# Finite modular symmetries and the strong CP problem 

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#### Abstract

Recently, it was shown that modular symmetry may solve the strong CP problem without axions, by producing a vanishing QCD angle while generating a large quark CP violation phase. We extend this framework to finite modular groups, systematically identifying the allowed mass textures. We find quark fields must furnish 1D representations and scan the minimal model landscape.


## I. INTRODUCTION

In an effort to emulate the undeniable success of the gauge principle in the Standard Model (SM), flavour symmetries have been put forward to explain the puzzling observed structures of masses and mixing in the quark and lepton sectors of the theory [1.8. The curious avenue of finite modular flavour symmetries [9] has attracted attention in the last years, due in part to the promising possibility of shaping flavour patterns without the need to invoke large numbers of exotic scalar fields, with aligned vacuum expectation values (VEVs). Modular flavour models allow one to determine fermion masses, mixing angles and Dirac and Majorana CP violation (CPV) phases in terms of a limited number of parameters; generate fermion mass hierarchies [10-15]; and provide a unified source of generalized CP (gCP) and flavour symmetry breaking [16, [17, to name key features of the framework (see [18, 19] for recent reviews).

More recently, it has been shown [20] that modular symmetries can play a role in unveiling yet another puzzle of the SM - the strong CP problem - related to the non-observation of an electric dipole moment for the neutron. Solutions to this conundrum usually fall into two categories: i) those promoting the relevant QCD angle $\bar{\theta}$ to a dynamical field - the axion [21-24] (see [25] for a review) ${ }^{1}$ and ii) those breaking an assumed CP symmetry in a way that produces a large CPV phase $\delta$ in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix while simultaneously ensuring $\bar{\theta}=0$. In what follows, we will focus on the latter possibility. Here,

$$
\begin{equation*}
\bar{\theta}=\theta_{\mathrm{QCD}}+\arg \operatorname{det} M_{u} M_{d} \tag{1}
\end{equation*}
$$

where $\theta_{\mathrm{QCD}}$ is the coefficient of the topological term in the QCD Lagrangian, while $M_{q}(q=u, d)$ is the $q$-type quark mass matrix. The assumed (g)CP symmetry of the theory can provide $\theta_{\mathrm{QCD}}=0$. The requirement on the flavour model is then to generate a viable $\delta$ from $M_{u}$ and $M_{d}$ while keeping the product of their determinants real. This could be achieved in models with vector-like quarks

[^0](see e.g. [27]), most famously via Nelson-Barr constructions [28, 29]. We instead follow the minimal approach put forward in Ref. [20], wherein modular symmetry is elegantly responsible for the necessary flavour textures, and extend this approach to the finite modular groups.

## II. SETUP

Consider the typical $\mathcal{N}=1$ global SUSY modular framework with a gCP symmetry [17] (see e.g. 30 32 for further details) ${ }^{2}$ The theory is taken to be invariant under the whole modular group $S L(2, \mathbb{Z})$. Matter fields $\psi$ instead transform in representations of a finite modular group of a certain level $N$ - which plays the role of a flavour symmetry - and are assigned modular weights $k_{\psi}$. The inhomogeneous finite groups $\Gamma_{N}$ are isomorphic to the well-known permutation groups $S_{3}$ [33, 34, $A_{4}$ [9, 35, 36, $S_{4}$ [10, 30, 31 and $A_{5}$ [10, 37, while the homogeneous finite groups $\Gamma_{N}^{\prime}$ are isomorphic to "double covers" of these groups [32, 38,41]. One can hence focus on $\Gamma_{N}^{\prime} \supseteq \Gamma_{N}$ without loss of generality.

We are interested in simple scenarios, where Yukawa and mass matrices depend only on modular forms of level $N$ - known functions of a single complex scalar field $\tau$ (the modulus) - and on a small number of coupling constants in the holomorphic superpotential $W$. The flavour structure is determined once $\tau$ acquires a VEV, which breaks both modular and CP symmetries. For simplicity, we consider a minimal-form Kähler potential,

$$
\begin{equation*}
K=-\Lambda_{K}^{2} \log (2 \operatorname{Im} \tau)+\sum_{\psi}|\psi|^{2}(2 \operatorname{Im} \tau)^{-k_{\psi}} \tag{2}
\end{equation*}
$$

where $\Lambda_{K}$ has mass dimension one. While not the most general choice 42, 43], it does not hinder the envisioned solution to the strong CP problem, as $\bar{\theta}$ is holomorphic and thus constrained by modular invariance, being insensitive to $K$, as shown in [20. We further take the MSSM Higgs doublets $H_{u, d}$ to transform trivially under the modular group.

[^1]The quark doublets $Q$ and quark singlets $u^{c}$ and $d^{c}$ furnish 3-dimensional representations of $\Gamma_{N}^{\prime}$, which may be reducible in general. Under a modular transformation $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in S L(2, \mathbb{Z})$, one has

$$
\begin{align*}
& Q_{i} \xrightarrow{\gamma}(c \tau+d)^{-k_{i}} \rho_{i j}(\gamma) Q_{j},  \tag{3}\\
& q_{i}^{c} \xrightarrow{\gamma}(c \tau+d)^{-k_{i}^{c}} \rho_{i j}^{c}(\gamma) q_{j}^{c},
\end{align*}
$$

for a given sector $q=u, d$. In the reducible case, the unitary representation matrices $\rho$ and $\rho^{c}$ are block diagonal.

It can be shown that the determinant of the quark mass matrix $M_{q}$ is a singlet (i.e. 1D) modular form of weight $k_{\text {det }}^{q}$, with $k_{\text {det }}^{q} \equiv \sum_{i} k_{i}+k_{i}^{c}$ (see Appendix A for explicit derivations). Immediately, it becomes clear that if $k_{\text {det }}^{q}<$ 0 , then $\operatorname{det} M_{q}=0$, as there are no non-zero negativeweight modular forms. In this study, we reject quark mass matrices with zero determinant, as they imply a massless quark ${ }^{3}$ The condition $\operatorname{det} M_{q} \neq 0$ then implies that both $k_{\text {det }}^{u} \geq 0$ and $k_{\text {det }}^{d} \geq 0$.

To implement the solution to the strong CP problem proposed in Ref. [20, one requires the cancellation of the QCD anomaly $A$ of the modular symmetry, which in our case simply reads $4^{4}$

$$
\begin{equation*}
A=k_{\mathrm{det}}^{u}+k_{\mathrm{det}}^{d} \stackrel{!}{=} 0 \tag{4}
\end{equation*}
$$

The vanishing of $A$ guarantees that $\operatorname{det} M_{u} \operatorname{det} M_{d}$ is a modular form of zero weight, hence a constant, independent of the only source of CP breaking, namely the VEV of $\tau$. Moreover, it is a real constant, given the assumed gCP invariance of the theory. Given the above, the vanishing of $A$ implies the separate vanishing of the modular weights of the determinants of both quark mass matrices,

$$
\begin{equation*}
k_{\mathrm{det}}^{u} \stackrel{!}{=} 0, \quad k_{\mathrm{det}}^{d} \stackrel{!}{=} 0 \tag{5}
\end{equation*}
$$

In the next section, we will look into the consequences of this restriction.

Note that these results already show that this mechanism of solving the strong CP problem is incompatible with the mechanism of generating fermion mass hierarchies put forward in Ref. [13]. This is because the latter is based on suppressing entries of fermion mass matrices via powers of a small $\tau$-dependent quantity, $\epsilon(\tau)$. The determinant of each mass matrix is then inevitably proportional to some power of $\epsilon(\tau)$ and is necessarily $\tau$ dependent, carrying a non-zero modular weight, in contradiction with Eq. (5).

[^2]
## III. ALLOWED TEXTURES

## A. Single sector

Asking for vanishing determinant weights while allowing for non-zero determinants is a rather restrictive requirement. We analyse this constraint systematically by first focusing on a single quark sector $q$. In Section IIIB, we discuss the compatibility of different structures for $M_{u}$ and $M_{d}$.

We start by associating each entry of the relevant mass matrix $M_{q}$ in the left-right (LR) convention to a weight $k_{i j}^{q} \equiv k_{i}+k_{j}^{q^{c}}$, according to

$$
M:\left(\begin{array}{lll}
k_{11} & k_{12} & k_{13}  \tag{6}\\
k_{21} & k_{22} & k_{23} \\
k_{31} & k_{32} & k_{33}
\end{array}\right)
$$

where we have omitted the label $q=u, d$ for readability. We then have

$$
\begin{equation*}
k_{\mathrm{det}}=k_{11}+k_{22}+k_{33}=k_{12}+k_{23}+k_{31}=\ldots \stackrel{!}{=} 0 \tag{7}
\end{equation*}
$$

Note further that if $k_{i j}<0$, then the corresponding mass matrix element $M_{i j}$ vanishes, since there are no modular forms of negative weight, and therefore no modular forms available to produce a modular invariant which can contribute to this entry.

The trivial solution to Eq. (7) is to have all $k_{i j}=0$. Then, given the assumed gCP symmetry, broken only by the VEV of $\tau$, CP would be conserved in the quark sector as the elements of $M$ would be $\tau$-independent CPconserving constants. At the same time, however, we must generate a non-trivial CPV CKM phase $\delta$. Therefore, some $k_{i j}$ must be non-zero. The non-zero ones cannot all be positive, otherwise there is no way of satisfying Eq. (7). Hence, a way to progress methodically is to analyse one by one the cases with a definite number of negative weights $k_{i j}$.

1 negative $k_{i j}$. We are free to permute representations and thus rows and columns such that the negative element is $k_{11} 5$ Now, since $k_{\text {det }}=k_{13}+k_{22}+k_{31}=$ $k_{12}+k_{21}+k_{33} \stackrel{!}{=} 0$ and none of these quantities can be negative, one has $k_{22}=k_{33}=0$. However, since also $k_{11}+k_{22}+k_{33} \stackrel{!}{=} 0$, we find in particular that $k_{11}=0$, in contradiction with our original assumption.

