{\sum_{n=1}^{\infty} n^{-\frac{5}{2}} K_{\frac{5}{2}}(2\pi n \bar{M}_2)}{\sum_{n=1}^{\infty} n^{-\frac{5}{2}} K_{\frac{5}{2}}(2\pi n \bar{M}_1)}.$$
(4.15)

The function  $\Delta(\bar{M}_1, \bar{M}_2)$  is exponentially suppressed when  $\bar{M}_1 \ll \bar{M}_2$ , but becomes non-negligible when  $\bar{M}_2 = \mathcal{O}(\bar{M}_1)$ . Since the infinite summation in (4.13) or (4.15) converges much faster than the KK summation, this expression is convenient to the numerical computation.

#### 5 Discussions and Conclusions

We studied the dependence of the Casimir energy density on the UV dynamics in the context of a 5D model with a compact dimension. In contrast to renormalizable theories, a non-renormalizable theory, such as our 5D model, should be regarded as an effective theory, and be replaced by a more fundamental theory at some high energy scale  $M_{\rm UV}$ . A typical situation is that some new particles appear at a scale around  $M_{\rm UV}$ , and cancel quantum corrections from the light fields in the 5D effective theory.

If  $M_{\rm UV}$  is not far from  $m_{\rm KK}$ , the existence of the new particles can affect lowenergy observables, such as the Casimir energy density. We have evaluated such effects on the Casimir energy density (and pressure). The most popular way of calculating the Casimir energy is the method using the analytic regularization because

the resultant expression is convenient for the numerical evaluation and the regularization preserves various symmetries, including the Lorentz symmetry. However, this regularization removes the power-law divergences by hand, and thus is inappropriate for our purpose, as we showed in Sec. 2.2.2. Instead of this, we work in the Pauli-Villars regularization, which mimics the situation that new particles cancel the quantum corrections from the light particles. To preserve the Lorentz symmetry of the vacuum, we have to prepare more than one regulator for each mode, and their masses and the degrees of freedom have to satisfy some conditions (see (3.3) and (3.6)). It should be noticed that two of them are automatically satisfied in a (spontaneously broken) SUSY model. The result in (3.14) indicates that the Pauli-Villars regulators can be regarded as the KK modes for 5D bulk fields. In a case that the model is embedded into a (spontaneously broken) SUSY 5D theory, the scalar field  $\Phi$ and the bulk regulators should be embedded into a 5D SUSY multiplet. Thus, the example of the regulators considered in Sec. 3 must be modified. Needless to say, the deviation from the result in the analytic regularization depends on the choice of the Pauli-Villars regulators. Still, our example shows a typical order of magnitude for the deviation.

If we do not impose the condition (3.17), it is not guaranteed that the resultant Casimir energy density (or pressure) agrees with the one obtained by the analytic regularization. We numerically evaluate them and confirm that they well agree with each other even if (3.17) is not satisfied, as loong as the KK mass  $m_{\rm KK}$  and the bulk mass  $M_{\rm bulk}$  are smaller than all the regulator masses.

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#### A Complete and incomplete beta and gamma functions

#### A.1 Definitions and properties

The integral expressions of the complete beta and gamma functions are given by

$$B(\alpha,\beta) \equiv \int_0^\infty dx \ x^{\alpha-1} (1-x)^{\beta-1} = B(\beta,\alpha),$$
  

$$\Gamma(\alpha) \equiv \int_0^\infty dt \ t^{\alpha-1} e^{-t},$$
(A.1)

which are valid only for  $\operatorname{Re} \alpha > 0$  and  $\operatorname{Re} \beta > 0$ . They are related as

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$
(A.2)

This relation holds over the whole domain of the beta function. The gamma function behaves near  $\alpha = 0, -1, -2$  as

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_{\rm E} + \mathcal{O}(\epsilon),$$
  

$$\Gamma(-1+\epsilon) = -\frac{1}{\epsilon} - 1 + \gamma_{\rm E} + \mathcal{O}(\epsilon),$$
  

$$\Gamma(-2+\epsilon) = \frac{1}{2\epsilon} + \frac{3}{4} - \frac{\gamma_{\rm E}}{2} + \mathcal{O}(\epsilon),$$
(A.3)

where  $\gamma_{\rm E}$  is the Euler-Mascheroni constant.

