IMAGINARITY MEASURE INDUCED BY RELATIVE ENTROPY

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Quantum resource theories provide a new perspective and method for the development and application of science. Imaginary numbers are used to describe and solve many complex problems. Therefore imaginarity resource theory proposed recently has become increasingly important and it is worthy of research. In this paper, we find two imaginarity measures, one of which is induced by α -z-Rényi relative entropy and the other defined for positive definite density matrices is induced by Tsallis relative operator entropy. The relationships between different imaginarity measures and their properties are also discussed.

Keywords: imaginarity measures, quantum resource theories, α -z-Rényi relative entropy, Tsallis relative operator entropy

1. Introduction

Quantum resource theories provide an operational way dealing with the quantification and manipulation of resources in quantum systems, where resources can refer to various quantum properties such as entanglement [1] or coherence [2]. It is considered an important branch in quantum information theory. Recently, a novel resource theory has been proposed in [3], that is imaginarity resource theory, which has garnered significant attention in recent years.

In the framework of the imaginarity resource theory, let \mathcal{H} be a *d*-dimensional Hilbert space, and consider $\{|m\rangle\}_{m=0}^{d-1}$ as a fixed set of orthonormal basis on \mathcal{H} , $L(\mathcal{H})$ denotes the set of density matrices on \mathcal{H} . In the imaginarity resource theory, free states are defined as real states, and the set of free states denoted by \mathcal{R} is defined as:

$$\mathcal{R} = \{ \rho \in L(\mathcal{H}) : \langle m | \rho | n \rangle \in \mathbb{R} \}.$$

Free operations Λ are defined as real operations. that is $\Lambda(\cdot) = \sum_{j} K_j \cdot K_j^{\dagger}$, where Kraus operators $\{K_j\}$ satisfy $\langle m|K_j|n\rangle \in \mathbb{R}$.

After defining free states and free operations, an appropriate imaginarity measure \mathcal{M} should satisfy the following conditions [3]:

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 $(I_1): \mathcal{M}(\rho) \ge 0$ for any quantum state ρ , and $\mathcal{M}(\rho) = 0 \iff \rho \in \mathcal{R}$, that is, ρ equals its conjugate ρ^* .

 (I_2) : $\mathcal{M}(\Lambda(\rho)) \leq \mathcal{M}(\rho)$, where Λ is a real operation.

(I₃): $\mathcal{M}(\rho) \ge \sum_j p_j \mathcal{M}(\rho_j)$, where $p_j = \text{Tr}[K_j \rho K_j^{\dagger}]$, $\rho_j = K_j \rho K_j^{\dagger}/p_j$, K_j are Kraus operators of a real operation Λ .

 (I_4) : $\sum_j p_j \mathcal{M}(\rho_j) \ge \mathcal{M}(\sum_j p_j \rho_j)$ for any $\{\rho_j\}$ and $\{p_j\}$ satisfied with $p_j \ge 0$ and $\sum_{j} p_{j} \stackrel{\frown}{=} 1.$

In fact, similar to alternative framework for quantifying coherence [4], under the premises of (I_1) and (I_2) , the condition $(I_3) + (I_4)$ is equivalent to the following condition under [5] as:

(I₅): There is $\mathcal{M}(p_1\rho_1\oplus p_2\rho_2) = p_1\mathcal{M}(\rho_1) + p_2\mathcal{M}(\rho_2)$ for any p_1, p_2 satisfied with $p_1+p_2 =$ 1 and $\rho_1, \rho_2 \in \mathcal{R}$.

Hence, it's easier to verify (I_1) , (I_2) , and (I_5) than a common method when verifying whether a certain function can serve as an imaginarity measure.

In addition to the robustness of imaginarity and the l_1 measure which were born along with the imaginarity resource theory [3], many imaginarity measures have been given recently, such as the fidelity of imaginarity [6], the geometric imaginarity measure [7] and convex roof [8]. Next, we will introduce some some imaginarity measures that may be helpful for the subsequent study. Suppose ρ and σ are density matrices, then the relative entropy of ρ with respect to σ is defined as:

$$S(\rho||\sigma) = \operatorname{Tr}[\rho \log(\rho)] - \operatorname{Tr}[\sigma \log(\sigma)].$$

The imaginarity measure $\mathcal{M}^{V}(\rho)$ induced by $S(\rho||\sigma)$ is defined by [5] as follows:

$$\mathcal{M}^{V}(\rho) = \min_{\sigma \in \mathcal{R}} S(\rho || \sigma) = S[\frac{1}{2}(\rho + \rho^{*})] - S(\rho),$$

The advantage of the second representing is more convenient to measure without using nonlinear programming.

