

# Entanglement Coordination Rates in Multi-User Networks

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## Abstract

The optimal coordination rates are determined in three primary settings of multi-user quantum networks, thus characterizing the minimal resources required in order to simulate a joint quantum state among multiple parties. We study the following models: (1) a cascade network with rate-limited entanglement, (2) a broadcast network, which consists of a single sender and two receivers, (3) a multiple-access network with two senders and a single receiver. We establish the necessary and sufficient conditions on the asymptotically-achievable communication and entanglement rates in each setting. At last, we show the implications of our results on nonlocal games with quantum strategies.

## Index Terms

Reverse Shannon theorem, quantum coordination, entanglement distribution.

## I. INTRODUCTION

State distribution and coordination are important in quantum communication [1], computation [2], and cryptography [3]. The quantum coordination problem can be described as follows. Consider a network that consists of  $N$  nodes, where Node  $i$  can perform an encoding operation  $\mathcal{E}_i$  on a quantum system  $A_i$ , and its state should be in a certain correlation with the rest of the network nodes. The objective is to simulate a specific joint state  $\omega_{A_1, A_2, \dots, A_N}$ . Some of the nodes are not free to choose their encoding operation, but rather have their quantum state dictated by Nature via a physical process. Node  $i$  can send qubits to node  $j$  via a quantum channel at a limited communication rate  $Q_{i,j}$ . In addition, the nodes may share limited entanglement resources, prior to their communication. The optimal performance of the communication network is characterized by the quantum communication rates  $Q_{i,j}$  that are necessary and sufficient for simulating the desired quantum correlation.

Instances of the network coordination problem include channel simulation [5–10], state merging [11, 12], state redistribution [13–15], entanglement dilution [16–19], randomness extraction [20, 21], source coding [22–24], distributed source simulation [25, 26], and many others. Recently, we have also considered networks with classical communication links [4].

*Two-node classical coordination:* In classical coordination, the goal is to simulate a prescribed joint probability distribution. In the basic two-node network, described in Figure 1, the simulation of a given joint distribution  $p_{XY}$  is possible if and only if the *classical* communication rate  $R_{1,2}$  is above Wyner’s common information [27], defined as:

$$C(X; Y) \triangleq \min I(U; XY), \quad (1)$$

where the minimum is taken over all auxiliary variables  $U$  that satisfy the Markov relation  $X \ominus U \ominus Y$ , and  $I(U; XY)$  is the mutual information between  $U$  and the pair  $(X, Y)$ . In the classical model, one may also consider the case where the nodes share classical correlation resources, a priori, in the form of common randomness, i.e., a sequence of pre-shared random bits. Given a sufficient amount of pre-shared common randomness between the nodes, the desired distribution can be simulated if and only if the classical communication rate is above the mutual information, i.e.,  $R_{1,2} \geq I(X; Y)$  [28, 29].

*Two-node quantum coordination:* In the quantum setting, the goal is to simulate a prescribed joint state. A bipartite state  $\omega_{AB}$  can be simulated if and only if the quantum communication rate is above the von Neumann entropy [17], i.e.,

$$Q_{1,2} \geq H(\omega_B), \quad (2)$$

with  $H(\rho) = -\text{Tr}(\rho \log_2(\rho))$ , where  $\omega_B$  is the reduced state of  $\omega_{AB}$ . Now, suppose that the nodes share entanglement resources, a priori, before communication begins, as illustrated in Figure 2. Given a sufficient amount of pre-shared entanglement between the nodes, the desired state can be simulated if and only if the quantum communication rate satisfies

$$Q_{1,2} \geq \frac{1}{2} I(A; B)_\omega, \quad (3)$$

where  $I(A; B)_\omega$  denotes the quantum mutual information, by the quantum reverse Shannon theorem [7].

*Multi-node quantum coordination:* Here, we consider quantum coordination in three multi-user networks. First, we examine a cascade network that consists of three users, Alice, Bob, and Charlie, as depicted in Figure 3. The cascade network is of particular importance for quantum repeater systems, which are crucial for the development of the quantum Internet [30, 31] and long-range quantum key distribution (QKD) [32, 33]. In this network, Alice, Bob, and Charlie wish to simulate a joint quantum state  $\omega_{ABC}$ . Let  $|\omega_{ABCR}\rangle$  be a purification of the desired state. Before communication begins, each party shares entanglement with their nearest neighbor, at a limited rate. Now, Alice sends qubits to Bob at a rate  $Q_{1,2}$ , and thereafter, Bob

sends qubits to Charlie at a rate  $Q_{2,3}$ . We show that coordination can be achieved if and only if the quantum communication rates  $Q_{1,2}$  and  $Q_{2,3}$  satisfy

$$Q_{1,2} \geq \frac{1}{2}I(BC; R)_\omega, \quad (4)$$

$$Q_{1,2} + E_{1,2} \geq H(BC)_\omega, \quad (5)$$

$$Q_{2,3} \geq \frac{1}{2}I(C; RA)_\omega, \quad (6)$$

$$Q_{2,3} + E_{2,3} \geq H(C)_\omega, \quad (7)$$

where  $E_{i,j}$  is the entanglement rate between Node  $i$  and Node  $j$ .

Next, we consider a broadcast network with one sender, Alice, and two receivers, Bob and Charlie, where the receivers are provided with classical sequences of information  $X^n$  and  $Y^n$ . See Figure 4. Among others, this is motivated by applications that are based on entanglement distribution [34]. We show that coordination can be achieved if and only if the quantum communication rates  $Q_{1,2}$  and  $Q_{1,3}$  satisfy

$$Q_{1,2} \geq H(B|X)_\omega, \quad (8)$$

$$Q_{1,3} \geq H(C|Y)_\omega. \quad (9)$$

In the third setting, we consider a multiple-access network, with two transmitters, Alice and Bob, and one receiver, Charlie, as illustrated in Figure 5. This network is relevant for multiple access QKD [35, 36] and IoT [37].

We observe that since there is no cooperation between the transmitters, a joint state  $\omega_{ABC}$  can only be simulated if it is isometrically equivalent to a state of the form  $\phi_{AC_1} \otimes \chi_{BC_2}$ . Then, we show that coordination can be achieved if and only if

$$Q_{1,3} \geq H(C_1)_\phi, \quad (10)$$

$$Q_{2,3} \geq H(C_2)_\chi. \quad (11)$$

Furthermore, we discuss the implications of our results on nonlocal quantum games. In particular, coordination in the broadcast network in Figure 4 can be viewed as a sequential game, where Alice is a coordinator that provides the players, Bob and Charlie, with quantum resources. In the course of the game, the referee sends questions,  $X^n$  and  $Y^n$ , to Bob and Charlie respectively, and they send their response through  $B^n$  and  $C^n$ . In order to win the game with a certain probability, the communication rates must satisfy the constraints (8)-(9) with respect to an appropriate correlation.

In the analysis, we use different techniques for each setting. For the cascade network, we use the state redistribution theorem [38]. In the broadcast network, we assume that Alice does not have prior correlation with Bob and Charlie's resources  $X^n$  and  $Y^n$ . Therefore, the standard techniques of state redistribution [38] or quantum source coding with side information [39] are not suitable for our purposes. Instead, we use a quantum version of binning. In the analysis of the multiple-access network, we use the Schumacher compression protocol and the isometric relation that is dictated by the network topology.

The paper is organized as follows. In Section II, we define three coordination models and present our main results for each network. In Section III, we discuss the implications of our results on nonlocal games, and in Sections IV, V, and VI, we provide the analysis for the cascade, broadcast, and multiple-access networks, respectively.

## II. MODEL DEFINITIONS AND RESULTS

We introduce three quantum coordination models and provide the required definitions.

The following notation conventions are used throughout. We use uppercase letters  $X, Y, Z, \dots$  for discrete random variables on finite alphabets  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \dots$ , and lowercase  $x, y, z, \dots$  for their realization, respectively. Let  $x^n = (x_i)_{i \in [n]}$  represent a sequence of letters from  $\mathcal{X}$ . A quantum state is specified by a density operator,  $\rho_A$ , on the Hilbert space  $\mathcal{H}_A$ . Let  $\Delta(\mathcal{H}_A)$  denote the set of all density operators on  $\mathcal{H}_A$ . Then,  $A^n = A_1 \cdots A_n$  is a sequence of quantum systems whose joint state  $\rho_{A^n}$  belongs to  $\Delta(\mathcal{H}_A^{\otimes n})$ . A quantum channel is a completely-positive trace-preserving map  $\mathcal{E}_{A \rightarrow B} : \Delta(\mathcal{H}_A) \rightarrow \Delta(\mathcal{H}_B)$ . The quantum mutual information is defined as  $I(A; B)_\rho = H(\rho_A) + H(\rho_B) - H(\rho_{AB})$ , where  $H(\rho) \equiv -\text{Tr}[\rho \log(\rho)]$  is the von Neumann entropy. The conditional quantum entropy is defined as  $H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B)$ , and the conditional quantum mutual information as  $I(A; B|C)_\rho = H(A|C)_\rho + H(B|C)_\rho - H(AB|C)_\rho$ .