2 negative $k_{i j}$. If the negative-weight entries do not share a row or column, we can choose $k_{11}, k_{22}<0$ without loss of generality. Then, one is forced to have $k_{33}>0$.

[^3]However, $k_{12}+k_{21}+k_{33} \stackrel{!}{=} 0$ implies negative $k_{12}$ or $k_{21}$, in contradiction with our original assumption.

Instead, one may have the negative-weight entries sharing a row or column. Without loss of generality, we can choose $k_{11}, k_{12}<0 .{ }^{6}$ Then, since $k_{\text {det }}=k_{13}+k_{22}+k_{31}=$ $k_{13}+k_{21}+k_{32} \stackrel{!}{=} 0$ and none of these quantities can be negative, we have $k_{13}=k_{21}=k_{22}=k_{31}=k_{32}=0$. Then, $k_{11}+k_{22}+k_{33} \stackrel{!}{=} 0$ implies $k \equiv k_{11}=-k_{33}$, while $k_{12}+k_{21}+k_{33} \stackrel{!}{=} 0$ leads to $k_{12}=-k_{33}=k$. Finally, $k_{11}+k_{23}+k_{32} \stackrel{!}{=} 0$ fixes $k_{23}=-k_{11}=-k$, so that we are left with the viable (and novel) mass matrix weight structure

$$
M:\left(\begin{array}{ccc}
k & k & 0  \tag{8}\\
0 & 0 & -k \\
0 & 0 & -k
\end{array}\right), \quad k<0
$$

up to permutations of rows and columns and up to transposition. We can do away with the transposition ambiguity by simply allowing for positive values of $k$.

Equation (8) implies the mass matrix has the form

$$
M=v_{q}\left(\begin{array}{ccc}
0 & 0 & \alpha_{13}  \tag{9}\\
\alpha_{21} & \alpha_{22} & \alpha_{23} \mathcal{Y}_{1}^{(|k|)} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} \mathcal{Y}_{2}^{(|k|)}
\end{array}\right),
$$

also up to permutations and transposition. Here, $v_{q}$ is the VEV of $H_{q}, \alpha_{i j}$ are real CP-conserving constants (under our working assumption that the only source of CPV is the VEV of $\tau$ ), and the $\mathcal{Y}_{i}$ are obtained from modular forms of weight $|k|$. The determinant of the mass matrix in Eq. (9) is given by

$$
\begin{equation*}
\operatorname{det} M=v_{q}^{3} \alpha_{13}\left(\alpha_{21} \alpha_{32}-\alpha_{22} \alpha_{31}\right) \tag{10}
\end{equation*}
$$

It is non-zero in general, as well as $\tau$-independent, as anticipated. The form of $M_{q}$ in Eq. (9) is directly obtained, for example, if $Q_{i}$ and $q_{i}^{c}$ are trivial singlets of the finite modular group $\Gamma_{N}^{\prime}$, with $\rho_{i j}=\rho_{i j}^{c}=\delta_{i j}$ in Eq. (3), and if trivial-singlet modular forms are available at weight $|k|$. In the presence of other representations, additional constraints are imposed on the $\alpha_{i j}$.

3 negative $k_{i j}$. There are four possibilities to consider, up to permutations and transposition. Either i) all negative-weight entries share a row or column, ii) none of these entries share a row or column, iii) exactly one pair of these entries shares a row or column, or iv) exactly two pairs of these entries shares a row or column.

We exclude case i), since in this case $M$ has a vanishing row or column, leading to a massless quark. In case ii), we can take $k_{11}, k_{22}, k_{33}<0$ without loss of generality.

[^4]Immediately, we find $k_{\text {det }}=k_{11}+k_{22}+k_{33}<0$, excluding this scenario. In case iii), we can take $k_{11}, k_{21}, k_{32}<0$ without loss of generality. Now, since $k_{\text {det }}=k_{13}+k_{22}+$ $k_{31} \stackrel{!}{=} 0$ and none of these quantities can be negative, one has $k_{13}=0$. This leads to a contradiction, as it implies $k_{\text {det }}=k_{13}+k_{21}+k_{32}<0$, excluding this scenario.

The only viable option is case iv), for which we can take $k_{11}, k_{12}, k_{21}<0$ without loss of generality. Then, we have $k_{13}+k_{22}+k_{31}=\ldots \stackrel{!}{=} 0$ and the following relations:

$$
\begin{align*}
& k_{13}=k_{22}=k_{31}=0 \\
& k_{21}=-k_{32} \equiv k \\
& k_{12}=-k_{23} \equiv k^{\prime}  \tag{11}\\
& k_{12}+k_{21}=-k_{33}=k_{11}=k+k^{\prime}
\end{align*}
$$

These relations define, up to permutations of rows and columns, a viable family of (weight) structures

$$
M:\left(\begin{array}{ccc}
k+k^{\prime} & k^{\prime} & 0  \tag{12}\\
k & 0 & -k^{\prime} \\
0 & -k & -k-k^{\prime}
\end{array}\right), \quad k, k^{\prime}<0
$$

corresponding to a mass matrix of the form

$$
M=v_{q}\left(\begin{array}{ccc}
0 & 0 & \alpha_{13}  \tag{13}\\
0 & \alpha_{22} & \alpha_{23} \mathcal{Y}_{1}^{\left(\left|k^{\prime}\right|\right)} \\
\alpha_{31} & \alpha_{32} \mathcal{Y}_{2}^{(|k|)} & \alpha_{33} \mathcal{Y}_{3}^{\left(\left|k+k^{\prime}\right|\right)}
\end{array}\right)
$$

Also here, the $\alpha_{i j}$ are real CP-conserving constants, while the $\mathcal{Y}_{i}^{(w)}$ are necessarily obtained from singlet (1D) modular forms of weight $w$ (see also Section III C). In this case, one has

$$
\begin{equation*}
\operatorname{det} M=-v_{q}^{3} \alpha_{13} \alpha_{22} \alpha_{31} \tag{14}
\end{equation*}
$$

which is $\tau$-independent and does not vanish in general. As before, if both quark fields and the available modular forms furnish trivial singlets of $\Gamma_{N}^{\prime}$, the form in Eq. (13) directly follows. Instead, in the presence of other representations or in the absence of modular forms of the requisite weight, one or several of the $\alpha_{i j}$ may vanish.

Note that, given the permutation freedom, we can relax the assumption of negative $k, k^{\prime}$ and take $k, k^{\prime} \in \mathbb{R}$, with the caveat that at least one of them must be nonzero. If exactly one of them vanishes or if $k=-k^{\prime}$, one can check that this structure reduces to that of Eq. (8). The textures considered in Ref. [20] are contained in the family of structures described by Eqs. (12) and 13 with non-zero $k, k^{\prime}$ and $k+k^{\prime}$, while the example explored numerically therein corresponds to a subcase with $k=k^{\prime}$.

4 negative $k_{i j}$. Since in this case some pair of negativeweight entries must share either a row or column, we take $k_{11}, k_{12}<0$ without loss of generality. Then, $k_{13} \geq 0$ to avoid a massless quark. A third negative-weight entry may either i) share a column with $k_{11}$ or $k_{12}$, say $k_{21}<0$ without loss of generality, or ii) be in the third column.

In case i), $k_{22}, k_{31} \geq 0$ to avoid a massless quark, and in fact, due to $k_{13}+k_{22}+k_{31} \stackrel{!}{=} 0$, we have $k_{13}=k_{22}=$ $k_{31}=0$. Then, no matter where we choose to place the fourth negative-weight entry $\left(k_{23}, k_{33}\right.$ or $\left.k_{32}\right)$, we will run into a conflict with the condition of vanishing determinant, since either $k_{12}+k_{23}+k_{31}, k_{12}+k_{21}+k_{33}$ or $k_{13}+k_{21}+k_{32}$ will be negative.

Case ii) is similarly excluded. It corresponds to $k_{23}, k_{33}<0$, since any other choice is covered by case i). Again, due to $k_{13}+k_{22}+k_{31} \stackrel{!}{=} 0$, we have $k_{13}=$ $k_{22}=k_{31}=0$. This immediately conflicts with the condition $k_{11}+k_{22}+k_{33} \stackrel{!}{=} 0$ of vanishing determinant.

5 (or more) negative $k_{i j}$. In this case, the reasoning follows that of case i) for 4 negative $k_{i j}$. Without loss of generality, one has $k_{11}, k_{12}, k_{21}<0$ and $k_{13}=k_{22}=$ $k_{31}=0$. For the same reasons as above, it is not possible to place the fourth negative-weight entry without running into a contradiction. This reasoning excludes any higher number of negative-weight entries.

6 negative $k_{i j}$. An alternative way to discard this scenario follows. To avoid a massless quark, the entries with non-negative weight cannot share any row or column. Then, one can take the off-diagonal entries to be the negative-weight ones without loss of generality. Then $k_{12}+k_{23}+k_{31}<0$ conflicts with the condition of vanishing determinant weight.

7 or more negative $k_{i j}$. An alternative way to discard these cases: $M$ has too many vanishing entries and a massless quark is unavoidable.