The incomplete beta functions are defined as

$$B_{z}(\alpha,\beta) \equiv \int_{0}^{z} dx \ x^{\alpha-1} (1-x)^{\beta-1},$$
 (A.4)

for  $\operatorname{Re} \alpha > 0$ , and the upper and the lower incomplete gamma functions are defined as

$$\Gamma_{z}(\alpha) \equiv \int_{z}^{\infty} dt \ t^{\alpha-1} e^{-t},$$
  

$$\gamma_{z}(\alpha) \equiv \Gamma(\alpha) - \Gamma_{z}(\alpha) = \int_{0}^{z} dt \ t^{\alpha-1} e^{-t}.$$
(A.5)

where the integral expression of  $\gamma_z(\alpha)$  is valid only for  $\operatorname{Re} \alpha > 0$ . From (A.4), we obtain

$$B_{1-\epsilon}(\alpha,\beta) = \int_{\epsilon}^{1} dy \ y^{\beta-1}(1-y)^{\alpha-1} \qquad (y \equiv 1-x)$$
  
=  $\int_{0}^{1} dy \ y^{\beta-1}(1-y)^{\alpha-1} - \int_{0}^{\epsilon} dy \ y^{\beta-1}(1-y)^{\alpha-1}$   
=  $B(\beta,\alpha) - \int_{0}^{\epsilon} dy \ y^{\beta-1} \left\{ 1 - (\alpha-1)y + \frac{2-3\alpha+\alpha^{2}}{2}y^{2} + \mathcal{O}(y^{3}) \right\}$   
=  $B(\alpha,\beta) - \frac{\epsilon^{\beta}}{\beta} + \frac{\alpha-1}{\beta+1}\epsilon^{\beta+1} - \frac{\alpha^{2}-3\alpha+2}{2(\beta+2)}\epsilon^{\beta+2} + \mathcal{O}(\epsilon^{\beta+3}), \quad (A.6)$ 

for  $\operatorname{Re} \alpha > 0$  and  $\operatorname{Re} \beta > 0$ .

Similarly, the incomplete gamma function can be expanded as

$$\Gamma_{\delta}(\alpha) = \int_{0}^{\infty} dt \ t^{\alpha-1} e^{-t} - \int_{0}^{\delta} dt \ t^{\alpha-1} e^{-t}$$
$$= \Gamma(\alpha) - \int_{0}^{\delta} dt \ t^{\alpha-1} \left\{ 1 - t + \frac{t^{2}}{2} + \mathcal{O}(t^{3}) \right\}$$
$$= \Gamma(\alpha) - \frac{\delta^{\alpha}}{\alpha} + \frac{\delta^{\alpha+1}}{\alpha+1} - \frac{\delta^{\alpha+2}}{2(\alpha+2)} + \mathcal{O}(\delta^{\alpha+3}).$$
(A.7)

The incomplete beta function is also expanded as

$$B_{1-\epsilon}(\alpha,\beta) = \frac{1}{\alpha\Gamma(\alpha+\beta)} \int_0^\infty dx \; x^\alpha e^{-x} \Gamma_{\frac{x\epsilon}{1-\epsilon}}(\beta) + \frac{1}{\alpha} \epsilon^\beta (1-\epsilon)^\alpha. \tag{A.8}$$

We can show this by differentiating both hand sides concerning  $\epsilon$ , and checking that they coincide. For  $\beta > 0$ , (A.8) reduces to (A.2) in the limit of  $\epsilon \to 0$ .