### A. Cascade network

Consider the cascade network with rate-limited entanglement, as depicted in Figure 6. In the Introduction section, we used the notation  $Q_{i,j}$  for the communication rate from Node  $i$  to Node  $j$ . Here, we simplify the notation, and write  $Q_1 \equiv Q_{1,2}$  and  $Q_2 \equiv Q_{2,3}$ , for convenience.

Alice, Bob, and Charlie would like to simulate a joint state  $\omega_{ABC}^{\otimes n}$ , where  $\omega_{ABC} \in \Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ . Before communication begins, each party shares bipartite entanglement with their nearest neighbor. The bipartite state  $|\Psi_{TA'T'_B}\rangle$  indicates the entanglement resource shared between Alice and Bob, while  $|\Theta_{T''_BT_C}\rangle$  is shared between Bob and Charlie. The

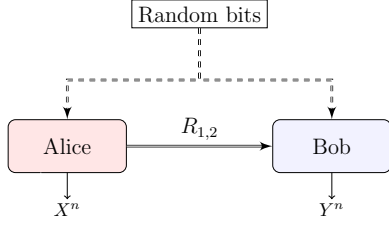


Fig. 1. Classical two-node network

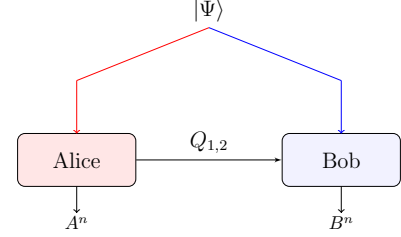


Fig. 2. Quantum two-node network

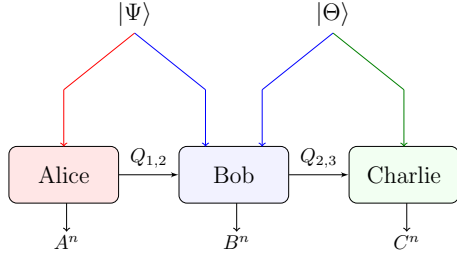


Fig. 3. Cascade network

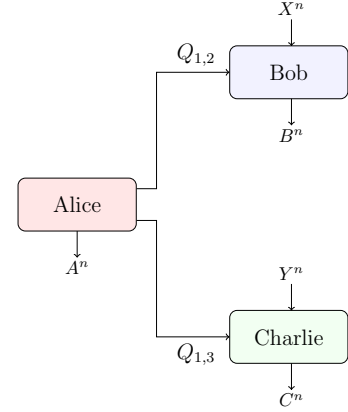


Fig. 4. Broadcast network

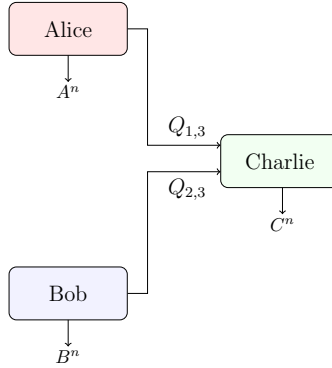


Fig. 5. Multiple access network

coordination protocol begins with Alice preparing the state of her output system  $A^n$ , as well as a “quantum description”  $M_1$ . She sends  $M_1$  to Bob. As Bob receives  $M_1$ , he encodes the output  $B^n$ , along with his own quantum description,  $M_2$ . Next, Bob sends  $M_2$  to Charlie. Upon receiving  $M_2$ , Charlie prepares the output state for  $C^n$ .

The transmissions  $M_1$  and  $M_2$  are limited to the quantum communication rates  $Q_1$  and  $Q_2$ , while the pre-shared resources between Alice and Bob and between Bob and Charlie are limited to the entanglement rates  $E_1$  and  $E_2$ , respectively.

*Definition 1.* A  $(2^{\ell_1}, 2^{\ell_2}, 2^{k_1}, 2^{k_2}, n)$  coordination code for the cascade network in Figure 6 consists of:

- 1) Two bipartite states  $|\Psi_{T_A T'_B}\rangle$  and  $|\Theta_{T'_B T_C}\rangle$  on Hilbert spaces of dimension  $2^{k_1}$  and  $2^{k_2}$ , respectively, i.e.,

$$\dim(\mathcal{H}_{T_A}) = \dim(\mathcal{H}_{T'_B}) = 2^{k_1}, \quad (12)$$

$$\dim(\mathcal{H}_{T'_B}) = \dim(\mathcal{H}_{T_C}) = 2^{k_2}, \quad (13)$$

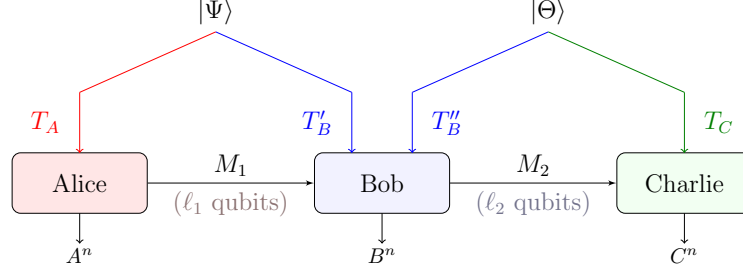


Fig. 6. Cascade network with rate-limited entanglement.

2) two Hilbert spaces,  $\mathcal{H}_{M_1}$  and  $\mathcal{H}_{M_2}$ , of dimension

$$\dim(\mathcal{H}_{M_j}) = 2^{\ell_j} \quad \text{for } j \in \{1, 2\}, \quad (14)$$

and

3) three encoding maps,

$$\mathcal{E}_{T_A \rightarrow A^n M_1} : \Delta(\mathcal{H}_{T_A}) \rightarrow \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1}), \quad (15)$$

$$\mathcal{F}_{M_1 T'_B T''_B \rightarrow B^n M_2} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{T'_B T''_B}) \rightarrow \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2}), \quad (16)$$

and

$$\mathcal{D}_{M_2 T_C \rightarrow C^n} : \Delta(\mathcal{H}_{M_2} \otimes \mathcal{H}_{T_C}) \rightarrow \Delta(\mathcal{H}_C^{\otimes n}), \quad (17)$$

corresponding to Alice, Bob, and Charlie, respectively.

The coordination protocol has limited communication rates  $Q_j$  and entanglement rates  $E_j$ , for  $j \in \{1, 2\}$ . That is, before the protocol begins, Alice and Bob are provided with  $k_1 = nE_1$  qubit pairs, while Bob and Charlie share  $k_2 = nE_2$  pairs. During the protocol, Alice transmits  $\ell_1 = nQ_1$  qubits to Bob, and then Bob transmits  $\ell_2 = nQ_2$  qubits to Charlie. See Figure 6. Hence,  $E_j = \frac{k_j}{n}$  is the rate of entanglement pairs per output, and  $Q_j = \frac{\ell_j}{n}$  is the rate of qubit transmissions per output, for  $j \in \{1, 2\}$ . A detailed description of the protocol is given below.

The coordination protocol works as follows. Alice applies the encoding map  $\mathcal{E}_{T_A \rightarrow A^n M_1}$  on her share  $T_A$  of the entanglement resources. This results in the output state

$$\rho_{A^n M_1 T'_B}^{(1)} = (\mathcal{E}_{T_A \rightarrow A^n M_1} \otimes \text{id}_{T'_B})(\Psi_{T_A T'_B}). \quad (18)$$

She sends  $M_1$  to Bob. Having received  $M_1$ , Bob uses it along with his share  $T'_B T''_B$  of the entanglement resources to encode the systems  $B^n$  and  $M_2$ . To this end, he uses the map  $\mathcal{F}_{M_1 T'_B T''_B \rightarrow B^n M_2}$ , hence

$$\rho_{A^n B^n M_2 T_C}^{(2)} = (\text{id}_{A^n} \otimes \mathcal{F}_{M_1 T'_B T''_B \rightarrow B^n M_2} \otimes \text{id}_{T_C})(\rho_{A^n M_1 T'_B}^{(1)} \otimes \Theta_{T'_B T''_B T_C}). \quad (19)$$

Bob sends  $M_2$  to Charlie, who applies the encoding channel  $\mathcal{D}_{M_2 T_C \rightarrow C^n}$ . This results in the final joint state,

$$\hat{\rho}_{A^n B^n C^n} = (\text{id}_{A^n B^n} \otimes \mathcal{D}_{M_2 T_C \rightarrow C^n}) \left( \rho_{A^n B^n M_2 T_C}^{(2)} \right). \quad (20)$$

The objective is that the final state  $\hat{\rho}_{A^n B^n C^n}$  is arbitrarily close to the desired state  $\omega_{ABC}^{\otimes n}$ .