## B. Both sectors

We have seen that, for three families, the viable weight textures in a certain quark sector (up or down) are summarized by Eq. 12 , but with real $k, k^{\prime}$ and $\left(k, k^{\prime}\right) \neq$ $(0,0)$. Let us now discuss the compatibility between possible $M_{u}$ and $M_{d}$ textures. The two quark sectors are connected by the doublets $Q_{i}$ of field weight $k_{i}$. One has $k_{i j}^{q}=k_{i}+k_{j}^{q^{c}}(q=u, d)$, which implies that

$$
\begin{equation*}
k_{i k}^{u}-k_{j k}^{u}=k_{i k}^{d}-k_{j k}^{d}=k_{i}-k_{j}, \tag{15}
\end{equation*}
$$

i.e. the differences $k_{i k}^{q}-k_{j k}^{q}$ are $q$-independent. Therefore, suppose that we fix

$$
M_{u}:\left(\begin{array}{ccc}
k_{u}+k_{u}^{\prime} & k_{u}^{\prime} & 0  \tag{16}\\
k_{u} & 0 & -k_{u}^{\prime} \\
0 & -k_{u} & -k_{u}-k_{u}^{\prime}
\end{array}\right)
$$

in the LR convention, using up our permutation freedom of the $Q_{i}$ and $u_{i}^{c}$. The only remaining freedom is in permuting the columns of $M_{d}$, which will however not
affect quark masses or standard (left-handed) quark mixing. Then, using the $q$-independence of the differences in Eq. (15), one constrains the weight structure in the down-quark sector to

$$
M_{d}:\left(\begin{array}{ccc}
a+k_{u}^{\prime} & b+k_{u}^{\prime} & c+k_{u}^{\prime}  \tag{17}\\
a & b & c \\
a-k_{u} & b-k_{u} & c-k_{u}
\end{array}\right)
$$

with $a, b, c$ as yet undetermined. This structure must match some permutation of the most general viable one in Eq. 12. This means that each of the rows in Eq. (17) will contain at least one vanishing entry. We can set $c+k_{u}^{\prime}=0, b=0$ and $a-k_{u}=0$ without loss of generality, since we are still free to permute the columns. We thus find that $M_{d}$ necessarily has the same weight structure as $M_{u}$ in Eq. 16, i.e.

$$
\begin{align*}
& M_{u}:\left(\begin{array}{ccc}
k+k^{\prime} & k^{\prime} & 0 \\
k & 0 & -k^{\prime} \\
0 & -k & -k-k^{\prime}
\end{array}\right),  \tag{18}\\
& M_{d}:\left(\begin{array}{ccc}
k+k^{\prime} & k^{\prime} & 0 \\
k & 0 & -k^{\prime} \\
0 & -k & -k-k^{\prime}
\end{array}\right)
\end{align*}
$$

with shared $k, k^{\prime}$ (real, at least one non-zero).
The structures of $M_{u}$ and $M_{d}$ in Eq. 18) correspond to the most general result. They are valid up weak basis changes, i.e. up to sector-independent permutations of columns and up to simultaneous permutations of rows in both sectors. Importantly, all such permutations are physically inconsequential. Indeed, in the LR convention we use, we have the singular value decomposition $M_{q}=V_{L}^{q} D_{q} V_{R}^{q \dagger}$, where $V_{L, R}^{q}$ are unitary matrices and $D_{q}$ are diagonal matrices of quark masses. Note that the unitary matrices $V_{L}^{q}$ diagonalize $M_{q} M_{q}^{\dagger}$, whose eigenvalues are the squares of the quark masses. The quark mixing matrix is given by $U_{\mathrm{CKM}}=V_{L}^{u \dagger} V_{L}^{d}$. The permutation of columns $i$ and $j$ of $M_{q}$ can be realized by a real orthogonal matrix $P_{i j}$ acting on $M_{q}$ on the right: $M_{q} \rightarrow M_{q} P_{i j}$. However, the products $M_{q} M_{q}^{\dagger}$ determining $V_{L}^{q}$ (and thus $\left.U_{\mathrm{CKM}}\right)$ clearly do not depend on $P_{i j}$. The simultaneous permutation of the $i$ and $j$ rows of $M_{u}$ and $M_{d}$ is instead realized by the same matrix $P_{i j}$ acting on both $M_{u}$ and $M_{d}$ on the left: $M_{q} \rightarrow P_{i j} M_{q}$. However, $M_{q} M_{q}^{\dagger}$ and $P_{i j} M_{q} M_{q}^{\dagger} P_{i j}^{T}$ have the same eigenvalues, and both $V_{L}^{u}$ and $V_{L}^{d}$ are modified by the additional factor $P_{i j}$ on the left, $V_{L}^{q} \rightarrow P_{i j} V_{L}^{q}$, which cancels in the expression for $U_{\text {CKM }}$.

## C. Compatibility with field representations

The structures summarized in Eq. (18) can of course be realized by appropriately choosing the weights of quark fields furnishing only 1 D representations of $\Gamma_{N}^{\prime}$,
i.e. $Q, u^{c}, d^{c} \sim \mathbf{1}^{*} \oplus \mathbf{1}^{*} \oplus \mathbf{1}^{*} \cdot 7$ with all irreducible representations (irreps) a priori unrelated, apart from the requirement of having enough invariants to populate the mass matrix and avoid massless quarks (see also Section V). The presence of irreducible representations of dimension higher than 1 translates into equating the weights $k_{i j}$ within some blocks of the mass matrices (see also Footnote 5. For instance, if $Q \sim \mathbf{3}^{*}$, then $k_{1 j}^{q} \stackrel{!}{=} k_{2 j}^{q} \stackrel{!}{=} k_{3 j}^{q}$ $(j=1,2,3)$. At once we see that this is incompatible with the general result in Eq. 18, since it would require $k=k^{\prime}=0$. The same reasoning excludes any isosinglet triplet representations $u^{c}, d^{c} \sim 3^{*}$. Therefore, avoiding the all-singlet case, one concludes that a doublet irrep must be present. The associated constraint of equal weights will imply that one of the following happens, independently of whether $Q, u^{c}$ or $d^{c}$ furnishes $\mathbf{2}^{*} \oplus \mathbf{1}^{*}$ : either $k=0, k^{\prime}=0$ or $k+k^{\prime}=0$. In any case, we will be in the situation of Eq. (8). More explicitly, one has

$$
M_{u}:\left(\begin{array}{cc|c}
0 & 0 & k  \tag{19}\\
0 & 0 & k \\
\hline-k & -k & 0
\end{array}\right), \quad M_{d}:\left(\begin{array}{cc|c}
0 & 0 & k \\
0 & 0 & k \\
\hline-k & -k & 0
\end{array}\right)
$$

with real non-zero $k$, up to (physically irrelevant) sectorindependent permutations of column blocks and simultaneous permutations of row blocks. Note that one cannot rely on non-trivial modular forms to populate the zeroweight $2 \times 2$ block. It follows that if there is a flavour doublet among the isodoublet fields, one must have the conjugate doublet in the isosinglet fields (and vice versa) as otherwise there is no way of producing an invariant in that block, leading to a massless quark. Therefore, the only potentially viable case not involving only 1D irreps is $Q, u^{c}, d^{c} \sim \mathbf{2}^{*} \oplus \mathbf{1}^{*}$, with the weight structure given in Eq. 19. We assess its viability in the following section. Potentially viable cases involving only 1D irreps, taking Eq. 18) as a starting point, are analyzed in Section $\nabla$.

## IV. THE 2* $\oplus \mathbf{1}^{*}$ CASE

As explained in Section IIIC, the only potentially viable scenario within this setup that does not rely on using only 1-dimensional irreps corresponds to having all three $Q, u^{c}, d^{c} \sim \mathbf{2}^{*} \oplus \mathbf{1}^{*}$. Accordingly, we denote $Q=Q_{2} \oplus Q_{1}$, $u^{c}=u_{2}^{c} \oplus u_{1}^{c}$ and $d^{c}=d_{2}^{c} \oplus d_{1}^{c}$. Then, to avoid a massless quark, we require $\mathbf{r}_{Q_{2}} \otimes \mathbf{r}_{u_{2}^{c}} \supset \mathbf{1}$ and $\mathbf{r}_{Q_{1}} \otimes \mathbf{r}_{u_{1}^{c}}=1$. In other words, $\mathbf{r}_{u_{2}^{c}}=\overline{\mathbf{r}_{Q_{2}}}$ and $\mathbf{r}_{u_{1}^{c}}=\overline{\mathbf{r}_{Q_{1}}}$, where the bar denotes the conjugate representation. The analogous consideration applies to the down sector. In short, we are

[^5]| Weight $w$ | $S_{3}$ | $A_{4}^{\prime}$ | $S_{4}^{\prime}$ | $A_{5}^{\prime}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | - | $\hat{2}$ | - | - |
| 2 | 2 | - | 2 | - |
| 3 | - | $\hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime \prime}$ | - | - |
| 4 | 2 | - | 2 | - |
| 5 | - | $\hat{\mathbf{2}}, \hat{\mathbf{\Sigma}}^{\prime}, \hat{\mathbf{2}}^{\prime \prime}$ | $\hat{2}$ | $\hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}$ |
| 6 | 2 | - | 2 | - |
| 7 | - | $\hat{\mathbf{2}}, \hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}, \hat{\mathbf{2}}^{\prime \prime}$ | $\hat{2}$ | $\hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}$ |
| 8 | 2, 2 |  | 2, 2 |  |
| 9 | - | $\hat{\mathbf{2}}, \hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}, \hat{\mathbf{2}}^{\prime \prime}, \hat{\mathbf{\Sigma}}^{\prime \prime}$ | $\hat{2}$ | $\hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}$ |
| 10 | 2, 2 | - | 2, 2 |  |
| 11 | - | $\hat{\mathbf{2}}, \hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}, \hat{\mathbf{2}}^{\prime}, \hat{\mathbf{2}}^{\prime \prime}, \hat{\mathbf{2}}^{\prime \prime}$ | $\hat{2}, \hat{2}$ | $\hat{\mathbf{2}}, \hat{\mathbf{2}}, \hat{\mathbf{2}}^{\prime}, \hat{\mathbf{2}}^{\prime}$ |