Tsallis entropy as a generalization form of Shannon entropy was extended to the quantum field in [9]. The definition of Tsallis relative entropy is written as:

$$K_q(\rho||\sigma) = \frac{1}{1-q} [1 - \text{Tr}(\rho^q \sigma^{1-q})].$$

 $M_a^T(\rho)$ is a imaginarity measure given by Xu [10] as follows:

$$\mathcal{M}_q^T(\rho) = (1-q)K_q(\rho||\rho^*).$$

The value of the parameter q required in the above two expressions should be chosen in the interval (0, 1).

Of course, in addition to these two relative entropies above, various other quantifiers have been discovered, such as Rényi entropy [11, 12], Tsallis entropy [13, 9, 14], Brègman distances [15, 16], and so on. Moreover, recent research [17, 18, 19] has shown that some linear combinations or functions of different quantifiers can also emerge as new quantifiers.

With the development of resource theory, some quantifiers have indeed proven to be effective measures, while others still require further investigation.

In this paper, we focus on the imaginarity measure induced by relative entropy. In section 2., we demonstrate that the family of quantifiers induced by α -z-Rényi relative entropy is indeed a measure and provide Properties and examples. In section 3., we discuss the relationship between different imaginarity measures. In section 4., we give an imaginarity measure that holds only for positive definite density matrices. Finally, in section 5., we provide a brief summary. Remarks made in certain places represent our own understanding.

2. Imaginarity measure induced by α -z-Rényi relative entropy

Different from the two specific single-parameter relative entropies $S(\rho||\sigma)$ and $K_q(\rho||\sigma)$ mentioned in section 1., the α -z-Rényi relative entropy defined with two parameters α and z was introduced in [20], regarded as:

$$D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha - 1} \log f_{\alpha,z}(\rho, \sigma),$$

where $f_{\alpha,z}(\rho,\sigma) = \text{Tr}(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}})^z$. It is observed that when $z = \alpha$, $D_{\alpha,z}(\rho||\sigma)$ reduces to the quantum Rényi divergence [12]. It can be seen that, just like \mathcal{M}^V or M_q^T , the measurements only depend on the

It can be seen that, just like \mathcal{M}^V or M_q^T , the measurements only depend on the quantum state ρ itself, without the need for any σ as an auxiliary. This is understandable, in fact, since the imaginarity resource theory delineates the boundary between the real and the imaginary. The requirement in (I_1) that $\rho \in \mathcal{R}$ prompts us to link the quantum state itself with its conjugate, thus choosing σ as ρ^* is a good choice.

2.1. Establishment of the measure

In the coherence resource theory, the quantifier inducing coherence measure can be extended from Rényi relative entropy to α -z-Rényi relative entropy [21]. Therefore, we can also conduct the following research in the imaginarity resource theory. For the smooth flow of the text, we need to introduce a lemma that will be used subsequently.

LEMMA 1. Suppose A, B are two positive semidefinite matrices, 0 < t < 1, $r \ge 1$, and $q \ge 0$, then the following inequality holds:

 $(1)Tr(A^{\frac{1-t}{2t}}BA^{\frac{1-t}{2t}})^t \leqslant Tr[(1-t)A+tB],$ (2)Tr(A^rB^rA^r)^q \ge Tr(ABA)^{rq}.

These two inequalities respectively appear in the literature [22] and [23].

We have already saw that choosing σ as ρ^* is a wise move. Therefore, what we are going to do next is exactly this. Based on the α -z-Rényi relative entropy, when the parameters α and z satisfy $0 < \max(\alpha, 1 - \alpha) \leq z < 1$, we present the following results:

THEOREM 1. The parametrized function $\mathcal{M}_{\alpha,z}^R$ of the state ρ is given as an imaginarity measure:

$$\mathcal{M}^{R}_{\alpha,z}(\rho) = 1 - f_{\alpha,z}(\rho, \rho^*),$$

where $0 < max(\alpha, 1 - \alpha) \leq z < 1$.

Proof. We only need to prove $(I_1) + (I_2) + (I_5)$. (I_1): The trace of the density matrix ρ is 1, and the same applies to ρ^* , then we can know from Lemma 1:

$$\operatorname{Tr}[(\rho^*)^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}(\rho^*)^{\frac{1-\alpha}{2z}}]^z = \operatorname{Tr}[(\rho^*)^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}(\rho^*)^{\frac{1-\alpha}{2z}}]^{\frac{z}{\alpha}\alpha}$$
$$\leqslant \operatorname{Tr}[(\rho^*)^{\frac{1-\alpha}{2\alpha}}\rho^{\frac{\alpha}{\alpha}}(\rho^*)^{\frac{1-\alpha}{2\alpha}}]^{\alpha}$$
$$\leqslant \operatorname{Tr}[(1-\alpha)(\rho^*) + \alpha\rho]$$
$$= 1.$$