**Definition 2.** A rate tuple  $(Q_1, Q_2, E_1, E_2)$  is achievable, if for every  $\varepsilon, \delta > 0$  and sufficiently large  $n$ , there exists a  $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, 2^{n(E_1+\delta)}, 2^{n(E_2+\delta)}, n)$  coordination code satisfying

$$\|\hat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n}\|_1 \leq \varepsilon. \quad (21)$$

**Remark 1.** Coordination in the cascade network can also be represented as as a resource inequality [40]

$$Q_1[q \rightarrow q]_{A \rightarrow B} + E_1[qq]_{AB} + Q_2[q \rightarrow q]_{B \rightarrow C} + E_2[qq]_{BC} \geq \langle \omega_{ABC} \rangle \quad (22)$$

where the resource units  $[q \rightarrow q]$ ,  $[qq]$ , and  $\langle \omega_{ABC} \rangle$  represent a single use of a noiseless qubit channel, an EPR pair, and the desired state  $\omega_{ABC}$ , respectively.

The optimal coordination rates for the cascade network are established below.

*Theorem 1.* Let  $|\omega_{RABC}\rangle$  be a purification of  $\omega_{ABC}$ . A rate tuple  $(Q_1, Q_2, E_1, E_2)$  is achievable for coordination in the cascade network in Figure 6, if and only if

$$Q_1 \geq \frac{1}{2}I(BC; R)_\omega, \quad (23)$$

$$Q_1 + E_1 \geq H(BC)_\omega, \quad (24)$$

$$Q_2 \geq \frac{1}{2}I(C; RA)_\omega, \quad (25)$$

$$Q_2 + E_2 \geq H(C)_\omega. \quad (26)$$

The proof for Theorem 1 is provided in Section IV.

### B. Broadcast network

Consider the broadcast network in Figure 7. This network, can be useful in analyzing refereed games and the required resources for achieving certain performances as described in section III. As before, we simplify the notation  $Q_{i,j}$  from the Introduction section, and write  $Q_1 \equiv Q_{1,2}$  and  $Q_2 \equiv Q_{1,3}$ , for convenience. Consider a classical-quantum state,

$$\omega_{XYABC} = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{XY}(x, y) |x, y\rangle\langle x, y|_{X,Y} \otimes \left| \sigma_{ABC}^{(x,y)} \right\rangle\left\langle \sigma_{ABC}^{(x,y)} \right| \quad (27)$$

corresponding to a given ensemble of states  $\left\{ p_{XY}, \left| \sigma_{ABC}^{(x,y)} \right\rangle \right\}$  in  $\Delta(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ .

Alice, Bob, and Charlie would like to simulate  $\omega_{XYABC}$ . Before communication takes place, the classical sequences  $X^n$  and  $Y^n$  are drawn from a common source  $p_{XY}^{\otimes n}$ . The sequence  $X^n$  is given to Bob, while  $Y^n$  is given to Charlie (see Figure 7).

Initially, Alice encodes her output  $A^n$ , along with two quantum descriptions,  $M_1$  and  $M_2$ . She then transmits  $M_1$  and  $M_2$ , to Bob and Charlie, respectively, at limited qubit transmission rates,  $Q_1$  and  $Q_2$ . As Bob receives the quantum description  $M_1$ , he uses it together with the classical sequence  $X^n$  to encode the output  $B^n$ . Similarly, Charlie receives  $M_2$  and  $Y^n$ , and encodes his output  $C^n$ .

*Definition 3.* A  $(2^{\ell_1}, 2^{\ell_2}, n)$  coordination code for the broadcast network with side information described in Figure 7, consists of two Hilbert spaces,  $\mathcal{H}_{M_1}$  and  $\mathcal{H}_{M_2}$ , of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{\ell_j} \text{ for } j \in \{1, 2\}, \quad (28)$$

and three encoding maps,

$$\mathcal{E}_{A^n \rightarrow A^n M_1 M_2} : \Delta(\mathcal{H}_A^{\otimes n}) \rightarrow \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}), \quad (29)$$

$$\mathcal{F}_{X^n M_1 \rightarrow B^n} : \mathcal{X}^n \otimes \Delta(\mathcal{H}_{M_1}) \rightarrow \Delta(\mathcal{H}_B^{\otimes n}), \quad (30)$$

and

$$\mathcal{D}_{Y^n M_2 \rightarrow C^n} : \mathcal{Y}^n \otimes \Delta(\mathcal{H}_{M_2}) \rightarrow \Delta(\mathcal{H}_C^{\otimes n}). \quad (31)$$

corresponding to Alice, Bob, and Charlie, respectively. In the course of the protocol, Alice transmits  $\ell_1 = nQ_1$  qubits to Bob, and  $\ell_2 = nQ_2$  qubits to Charlie, as illustrated in Figure 7. Thus, the qubit transmission rates are  $Q_j = \frac{\ell_j}{n}$  for  $j \in \{1, 2\}$ .

*Remark 2.* In the quantum world, broadcasting a quantum state among multiple receivers is impossible by the no-cloning theorem. However, in the broadcast network in Figure 7, Alice sends two different “quantum messages”  $M_1$  and  $M_2$  to Bob and Charlie, respectively. Roughly speaking, Alice is broadcasting correlation. Since Alice prepares both quantum descriptions,  $M_1$  and  $M_2$ , she can create correlation and generate tripartite entanglement between her, Bob, and Charlie.

The coordination protocol is described below. Alice applies her encoding map and prepares

$$\rho_{A^n M_1 M_2}^{(1)} = \mathcal{E}_{A^n \rightarrow A^n M_1 M_2}(\omega_A^{\otimes n}). \quad (32)$$

She sends  $M_1$  and  $M_2$  to Bob and Charlie, respectively. Once Bob receives  $M_1$  and the classical assistance,  $X^n$ , he applies his encoding map  $\mathcal{F}_{X^n M_1 \rightarrow B^n}$ . Similarly, Charlie receives  $M_2$  and  $Y^n$ , and applies  $\mathcal{D}_{Y^n M_2 \rightarrow C^n}$ . Their encoding operations result in the following extended state:

$$\begin{aligned} \hat{\rho}_{X^n Y^n A^n B^n C^n} &= \sum_{x^n \in \mathcal{X}^n} \sum_{y^n \in \mathcal{Y}^n} p_{XY}^{\otimes n}(x^n, y^n) |x^n, y^n\rangle\langle x^n, y^n|_{X^n Y^n} \otimes \\ &\quad (\text{id}_{A^n} \otimes \mathcal{F}_{X^n M_1 \rightarrow B^n} \otimes \mathcal{D}_{Y^n M_2 \rightarrow C^n}) \left( |x^n, y^n\rangle\langle x^n, y^n|_{\bar{X}^n \bar{Y}^n} \otimes \rho_{A^n M_1 M_2}^{(1)} \right), \quad (33) \end{aligned}$$

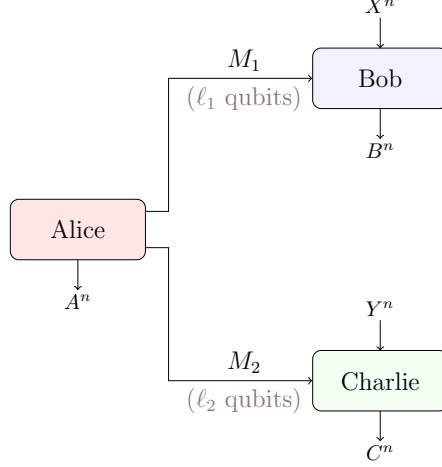


Fig. 7. Broadcast network.

where  $\bar{X}^n \bar{Y}^n$  are classical registers that store a copy of the (classical) sequences  $X^n Y^n$ , respectively. The goal is to encode such that the final state  $\hat{\rho}_{X^n Y^n A^n B^n C^n}$  is arbitrarily close to the desired state  $\omega_{XYABC}^{\otimes n}$ .

**Definition 4.** A rate pair  $(Q_1, Q_2)$  is achievable, if for every  $\varepsilon, \delta > 0$  and sufficiently large  $n$ , there exists a  $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$  coordination code satisfying

$$\|\hat{\rho}_{X^n Y^n A^n B^n C^n} - \omega_{XYABC}^{\otimes n}\|_1 \leq \varepsilon. \quad (34)$$

**Remark 3.** Notice that Alice has no access to  $X^n$  nor  $Y^n$ . Therefore, coordination can only be achieved for states  $\omega_{XYABC}$  such that there is no correlation between  $A$  and  $XY$ , on their own. That is, the reduced state  $\omega_{XYA}$  must have a product form,

$$\omega_{XYA} = \omega_{XY} \otimes \omega_A. \quad (35)$$

Since Alice does not share prior correlation with Bob and Charlie's resources  $X^n$  and  $Y^n$ , standard techniques, such as state redistribution [38] and quantum source coding with side information [39], are not suitable for our purposes. Instead, we introduce a quantum version of binning.