TABLE I. Doublet modular form irreps available at a given weight $w \leq 11$ for the $\Gamma_{N}^{\prime}$ groups ( $N \leq 5$ ). Hatted (unhatted) irreps are in correspondence with odd (even) weights.
constrained to the assignments (cf. Eq. 19p):

$$
\begin{align*}
Q & \sim\left(\mathbf{2}_{Q}, k_{2}\right) \oplus\left(\mathbf{1}_{Q}, k_{1}\right), \\
u^{c} & \sim\left(\overline{\mathbf{2}_{Q}},-k_{2}\right) \oplus\left(\overline{\mathbf{1}_{Q}},-k_{1}\right),  \tag{20}\\
d^{c} & \sim\left(\overline{\mathbf{2}_{Q}},-k_{2}\right) \oplus\left(\overline{\mathbf{1}_{Q}},-k_{1}\right),
\end{align*}
$$

where we have defined $\mathbf{2}_{Q} \equiv \mathbf{r}_{Q_{2}}, \mathbf{1}_{Q} \equiv \mathbf{r}_{Q_{1}}$. Moreover, we have $k=k_{2}-k_{1}$ and must have $k_{1} \neq k_{2}$, since $k \neq 0$ is needed to generate a non-trivial CKM phase. Finally, only two entries of the mass matrix - those corresponding to weight $|k|$ - will be populated by modular forms. These forms are necessarily (one or more) doublets furnishing a shared representation, $Y \sim\left(\mathbf{2}_{Y},|k|\right)$. One has

$$
\mathbf{2}_{Y}= \begin{cases}\overline{\mathbf{2}_{Q}} \otimes \mathbf{1}_{Q}, & \text { if } k>0  \tag{21}\\ \mathbf{2}_{Q} \otimes \overline{\mathbf{1}_{Q}}, & \text { if } k<0\end{cases}
$$

In Table $\square$ we collect the available doublet modular form irreps for each of the finite modular groups $\Gamma_{N}^{\prime}(N \leq 5)$, up to weight $w=11$. Beyond this weight, one is sure to have at least two independent modular forms $Y_{\mathbf{r}, j}^{(w)}$ $(j=1,2, \ldots)$ furnishing each possible doublet irrep r.

Going over each of the finite modular groups $\Gamma_{2}^{(\prime)} \simeq S_{3}$, $\Gamma_{3}^{\prime} \simeq A_{4}, \Gamma_{4}^{\prime} \simeq S_{4}$, and $\Gamma_{5}^{\prime} \simeq A_{5}$, one finds only two inequivalent families of textures are allowed. Namely, for a given non-zero integer $k$ and doublet irrep $\mathbf{2}_{Y}$ available at weight $|k|$ (see Table $\rrbracket$ ), the viable mass matrices read $(q=u, d)$

$$
M_{q}=\left(\begin{array}{ccc}
\alpha_{1}^{q} & 0 & 0  \tag{22}\\
0 & \alpha_{1}^{q} & 0 \\
0 & 0 & \alpha_{2}^{q}
\end{array}\right)+\alpha_{3}^{q} \begin{cases}\left(\begin{array}{ccc}
0 & 0 & \mathcal{Y}_{1}^{q} \\
0 & 0 & \mathcal{Y}_{2}^{q} \\
0 & 0 & 0
\end{array}\right), & \text { if } k>0 \\
\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\mathcal{Y}_{1}^{q} & \mathcal{Y}_{2}^{q} & 0
\end{array}\right), & \text { if } k<0\end{cases}
$$

in an appropriate weak basis, with

$$
\begin{equation*}
\mathcal{Y}_{i}^{q} \equiv \sum_{j=1} g_{j}^{q}\left(Y_{\mathbf{2}_{Y}, j}^{(|k|)}\right)_{i} \quad \text { and } \quad g_{j}^{q} \equiv \frac{\alpha_{j+2}^{q}}{\alpha_{3}^{q}} \tag{23}
\end{equation*}
$$

Here, the $\alpha_{i}^{q}$ are real superpotential parameters, given the imposed gCP symmetry, while the sum in $j$ goes over all available independent modular forms of the selected weight and representation. Note the correspondence with the structure in Eq. (9), following a row permutation (and transposition, if $k<0$ ). As anticipated, the presence of doublet representations leads to constraints on the $\alpha_{i j}$, which are dictated by tensor products. The form in Eq. 22 may not follow immediately from the ClebschGordan coefficients used in the literature, but is always reachable via field redefinitions (weak basis transformations).

Finally, Eq. (22) must be modified to include the effects of canonical normalization. Given the assumed minimalform Kähler of Eq. (2), bringing the kinetic terms to a standard form leads to a renormalization of the quark fields. As a result, the couplings multiplying the two components of the doublet modular forms are modified as follows:

$$
\begin{equation*}
\alpha_{i}^{q} \rightarrow \tilde{\alpha}_{i}^{q}=\alpha_{i}^{q}(2 \operatorname{Im} \tau)^{|k| / 2} \quad(i \geq 3) \tag{24}
\end{equation*}
$$

Correspondingly, one has $\alpha_{3}^{q} \rightarrow \tilde{\alpha}_{3}^{q}$ in Eq. 22, while keeping the ratios $g_{j}^{q}$ and thus the $\mathcal{Y}_{i}^{q}$ unchanged.

The found $\mathbf{2}^{*} \oplus \mathbf{1}^{*}$ textures have a chance of being highly predictive. Aside from the VEV of the modulus, the mass matrix $M_{q}$ in each of the two sectors depends only on three real constants, in the minimal case where a single modular form is present in the sum of Eq. (23). Nevertheless, one must be able to obtain a viable quark CPV phase $\delta$. Recall that values of the modulus lying on the border of the fundamental domain $\mathcal{D}$ of the modular group and on the line $\operatorname{Re} \tau=0$ conserve CP [17], while all other values of $\tau$ in $\mathcal{D}$ violate CP. The latter lead to complex, CP-violating modular forms in the quark mass matrices. Writing the components of the $\mathcal{Y}^{q}$ as

$$
\begin{equation*}
\mathcal{Y}_{i}^{q}=\left|\mathcal{Y}_{i}^{q}\right| e^{i \phi_{i}^{q}} \quad(i=1,2) \tag{25}
\end{equation*}
$$

one may cast each mass matrix in the form

$$
\begin{equation*}
M_{q}=P_{q} \tilde{M}_{q} P_{q}^{*}, \quad P_{q}=\operatorname{diag}\left(e^{ \pm i \phi_{1}^{q}}, e^{ \pm i \phi_{2}^{q}}, 1\right) \tag{26}
\end{equation*}
$$

where $\tilde{M}_{q}$ is a real matrix and signs agree with the sign of $k$. The unitary matrix diagonalizing $M_{q} M_{q}^{\dagger}$ is then given by $V_{L}^{q}=P_{q} O_{L}^{q}$, where $O_{L}^{q}$ is the orthogonal matrix diagonalizing $\tilde{M}_{q} \tilde{M}_{q}^{\dagger}$. For the CKM matrix we obtain

$$
\begin{equation*}
U_{\mathrm{CKM}}=V_{L}^{u \dagger} V_{L}^{d}=O_{L}^{u T} P_{u}^{*} P_{d} O_{L}^{d} \tag{27}
\end{equation*}
$$

with

$$
\begin{equation*}
P_{u}^{*} P_{d}=\operatorname{diag}\left(e^{ \pm i\left(\phi_{1}^{d}-\phi_{1}^{u}\right)}, e^{ \pm i\left(\phi_{2}^{d}-\phi_{2}^{u}\right)}, 1\right) \tag{28}
\end{equation*}
$$

Thus, a necessary condition for having a complex, CPviolating CKM matrix reads

$$
\begin{equation*}
\phi_{1}^{d}-\phi_{1}^{u} \neq 0 \quad \text { or } \quad \phi_{2}^{d}-\phi_{2}^{u} \neq 0 \tag{29}
\end{equation*}
$$

This condition is not fulfilled if a single modular form contributes to the mass matrix, as it must be shared by both the up and down sectors, implying $\phi_{i}^{d}=\phi_{i}^{u}$. This problem is generically bypassed if more than one doublet form contributes to the $\mathcal{Y}_{i}^{q}$. Unfortunately, as shown in the remainder of this section, even in this case these textures turn out to be too restrictive and cannot lead to phenomenologically viable quark mass matrices.

## A. Texture limitations

One can show that the potentially viable $\mathbf{2}^{*} \oplus \mathbf{1}^{*}$ textures, summarized in Eq. 22, cannot fit quark data. To start, notice that quark mass ratios severely constrain the $\alpha_{i}^{q}$. Namely, given the hierarchical structure of quark masses, one finds, independently of the sign of $k$,

$$
\begin{align*}
& \operatorname{det} M_{q}=v_{q}^{3}\left(\alpha_{1}^{q}\right)^{2} \alpha_{2}^{q}=m_{1}^{q} m_{2}^{q} m_{3}^{q}  \tag{30}\\
& \begin{aligned}
\operatorname{Tr}\left(M_{q} M_{q}^{\dagger}\right) & =v_{q}^{2}\left[2\left(\alpha_{1}^{q}\right)^{2}+\left(\alpha_{2}^{q}\right)^{2}+\beta_{q}^{2}\right] \\
& =\left(m_{1}^{q}\right)^{2}+\left(m_{2}^{q}\right)^{2}+\left(m_{3}^{q}\right)^{2} \simeq\left(m_{3}^{q}\right)^{2}
\end{aligned} \\
& \begin{aligned}
& \frac{1}{2}\left[\left(\operatorname{Tr}\left(M_{q} M_{q}^{\dagger}\right)\right)^{2}-\operatorname{Tr}\left(\left(M_{q} M_{q}^{\dagger}\right)^{2}\right)\right] \\
& \quad=v_{q}^{4}\left(\alpha_{1}^{q}\right)^{2}\left[\left(\alpha_{1}^{q}\right)^{2}+2\left(\alpha_{2}^{q}\right)^{2}+\beta_{q}^{2}\right] \\
& \quad=\left(m_{1}^{q}\right)^{2}\left(m_{2}^{q}\right)^{2}+\left(m_{1}^{q}\right)^{2}\left(m_{3}^{q}\right)^{2}+\left(m_{2}^{q}\right)^{2}\left(m_{3}^{q}\right)^{2} \\
& \quad \simeq\left(m_{2}^{q}\right)^{2}\left(m_{3}^{q}\right)^{2}
\end{aligned} \tag{31}
\end{align*}
$$

with $\beta_{q} \equiv \tilde{\alpha}_{3}^{q} \sqrt{\left|\mathcal{Y}_{1}^{q}\right|^{2}+\left|\mathcal{Y}_{2}^{q}\right|^{2}}$. Here, $m_{i}^{q}$ are the $q$-type quark masses, with $m_{3}^{q} \gg m_{2}^{q} \gg m_{1}^{q}$. Omitting the index $q$ for readability, Eqs. (30) to (32) imply

$$
\begin{align*}
m_{3} & \simeq v \sqrt{2 \alpha_{1}^{2}+\alpha_{2}^{2}+\beta^{2}} \\
m_{2} & \simeq v\left|\alpha_{1}\right| \sqrt{\frac{\alpha_{1}^{2}+2 \alpha_{2}^{2}+\beta^{2}}{2 \alpha_{1}^{2}+\alpha_{2}^{2}+\beta^{2}}} \equiv v\left|\alpha_{1}\right| f\left(\alpha_{i}, \beta\right)  \tag{33}\\
m_{1} & \simeq v \frac{\left|\alpha_{1} \alpha_{2}\right|}{\sqrt{\alpha_{1}^{2}+2 \alpha_{2}^{2}+\beta^{2}}}
\end{align*}
$$