#### A.2 Explicit forms

Here we show the explicit forms of the incomplete beta functions that appear in Sec. 2.3,

$$B_{1-\epsilon_n}\left(\frac{3}{2},-2\right), \quad B_{1-\epsilon_n}\left(\frac{5}{2},-2\right).$$
 (A.9)

Since

$$1 - \epsilon_n = 1 - \frac{m_n^2}{\Lambda_{\text{cut}}^2 + m_n^2} = \frac{\Lambda_{\text{cut}}^2}{\Lambda_{\text{cut}}^2 + m_n^2} = \frac{X^2}{X^2 + 1},$$
 (A.10)

where  $X \equiv \Lambda_{\rm cut}/m_n$ , the above functions can be expressed in the form of

$$B_{\frac{X^2}{X^2+1}}(\alpha,\beta) = \int_0^{\frac{X^2}{X^2+1}} dx \ x^{\alpha-1}(1-x)^{\beta-1}, \tag{A.11}$$

when  $\operatorname{Re} \alpha > 0$ . By differentiating this concerning X, we have

$$\partial_X B_{\frac{X^2}{X^2+1}}(\alpha,\beta) = \left(\frac{X^2}{X^2+1}\right)^{\alpha-1} \left(1 - \frac{X^2}{X^2+1}\right)^{\beta-1} \partial_X \left(\frac{X^2}{X^2+1}\right)$$
$$= \frac{2X^{2\alpha-1}}{(X^2+1)^{\alpha+\beta}}.$$
(A.12)

Since  $B_{\frac{X^2}{X^2+1}}(\alpha,\beta)|_{X=0} = 0$ , (A.11) is reexpressed as

$$B_{\frac{X^2}{X^2+1}}(\alpha,\beta) = \int_0^X dY \; \frac{2Y^{2\alpha-1}}{(Y^2+1)^{\alpha+\beta}}.$$
 (A.13)

From this expression, we obtain

$$B_{\frac{X^2}{X^2+1}}\left(\frac{3}{2},-2\right) = \int_0^X dY \, 2Y^2 \sqrt{Y^2+1}$$
  
=  $\frac{1}{4} \left\{ X\sqrt{X^2+1}(2X^2+1) - \ln\left(X+\sqrt{X^2+1}\right) \right\},$   
$$B_{\frac{X^2}{X^2+1}}\left(\frac{5}{2},-2\right) = \int_0^X dY \, \frac{2Y^4}{\sqrt{Y^2+1}}$$
  
=  $\frac{1}{4} \left\{ X\sqrt{X^2+1}(2X^2-3) + 3\ln\left(X+\sqrt{X^2+1}\right) \right\}.$  (A.14)

## **B** Formulae for zeta function regularization

In order to evaluate the regularized sums in (2.9), we define

$$U(\alpha,\beta;M^2) \equiv \sum_{n=1}^{\infty} B_{1-\epsilon_n}(\alpha,\beta) \left(M^2 + n^2\right)^{-\beta} e^{-a^2n^2}, \tag{B.1}$$

where a is a tiny positive constant, and

$$\epsilon_n \equiv \frac{M^2 + n^2}{a^{-2} + M^2 + n^2}.$$
 (B.2)

Using the formula (A.8), this is expressed as

$$U(\alpha,\beta;M^2) = \sum_{n=1}^{\infty} \left\{ \int_0^\infty dx \, \frac{x^{\alpha} e^{-x}}{\alpha \Gamma(\alpha+\beta)} \Gamma_{(M^2+n^2)a^2x}(\beta) + \frac{\epsilon_n^{\beta}(1-\epsilon_n)^{\alpha}}{\alpha} \right\}$$
$$\times \left(M^2 + n^2\right)^{-\beta} e^{-a^2n^2}$$
$$= \frac{U_1(\alpha,\beta;M^2)}{\alpha(\alpha+\beta)} + \frac{U_2(\alpha,\beta;M^2)}{\alpha a^{2\alpha}}, \tag{B.3}$$

where

$$U_{1}(\alpha,\beta;M^{2}) \equiv \int_{0}^{\infty} dx \ x^{\alpha} e^{-x} S_{a^{2}x}(\beta;M^{2}),$$
  

$$S_{\delta}(\beta;M^{2}) \equiv \sum_{n=1}^{\infty} \left(M^{2} + n^{2}\right)^{-\beta} \Gamma_{(M^{2} + n^{2})\delta}(\beta) e^{-a^{2}n^{2}},$$
  

$$U_{2}(\alpha,\beta;M^{2}) \equiv \sum_{n=1}^{\infty} \frac{e^{-a^{2}n^{2}}}{\left(a^{-2} + M^{2} + n^{2}\right)^{\alpha+\beta}}.$$
(B.4)