The optimal coordination rates for the broadcast network are established below.

**Theorem 2.** A rate pair  $(Q_1, Q_2)$  for the broadcast network in Figure 7 is achievable if and only if

$$Q_1 \geq H(B|X)_\omega, \quad (36)$$

$$Q_2 \geq H(C|Y)_\omega \quad (37)$$

The proof for Theorem 2 is provided in Section V.

### C. Multiple access network

Consider the multiple-access network in Figure 8. Alice, Bob, and Charlie would like to simulate a pure state  $|\omega_{ABC}\rangle^{\otimes n}$ , where  $|\omega_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ . We simplify the notation and write  $Q_1 \equiv Q_{1,3}$  and  $Q_2 \equiv Q_{2,3}$ . At first, Alice prepares the state of the quantum systems  $A^n$  and  $M_1$ , and Bob prepares the states of the quantum systems  $B^n$  and  $M_2$ . Alice and Bob send  $M_1$  and  $M_2$  to Charlie. Charlie then uses  $M_1$  and  $M_2$  to encode the system  $C^n$ . As in the previous settings,  $M_1$  and  $M_2$  are referred to as quantum descriptions, which are limited to the qubit transmission rates,  $Q_1$  and  $Q_2$ , respectively.

**Definition 5.** A  $(2^{\ell_1}, 2^{\ell_2}, n)$  coordination code for the multiple-access network described in Figure 8, consists of two Hilbert spaces,  $\mathcal{H}_{M_1}$  and  $\mathcal{H}_{M_2}$ , of dimensions

$$\dim(\mathcal{H}_{M_j}) = 2^{\ell_j} \text{ for } j \in \{1, 2\}, \quad (38)$$

and three encoding maps,

$$\mathcal{E}_{A^n \rightarrow A^n M_1} : \Delta(\mathcal{H}_A^{\otimes n}) \rightarrow \Delta(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{M_1}), \quad (39)$$

$$\mathcal{F}_{B^n \rightarrow B^n M_2} : \Delta(\mathcal{H}_B^{\otimes n}) \rightarrow \Delta(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_{M_2}) \quad (40)$$

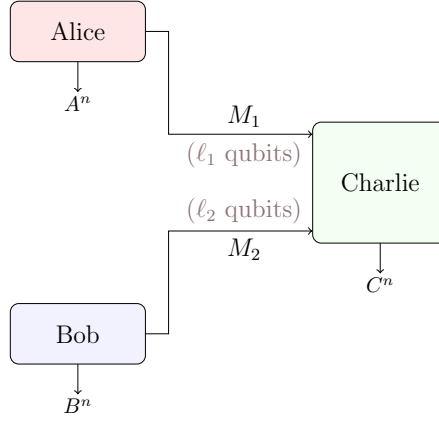


Fig. 8. Multiple access network.

and

$$\mathcal{D}_{M_1 M_2 \rightarrow C^n} : \Delta(\mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}) \rightarrow \Delta(\mathcal{H}_C^{\otimes n}), \quad (41)$$

corresponding Alice, Bob, and Charlie, respectively. The protocol has limited communication rates  $Q_j$  for  $j \in \{1, 2\}$ . That is, Alice sends  $\ell_1 = nQ_1$  qubits to Charlie, while Bob sends  $\ell_2 = nQ_2$  qubits to Charlie.

Alice and Bob apply the encoding maps, preparing  $\rho_{A^n M_1}^{(1)} \otimes \rho_{B^n M_2}^{(1)}$ , where

$$\rho_{A^n M_1}^{(1)} = \mathcal{E}_{A^n \rightarrow A^n M_1}(\omega_A^{\otimes n}), \quad \rho_{B^n M_2}^{(2)} = \mathcal{F}_{B^n \rightarrow B^n M_2}(\omega_B^{\otimes n}). \quad (42)$$

As Charlie receives  $M_1$  and  $M_2$ , he applies his encoding map, which yields the final state,

$$\hat{\rho}_{A^n B^n C^n} = (\text{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \rightarrow C^n})(\rho_{A^n B^n M_1 M_2}) \quad (43)$$

The ultimate goal of the coordination protocol is that the final state of  $\hat{\rho}_{A^n B^n C^n}$ , is arbitrarily close to the desired state  $\omega_{ABC}^{\otimes n}$ .

*Remark 4.* Notice that since Charlie only acts on  $M_1$  and  $M_2$  which are encoded separately without coordination, we have  $\hat{\rho}_{A^n B^n} = \rho_{A^n}^{(1)} \otimes \rho_{B^n}^{(2)}$ . Therefore, it is only possible to simulate states  $\omega_{ABC}$  such that  $\omega_{AB} = \omega_A \otimes \omega_B$ . Since all purifications are isometrically equivalent [41, Theorem 5.1.1] there exists an isometry  $V_{C \rightarrow C_1 C_2}$  such that

$$(\mathbb{1} \otimes V_{C \rightarrow C_1 C_2})|\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \quad (44)$$

where  $|\phi_{AC_1}\rangle$  and  $|\chi_{BC_2}\rangle$  are purifications of  $\omega_A$  and  $\omega_B$ , respectively. If  $\omega_{ABC}$  cannot be decomposed as in (44), then coordination is impossible in the multiple-access network.

*Definition 6.* A rate pair  $(Q_1, Q_2)$  is achievable, if for every  $\varepsilon, \delta > 0$  and a sufficiently large  $n$ , there exists a  $(2^{n(Q_1+\delta)}, 2^{n(Q_2+\delta)}, n)$  coordination code satisfying

$$\|\hat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n}\|_1 \leq \varepsilon. \quad (45)$$

*Remark 5.* The resource inequality for coordination in the multiple-access network is

$$Q_1[q \rightarrow q]_{A \rightarrow C} + Q_2[q \rightarrow q]_{B \rightarrow C} \geq \langle \omega_{ABC} \rangle. \quad (46)$$

See resource definitions in Remark 1.

The optimal coordination rates for the multiple-access network are provided below.

*Theorem 3.* Let  $|\omega_{ABC}\rangle$  be a pure state as in (44). Then, a rate pair  $(Q_1, Q_2)$  for coordination in the multiple-access network in Figure 8 is achievable if and only if

$$Q_1 \geq H(A)_\omega, \quad (47)$$

$$Q_2 \geq H(B)_\omega. \quad (48)$$

The proof for Theorem 3 is provided in Section VI.



### III. NONLOCAL GAMES

The broadcast network model presented in Figure 7 represents refereed games that have a quantum advantage. Such games are often referred to as nonlocal games [42]. The quantum advantage can be attributed to the quantum coordination between the users, before the beginning of the game.

First, we discuss the single-shot cooperative game, and then move on to sequential games with an asymptotic payoff. Consider Figure 7. Here, we assume that  $B$  and  $C$  are classical, while  $A$  is void. Here, Alice is a coordinator that generates correlation between the players, Bob and Charlie. In the sequential game, we denote the number of rounds by  $n$ .

*Single shot game:* The game involves a single round, hence  $n = 1$ . A referee provides two queries  $X$  and  $Y$ , drawn at random, one for Bob and the other for Charlie, respectively. The players, Bob and Charlie, provide responses,  $B$  and  $C$ , respectively. The players win the game (together) if the tuple  $(X, Y, B, C)$  satisfies a particular condition,  $\mathcal{W}$ . A well known example is the CHSH game [43], where  $X, Y, B, C \in \{0, 1\}$ , and the winning condition is

$$X \wedge Y = B \oplus C. \quad (49)$$

Using classical correlations, the game can be won with probability of at most 0.75.

If the players, Bob and Charlie, share a bipartite state  $\rho_{M_1 M_2}$ , then they can generate a quantum correlation,

$$P_{BC|XY}(b, c|xy) = \text{Tr} \left[ \left( F_b^{(x)} \otimes D_c^{(y)} \right) \rho_{M_1 M_2} \right] \quad (50)$$

by performing local measurements  $\{F_b^{(x)}\}$  and  $\{D_c^{(y)}\}$ , respectively. Such correlations can improve the players' performance. In particular, in the CHSH game, if the coordinator, Alice, provides the players with an EPR pair,

$$|\Phi_{M_1 M_2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad (51)$$

then their chance of winning improves to  $\cos^2(\frac{\pi}{8}) \approx 0.8535$ . As Alice sends one qubit to each player,  $(Q_1, Q_2) = (1, 1)$  is optimal.

In pseudo-telepathy games, quantum strategies guarantee winning with probability 1. One example is the magic square game [44], where  $(X, Y)$  are the coordinates of a cell in the square, and the players win the game if they can provide 3 bits each that satisfy a parity condition. In this case, the game can be won with  $Q_1 = Q_2 = 2$  qubits for each player. Slofstra and Vidick [45] presented a game where coordination of a correlation that could win with probability  $(1 - e^{-T})$  requires  $Q_j \propto T$  qubits for each user.