Since the function $f\left(\alpha_{i}, \beta\right)$ is bounded between $1 / \sqrt{2}$ and $\sqrt{2}$ for any values of the parameters, one has $m_{2} \sim v\left|\alpha_{1}\right|$ up to an $\mathcal{O}(1)$ factor. Hence, given the strongly hierarchical pattern of quark masses, $m_{2}^{2} / m_{3}^{2} \ll 1$ implies $\alpha_{1}^{2} \ll \alpha_{2}^{2}+\beta^{2}$, while

$$
\begin{equation*}
\frac{m_{1}^{2}}{m_{2}^{2}} \sim \frac{\alpha_{2}^{2}}{2 \alpha_{2}^{2}+\beta^{2}} \ll 1 \tag{34}
\end{equation*}
$$

implies $\alpha_{2}^{2} \ll \beta^{2}$. This in turn implies $f\left(\alpha_{i}, \beta\right) \simeq 1$ and $m_{2} \simeq v\left|\alpha_{1}\right|$. Then, $m_{3} \simeq v|\beta|$ and finally

$$
\begin{equation*}
m_{1} \simeq v\left|\frac{\alpha_{1} \alpha_{2}}{\beta}\right| \simeq v\left|\alpha_{2}\right| \frac{m_{2}}{m_{3}} \tag{35}
\end{equation*}
$$

to a good approximation. Using the best-fit values of quark mass ratios summarized in [45, obtained at an energy scale $M_{\mathrm{GUT}}=2 \times 10^{16} \mathrm{GeV}$ for $v_{u} / v_{d}=5$ (see also [46-48), one can estimate the ratios of the relevant parameters. For the down sector, one finds

$$
\begin{align*}
\left|\frac{\alpha_{1}^{d}}{\beta_{d}}\right| & \simeq \frac{m_{s}}{m_{b}} \simeq 1.37 \times 10^{-2} \\
\left|\frac{\alpha_{2}^{d}}{\beta_{d}}\right| & \simeq \frac{m_{d}}{m_{s}} \simeq 5.05 \times 10^{-2} \tag{36}
\end{align*}
$$

while for the up sector one has

$$
\begin{align*}
& \left|\frac{\alpha_{1}^{u}}{\beta_{u}}\right| \simeq \frac{m_{c}}{m_{t}} \simeq 2.69 \times 10^{-3}, \\
& \left|\frac{\alpha_{2}^{u}}{\beta_{u}}\right| \simeq \frac{m_{u}}{m_{c}} \simeq 2.04 \times 10^{-3} . \tag{37}
\end{align*}
$$

Let us now focus on the case $k>0$. By flipping the signs of quark fields, one can parameterize the mass matrices as

$$
M_{q} \propto\left(\begin{array}{ccc}
\left|\alpha_{1}^{q} / \beta_{q}\right| & 0 & \cos \theta_{q} e^{i \phi_{1}^{q}}  \tag{38}\\
0 & \left|\alpha_{1}^{q} / \beta_{q}\right| & \sin \theta_{q} e^{i \phi_{2}^{q}} \\
0 & 0 & \left|\alpha_{2}^{q} / \beta_{q}\right|
\end{array}\right)
$$

up to an overall normalization which does not affect quark mixing. Even though only two of the four $\phi_{i}^{q}$ phases are physical (see Eq. (29)), we keep all of them for simplicity. The columns of the matrices $V_{L}^{(q)}$ determine the elements of the CKM matrix, cf. Eq. 27. For $k>0$, one has

$$
\begin{align*}
& \left(V_{L}\right)_{i 1} \simeq\left(\left|\frac{\alpha_{2}}{\beta}\right| \cos \theta e^{i \phi_{1}},\left|\frac{\alpha_{2}}{\beta}\right| \sin \theta e^{i \phi_{2}},-1\right)_{i}  \tag{39}\\
& \left(V_{L}\right)_{i 2}=\left(\sin \theta e^{i\left(\phi_{1}-\phi_{2}\right)},-\cos \theta, 0\right)_{i}
\end{align*}
$$

The first column is obtained at leading order in the small expansion parameters $\left|\alpha_{i} / \beta\right|$, while the result for the second column is exact. It follows that

$$
\begin{align*}
\left|V_{u s}\right| & =\left|\left(V_{L}^{u}\right)_{i 1}^{*}\left(V_{L}^{d}\right)_{i 2}\right| \\
& \simeq\left|\frac{\alpha_{2}^{u}}{\beta_{u}}\right|\left|\cos \theta_{u} \sin \theta_{d} e^{i \varphi}-\cos \theta_{d} \sin \theta_{u}\right| \tag{40}
\end{align*}
$$

with $\varphi \equiv \phi_{1}^{d}-\phi_{1}^{u}-\phi_{2}^{d}+\phi_{2}^{u}$. The second factor varies in the interval $[0,1]$, implying the upper bound $\left|V_{u s}\right| \lesssim$ $m_{u} / m_{c} \simeq 0.002$. This is two orders of magnitude smaller than the value $\left|V_{u s}\right| \simeq 0.225$ [45] required by quark data. By a similar procedure, one can also show that

$$
\begin{equation*}
\left|\left|V_{u b}\right|-\frac{m_{d}}{m_{s}}\right| \lesssim \frac{m_{u}}{m_{c}} \tag{41}
\end{equation*}
$$

i.e. that $\left|V_{u b}\right| \in[0.048,0.053]$, again in contradiction with the data, which requires the much smaller $\left|V_{u b}\right| \simeq 0.003$.

Finally, the case $k<0$ is also excluded. For negative $k$, one must replace $M_{q} \rightarrow M_{q}^{T}$ in Eq. (38), leading to

$$
\begin{align*}
& \left(V_{L}\right)_{i 2}=\left(-\sin \theta e^{i\left(\phi_{2}-\phi_{1}\right)}, \cos \theta, 0\right)_{i} \\
& \left(V_{L}\right)_{i 3} \simeq\left(\left|\frac{\alpha_{1}}{\beta}\right| \cos \theta e^{-i \phi_{1}},\left|\frac{\alpha_{1}}{\beta}\right| \sin \theta e^{-i \phi_{2}}, 1\right)_{i} \tag{42}
\end{align*}
$$

This implies that

$$
\begin{align*}
\left|V_{c b}\right| & =\left|\left(V_{L}^{u}\right)_{i 2}^{*}\left(V_{L}^{d}\right)_{i 3}\right| \\
& \simeq\left|\frac{\alpha_{1}^{d}}{\beta_{d}}\right|\left|\cos \theta_{u} \sin \theta_{d} e^{i \varphi}-\cos \theta_{d} \sin \theta_{u}\right| \tag{43}
\end{align*}
$$

once again leading to an upper bound on the magnitude of a mixing matrix element. In this case one has $\left|V_{c b}\right| \lesssim$ $m_{s} / m_{b} \simeq 0.014$, while data requires $\left|V_{c b}\right| \simeq 0.036$, a value more than twice as large.

The above no-go analytical results are corroborated by numerical scans of the $\mathbf{2}^{*} \oplus \mathbf{1}^{*}$ parameter space, treating the $\mathcal{Y}_{i}^{q}$ as free quantities. Indeed, the details of the modular forms play essentially no role in the exclusion of this case, as evidenced by the general parameterization of Eq. (38). These results may then hold true even if we one is dealing with vector-valued modular forms [49]. For instance, the modular binary dihedral group $2 D_{3}$, recently studied in Ref. [50, admits only singlet and doublet irreps (four and two of each, respectively). The corresponding tensor products are such that there always exists a weak basis in which mass matrices take the form of Eq. 22. A $2 D_{3}$ generalization of the $\mathbf{2}^{*} \oplus \mathbf{1}^{*}$ case is then prevented by the analyzed restrictive texture.

## V. THE $\mathbf{1}^{*} \oplus \mathbf{1}^{*} \oplus \mathbf{1}^{*}$ CASE

As shown in the previous sections, only scenarios where quarks furnish 1D irreps of the finite modular group have a chance to be phenomenologically viable. We start from the general weight structure of Eq. 18. Without loss of generality, it can be made to read

$$
\begin{align*}
& M_{q}:\left(\begin{array}{ccc}
0 & 0 & k \\
0 & 0 & k \\
-k & -k & 0
\end{array}\right), \quad k>0  \tag{44}\\
& M_{q}:\left(\begin{array}{ccc}
0 & k^{\prime} & k^{\prime} \\
-k^{\prime} & 0 & 0 \\
-k^{\prime} & 0 & 0
\end{array}\right), \quad k^{\prime}>0, \quad \text { or }  \tag{45}\\
& M_{q}:\left(\begin{array}{ccc}
0 & k^{\prime} & k+k^{\prime} \\
-k^{\prime} & 0 & k \\
-k-k^{\prime} & -k & 0
\end{array}\right), \quad k, k^{\prime}>0 \tag{46}
\end{align*}
$$

in both sectors $\left(k_{i j}^{q}=k_{i j}\right)$, depending on the number of negative-weight entries in $M_{q}$ and, in the case of 2 such entries, on whether these share a row or a column.