This leads to $\mathcal{M}^{R}_{\alpha,z}(\rho) \ge 0$. When the equality sign holds, an equivalence chain exists:

$$\mathcal{M}^{R}_{\alpha,z}(\rho) = 0 \iff f_{\alpha,z}(\rho,\rho^{*}) = 1$$
$$\iff D_{\alpha,z}(\rho||\rho^{*}) = 0 \iff \rho = \rho^{*},$$

This implies $\rho \in \mathcal{R}$, thus the condition (I_1) is satisfied.

(I₂): Since the density matrix ρ can be written as $\operatorname{Re}(\rho) + i\operatorname{Im}(\rho)$, we can know that $\Lambda(\rho^*)$ is equal to the conjugate of $\Lambda(\rho)$ by using Kraus operators. Then, α -z-Rényi relative entropy satisfies the property of Data Processing Inequality (DPI), that is, for any completely positive trace-preserving (CPTP) map Λ , the inequality between Λ and ρ, σ :

$$D_{\alpha,z}(\Lambda(\rho)||\lambda(\sigma)) \leq D_{\alpha,z}(\rho,\sigma).$$

The condition (I_2) holds, just notice

$$\mathcal{M}^{R}_{\alpha,z}(\rho) = 1 - e^{(\alpha - 1)D_{\alpha,z}(\rho,\rho^*)}.$$

(I₅): set $\rho = p_1 \rho_1 \oplus p_2 \rho_2$, and verify $f_{\alpha,z}(\rho, \rho^*)$ directly, we have the following derivation:

$$f_{\alpha,z}(p_{1}\rho_{1} \oplus p_{2}\rho_{2}, p_{1}\rho_{1}^{*} \oplus p_{2}\rho_{2}^{*})$$

$$= \operatorname{Tr}\left[\bigoplus_{j=1}^{2} p_{j}^{\frac{1-\alpha}{2z}} p_{j}^{\frac{\alpha}{z}} p_{j}^{\frac{1-\alpha}{2z}} (\rho_{j}^{*})^{\frac{1-\alpha}{2z}} \rho_{j}^{\frac{\alpha}{z}} (\rho_{j}^{*})^{\frac{1-\alpha}{2z}}\right]^{z}$$

$$= \sum_{j=1}^{2} p_{j} \operatorname{Tr}\left[(\rho_{j}^{*})^{\frac{1-\alpha}{2z}} \rho_{j}^{\frac{\alpha}{z}} (\rho_{j}^{*})^{\frac{1-\alpha}{2z}}\right]^{z}$$

$$= p_{1} f_{\alpha,z}(\rho_{1}, \rho_{1}^{*}) + p_{2} f_{\alpha,z}(\rho_{2}, \rho_{2}^{*}),$$

Note that $p_1 + p_2 = 1$, we obtain:

$$\mathcal{M}_{\alpha,z}^R(p_1\rho_1\oplus p_2\rho_2)=p_1\mathcal{M}_{\alpha,z}^R(\rho_1)+p_2\mathcal{M}_{\alpha,z}^R(\rho_2),$$

Thus $\mathcal{M}^{R}_{\alpha,z}$ satisfies (I_5) .

2.2. Properties of the measure

Some axiomatic properties existed at the beginning of the definition of relative entropy [20], and they can now be applied to the imaginariyt measures.

THEOREM 2. The imaginarity measures $\mathcal{M}_{\alpha,z}^R$ and \mathcal{M}_q^T have the following properties: (1) $\mathcal{M}_{\alpha,z}^R$ and \mathcal{M}_q^T are invariant under any unitary matrix U.

(2) For any density matrix τ , we have $\mathcal{M}_{\alpha,z}^R(\rho \otimes \tau) \ge \mathcal{M}_{\alpha,z}^R(\rho)\mathcal{M}_{\alpha,z}^R(\tau)$, which means that imaginarity measure increases under the tensor product. It's the same case for \mathcal{M}_q^T .