*Sequential game:* In the sequential version, the players repeat the game  $n$  times, and they can thus use a coordination code in order to play the game. Let  $\mathcal{S}(\gamma)$  denote the set of correlations  $P_{BC|XY}$  that win the game with probability  $\gamma$ . Based on our results, the game can be won with probability  $\gamma$  if and only if Alice can send qubits to Bob and Charlie at rates  $Q_1$  and  $Q_2$  that satisfy the constraints in Theorem 2 with respect to some correlation  $P_{BC|XY} \in \mathcal{S}(\gamma)$ .

### IV. CASCADE NETWORK ANALYSIS

We prove the rate characterization in Theorem 1. Consider the cascade network in Figure 6.

#### A. Achievability proof

The proof for the direct part exploits the state redistribution result by Yard and Devetak in [38]. We first describe the state redistribution problem. Consider two parties, Alice and Bob. Let their systems be described by the joint state  $\psi_{ABG}$ , where  $A$  and  $B$  belong to Alice, and  $G$  belongs to Bob. Let the state  $|\psi_{ABGR}\rangle$  be a purification of  $\psi_{ABG}$ . Alice and Bob would like to redistribute the state  $\psi_{ABG}$  such that  $B$  is transferred from Alice to Bob. Alice can send quantum description systems at rate  $Q$  and they share maximally entangled pairs of qubits at a rate  $E$ .

*Theorem 4 (State Redistribution [38]).* The optimal rates for state redistribution of  $|\psi_{ABGR}\rangle$  with rate-limited entanglement are

$$Q \geq \frac{1}{2}I(B; R|G)_\psi, \quad (52)$$

$$Q + E \geq H(B|G)_\psi. \quad (53)$$

We go back to the coordination setting for the cascade network (see Figure 6). Alice, Bob, and Charlie would like to simulate the joint state  $\omega_{ABC}^{\otimes n}$ . Let  $|\omega_{ABCR}\rangle$  be a purification. Suppose that Alice prepares the desired state  $|\omega_{ABCR}\rangle^{\otimes n}$  locally in her lab, where  $\bar{B}^n$ ,  $\bar{C}^n$ , and  $\bar{R}^n$  are her ancillas. Let  $\varepsilon > 0$  be arbitrarily small. By the state redistribution theorem, Theorem 4, Alice can transmit  $\bar{B}^n \bar{C}^n$  to Bob at communication rate  $Q_1$  and entanglement rate  $E_1$ , provided that

$$Q_1 \geq \frac{1}{2}I(\bar{B}\bar{C}; \bar{R})_\omega = \frac{1}{2}I(BC; R)_\omega, \quad (54)$$

$$Q_1 + E_1 \geq H(\bar{B}\bar{C})_\omega = H(BC)_\omega \quad (55)$$



(see [38]). That is, there exist a bipartite state  $\Psi_{T_A T'_B}$  and encoding maps,  $\mathcal{E}_{\bar{B}^n \bar{C}^n T_A \rightarrow M_1}^{(1)}$  and  $\mathcal{F}_{M_1 T'_B \rightarrow B^n \bar{C}^n}^{(1)}$ , such that

$$\left\| \tau_{\bar{R}^n A^n B^n \bar{C}^n}^{(1)} - \omega_{RABC}^{\otimes n} \right\|_1 \leq \varepsilon, \quad (56)$$

for sufficiently large  $n$ , where

$$\tau_{\bar{R}^n A^n B^n \bar{C}^n}^{(1)} = \left[ \text{id}_{\bar{R}^n A^n} \otimes \mathcal{F}_{M_1 T'_B \rightarrow B^n \bar{C}^n}^{(1)} \circ \left( \mathcal{E}_{\bar{B}^n \bar{C}^n T_A \rightarrow M_1}^{(1)} \otimes \text{id}_{T'_B} \right) \right] (\omega_{RABC}^{\otimes n} \otimes \Psi_{T_A T'_B}). \quad (57)$$

Similarly,  $\bar{C}^n$  can be compressed and transmitted with rates

$$Q_2 \geq \frac{1}{2} I(\bar{C}; A\bar{R})_\omega = \frac{1}{2} I(C; AR)_\omega, \quad (58)$$

$$Q_2 + E_2 \geq H(\bar{C})_\omega = H(C)_\omega, \quad (59)$$

by Theorem 4. Namely, there exists a bipartite state  $\Theta_{T''_B T_C}$  and encoding maps,  $\mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)}$  and  $\mathcal{D}_{M_2 T_C \rightarrow C^n}^{(2)}$ , such that

$$\left\| \tau_{\bar{R}^n A^n B^n C^n}^{(2)} - \omega_{RABC}^{\otimes n} \right\|_1 \leq \varepsilon, \quad (60)$$

where

$$\tau_{\bar{R}^n A^n B^n C^n}^{(2)} = \left[ \left( \text{id}_{\bar{R}^n A^n \bar{B}^n} \otimes \mathcal{D}_{M_2 T_C \rightarrow C^n}^{(2)} \right) \circ \left( \mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)} \otimes \text{id}_{T_C} \right) \right] (\omega_{RABC}^{\otimes n} \otimes \Theta_{T''_B T_C}). \quad (61)$$

The coding operations for the cascade network are described below.

*Encoding:* A)

- 1) Alice prepares  $|\omega_{A\bar{B}\bar{C}\bar{R}}\rangle^{\otimes n}$  locally. She applies  $\text{id}_{\bar{R}^n A^n} \otimes \mathcal{E}_{\bar{B}^n \bar{C}^n T_A \rightarrow M_1}^{(1)}$ , and sends  $M_1$  to Bob.
- 2) As Bob receives  $M_1$ , he applies

$$\mathcal{F}_{M_1 T'_B T'_B \rightarrow B^n M_2} \equiv \left( \text{id}_{B^n} \otimes \mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)} \right) \circ \mathcal{F}_{M_1 T'_B \rightarrow B^n \bar{C}^n}^{(1)}. \quad (62)$$

- 3) Charlie receives  $M_2$  from Bob and applies  $\mathcal{D}_{M_2 T_C \rightarrow C^n}^{(2)}$ .

*Error analysis:* We trace out the reference system  $R$  and write the analysis with respect to the reduced states. The joint state after Alice's encoding is

$$\rho_{A^n M_1 T'_B}^{(1)} = \left[ \text{id}_{A^n} \otimes \mathcal{E}_{\bar{B}^n \bar{C}^n T_A \rightarrow M_1}^{(1)} \otimes \text{id}_{T'_B} \right] (\omega_{A\bar{B}\bar{C}}^{\otimes n} \otimes \Psi_{T_A T'_B}). \quad (63)$$

After Bob applies his encoder, this results in

$$\begin{aligned} \rho_{A^n B^n M_2 T_C}^{(2)} &= \left[ \left( \text{id}_{A^n B^n} \otimes \mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)} \otimes \text{id}_{T_C} \right) \circ \left( \text{id}_{A^n} \otimes \mathcal{F}_{M_1 T'_B \rightarrow B^n \bar{C}^n}^{(1)} \otimes \text{id}_{T''_B T_C} \right) \right] (\rho_{A^n M_1 T'_B}^{(1)} \otimes \Theta_{T''_B T_C}) \\ &= \left( \text{id}_{A^n B^n} \otimes \mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)} \otimes \text{id}_{T_C} \right) (\tau_{A^n B^n \bar{C}^n}^{(1)} \otimes \Theta_{T''_B T_C}) \end{aligned} \quad (64)$$

by (62), and based on the definition of  $\tau^{(1)}$  in (57). According to (56),  $\tau^{(1)}$  and  $\omega^{\otimes n}$  are close in trace distance. By trace monotonicity under quantum channels, we have

$$\left\| \rho_{A^n B^n M_2 T_C}^{(2)} - \left( \text{id}_{A^n B^n} \otimes \mathcal{F}_{\bar{C}^n T''_B \rightarrow M_2}^{(2)} \otimes \text{id}_{T_C} \right) (\omega_{A\bar{B}\bar{C}}^{\otimes n} \otimes \Theta_{T''_B T_C}) \right\|_1 \leq \varepsilon. \quad (65)$$

As Charlie receives  $M_2$  and encodes, the final state, at the output of the cascade network, is given by

$$\hat{\rho}_{A^n B^n C^n} = \left[ \text{id}_{A^n B^n} \otimes \mathcal{D}_{M_2 T_C \rightarrow C^n}^{(2)} \right] (\rho_{A^n B^n M_2 T_C}^{(2)}). \quad (66)$$

Once more, by trace monotonicity,

$$\left\| \hat{\rho}_{A^n B^n C^n} - \tau_{A^n B^n C^n}^{(2)} \right\|_1 \leq \varepsilon. \quad (67)$$

(see (60) and (61)). Thus, using (60), (67), and the triangle inequality, we have

$$\begin{aligned} \left\| \hat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 &\leq \left\| \tau_{A^n B^n C^n}^{(2)} - \omega_{ABC}^{\otimes n} \right\|_1 + \left\| \hat{\rho}_{A^n B^n C^n} - \tau_{A^n B^n C^n}^{(2)} \right\|_1 \\ &\leq 2\varepsilon. \end{aligned} \quad (68)$$

This completes the achievability proof for the cascade network.