One can then classify potentially viable models in terms of their number of parameters. We denote $Q=$ $Q_{1} \oplus Q_{2} \oplus Q_{3}$ and $q^{c}=q_{1}^{c} \oplus q_{2}^{c} \oplus q_{3}^{c}(q=u, d)$. To avoid a massless quark, one must have

$$
\begin{align*}
Q & \sim\left(\mathbf{1}_{Q_{1}}, k_{1}\right) \oplus\left(\mathbf{1}_{Q_{2}}, k_{2}\right) \oplus\left(\mathbf{1}_{Q_{3}}, k_{3}\right), \\
u^{c} & \sim\left(\overline{\mathbf{1}_{Q_{1}}},-k_{1}\right) \oplus\left(\overline{\mathbf{1}_{Q_{2}}},-k_{2}\right) \oplus\left(\overline{\mathbf{1}_{Q_{3}}},-k_{3}\right),  \tag{47}\\
d^{c} & \sim\left(\overline{\mathbf{1}_{Q_{1}}},-k_{1}\right) \oplus\left(\overline{\mathbf{1}_{Q_{2}}},-k_{2}\right) \oplus\left(\overline{\mathbf{1}_{Q_{3}}},-k_{3}\right),
\end{align*}
$$

with $k^{\prime}=k_{12}=k_{1}-k_{2}$ and $k=k_{23}=k_{2}-k_{3}$.

## A. Model landscape

Given the above, there are five inequivalent textures one may consider for the $M_{q}$. In an appropriate weak basis, they read, up to an overall factor $v_{q} \|^{8}$

$$
\begin{align*}
& \text { I : }\left(\begin{array}{ccc}
\alpha_{1}^{q} & 0 & \tilde{\alpha}_{13}^{q} \mathcal{Y}_{q, 13}^{\left(k+k^{\prime}\right)} \\
0 & \alpha_{2}^{q} & \tilde{\alpha}_{23}^{q} \mathcal{Y}_{q, 23}^{(k)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right), k>0, k^{\prime} \geq 0,  \tag{48a}\\
& \text { II : }\left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{12}^{q} \mathcal{Y}_{q, 12}^{\left(k^{\prime}\right)} & \tilde{\alpha}_{13}^{q} \mathcal{Y}_{q, 13}^{\left(k+k^{\prime}\right)} \\
0 & \alpha_{2}^{q} & 0 \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right), k \geq 0, k^{\prime}>0,  \tag{48b}\\
& \text { III : }\left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{12}^{q} \mathcal{Y}_{q, 12}^{\left(k^{\prime}\right)} & \tilde{\alpha}_{13}^{q} \mathcal{Y}_{q, 13}^{\left(k+k^{\prime}\right)} \\
0 & \alpha_{2}^{q} & \tilde{\alpha}_{23}^{q} \mathcal{Y}_{q, 23}^{(k)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right), k, k^{\prime}>0, \quad(48 \mathrm{c}  \tag{48c}\\
& \text { IV : }  \tag{48~d}\\
& \left(\begin{array}{ccc}
\alpha_{11}^{q} & \alpha_{12}^{q} & \tilde{\alpha}_{13}^{q} \mathcal{Y}_{q, 13}^{(k)} \\
\alpha_{21}^{q} & \alpha_{22}^{q} & \tilde{\alpha}_{23}^{q} \mathcal{Y}_{q, 23}^{(k)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right), k>0,  \tag{48e}\\
& \text { V : } \begin{array}{l}
\text { (48d } \\
0
\end{array}\left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{12}^{q} \mathcal{Y}_{q, 12}^{\left(k^{\prime}\right)} & \tilde{\alpha}_{13}^{q} \mathcal{Y}_{q, 13}^{\left(k^{\prime}\right)} \\
0 & \alpha_{22}^{q} & \alpha_{23}^{q} \\
0 & \alpha_{32}^{q} & \alpha_{33}^{q}
\end{array}\right), k^{\prime}>0,
\end{align*}
$$

with

$$
\begin{equation*}
\mathcal{Y}_{q, i j}^{(w)} \equiv \sum_{n=1} g_{n}^{q} Y_{\mathbf{1}_{i j}, n}^{(w)}, \quad g_{n}^{q} \equiv \frac{\alpha_{i j, n}^{q}}{\alpha_{i j}^{q}} \tag{49}
\end{equation*}
$$

[^6]and $\alpha_{i j, 1} \equiv \alpha_{i j}$. The sum in $n$ goes over all available independent 1D modular forms of weight $w=k_{i j}$ and irrep $\mathbf{1}_{i j} \equiv \overline{\mathbf{1}_{Q_{i}}} \otimes \mathbf{1}_{Q_{j}}$. Singlet modular form irreps available at a certain weight $w \leq 36$, for a given level $N \leq 5$, are listed in Table III of Appendix B. The effects of canonical normalization have been included in Eqs. (48a to 48e via the definitions
\[

$$
\begin{equation*}
\tilde{\alpha}_{i j}^{q} \equiv \alpha_{i j}^{q}(2 \operatorname{Im} \tau)^{k_{i j} / 2} \tag{50}
\end{equation*}
$$

\]

where $\alpha_{i j}^{q}$ are the original superpotential parameters. As before, the $g$ and $\mathcal{Y}$ remain unchanged by this normalization.

The vanishing of the $(i, j)=(1,2)$ and $(2,3)$ entries in textures I and II, respectively, can be consistently achieved in the following conditions:
i) for $k_{i j}>0$ : if there are no modular forms of the requisite irrep $\mathbf{1}_{i j}$ at the corresponding weight $k_{i j}$;
ii) for $k_{i j}=0$ : if $\mathbf{1}_{i j} \neq \mathbf{1}$, guaranteeing also that the $(j, i)$ zero-weight entry vanishes, as $\mathbf{1}_{j i}=\overline{\mathbf{1}_{i j}} \neq \mathbf{1}$.

No more entries may vanish, as this would lead either to a massless quark or to a vanishing CKM angle. In contrast, one cannot consistently forbid entry $(1,3)$ of texture III, since if forms $Y_{\mathbf{1}_{12}}^{\left(k^{\prime}\right)}$ and $Y_{\mathbf{1}_{23}}^{(k)}$ are available, their product is a modular form of the requisite weight $k+k^{\prime}$ and irrep $\mathbf{1}_{13}=\mathbf{1}_{12} \otimes \mathbf{1}_{23}$.

Note that by working with the finite modular groups $\Gamma_{N}^{\prime}$ we can access non-trivial singlet irreps and find mass matrices solving the strong CP problem beyond those considered in the context of the full modular group [20]. The key differences are:
i) one now has access to modular forms $Y \sim \mathbf{1}^{*}$ aside from trivial singlets 1 (polynomials in $E_{4}$ and $E_{6}$ );
ii) one can now forbid certain zero-weight entries of the mass matrices by appropriate choices of $\mathbf{1}_{Q_{i}}$.

It is this last possibility that allows access to textures I and II with vanishing $k^{\prime}$ and $k$, respectively. It requires a non-trivial multiplier system for the matter fields - an option alluded to in Appendix A of Ref. [20] - which, in the context of the finite modular groups, corresponds to assigning these fields to non-trivial 1D irreps of $\Gamma_{N}^{\prime}$.

Naïvely, the minimal scenarios would have only one singlet modular form contributing to each non-zero positive-weight entry and would feature 5 (cases I, II), 6 (case III), or 7 (cases IV, V) real parameters in each sector, not counting $\tau$. However, these simple possibilities are excluded, as they conserve CP. Indeed, recall that the textures for both up and down quark sectors coincide in an appropriate weak basis, as mass matrices have a common weight and irrep structure. This leads to a situation analogous to the one discussed in Section IV. Namely, for cases I and II, sector-independent phases factorize and cancel in the determination of $U_{\mathrm{CKM}}$. This is
also true for cases III-V. In case III, if only one form is available in each entry, the relevant phases are related by

$$
\begin{equation*}
\arg Y_{\mathbf{1}_{13}}^{\left(k+k^{\prime}\right)}=\arg Y_{\mathbf{1}_{12}}^{\left(k^{\prime}\right)}+\arg Y_{\mathbf{1}_{23}}^{(k)}, \tag{51}
\end{equation*}
$$

leading to CP conservation. In case IV, $k_{13}=k_{23}$ and the fact that the zero-weight $(1,2)$ entry is allowed by the symmetry implies $\mathbf{1}_{12}=\mathbf{1}$, forcing $\mathbf{1}_{Q_{1}}=\mathbf{1}_{Q_{2}}$ and thus $\mathbf{1}_{13}=\mathbf{1}_{23} \square^{9}$ A similar reasoning applies to case V. So, in these last two cases, if a single form contributes to each positive-weight entry, it must be the same form, implying that phases factorize and CP is conserved. As a cross-check, one may also compute a CP-odd weak-basis invariant like $\operatorname{Tr}\left[M_{u} M_{u}^{\dagger}, M_{d} M_{d}^{\dagger}\right]^{3}$ [51] and verify that it vanishes when phases are common to both sectors.

The problem of CP conservation due to common phases can be avoided if more than one form is present in some entry of the mass matrices. The minimal matrices are then obtained from textures I and II of Eqs. 48a) and (48b whenever two singlet modular forms may contribute to one of the positive-weight entries, while the other positive-weight entry depends on a single modular form. The minimal viable number of parameters thus corresponds to 6 real constants in each sector. Taking into account $\operatorname{Re} \tau$ and $\operatorname{Im} \tau$, this leads to a total of 14 parameters in the quark sector ${ }^{10}$ Next-to-minimal matrices instead correspond to:

- textures I and II Eqs. 48a and 48b if four independent modular forms are involved, and
- texture III of Eq. (48c) with two singlet modular forms contributing to one of the three positiveweight entries, while the other two depend on a single form each. By consistency, the entry populated with two independent forms must be $(1,3)$.