Proof. We only prove the case for $\mathcal{M}^R_{\alpha,z}$, the case for \mathcal{M}^R_q is entirely similar. Here, we denote ρ^* as σ and τ^* as γ for easier notation.

$$f_{\alpha,z}(U\rho U^*, U\sigma U^*)$$

$$= \operatorname{Tr} \left[(U\sigma U^*)^{\frac{1-\alpha}{2z}} (U\rho U^*)^{\frac{\alpha}{z}} (U\sigma U^*)^{\frac{1-\alpha}{2z}} \right]^z$$

$$= \operatorname{Tr} \left[U(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}) U^* \right]^z$$

$$= \operatorname{Tr} \left[U(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z U^* \right]$$

$$= f_{\alpha,z}(\rho, \sigma).$$

Therefore, $\mathcal{M}^{R}_{\alpha,z}(U\rho U^{*}) = \mathcal{M}^{R}_{\alpha,z}(\rho)$ holds.

$$\begin{aligned} &f_{\alpha,z}(\rho \otimes \tau, \sigma \otimes \gamma) \\ &= \operatorname{Tr} \left[(\sigma \otimes \gamma)^{\frac{1-\alpha}{2z}} (\rho \otimes \tau)^{\frac{\alpha}{z}} (\sigma \otimes \gamma)^{\frac{1-\alpha}{2z}} \right]^z \\ &= \operatorname{Tr} \left[(\sigma^{\frac{1-\alpha}{2z}} \otimes \gamma^{\frac{1-\alpha}{2z}}) (\rho^{\frac{\alpha}{z}} \otimes \tau^{\frac{\alpha}{z}}) (\sigma^{\frac{1-\alpha}{2z}} \otimes \gamma^{\frac{1-\alpha}{2z}}) \right]^z \\ &= \operatorname{Tr} \left[(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}}) \otimes (\gamma^{\frac{1-\alpha}{2z}} \tau^{\frac{\alpha}{z}} \gamma^{\frac{1-\alpha}{2z}}) \right]^z \\ &= \operatorname{Tr} \left[(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z \otimes (\gamma^{\frac{1-\alpha}{2z}} \tau^{\frac{\alpha}{z}} \gamma^{\frac{1-\alpha}{2z}})^z \right]^z \\ &= \operatorname{Tr} \left[(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z \operatorname{Tr} (\gamma^{\frac{1-\alpha}{2z}} \tau^{\frac{\alpha}{z}} \gamma^{\frac{1-\alpha}{2z}})^z \right]^z \\ &= f_{\alpha,z}(\rho,\sigma) f_{\alpha,z}(\tau,\gamma). \end{aligned}$$

Due to $f_{\alpha,z}(\tau) \leq 1$, $1 - f_{\alpha,z}(\rho)f_{\alpha,z}(\tau) \geq 1 - f_{\alpha,z}(\tau)$ holds, the proof of property (2) is finished.

2.3. Two examples of the measure

We can not obtain an intuitive sense from abstract concepts. Here, we provide two examples of $M_{\alpha,z}^R$.

EXAMPLE 1. In this example, we take $\alpha = z$ in $\mathcal{M}^{R}_{\alpha,z}$ and set its values to be 0.3 and 0.7 respectively. We randomly generate 25 density matrices for measurements, and the results are plotted below as Fig 1.

[Author and title]

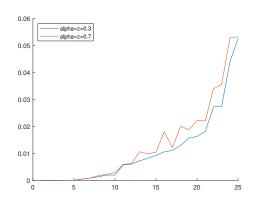


Fig. 1: $\mathcal{M}^{R}_{0,3,0,3}$ and $\mathcal{M}^{R}_{0,7,0,7}$

The sorting criterion in the figure is that the first group of measurements are sorted in ascending order, and the other groups are sorted simultaneously according to the corresponding index of this group, the same rule goes for all the following figures.

Ignoring the error introduced by precision, we can intuitively see that 25 randomly generated density matrices are non-negative under two different measurements. The values measured using $\mathcal{M}_{0.3,0.3}^R$ are always smaller than those measured using $\mathcal{M}_{0.7,0.7}^R$.

EXAMPLE 2. In this example, we fix $\alpha = 0.2$ in $\mathcal{M}^{R}_{\alpha,z}$ and set the values of z to be 0.6 and 0.8 respectively. We randomly generate 25 density matrices for measurements, and the results are plotted below as Fig 2.

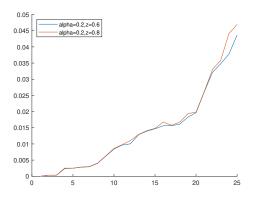


Fig. 2: $\mathcal{M}^{R}_{0.2,0.6}$ and $\mathcal{M}^{R}_{0.2,0.8}$

Ignoring the error introduced by precision, we can intuitively see that 25 randomly

generated density matrices are non-negative under two different measurements. The values measured using $\mathcal{M}_{0.2,0.6}^R$ are always smaller than those measured using $\mathcal{M}_{0.2,0.8}^R$.

From the two examples above, we can see that the measurement $\mathcal{M}^{R}_{\alpha,z}$ exhibits monotonicity with respect to its parameters. Next section, we will study this phenomenon.