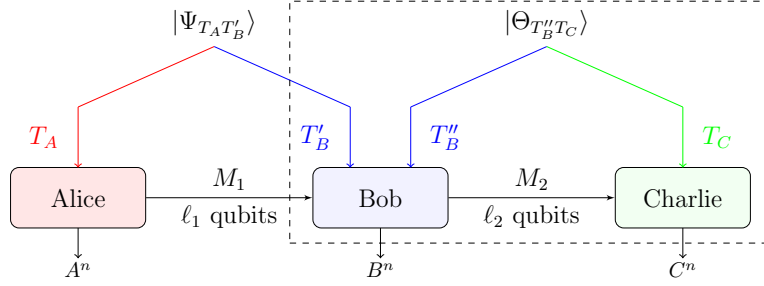


Fig. 9. At first, we treat the encoding operation of Bob and Charlie as a black box.

### B. Converse proof

We now prove the converse part for Theorem 1. Recall that in the cascade network, each party shares entanglement with their nearest neighbor a priori, i.e., Alice and Bob share  $|\Psi_{T_A T'_B}\rangle$ , while Bob and Charlie share  $|\Theta_{T''_B T_C}\rangle$  (see Figure 6 in subsection II-A). Alice applies an encoding map  $\mathcal{E}_{\bar{A}^n T_A \rightarrow A^n M_1}$  on her part, and sends the output  $M_1$  to Bob. As Bob receives  $M_1$ , he encodes using a map  $\mathcal{F}_{M_1 T'_B T''_B \rightarrow B^n M_2}$ , and sends  $M_2$ . As Charlie receives  $M_2$ , he applies an encoding channel  $\mathcal{D}_{M_2 T_C \rightarrow C^n}$ . Suppose that Alice prepares the state  $|\omega_{R\bar{A}BC}\rangle^{\otimes n}$  locally, and then encodes as explained above. The protocol can be described through the following relations:

$$\rho_{R^n A^n M_1 T'_B}^{(1)} = (\text{id}_{R^n} \otimes \mathcal{E}_{\bar{A}^n T_A \rightarrow A^n M_1} \otimes \text{id}_{T'_B})(\omega_{RA}^{\otimes n} \otimes \Psi_{T_A T'_B}), \quad (69)$$

$$\rho_{R^n A^n B^n M_2 T_C}^{(2)} = (\text{id}_{\bar{R}^n A^n} \otimes \mathcal{F}_{M_1 T'_B T''_B \rightarrow B^n M_2} \otimes \text{id}_{T_C})(\rho_{R^n A^n M_1 T'_B}^{(1)} \otimes \Theta_{T''_B T_C}), \quad (70)$$

$$\hat{\rho}_{R^n A^n B^n C^n} = (\text{id}_{R^n A^n B^n} \otimes \mathcal{D}_{M_2 T_C \rightarrow C^n}) \left( \rho_{R^n A^n B^n M_2 T_C}^{(2)} \right). \quad (71)$$

Let  $(Q_1, Q_2, E_1, E_2)$  be achievable rate tuple for coordination in the cascade network. Then, there exists a sequence of codes such that

$$\|\hat{\rho}_{R^n A^n B^n C^n} - \omega_{RABC}^{\otimes n}\|_1 \leq \varepsilon_n \quad (72)$$

where  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Consider Alice's communication and entanglement rates,  $Q_1$  and  $E_1$ . At this point, we may view the entire encoding operation of Bob and Charlie as a black box whose input and output are  $(M_1, T'_B)$  and  $(B^n, C^n)$ , respectively, as illustrated in Figure 9. Now,

$$\begin{aligned} 2n(Q_1 + E_1) &= 2[\log \dim(\mathcal{H}_{M_1}) + \log \dim(\mathcal{H}_{T'_B})] \\ &\geq I(M_1 T'_B; A^n R^n)_{\rho^{(1)}} \end{aligned} \quad (73)$$

since the quantum mutual information satisfies  $I(A; B)_\rho \leq 2 \log \dim(\mathcal{H}_A)$  in general. Therefore, by the data processing inequality,

$$\begin{aligned} I(M_1 T'_B; A^n R^n)_{\rho^{(1)}} &\geq I(B^n C^n; A^n R^n)_{\hat{\rho}} \\ &\geq I(B^n C^n; A^n R^n)_{\omega^{\otimes n}} - n\alpha_n \\ &= n[I(BC; AR)_\omega - \alpha_n], \end{aligned} \quad (74)$$

where  $\alpha_n \rightarrow 0$  when  $n \rightarrow \infty$ . The second inequality follows from (72) and the Alicki-Fannes-Winter (AFW) inequality [46] (entropy continuity). Since  $|\omega_{RABC}\rangle$  is pure, we have  $I(BC; AR)_\omega = 2H(BC)_\omega$  [41, Th. 11.2.1]. Therefore, combining (73)-(74), we have

$$Q_1 + E_1 \geq H(BC)_\omega - \frac{1}{2}\alpha_n. \quad (75)$$

To show the bound on  $Q_1$ , observe that a lower bound on the communication rate with unlimited entanglement resources also holds with limited resources. Therefore, the bound  $Q_1 \geq \frac{1}{2}I(BC; R)_\omega$  follows from the entanglement-assisted capacity theorem due to Bennett et al. [28]. It is easier to see this through resource inequalities, following the arguments in [7]. If the entanglement resources are unlimited, then the coordination code achieves

$$\begin{aligned} Q_1 [q \rightarrow q]_{A \rightarrow BC} &\geq \langle \omega_{RBC} \rangle \\ &\equiv \langle \text{Tr}_A : \omega_{RABC} \rangle \\ &\geq \frac{1}{2} I(BC; R)_\omega [q \rightarrow q]_{A \rightarrow BC} \end{aligned} \quad (76)$$

where the resource units  $[q \rightarrow q]$ ,  $[qq]$ , and  $\langle \omega_{ABC} \rangle$  represent a single use of a noiseless qubit channel, an EPR pair, and the desired state  $\omega_{ABC}$ , respectively, while the unit resource  $\langle \mathcal{N}_{A \rightarrow B} : \rho \rangle$  indicates a simulation of the channel output from  $\mathcal{N}_{A \rightarrow B}$  with respect to the input state  $\rho$ . The last inequality holds by [7, 28].

Similarly, we bound Bob's communication and entanglement rates as follows,

$$2n(Q_2 + E_2) = 2 [\log \dim(\mathcal{H}_{M_2}) + \log \dim(\mathcal{H}_{T_B''})] \quad (77)$$

$$\geq I(M_2 T_C; A^n B^n R^n)_{\rho^{(2)}} \quad (78)$$

$$\geq I(C^n; A^n B^n R^n T_B'')_{\hat{\rho}} \quad (79)$$

$$\geq n[I(C; ABR)_\omega - \beta_n] \quad (80)$$

$$= n[2H(C)_\omega - \beta_n] \quad (81)$$

where  $\beta_n \rightarrow 0$  when  $n \rightarrow \infty$ . As before, the last inequality follows from (72) and the AFW inequality [46]. Hence,

$$Q_2 + E_2 \geq H(C)_\omega - \frac{1}{2}\beta_n. \quad (82)$$

Furthermore,

$$\begin{aligned} Q_2 [q \rightarrow q]_{B \rightarrow C} &\geq \langle \omega_{RAC} \rangle \\ &\equiv \langle \text{Tr}_B : \omega_{RABC} \rangle \\ &\geq \frac{1}{2} I(C; AR)_\omega [q \rightarrow q]_{B \rightarrow C} \end{aligned} \quad (83)$$

which implies  $Q_2 \geq \frac{1}{2} I(C; AR)_\omega$ .

This completes the proof of Theorem 1 for the cascade network.

## V. BROADCAST ANALYSIS

We prove the rate characterization in Theorem 2. Consider the broadcast network in Figure 7.