Here note that  $S_{\delta}(\beta; M^2)$  can be rewritten as

$$S_{\delta}(\beta; M^2) = \int_{\delta}^{\infty} dt \ t^{\beta - 1} e^{-M^2 t} \vartheta(t + a^2), \tag{B.5}$$

where  $\vartheta(t) \equiv \sum_{n=1}^{\infty} e^{-n^2 t}$  is the Jacobi theta function, which has the property,

$$\vartheta(t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\pi}{t}} + \sqrt{\frac{\pi}{t}}\vartheta\left(\frac{\pi^2}{t}\right).$$
(B.6)

Using this property of  $\vartheta(t)$  and the definition of the gamma function,  $S_{a^2x}(\beta; M^2)$  is expanded as

$$S_{a^{2}x}(\beta; M^{2}) = -\frac{M^{-2\beta}}{2}\Gamma(\beta) + \frac{\sqrt{\pi}M^{-2\beta+1}}{2}\sum_{j=0}^{\infty}c_{j}a^{2j}\Gamma\left(\beta - \frac{1}{2} - j\right) + \frac{(a^{2}x)^{\beta}}{2\beta} - \frac{M^{2}(a^{2}x)^{\beta+1}}{2(\beta+1)} + \frac{M^{4}(a^{2}x)^{\beta+2}}{4(\beta+2)} - \frac{\sqrt{\pi}M^{-2\beta+1}}{2}\left\{a^{2\beta-1}H_{\beta-\frac{1}{2}}(M^{2}x) - a^{2\beta+1}H_{\beta+\frac{1}{2}}(M^{2}x) + \frac{a^{2\beta+3}}{2}H_{\beta+\frac{3}{2}}(M^{2}x)\right\} + \sqrt{\pi}\int_{\delta}^{\infty}dt \,\frac{t^{\beta-1}e^{-M^{2}t}}{\sqrt{t+a^{2}}}\vartheta\left(\frac{\pi^{2}}{t+a^{2}}\right) + \mathcal{O}\left(a^{2\beta+5}\right), \quad (B.7)$$

where the constants  $c_j$  are defined by

$$(1+x)^{-1/2} = \sum_{j=0}^{\infty} c_j x^j,$$
(B.8)

and the function  $H_b(z)$  is defined as

$$H_b(z) \equiv \sum_{j=0}^{\infty} \frac{c_j}{b-j} z^{b-j}.$$
 (B.9)

When  $\operatorname{Re} \beta > \frac{1}{2}$ , all the powers of a are positive, and we can take the limit of  $a \to 0$  and obtain

$$S_{0}(\beta; M^{2}) \equiv \lim_{a \to 0} S_{a^{2}x}(\beta; M^{2}) = \Gamma(\beta) \sum_{n=1}^{\infty} (M^{2} + n^{2})^{-\beta}$$
$$= -\frac{M^{-2\beta}}{2} \Gamma(\beta) + \frac{\sqrt{\pi}M^{-2\beta+1}}{2} \Gamma\left(\beta - \frac{1}{2}\right)$$
$$+ \sqrt{\pi} \sum_{n=1}^{\infty} \int_{0}^{\infty} dt \ t^{\beta - \frac{3}{2}} \exp\left(-M^{2}t - \frac{\pi^{2}n^{2}}{t}\right).$$
(B.10)

When  $\operatorname{Re} \beta < 1$ , the integral in the last term is expressed as

$$\int_{0}^{\infty} dt \ t^{\beta - \frac{3}{2}} \exp\left(-M^{2}t - \frac{\pi^{2}n^{2}}{t}\right) = 2\left(\frac{\pi n}{M}\right)^{\beta - \frac{1}{2}} K_{\frac{1}{2} - \beta}(2\pi nM).$$
(B.11)