3. Relationships between imaginarity measures

3.1. Monotonicity with respect to parameters

The example at the end of the previous section prompts us to study the monotonicity of the imaginarity measure with respect to its parameters. For simplicity, we denote $\mathcal{M}_{\alpha,z}^R$ as \mathcal{M}_{α}^R when $\alpha = z$. We introduce the following properties:

THEOREM 3. The imaginarity measure $\mathcal{M}^{R}_{\alpha,z}$ satisfies monotonicity with respect to its parameters:

 $\begin{array}{l} (1)\mathcal{M}_{\alpha,z}^{R} = \mathcal{M}_{1-\alpha,z}^{R}; \\ (2)\mathcal{M}_{\alpha_{1}}^{R} \leqslant \mathcal{M}_{\alpha_{2}}^{R} \quad \text{if } \alpha_{1} \leqslant \alpha_{2}; \\ (3)\mathcal{M}_{\alpha,z_{1}}^{R} \leqslant \mathcal{M}_{\alpha,z_{2}}^{R} \quad \text{if } z_{1} \leqslant z_{2}. \end{array}$

Proof. The proof is not difficult. Taking the conjugate in the trace gives (1). It is indicated in [12] that when $\alpha_1 \leq \alpha_2$, $\tilde{D}_{\alpha}(\rho || \sigma) \leq \tilde{D}_{\alpha}(\rho || \sigma)$ holds. Consider that $\alpha - 1 < 0$ and the monotonicity of the logarithmic function, we have $f_{\alpha_1,\alpha_1} \geq f_{\alpha_2,\alpha_2}$, which gives us the first conclusion. The proof of (ii) only needs to consider $\frac{z_2}{z_1} \geq 1$, which directly leads to $f_{\alpha,z_1} \geq f_{\alpha,z_2}$ by Lemma 1, thus the prove is finished.

Thus, we have explained why monotonicity appears in the example from the previous section. Additionally, we can also give similar properties for \mathcal{M}_{a}^{T} :

THEOREM 4. The imaginarity measure \mathcal{M}_q^T satisfies properties below: (1) $\mathcal{M}_q^T(\rho) = \mathcal{M}_{1-q}^T(\rho);$ (2)if $q_1 \leq q_2 \leq \frac{1}{2}$ holds, $\mathcal{M}_{q_1}^T(\rho) \leq \mathcal{M}_{q_2}^T(\rho)$ holds.

Proof. Property (1) is similar to theorem 3..(1), and it remains to prove Property (2). Notice that the research in [24] indicates $\operatorname{Tr}(\rho^q(\rho^*)^{1-q})$ is Log-convex function when $q \in (0,1)$. A Log-convex function must be convex [25], Combined with (i), we can conclude the inequation $\operatorname{Tr}(\rho^{q_1}(\rho^*)^{1-q_1}) \geq \operatorname{Tr}(\rho^{q_2}(\rho^*)^{1-q_2})$ holds, $\mathcal{M}_{q_1}^T(\rho) \leq \mathcal{M}_{q_2}^T(\rho)$ is followed.

In the following subsection, we begin to investigate the relationship between different imaginarity measures.

3.2. The magnitude relation between the two imaginarity measures

First, let's compare the relationship between $\mathcal{M}_{\alpha,z}^R$ and $\mathcal{M}_q^T(\rho)$. Since $\frac{1}{z} > 1$, we can know from Lemma 1 that:

$$\operatorname{Tr}(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}})^{z} \geq \operatorname{Tr}(\sigma^{\frac{1-\alpha}{2}}\rho^{\alpha}\sigma^{\frac{1-\alpha}{2}})^{\frac{1}{z}z}$$
$$= \operatorname{Tr}(\sigma^{\frac{1-\alpha}{2}}\rho^{\alpha}\sigma^{\frac{1-\alpha}{2}})$$
$$= \operatorname{Tr}(\rho^{\alpha}\sigma^{1-\alpha}).$$

Therefore, from the above content and in combination with Theorem 3 and Theorem 4, we can obtain the following theorem:

THEOREM 5. For any quantum state ρ , the order of imaginarity measures $\mathcal{M}_{\alpha,z}^R$, \mathcal{M}_{α}^R , and \mathcal{M}_{α}^T is as follows:

$$\mathcal{M}^R_{\alpha}(\rho) \leqslant \mathcal{M}^R_{\alpha,z}(\rho) \leqslant \mathcal{M}^T_{\alpha}(\rho).$$

This can also be seen in the following Fig 3.