We show achievability by using a quantum version of the binning technique. Let  $\varepsilon_i, \delta > 0$  be arbitrarily small. Define the average states,

$$\sigma_{AB}^{(x)} = \sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \sigma_{AB}^{(x,y)}, \quad (84)$$

$$\sigma_{AC}^{(y)} = \sum_{x \in \mathcal{X}} p_{X|Y}(x|y) \sigma_{AC}^{(x,y)}, \quad (85)$$

and consider a spectral decomposition of the reduced states of Bob and Charlie,

$$\sigma_B^{(x)} = \sum_{z \in \mathcal{Z}} p_{Z|X}(z|x) |\psi_{x,z}\rangle \langle \psi_{x,z}|, \quad (86)$$

$$\sigma_C^{(y)} = \sum_{w \in \mathcal{W}} p_{W|Y}(w|y) |\phi_{y,w}\rangle \langle \phi_{y,w}|, \quad (87)$$

where  $p_{Z|X}$  and  $p_{W|Y}$  are conditional probability distributions, and  $\{|\psi_{x,z}\rangle\}_z, \{|\phi_{y,w}\rangle\}_w$  are orthonormal bases for  $\mathcal{H}_B, \mathcal{H}_C$ , respectively, for every given  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ . We can also assume that the different bases are orthogonal to each other by requiring that Bob and Charlie encode on a different Hilbert space for every value of  $(x, y)$ .

We use the type class definitions and notations in [41, Chap. 14]. In particular,  $T_\delta^{X^n}$  denotes the  $\delta$ -typical set with respect to  $p_X$ , and  $T_\delta^{Z^n|x^n}$  is the conditional  $\delta$ -typical set with respect to  $p_{XZ}$ , given  $x^n \in T_\delta^{X^n}$ .

*Classical Codebook Generation:* For every sequence  $z^n \in \mathcal{Z}^n$ , assign an index  $m_1(z^n)$ , uniformly at random from  $[2^{nQ_1}]$ . A bin  $\mathfrak{B}_1(m_1)$  is defined as the subset of sequences in  $\mathcal{Z}^n$  that are assigned the same index  $m_1$ , for  $m_1 \in [2^{nQ_1}]$ . The codebook is revealed to all parties.

*Encoding:* A)

- 1) Alice prepares  $\omega_{ABC}^{\otimes n}$  locally, where  $\bar{B}^n \bar{C}^n$  are her ancillas, without any correlation with  $X^n$  and  $Y^n$  (see Remark 3). She applies the encoding channel  $\mathcal{E}_{\bar{B}^n \rightarrow M_1}^{(1)} \otimes \mathcal{E}_{\bar{C}^n \rightarrow M_2}^{(2)}$ ,

$$\mathcal{E}_{\bar{B}^n \rightarrow M_1}^{(1)}(\rho_1) = \sum_{x^n \in \mathcal{X}^n} p_X^{\otimes n}(x^n) \sum_{z^n \in \mathcal{Z}^n} \langle \psi_{x^n, z^n} | \rho_1 | \psi_{x^n, z^n} \rangle |m_1(z^n)\rangle \langle m_1(z^n)|, \quad (88)$$

$$\mathcal{E}_{\bar{C}^n \rightarrow M_2}^{(2)}(\rho_2) = \sum_{y^n \in \mathcal{Y}^n} p_Y^{\otimes n}(y^n) \sum_{w^n \in \mathcal{W}^n} \langle \phi_{y^n, w^n} | \rho_2 | \phi_{y^n, w^n} \rangle |m_2(w^n)\rangle \langle m_2(w^n)|, \quad (89)$$

for  $\rho_1 \in \Delta(\mathcal{H}_B^{\otimes n})$ ,  $\rho_2 \in \Delta(\mathcal{H}_C^{\otimes n})$ , and transmits  $M_1$  and  $M_2$  to Bob and Charlie, respectively.

2) First, Bob applies the following encoding channel,

$$\mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}(\rho_{M_1}) = \sum_{m_1=1}^{2^{nQ_1}} \langle m_1 | \rho_{M_1} | m_1 \rangle \left( \frac{1}{|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1)|} \sum_{z^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1)} |\psi_{x^n, z^n}\rangle \langle \psi_{x^n, z^n}| \right) \quad (90)$$

3) Charlie's decoder is defined in a similar manner.

*Error analysis:* Due to the code construction, it suffices to consider the individual errors of Bob and Charlie,

$$\frac{1}{2} \left\| \omega_{XAB}^{\otimes n} - \left( \mathcal{F}_{X^n M_1 \rightarrow X^n B^n} \circ \mathcal{E}_{B^n \rightarrow M_1}^{(1)} \right) (\omega_X^{\otimes n} \otimes \omega_{AB}^{\otimes n}) \right\|_1, \quad (91)$$

$$\frac{1}{2} \left\| \omega_{YAC}^{\otimes n} - \left( \mathcal{D}_{Y^n M_2 \rightarrow Y^n C^n} \circ \mathcal{E}_{C^n \rightarrow M_2}^{(2)} \right) (\omega_Y^{\otimes n} \otimes \omega_{AC}^{\otimes n}) \right\|_1, \quad (92)$$

respectively, where we use the short notation  $\mathcal{E}_{B^n \rightarrow M_1}^{(1)} \equiv \text{id}_{X^n A^n} \otimes \mathcal{E}_{B^n \rightarrow M_1}^{(1)}$ , and similarly for the other encoding maps.

We now focus on Bob's error. Consider a given codebook  $\mathcal{C}_1 = \{m_1(z^n)\}$ . Alice encodes  $M_1$  by

$$\mathcal{E}_{B^n \rightarrow M_1}^{(1)}(\omega_{AB}^{\otimes n}) = \sum_{\tilde{x}^n \in \mathcal{X}^n} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n \in \mathcal{Z}^n} \langle \psi_{\tilde{x}^n, z^n} | \omega_{AB}^{\otimes n} | \psi_{\tilde{x}^n, z^n} \rangle |m_1(z^n)\rangle \langle m_1(z^n)|, \quad (93)$$

where we use the short notation  $|\psi\rangle_{x^n, z^n} \equiv \bigotimes_{i=1}^n |\psi\rangle_{x_i, z_i}$ . By the weak law of large numbers, this state is  $\varepsilon_1$ -close in trace distance to

$$\begin{aligned} \rho_{A^n M_1}^{(1)} &= \sum_{\tilde{x}^n \in T_\delta^{X^n}} p_X^{\otimes n}(\tilde{x}^n) \sum_{z^n \in T_\delta^{Z^n|\tilde{x}^n}} \langle \psi_{\tilde{x}^n, z^n} | \sigma_{A^n \tilde{B}^n}^{(\tilde{x}^n)} | \psi_{\tilde{x}^n, z^n} \rangle |m_1(z^n)\rangle \langle m_1(z^n)| \\ &= \sum_{x^n \in T_\delta^{X^n}} p_X^{\otimes n}(x^n) \rho_{A^n M_1}^{(1|x^n)}, \end{aligned} \quad (94)$$

for sufficiently large  $n$ , where we have defined

$$\rho_{A^n M_1}^{(1|x^n)} = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n, z^n} | \sigma_{A^n \tilde{B}^n}^{(x^n)} | \psi_{x^n, z^n} \rangle |m_1(z^n)\rangle \langle m_1(z^n)|. \quad (95)$$

Let  $x^n \in T_\delta^{X^n}$ . After Bob encodes  $B^n$ , we have

$$\mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}(\rho_{A^n M_1}^{(1|x^n)}) = \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n, z^n} | \sigma_{A^n \tilde{B}^n}^{(x^n)} | \psi_{x^n, z^n} \rangle \mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}(|m_1(z^n)\rangle \langle m_1(z^n)|). \quad (96)$$

By the definition of Bob's encoding channel,  $\mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}$ , in (90),

$$\mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}(|m_1(z^n)\rangle \langle m_1(z^n)|) = \frac{1}{|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))} |\psi_{x^n, \tilde{z}^n}\rangle \langle \psi_{x^n, \tilde{z}^n}|. \quad (97)$$

Substituting in (96) yields

$$\begin{aligned} \mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)}(\rho_{A^n M_1}^{(1|x^n)}) &= \sum_{z^n \in T_\delta^{Z^n|x^n}} \langle \psi_{x^n, z^n} | \sigma_{A^n \tilde{B}^n}^{(x^n)} | \psi_{x^n, z^n} \rangle \\ &\quad \otimes \left[ \frac{1}{|T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))|} \sum_{\tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))} |\psi_{x^n, \tilde{z}^n}\rangle \langle \psi_{x^n, \tilde{z}^n}| \right]. \end{aligned} \quad (98)$$

Based on the classical result [47, Chapter 10.3], the random codebook  $\mathcal{C}_1$  satisfies that

$$\Pr_{\mathcal{C}_1} \left( \exists \tilde{z}^n \in T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n)) : \tilde{z}^n \neq z^n \right) \leq \varepsilon_2 \quad (99)$$

given  $z^n \in T_\delta^{Z^n|x^n}$ , for sufficiently large  $n$ , provided that the codebook size is at least  $2^{n(H(Z|X) + \varepsilon_3)}$ , where  $H(Z|X)$  denotes the classical conditional entropy. As  $|\mathcal{C}_1| = 2^{nQ_1}$ , this holds if

$$\begin{aligned} Q_1 &> H(Z|X) + \varepsilon_3 \\ &= H(B|X)_\omega + \varepsilon_3. \end{aligned} \quad (100)$$

Observe that if the summation set in (98),  $T_\delta^{Z^n|x^n} \cap \mathfrak{B}_1(m_1(z^n))$ , consists of the sequence  $z^n$  alone, then the overall state in (98) is identical to the post-measurement state after a typical subspace measurement on  $B^n$ , with respect to the conditional  $\delta$ -typical set  $T_\delta^{Z^n|x^n}$ . Based on the gentle measurement lemma [48], this state is  $\varepsilon_4$ -close to  $\sigma_{AB}^{(x^n)}$ , for sufficiently large  $n$ .