Substituting (B.7) into the first expression in (B.4), we obtain

$$\begin{aligned} U_1(\alpha,\beta;M^2) &= -\frac{M^{-2\beta}\alpha\Gamma(\alpha)\Gamma(\beta)}{2} + \frac{\sqrt{\pi}M^{-2\beta+1}\Gamma(\alpha)}{2}\sum_{j=0}^{\infty}c_j a^{2j}\left(\beta - \frac{1}{2} - j\right) \\ &+ \frac{\Gamma(\alpha+\beta+1)}{2\beta}a^{2\beta} - \frac{M^2\Gamma(\alpha+\beta+2)}{2(\beta+1)}a^{2\beta+2} + \frac{M^4\Gamma(\alpha+\beta+3)}{4(\beta+2)}a^{2\beta+4} \\ &+ C_1(\beta;M^2)a^{2\beta-1} + C_2(\beta;M^2)a^{2\beta+1} + C_3(\beta;M^2)a^{2\beta+3} \\ &+ \sqrt{\pi}\int_0^{\infty}dx \ x^{\alpha}e^{-x}\int_{a^2x}^{\infty}dt \ \frac{t^{\beta-1}e^{-M^2t}}{\sqrt{t+a^2}}\vartheta\left(\frac{\pi^2}{t+a^2}\right) + \mathcal{O}\left(a^{2\beta+5}\right), \end{aligned}$$
(B.12)

where

$$C_{1}(\beta; M^{2}) \equiv -\frac{\sqrt{\pi}M^{-2\alpha-2\beta}}{2} \int_{0}^{\infty} dy \ y^{\alpha} e^{-y/M^{2}} H_{\beta-\frac{1}{2}}(y),$$

$$C_{2}(\beta; M^{2}) \equiv \frac{\sqrt{\pi}M^{-2\alpha-2\beta}}{2} \int_{0}^{\infty} dy \ y^{\alpha} e^{-y/M^{2}} H_{\beta+\frac{1}{2}}(y),$$

$$C_{3}(\beta; M^{2}) \equiv -\frac{\sqrt{\pi}M^{-2\alpha-2\beta}}{4} \int_{0}^{\infty} dy \ y^{\alpha} e^{-y/M^{2}} H_{\beta+\frac{3}{2}}(y).$$
(B.13)

## C Derivatives of F(x)

Here we collect the explicit forms of derivatives of F(x) defined in (4.2).

$$\begin{aligned} F^{(1)}(x) &= 4x \left\{ \left( \hat{M}_{\text{bulk}}^2 + x^2 \right) \ln \left( \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &- \left( 1 + \hat{M}_{\text{bulk}}^2 + x^2 \right) \ln \left( 1 + \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &- 3 \left( \frac{1}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right) \ln \left( \frac{1}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &+ 3 \left( \frac{2}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right) \ln \left( \frac{2}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right) \right\}, \end{aligned}$$

$$F^{(2)}(x) &= 12 \left( \frac{\hat{M}_{\text{bulk}}^2}{3} + x^2 \right) \ln \left( \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &- 12 \left( \frac{1 + \hat{M}_{\text{bulk}}^2}{3} + x^2 \right) \ln \left( 1 + \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &- 36 \left( \frac{1}{9} + \frac{\hat{M}_{\text{bulk}}^2}{3} + x^2 \right) \ln \left( \frac{1}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right) \\ &+ 36 \left( \frac{2}{9} + \frac{\hat{M}_{\text{bulk}}^2}{3} + x^2 \right) \ln \left( \frac{2}{3} + \hat{M}_{\text{bulk}}^2 + x^2 \right). \end{aligned}$$
(C.1)

For  $x \gg 1$ ,  $F^{(2)}(x)$  is expanded as

$$F^{(2)}(x) = -\frac{4}{9x^4} + \frac{20\left(1+2\hat{M}_{\text{bulk}}^2\right)}{27x^6} - \frac{14}{81x^8}\left(5+18\hat{M}_{\text{bulk}}^2+18\hat{M}_{\text{bulk}}^4\right) + \frac{8}{9x^{10}}\left(1+5\hat{M}_{\text{bulk}}^2+9\hat{M}_{\text{bulk}}^4+6\hat{M}_{\text{bulk}}^6\right) - \frac{44}{2187x^{12}}\left\{43+135\hat{M}_{\text{bulk}}^2\left(1+\hat{M}_{\text{bulk}}^2\right)\left(2+3\hat{M}_{\text{bulk}}^2+3\hat{M}_{\text{bulk}}^4\right)\right\} + \mathcal{O}\left(x^{-14}\right).$$
(C.2)