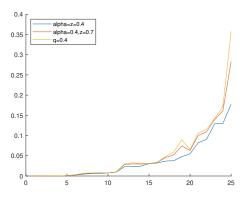


Fig. 3: $\mathcal{M}_{0.4,0.4}^R$, $\mathcal{M}_{0.4,0.7}^R$ and $\mathcal{M}_{0.4}^T$

Unfortunately, when $q \neq \alpha$, \mathcal{M}_q^T cannot be simply included in the above theorem. Let's choose an example to illustrate.

EXAMPLE 3. In this example, we take the density matrix:

$$\rho_0 = \frac{1}{10} \begin{pmatrix} 4 & 3-i \\ 3+i & 6 \end{pmatrix},$$

then the calculation shows:

$$\mathcal{M}_{0.3}^T(\rho_0) \leqslant \mathcal{M}_{0.5}^R(\rho_0) \leqslant \mathcal{M}_{0.5}^T(\rho_0),$$

this indicates that the situation is variable when $q \neq \alpha$.

REMARK 1. We codsider that the impact of multiple parameters may increase due to the matrix operations involved, which could lead to unnecessary delays, increasing consumption and propagation of errors. Since \mathcal{M}_{α}^{R} is a single-parameter quantifier and has relatively small measurement values according to the above theorem, we subjectively prefer to use \mathcal{M}_{α}^{R} .

We can also consider the combination of Theorem 2 and Theorem 4. We assert that, regardless of the relationship between α and q, the following theorem obviously holds:

THEOREM 6. For any density matrix ρ and τ , if $M^R_{\alpha,z}(\rho) \leq M^T_q(\rho)$ and $M^R_{\alpha,z}(\tau) \leq M^T_q(\tau)$ holds, then $M^R_{\alpha,z}(\rho \otimes \tau) \leq M^T_q(\rho \otimes \tau)$ holds.

The proof of the theorem is trivial, as it only requires following the definitions. Based on this, we can derive some inequalities, such as:

 $(1)M_{\alpha}^{R}(\rho \otimes \tau) \leqslant M_{\alpha}^{T}(\rho \otimes \tau);$ $(2)M_{\alpha}^{R}(\rho^{\otimes n}) \leqslant M_{\alpha}^{T}(\rho^{\otimes n}).$

Apart from the above theorems, based on the literature [12, 20] and so on, we can also introduce some other properties of $M_{\alpha,z}^R$, such as limit properties, which will not be listed here.

4. Imaginarity measure of positive definite density matrices

As mentioned in the introduction of section 1., some parameterized functions induced by quantifiers cannot be considered as a appropriate measure because they do not satisfy certain conditions or concepts in resource theory, as also suggested in other literatures [26]. In order to empower quantifiers that were previously ineffective, a new approach is presented here. We need to change the domain to positive definite density matrices.

In order to distinguish positive definite density matrices or not, we do not use ρ in this section but δ instead. Here, we present a imaginarity measure built on positive definite density matrices.

4.1. Imaginarity measure induced by Tsallis relative operator entropy

In [13], the authors introduced the Tsallis relative operator entropy, defined as follows:

$$T_{\lambda}(\delta||\eta) := \delta^{\frac{1}{2}} \ln_{\lambda}(\delta^{-\frac{1}{2}}\eta\delta^{-\frac{1}{2}})\delta^{\frac{1}{2}},$$

where δ and β are two invertible positive operators, $\ln_{\lambda} X = \frac{X^{\lambda} - I}{\lambda}$ for positive definite operator X and identity operator I with $\lambda \in (0, 1]$.

We can rewrite $T_{\lambda}(\delta||\eta)$ in another form by introducing the notation $\delta \sharp_{\lambda} \eta = \delta^{\frac{1}{2}} (\delta^{-\frac{1}{2}} \eta \delta^{-\frac{1}{2}})^{\lambda} \delta^{\frac{1}{2}}$.

$$T_{\lambda,\beta}(\delta||\eta) = \frac{1}{\lambda}(\delta \sharp_{\lambda} \eta - \delta).$$

In fact, we can observe that similar to the relationship between $\mathcal{M}^{R}_{\alpha,z}(\delta)$ and $\mathcal{M}^{V}(\delta)$, the extension of $\mathcal{M}^{T}_{a}(\delta)$ is also possible.

Now, return to the our work, literature [13] provides favorable properties of $T_{\lambda}(\delta || \eta)$, which facilitates our subsequent study.