Therefore, by the triangle inequality and total expectation formula,

$$\begin{aligned} & \left\| \omega_{XAB}^{\otimes n} - \mathbb{E}_{\mathcal{C}_1} \left( \mathcal{F}_{X^n M_1 \rightarrow X^n B^n} \circ \mathcal{E}_{B^n \rightarrow M_1}^{(1)} \right) (\omega_X^{\otimes n} \otimes \omega_{AB}^{\otimes n}) \right\|_1 \\ & \leq \sum_{x^n \in \mathcal{X}^n} p_X^{\otimes n}(x^n) \cdot \mathbb{E}_{\mathcal{C}_1} \left\| \sigma_{A^n B^n}^{(x^n)} - \left( \mathcal{F}_{M_1 \rightarrow B^n}^{(x^n)} \circ \mathcal{E}_{B^n \rightarrow M_1}^{(1)} \right) (\sigma_{A^n B^n}^{(x^n)}) \right\|_1 \\ & \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_4. \end{aligned} \quad (101)$$

By symmetry, Charlie's error tends to zero as well, provided that  $Q_2 \geq H(C|Y)_\omega + \varepsilon_5$ . Since the total error vanishes, when averaged over the class of binning codebooks, it follows that there exists a deterministic codebook with the same property. The achievability proof follows by taking  $n \rightarrow \infty$  and then  $\varepsilon_j, \delta \rightarrow 0$ .

The converse proof follows the lines of [38], and it is thus omitted. This completes the proof of Theorem 2 for the broadcast network.

## VI. MULTIPLE-ACCESS ANALYSIS

We prove the rate characterization in Theorem 3. Consider the multiple-access network in Figure 8. As explained in Remark 4, coordination in the multiple-access network is only possible if there exists an isometry  $V : \mathcal{H}_C \rightarrow \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2}$  such that

$$(\mathbb{1} \otimes V) |\omega_{ABC}\rangle = |\phi_{AC_1}\rangle \otimes |\chi_{BC_2}\rangle \quad (102)$$

where  $|\phi_{AC_1}\rangle$  and  $|\chi_{BC_2}\rangle$  are purifications of  $\omega_A$  and  $\omega_B$ , respectively. For this reason, Theorem 3 assumes that this property holds. Furthermore, since  $|\phi_{AC_1}\rangle$  and  $|\chi_{BC_2}\rangle$  purify  $\omega_A$  and  $\omega_B$ , respectively, we have  $H(C_1)_\phi = H(A)_\phi = H(A)_\omega$  and  $H(C_2)_\chi = H(B)_\chi = H(B)_\omega$ . Thus, it suffices to show that  $(Q_1, Q_2)$  is achievable if and only if

$$Q_1 \geq H(C_1)_\phi, \quad (103)$$

$$Q_2 \geq H(C_2)_\chi. \quad (104)$$

The achievability proof follows from the Schumacher compression protocol [49] [41, chap. 18] in a straightforward manner. Alice and Bob prepare  $\phi_{AC_1}^{\otimes n}$  and  $\chi_{BC_2}^{\otimes n}$ , respectively. Then, they send  $C_1^n$  and  $C_2^n$  using the Schumacher compression protocol, and finally, Charlie applies the isometry  $(V^\dagger)^{\otimes n}$  in order to simulate  $\omega_{ABC}^{\otimes n}$  (see (102)). The details are omitted.

It remains to show the converse part. Recall that in the multiple-access network, Alice and Bob each applies their respective encoding map,  $\mathcal{E}_{A^n \rightarrow A^n M_1}$  and  $\mathcal{F}_{B^n \rightarrow B^n M_2}$ , and send the quantum descriptions  $M_1$  and  $M_2$ . Then, Charlie encodes by  $\mathcal{D}_{M_1 M_2 \rightarrow C^n}$ . The protocol can be described through the following relations:

$$\rho_{A^n M_1}^{(1)} = \mathcal{E}_{A^n \rightarrow A^n M_1}(\omega_A^{\otimes n}), \quad \rho_{B^n M_2}^{(2)} = \mathcal{F}_{B^n \rightarrow B^n M_2}(\omega_B^{\otimes n}), \quad (105)$$

$$\hat{\rho}_{A^n B^n C^n} = (\text{id}_{A^n B^n} \otimes \mathcal{D}_{M_1 M_2 \rightarrow C^n})(\rho_{A^n M_1}^{(1)} \otimes \rho_{B^n M_2}^{(2)}). \quad (106)$$

Let  $(Q_1, Q_2)$  be an achievable rate pair for coordination in the multiple-access network in Figure 8. Then, there exists a sequence of  $(2^{nQ_1}, 2^{nQ_2}, n)$  coordination codes such that

$$\left\| \hat{\rho}_{A^n B^n C^n} - \omega_{ABC}^{\otimes n} \right\|_1 \leq \varepsilon_n \quad (107)$$

tends to zero as  $n \rightarrow \infty$ . Applying the isometry  $V^{\otimes n}$  yields

$$\left\| \hat{\sigma}_{A^n B^n C_1^n C_2^n} - \phi_{AC_1}^{\otimes n} \otimes \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n, \quad (108)$$

by (102), where

$$\hat{\sigma}_{A^n B^n C_1^n C_2^n} = (\mathbb{1}_{AB} \otimes V)^{\otimes n} \hat{\rho}_{A^n B^n C^n} (\mathbb{1}_{AB} \otimes V^\dagger)^{\otimes n}. \quad (109)$$

It thus follows that

$$\left\| \hat{\sigma}_{A^n C_1^n} - \phi_{AC_1}^{\otimes n} \right\|_1 \leq \varepsilon_n \quad (110)$$

and

$$\left\| \hat{\sigma}_{B^n C_2^n} - \chi_{BC_2}^{\otimes n} \right\|_1 \leq \varepsilon_n. \quad (111)$$

Now, Alice's communication rate is bounded by

$$2nQ_1 \stackrel{(a)}{\geq} I(M_1; A^n | M_2)_{\rho^{(1)} \otimes \rho^{(2)}}$$

$$\begin{aligned}
&\stackrel{(b)}{=} I(M_1 M_2; A^n)_{\rho^{(1)} \otimes \rho^{(2)}} \\
&\stackrel{(c)}{\geq} I(C^n; A^n)_{\hat{\rho}} \\
&\stackrel{(d)}{=} I(C_1^n C_2^n; A^n)_{\hat{\sigma}} \\
&\stackrel{(e)}{\geq} I(C_1^n C_2^n; A^n)_{\omega} - n\alpha_n \\
&\stackrel{(f)}{=} 2nH(C_1)_{\phi} - n\alpha_n \\
&\stackrel{(g)}{=} 2nH(A)_{\omega} - n\alpha_n,
\end{aligned} \tag{112}$$

where (a) holds because  $M_1$  is of dimension  $2^{nQ_1}$ , (b) since  $I(M_2; A^n)_{\rho^{(1)} \otimes \rho^{(2)}} = 0$ , (c) follows from the data processing inequality, (d) holds since the von Neumann entropy is isometrically invariant, (e) by the AFW inequality [46], (f) since the mutual information is calculated with respect to the product state  $|\phi_{AC_1}\rangle^{\otimes n} \otimes |\chi_{BC_2}\rangle^{\otimes n}$ , and (g) holds since  $|\phi_{AC_1}\rangle$  is a purification of  $\omega_A$ . The bound on Bob's communication rate follows by symmetry. This completes the proof of Theorem 3.  $\square$

## VII. SUMMARY

We have considered quantum coordination in three models, the cascade, broadcast, and multiple-access networks. In the cascade network, three users simulate a joint state using limited communication and entanglement rates. Next, we considered a broadcast network, where a single sender broadcasts correlation to two receivers. We have discussed the implications of our results on nonlocal games with quantum strategies. At last, we considered a multiple-access network where two senders transmit qubits to a single receiver. We observe that the network topology dictates the type of states that can be simulated. Our results generalize various results in the literature, e.g., by Cuff et al. [50] and by George et al. [26], and can easily be extended to more than three nodes.

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