#### References

- H.B.G. Casimir, On the attraction between two perfectly conducting plates, Indag. Math. 10 (1948) 261.
- [2] S.K. Lamoreaux, Demonstration of the Casimir force in the 0.6 to 6 micrometers range, Phys. Rev. Lett. 78 (1997) 5.
- U. Mohideen and A. Roy, Precision measurement of the Casimir force from 0.1 to 0.9 micrometers, Phys. Rev. Lett. 81 (1998) 4549 [physics/9805038].
- [4] A. Roy, C.-Y. Lin and U. Mohideen, Improved precision measurement of the casimir force, Phys. Rev. D 60 (1999) 111101 [quant-ph/9906062].
- [5] G. Bimonte, B. Spreng, P.A. Maia Neto, G.-L. Ingold, G.L. Klimchitskaya,
   V.M. Mostepanenko et al., Measurement of the Casimir Force between 0.2 and 8 μm: Experimental Procedures and Comparison with Theory, Universe 7 (2021) 93
   [2104.03857].
- [6] J. Garriga, O. Pujolas and T. Tanaka, Radion effective potential in the brane world, Nucl. Phys. B 605 (2001) 192 [hep-th/0004109].
- [7] D.J. Toms, Quantized bulk fields in the Randall-Sundrum compactification model, Phys. Lett. B 484 (2000) 149 [hep-th/0005189].
- [8] W.D. Goldberger and I.Z. Rothstein, Quantum stabilization of compactified AdS(5), Phys. Lett. B 491 (2000) 339 [hep-th/0007065].
- [9] I.H. Brevik, K.A. Milton, S. Nojiri and S.D. Odintsov, Quantum (in)stability of a brane world AdS(5) universe at nonzero temperature, Nucl. Phys. B 599 (2001) 305 [hep-th/0010205].
- [10] C.G. Beneventano and E.M. Santangelo, Connection between zeta and cutoff regularizations of Casimir energies, Int. J. Mod. Phys. A 11 (1996) 2871
   [hep-th/9501122].
- [11] V. Moretti, Local zeta function techniques versus point splitting procedure: A Few rigorous results, Commun. Math. Phys. 201 (1999) 327 [gr-qc/9805091].
- [12] C.R. Hagen, Cutoff dependence of the Casimir effect, Eur. Phys. J. C 19 (2001) 677 [quant-ph/0003108].