Such next-to-minimal scenarios feature 7 real constants in each sector, for a total of 16 parameters.

After going through the available forms and irreps (see Table III of Appendix B), we summarize all possible minimal and next-to-minimal models in Table II. Each pair of numbers denotes the weights of the two lowestweight modular forms involved, while primes and hats denote the corresponding singlet irreps. For instance,

[^7]the $S_{4}^{\prime}$-specific $\left(\widehat{7}^{\prime}, 12\right)$ refers to the minimal textures
\[

$$
\begin{align*}
& \left(\begin{array}{ccc}
\alpha_{1}^{q} & 0 & \tilde{\alpha}_{13,1}^{q} Y_{\mathbf{1}, 1}^{(12)}+\tilde{\alpha}_{13,2}^{q} Y_{\mathbf{1}, 2}^{(12)} \\
0 & \alpha_{2}^{q} & \tilde{\alpha}_{23}^{q} Y_{\hat{\mathbf{1}}^{\prime}}^{(7)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right) \text { and }  \tag{52}\\
& \left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{23}^{q} Y_{\hat{\mathbf{1}}^{\prime}}^{(7)} & \tilde{\alpha}_{13,1}^{q} Y_{\mathbf{1}, 1}^{(12)}+\tilde{\alpha}_{13,2}^{q} Y_{\mathbf{1}, 2}^{(12)} \\
0 & \alpha_{2}^{q} & 0 \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right)
\end{align*}
$$
\]

of type I and II, respectively. Here and in what follows, the level $N$ is implied. Instead, the $A_{4}^{\prime}$-specific $\left(4^{\prime}, 8^{\prime \prime}\right)$ refers to the next-to-minimal textures

$$
\begin{align*}
& \left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{12}^{q} Y_{\mathbf{1}^{\prime \prime}}^{(8)} & \tilde{\alpha}_{13,1}^{q} Y_{\mathbf{1}, 1}^{(12)}+\tilde{\alpha}_{13,2}^{q} Y_{\mathbf{1}, 2}^{(12)} \\
0 & \alpha_{2}^{q} & \tilde{\alpha}_{23}^{q} Y_{\mathbf{1}^{\prime}}^{(4)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right) \text { and } \\
& \left(\begin{array}{ccc}
\alpha_{1}^{q} & \tilde{\alpha}_{12}^{q} Y_{\mathbf{1}^{\prime}}^{(4)} & \tilde{\alpha}_{13,1}^{q} Y_{\mathbf{1}, 1}^{(12)}+\tilde{\alpha}_{13,2}^{q} Y_{\mathbf{1}, 2}^{(12)} \\
0 & \alpha_{2}^{q} & \tilde{\alpha}_{23}^{q} Y_{\mathbf{1}^{\prime \prime}}^{(8)} \\
0 & 0 & \alpha_{3}^{q}
\end{array}\right) \tag{53}
\end{align*}
$$

of type III. In general, each pair refers to two inequivalent models related by the transposition of mass matrices, as in the above examples. The exceptions are symmetric pairs arising for type-III next-to-minimal models, e.g. $(6,6)$, which refer to a single texture.

We find a total of 462 minimal and 875 next-tominimal inequivalent models. Out of the minimal models, 6 are available for all the $\Gamma_{N}^{\prime}$, as these can be realized using polynomials in $Y_{1}^{(4)}=E_{4}$ and $Y_{1}^{(6)}=E_{6}$, the (trivial) singlets of $S L(2, \mathbb{Z})$ common to all finite modular groups. There are additionally 48,138 and 270 minimal models specific to $S_{3}, A_{4}^{\prime}$ and $S_{4}^{\prime}$, respectively ( $A_{5}^{\prime}$ does not have non-trivial singlets). As for the next-to-minimal models of type I and II, 6 are available for all $\Gamma_{N}^{\prime}$, while there are 48,116 and 270 minimal models specific to $S_{3}$, $A_{4}^{\prime}$ and $S_{4}^{\prime}$, respectively. Out of these 440 models, 60 involve weights and irreps corresponding to three available independent modular forms. Finally, there are 435 next-to-minimal models of type III, 15 of which are available for all $\Gamma_{N}^{\prime}$, while 48, 129 and 243 are specific to $S_{3}, A_{4}^{\prime}$ and $S_{4}^{\prime}$, respectively. We stress that these lists are complete as no weight limit is being applied ${ }^{11}$ Any other viable choice of weights will result in more than 7 parameters in each sector (at least 18 parameters in total).

[^8]TABLE II. Complete landscape of minimal and next-to-minimal models.