LEMMA 2. For positive definite density matrices δ and η , any real parameter a satisfied a > 0 and $\lambda \in (0, 1]$, the following inequality holds:

$$\begin{cases} T_{\lambda}(\delta||\eta) \ge \delta \sharp_{\lambda} \eta - \frac{1}{a} \delta \sharp_{\lambda-1} \eta + (ln_{\lambda} \frac{1}{a}) \delta, \\ T_{\lambda}(\delta||\eta) \le \frac{1}{a} \eta - \delta - (ln_{\lambda} \frac{1}{a}) \delta \sharp_{\lambda} \eta, \end{cases}$$

where $\ln_{\lambda \frac{1}{a}} = \frac{(\frac{1}{a})^{\lambda} - 1}{\lambda}$. In particular, we can choose the parameters a = 1 such that, when the condition $T_{\lambda}(\delta || \eta) = 0$ holds, $\delta = \eta$ simultaneously.

The lemma ingeniously provided by the authors in this paper aims to demonstrate the equivalence between $\delta = \eta$ and the vanishing of $T_{\lambda}(\delta||\eta)$, which is precisely what we need. From [27], a useful lemma is:

LEMMA 3. If Φ is a positive linear map, then for any positive definite matrices A and B, there is:

$$\Phi(A\sharp_{\lambda}B) \leqslant \Phi(A)\sharp_{\lambda}\Phi(B)$$

Next, similar to Theorem 1, we can still verify the following theorem by proving $(I_1) + (I_2) + (I_5)$.

THEOREM 7. The parametrized function $\mathcal{M}_{\lambda}^{O}$ of the state δ is given as an imaginarity measure:

$$\mathcal{M}_{\lambda}^{O}(\delta) = 1 - Tr(\delta \sharp_{\lambda} \delta^{*}),$$

where $\lambda \in (0, 1)$.

Proof. Let us start the proof.

First of all, we can see $\operatorname{Tr}(A\sharp_{\alpha}B) \leq \operatorname{Tr}[(1-\alpha)A + \alpha B]$ in [23], thus $\mathcal{M}^{O}_{\alpha}(\delta) \geq 0$, in conjunction with the discussion in subsection above, the condition (I_{1}) is satisfied.

From Lemma 3, $\Phi(\delta^*)$ is equal to the conjugate of $\Phi(\delta)$ and the trace-preserving of Φ , we can see:

$$\operatorname{Tr}(\delta\sharp_{\alpha}\delta^{*}) = \operatorname{Tr}[\Phi(\delta\sharp_{\alpha}\delta^{*})] \leqslant \operatorname{Tr}[\Phi(\delta)\sharp_{\alpha}\Phi(\delta^{*})] = \operatorname{Tr}[\Phi(\delta)\sharp_{\alpha}\Phi(\delta)^{*}]$$

In this way, it can be known that the condition (I_2) is satisfied.

Suppose $\delta = d_1 \delta_1 \oplus d_2 \delta_2$ with $d_1 + d_2 = 1$, there will be $\delta^* = d_1 \delta_1^* \oplus d_2 \delta_2^*$, next:

$$\mathcal{M}_{\alpha}^{O}(\delta) = 1 - \operatorname{Tr}(\delta \sharp_{\alpha} \delta^{*}) = 1 - \operatorname{Tr}[(d_{1} \delta_{1} \oplus d_{2} \delta_{2})\sharp_{\alpha}(d_{1} \delta_{1}^{*} \oplus d_{2} \delta_{2}^{*})]$$

$$= 1 - \operatorname{Tr}\left\{ (d_{1} \delta_{1} \oplus d_{2} \delta_{2})^{\frac{1}{2}} \left[(d_{1} \delta_{1} \oplus d_{2} \delta_{2})^{-\frac{1}{2}} (d_{1} \delta_{1}^{*} \oplus d_{2} \delta_{2}^{*})(d_{1} \delta_{1} \oplus d_{2} \delta_{2})^{-\frac{1}{2}} \right]^{\alpha} (d_{1} \delta_{1} \oplus d_{2} \delta_{2})^{\frac{1}{2}} \right\}$$

$$= 1 - \operatorname{Tr}\left[\bigoplus_{j=1}^{2} d_{j}^{\frac{1}{2}} (d_{j}^{-\frac{1}{2}} d_{j} d_{j}^{-\frac{1}{2}})^{\alpha} d_{j}^{\frac{1}{2}} \delta_{j}^{\frac{1}{2}} (\delta_{j}^{-\frac{1}{2}} \delta_{j}^{*} \delta_{j}^{-\frac{1}{2}})^{\alpha} \delta_{j}^{\frac{1}{2}} \right]$$

$$= 1 - \operatorname{Tr}\left[\bigoplus_{j=1}^{2} d_{j} \delta_{j}^{\frac{1}{2}} (\delta_{j}^{-\frac{1}{2}} \delta_{j}^{*} \delta_{j}^{-\frac{1}{2}})^{\alpha} \delta_{j}^{\frac{1}{2}} \right]$$