- [13] M. Visser, Regularization versus Renormalization: Why Are Casimir Energy Differences So Often Finite?, Particles 2 (2018) 14 [1601.01374].
- [14] H. Matsui and Y. Matsumoto, Revisiting regularization with Kaluza-Klein states and Casimir vacuum energy from extra dimensional spaces, Phys. Rev. D 100 (2019) 016010 [1804.01052].
- [15] Y. Asai and Y. Sakamura, Ultraviolet sensitivity of Casimir energy, PTEP 2022 (2022) 033B07 [2112.04708].
- [16] E.K. Akhmedov, Vacuum energy and relativistic invariance, hep-th/0204048.
- [17] J.F. Koksma and T. Prokopec, The Cosmological Constant and Lorentz Invariance of the Vacuum State, 1105.6296.
- [18] J. Martin, Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask), Comptes Rendus Physique 13 (2012) 566 [1205.3365].
- [19] U. Danielsson, The quantum swampland, JHEP 04 (2019) 095 [1809.04512].
- [20] L.A. Escamilla, W. Giarè, E. Di Valentino, R.C. Nunes and S. Vagnozzi, The state of the dark energy equation of state circa 2023, 2307.14802.
- [21] S.W. Hawking, The Chronology protection conjecture, Phys. Rev. D 46 (1992) 603.
- [22] V.A. Rubakov, The Null Energy Condition and its violation, Phys. Usp. 57 (2014) 128 [1401.4024].
- [23] M. Visser, Lorentz invariance and the zero-point stress-energy tensor, Particles 1 (2018) 138 [1610.07264].
- [24] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge, UK (2, 1984), 10.1017/CBO9780511622632.
- [25] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective action in quantum gravity* (1992).
- [26] K.S. Stelle, Renormalization of Higher Derivative Quantum Gravity, Phys. Rev. D 16 (1977) 953.
- [27] G.T. Horowitz and R.M. Wald, Dynamics of Einstein's Equation Modified by a Higher Order Derivative Term, Phys. Rev. D 17 (1978) 414.
- [28] G.T. Horowitz, SEMICLASSICAL RELATIVITY: THE WEAK FIELD LIMIT, Phys. Rev. D 21 (1980) 1445.
- [29] J.B. Hartle and G.T. Horowitz, Ground State Expectation Value of the Metric in the 1/N or Semiclassical Approximation to Quantum Gravity, Phys. Rev. D 24 (1981) 257.
- [30] S. Randjbar-Daemi, Stability of the Minkowski Vacuum in the Renormalized Semiclassical Theory of Gravity, J. Phys. A 14 (1981) L229.

- [31] R.D. Jordan, Stability of Flat Space-time in Quantum Gravity, Phys. Rev. D 36 (1987) 3593.
- [32] W.-M. Suen, The Stability of the Semiclassical Einstein Equation, Phys. Rev. D 40 (1989) 315.
- [33] W.M. Suen, Minkowski Space-time Is Unstable in Semiclassical Gravity, Phys. Rev. Lett. 62 (1989) 2217.
- [34] P.R. Anderson, C. Molina-Paris and E. Mottola, Linear response, validity of semiclassical gravity, and the stability of flat space, Phys. Rev. D 67 (2003) 024026 [gr-qc/0209075].
- [35] H. Matsui and N. Watamura, Quantum Spacetime Instability and Breakdown of Semiclassical Gravity, Phys. Rev. D 101 (2020) 025014 [1910.02186].
- [36] J.Z. Simon, No Starobinsky inflation from selfconsistent semiclassical gravity, Phys. Rev. D 45 (1992) 1953.
- [37] L. Parker and J.Z. Simon, Einstein equation with quantum corrections reduced to second order, Phys. Rev. D 47 (1993) 1339 [gr-qc/9211002].
- [38] C.M. Will, The Confrontation between general relativity and experiment, Living Rev. Rel. 9 (2006) 3 [gr-qc/0510072].
- [39] C.M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Rel. 17 (2014) 4 [1403.7377].
- [40] S. Leseduarte and A. Romeo, Complete zeta function approach to the electromagnetic Casimir effect for spheres and circles, Annals Phys. 250 (1996) 448
   [hep-th/9605022].
- [41] S. Leseduarte and A. Romeo, Influence of a magnetic fluxon on the vacuum energy of quantum fields confined by a bag, Commun. Math. Phys. 193 (1998) 317
   [hep-th/9612116].
- [42] B.S. Kay, The Casimir Effect Without MAGIC, Phys. Rev. D 20 (1979) 3052.
- [43] S. Ferrara, L. Girardello and F. Palumbo, A General Mass Formula in Broken Supersymmetry, Phys. Rev. D 20 (1979) 403.
- [44] T.H. Boyer, Quantum electromagnetic zero point energy of a conducting spherical shell and the Casimir model for a charged particle, Phys. Rev. 174 (1968) 1764.
- [45] G. Mahajan, S. Sarkar and T. Padmanabhan, Casimir Effect confronts Cosmological Constant, Phys. Lett. B 641 (2006) 6 [astro-ph/0604265].
- [46] R. Saghian, M.A. Valuyan, A. Seyedzahedi and S.S. Gousheh, Casimir Energy For a Massive Dirac Field in One Spatial Dimension: A Direct Approach, Int. J. Mod. Phys. A 27 (2012) 1250038 [1204.3181].