$$= d_{1} \left\{ 1 - \operatorname{Tr}[\delta_{1}^{\frac{1}{2}} (\delta_{1}^{-\frac{1}{2}} \delta_{1}^{*} \delta_{1}^{-\frac{1}{2}})^{\alpha} \delta_{1}^{\frac{1}{2}}] \right\} + d_{2} \left\{ 1 - \operatorname{Tr}[\delta_{2}^{\frac{1}{2}} (\delta_{2}^{-\frac{1}{2}} \delta_{2}^{*} \delta_{2}^{-\frac{1}{2}})^{\alpha} \delta_{2}^{\frac{1}{2}}] \right\}$$

$$= d_{1} \operatorname{Tr}(\delta_{1} \sharp \delta_{1}^{*}) + d_{2} \operatorname{Tr}(\delta_{2} \sharp \delta_{2}^{*}) = d_{1} \mathcal{M}_{\alpha}^{O}(\delta_{1}) + d_{2} \mathcal{M}_{\alpha}^{O}(\delta_{2})$$

Combine all the above, we obtain the proof of the Theorem 7.

 $\mathcal{M}^{O}_{\lambda}$ also has many properties similar to $\mathcal{M}^{R}_{\alpha,z}$, which we skip here.

4.2. $\mathcal{M}^{O}_{\lambda}$ versus $\mathcal{M}^{R}_{\alpha,z}$ and $\mathcal{M}^{T}_{\lambda}$

What we need to declare is that only positive definite density matrices can be compared among the three measures at this moment.

With the help of [23], we have the inequality:

$$\operatorname{Tr}(\delta\sharp_{\lambda}\delta^{*}) \leqslant \operatorname{Tr}[\delta^{1-\lambda}(\delta^{*})^{\lambda}] = \operatorname{Tr}[\delta^{\lambda}(\delta^{*})^{1-\lambda}].$$

Therefore, we can establish a connection between $\mathcal{M}^{O}_{\lambda}$, $\mathcal{M}^{R}_{\alpha,z}$, and $\mathcal{M}^{T}_{\lambda}$.

THEOREM 8. For positive definite density matrices σ , the order of imaginarity measures $\mathcal{M}_{\alpha,z}^R$, \mathcal{M}_{α}^T and \mathcal{M}_{λ}^O is as follows:

$$\mathcal{M}^{R}_{\alpha}(\delta) \leqslant \mathcal{M}^{R}_{\alpha,z}(\delta) \leqslant \mathcal{M}^{T}_{\alpha}(\delta) \leqslant \mathcal{M}^{O}_{\alpha}(\delta).$$

As the end of this subsection, we intuitively present a comparison as Fig 4 obtained after randomly selecting 25 positive definite density matrices in.

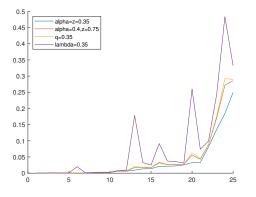


Fig. 4: $\mathcal{M}_{0.35,0.35}^R$, $\mathcal{M}_{0.35,0.75}^R$, $\mathcal{M}_{0.35}^T$ and $\mathcal{M}_{0.35}^O$

However, please attention, when $\lambda \neq \alpha$, the situation becomes more complicated, See the following example.

EXAMPLE 4. Take the positive definite density matrix δ_0 :

$$\delta_0 = \frac{1}{10} \begin{pmatrix} 6 & 1+i\\ 1+i & 4 \end{pmatrix},$$

the result of the calculation shows:

$$\mathcal{M}_{0.3}^O(\delta_0) \leqslant \mathcal{M}_{0.5}^R(\delta_0) \leqslant \mathcal{M}_{0.5}^T(\delta_0)$$

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REMARK 2. From the special case above, we can see that in the case of positive definite density matrices, $\mathcal{M}_{\lambda}^{O}$ may have lower measurements than general imaginarity measures. So, we need to consider comprehensively when choosing the appropriate measure.

5. Conclusion

In conclusion, we present a new definition of imaginarity measure $\mathcal{M}_{\alpha,z}^R$ which is shown to be an appropriate imaginarity measure, and some of its properties are also given. We also supplement some properties of \mathcal{M}_q^T , and since the similarity between \mathcal{M}_q^T and $\mathcal{M}_{\alpha,z}^R$, we compare them. An imaginarity measure \mathcal{M}_{λ}^O that only applies in positive definite density matrices was introduced, under the assumption of a positive definite density matrices, we present relations between $\mathcal{M}_{\alpha,z}^R$, \mathcal{M}_{λ}^O and \mathcal{M}_q^T . Remarks in this paper indicate that our idea is there should be different choices in different situations.

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