|  | Minimal models (I and II) | Next-to-minimal models (I and II) | Next-to-minimal models (III) |
| :---: | :---: | :---: | :---: |
| All $\Gamma_{N}^{\prime}$ | $(10,12),(12,14),(14,16)$ | $(16,18),(18,20),(20,22)$ | $\begin{gathered} (4,8),(4,14),(6,6),(6,10),(6,14),(8,8), \\ (8,10),(8,14),(10,10) \end{gathered}$ |
| $S_{3}$ only | $\begin{aligned} & \left(10^{\prime}, 12\right),\left(10,18^{\prime}\right),\left(10^{\prime}, 18\right),\left(12,12^{\prime}\right), \\ & \left(12,14^{\prime}\right),\left(12,16^{\prime}\right),\left(12^{\prime}, 16\right),\left(12,20^{\prime}\right), \\ & \left(12^{\prime}, 20\right),\left(14^{\prime}, 16\right),\left(14,18^{\prime}\right),\left(14^{\prime}, 18\right), \\ & \left(14,22^{\prime}\right),\left(14^{\prime}, 22\right),\left(16,16^{\prime}\right),\left(16^{\prime}, 18\right), \\ & \left(16^{\prime}, 18^{\prime}\right),\left(16,20^{\prime}\right),\left(16^{\prime}, 2\right),\left(18,20^{\prime}\right), \\ & \left(18^{\prime}, 20^{\prime}\right),\left(20,20^{\prime}\right),\left(20^{\prime}, 22\right),\left(20^{\prime}, 22^{\prime}\right) \end{aligned}$ | $\begin{aligned} & \left(16,18^{\prime}\right),\left(16,24^{\prime}\right),\left(16^{\prime}, 24\right),\left(18,18^{\prime}\right), \\ & \left(18^{\prime}, 20\right),\left(18,22^{\prime}\right),\left(18^{\prime}, 22\right),\left(18,26^{\prime}\right), \\ & \left(18^{\prime}, 26\right),\left(20,22^{\prime}\right),\left(20,24^{\prime}\right),\left(20^{\prime}, 24\right), \\ & \left(20,28^{\prime}\right),\left(20^{\prime}, 28\right),\left(22,22^{\prime}\right),\left(22,24^{\prime}\right), \\ & \left(22^{\prime}, 24^{\prime}\right),\left(22,26^{\prime}\right),\left(22^{\prime}, 26\right),\left(24^{\prime}, 26\right), \\ & \left(24^{\prime}, 26^{\prime}\right),\left(26,26^{\prime}\right),\left(26,28^{\prime}\right),\left(26^{\prime}, 28^{\prime}\right) \end{aligned}$ | $\begin{gathered} \left(4,14^{\prime}\right),\left(4,20^{\prime}\right),\left(6^{\prime}, 6^{\prime}\right),\left(6^{\prime}, 10^{\prime}\right),\left(6,12^{\prime}\right), \\ \left(6^{\prime}, 12^{\prime}\right),\left(6^{\prime}, 14^{\prime}\right),\left(6,16^{\prime}\right),\left(6^{\prime}, 16^{\prime}\right), \\ \left(6,20^{\prime}\right),\left(6^{\prime}, 20^{\prime}\right),\left(8,10^{\prime}\right),\left(8,14^{\prime}\right),\left(8,16^{\prime}\right), \\ \left(8,20^{\prime}\right),\left(10^{\prime}, 10^{\prime}\right),\left(10,12^{\prime}\right),\left(10^{\prime}, 12^{\prime}\right), \\ \left(10,14^{\prime}\right),\left(10^{\prime}, 14\right),\left(10,16^{\prime}\right),\left(10^{\prime}, 16^{\prime}\right), \\ \left(12^{\prime}, 14\right),\left(12^{\prime}, 14^{\prime}\right),\left(14,14^{\prime}\right) \end{gathered}$ |
| $A_{4}^{\prime}$ <br> only | $\begin{gathered} \left(8^{\prime}, 12\right),\left(8^{\prime}, 18\right),\left(10^{\prime}, 12\right),\left(10,16^{\prime}\right), \\ \left.\left(10^{\prime}, 16\right),\left(10,20^{\prime \prime}\right),\left(10^{\prime}\right), 20\right),\left(12,12^{\prime}\right), \\ \left(12,12^{\prime \prime}\right),\left(12,14^{\prime}\right),\left(12,14^{\prime \prime}\right),\left(12,16^{\prime \prime}\right), \\ \left(12^{\prime}, 16\right),\left(12^{\prime \prime}, 1^{\prime}\right),\left(12,18^{\prime}\right),\left(12,18^{\prime \prime}\right), \\ \left(12^{\prime}, 18\right),\left(12^{\prime \prime}, 18\right),\left(12,22^{\prime \prime}\right),\left(12^{\prime}, 22\right), \\ \left(12^{\prime \prime}, 22^{\prime}\right),\left(14,16^{\prime}\right),\left(14^{\prime}, 16\right),\left(14^{\prime}, 16^{\prime}\right), \\ \left(14^{\prime \prime}, 16\right),\left(14^{\prime \prime}, 16^{\prime}\right),\left(14^{\prime}, 18\right),\left(14,20^{\prime}\right), \\ \left(14,20^{\prime \prime}\right),\left(14^{\prime}, 20\right),\left(14^{\prime}, 20^{\prime \prime}\right),\left(14^{\prime \prime}, 20\right), \\ \left(14^{\prime \prime}, 20^{\prime}\right),\left(14,24^{\prime \prime}\right),\left(14^{\prime \prime}, 24^{\prime}\right),\left(16,16^{\prime \prime}\right), \\ \left(16^{\prime}, 16^{\prime \prime}\right),\left(16,18^{\prime}\right),\left(16,18^{\prime \prime}\right),\left(16^{\prime}, 18^{\prime}\right), \\ \left(16^{\prime}, 18^{\prime \prime}\right),\left(16^{\prime \prime}, 18\right),\left(16^{\prime \prime}, 20^{\prime}\right),\left(16,, 22^{\prime \prime}\right), \\ \left(16^{\prime}, 22^{\prime \prime}\right),\left(16^{\prime \prime}, 22\right),\left(16^{\prime \prime}, 22^{\prime}\right), \\ \left(16^{\prime \prime}, 26^{\prime}\right),\left(18,18^{\prime}\right),\left(18,18^{\prime \prime}\right),\left(18^{\prime}, 20\right), \\ \left(18^{\prime}, 20^{\prime}\right),\left(18^{\prime}, 20^{\prime \prime}\right),\left(18^{\prime \prime}, 20\right),\left(18^{\prime \prime}, 20^{\prime}\right), \\ \left.\left(18^{\prime \prime}, 0^{\prime \prime}\right),\left(18,, 2^{\prime \prime}\right),\left(18^{\prime}, 22\right)\right),\left(18^{\prime \prime}, 22^{\prime}\right), \\ \left(18^{\prime}, 24^{\prime \prime}\right),\left(18^{\prime \prime}, 24^{\prime}\right),\left(20,22^{\prime \prime}\right),\left(20^{\prime}, 22^{\prime \prime}\right), \\ \left(20_{\prime \prime \prime}^{\prime \prime}, 22^{\prime \prime}\right),\left(22,22^{\prime \prime}\right),\left(22^{\prime}, 22^{\prime \prime}\right), \\ \left(22^{\prime \prime}, 24^{\prime}\right),\left(22^{\prime \prime}, 24^{\prime \prime}\right),\left(22^{\prime \prime}\right),\left(22^{\prime \prime}\right. \end{gathered},$ | $\left(14^{\prime}, 24\right),\left(16,16^{\prime}\right),\left(16^{\prime}, 18\right),\left(16,20^{\prime \prime}\right)$, $\left(16^{\prime}, 20\right),\left(16,22^{\prime}\right),\left(16^{\prime}, 22\right),\left(16,26^{\prime \prime}\right)$, $\left(16^{\prime}, 26\right),\left(18,20^{\prime}\right),\left(18,20^{\prime \prime}\right),\left(18,24^{\prime}\right)$, $\left(18,24^{\prime \prime}\right),\left(18^{\prime}, 24\right),\left(18^{\prime \prime}, 24\right),\left(18,28^{\prime \prime}\right)$, $\left(18^{\prime}, 28\right),\left(18^{\prime \prime}, 28^{\prime}\right),\left(20,20^{\prime}\right),\left(20,20^{\prime \prime}\right)$, $\left(20^{\prime}, 20^{\prime \prime}\right),\left(20,22^{\prime}\right),\left(20^{\prime}, 22\right),\left(20^{\prime}, 22^{\prime}\right)$, $\left(20^{\prime \prime}, 22\right),\left(20^{\prime \prime}, 22^{\prime}\right),\left(20,24^{\prime \prime}\right),\left(20^{\prime \prime}, 24^{\prime}\right)$, $\left(20,26^{\prime}\right),\left(20,26^{\prime \prime}\right),\left(20^{\prime}, 26\right),\left(20^{\prime}, 26^{\prime \prime}\right)$, $\left(20^{\prime \prime}, 26\right),\left(20^{\prime \prime}, 26^{\prime}\right),\left(20,30^{\prime \prime}\right),\left(20^{\prime \prime}, 30^{\prime}\right)$, $\left(22,22^{\prime}\right),\left(22,24^{\prime}\right),\left(22,24^{\prime \prime}\right),\left(22^{\prime}, 24^{\prime}\right)$, $\left(22^{\prime}, 24^{\prime \prime}\right),\left(22^{\prime \prime}, 24\right),\left(22,2^{\prime \prime}\right),\left(22^{\prime}, 26\right)$, $\left(22,22^{\prime \prime}\right),\left(22^{\prime}, 28^{\prime \prime}\right),\left(22^{\prime \prime}, 28\right),\left(22^{\prime \prime}, 28^{\prime}\right)$, $\left(22^{\prime \prime}, 32^{\prime}\right),\left(24^{\prime}, 24^{\prime \prime}\right),\left(24^{\prime}, 26\right),\left(24^{\prime}, 26^{\prime}\right)$, $\left(24^{\prime}, 26^{\prime \prime}\right),\left(24^{\prime \prime}, 26\right),\left(24^{\prime \prime}, 26^{\prime}\right)$, $\left(24^{\prime \prime}, 26^{\prime \prime}\right),\left(24^{\prime}, 30^{\prime \prime}\right),\left(24^{\prime \prime}, 30^{\prime}\right)$ | $\left(4^{\prime}, 8^{\prime \prime}\right),\left(4,12^{\prime}\right),\left(4^{\prime}, 12^{\prime \prime}\right),\left(4^{\prime}, 14^{\prime \prime}\right)$, $\left(4,16^{\prime \prime}\right),\left(4^{\prime}, 16^{\prime \prime}\right),\left(4,18^{\prime}\right),\left(4^{\prime}, 18^{\prime \prime}\right)$, $\left(4,22^{\prime \prime}\right),\left(4^{\prime}, 22^{\prime \prime}\right),\left(6,10^{\prime}\right),\left(6,14^{\prime}\right)$, $\left(6,14^{\prime \prime}\right),\left(6,18^{\prime}\right),\left(6,18^{\prime \prime}\right),\left(6,22^{\prime \prime}\right),\left(8,8^{\prime}\right)$, $\left(8^{\prime}, 8^{\prime \prime}\right),\left(8^{\prime \prime}, 8^{\prime \prime}\right),\left(8^{\prime \prime}, 10^{\prime}\right),\left(8,12^{\prime}\right)$, $\left(8,12^{\prime \prime}\right),\left(8^{\prime}, 12^{\prime}\right),\left(8^{\prime}, 12^{\prime \prime}\right),\left(8^{\prime \prime}, 12^{\prime}\right)$, $\left(8^{\prime \prime}, 12^{\prime \prime}\right),\left(8,14^{\prime}\right),\left(8^{\prime}, 14\right),\left(8^{\prime}, 14^{\prime \prime}\right)$, $\left(8^{\prime \prime}, 14^{\prime}\right),\left(8^{\prime \prime}, 14^{\prime \prime}\right),\left(8,16^{\prime \prime}\right),\left(8^{\prime \prime}, 16^{\prime \prime}\right)$, $\left(8,18^{\prime}\right),\left(8,18^{\prime \prime}\right),\left(8^{\prime}, 18^{\prime}\right),\left(8^{\prime}, 18^{\prime \prime}\right)$, $\left(8^{\prime \prime}, 18^{\prime}\right),\left(8^{\prime \prime}, 18^{\prime \prime}\right),\left(8,22^{\prime \prime}\right),\left(8^{\prime \prime}, 22^{\prime \prime}\right)$, $\left(10,10^{\prime}\right),\left(10^{\prime}, 10^{\prime}\right),\left(10,12^{\prime}\right),\left(10^{\prime}, 12^{\prime \prime}\right)$, $\left(10,14^{\prime}\right),\left(10,14^{\prime \prime}\right),\left(10^{\prime}, 14\right),\left(10^{\prime}, 14^{\prime}\right)$, $\left(10,16^{\prime \prime}\right),\left(10^{\prime}, 11^{\prime \prime}\right),\left(10,11^{\prime \prime}\right),\left(10^{\prime}, 18^{\prime}\right)$, $\left(12^{\prime}, 12^{\prime}\right),\left(12^{\prime \prime}, 12^{\prime \prime}\right),\left(12^{\prime}, 14\right),\left(12^{\prime}, 14^{\prime}\right)$, $\left(12^{\prime}, 14^{\prime \prime}\right),\left(2^{\prime \prime}, 14\right),\left(2^{\prime \prime}, 14^{\prime}\right)$, $\left(12^{\prime \prime}, 111^{\prime \prime}\right),\left(2^{\prime}, 18^{\prime}\right),\left(12^{\prime \prime}, 18^{\prime \prime}\right)$, $\left(14,14^{\prime \prime}\right),\left(14^{\prime}, 14^{\prime}\right),\left(14,16^{\prime \prime}\right),\left(14^{\prime \prime}, 16^{\prime \prime}\right)$ |
| $\begin{aligned} & S_{4}^{\prime} \\ & \text { only } \end{aligned}$ |  |  |  |

## B. Scan of minimal models

Focusing on the minimal model landscape, we aim at confronting models with quark data, namely GUT-scale mass ratios, mixing angles and CPV phase [45]. To this end, we start by fitting the textures themselves to data, i.e. we consider textures I and II of Eqs. (48a) and 48b with their non-zero off-diagonal ( $\tau$-dependent) entries replaced by free complex parameters. Thanks to weak basis transformations, it turns out that only one complex phase is relevant for the scan. Textures are fitted up to an overall factor in each sector, which dictates the up and down quark mass scales but does not affect mass ratios. Following the procedure outlined in Appendix C of Ref. 31, we first search for local minima of the Gaussian loss function and then explore $3 \sigma$ viable regions around them via the Metropolis algorithm.

We find that both textures I and II provide enough freedom to fit the central values of quark mass ratios and mixing parameters, resulting in a negligible loss. This means that all models in Table II - including type-III next-to-minimal ones - can fit the data, since texture III reduces to either texture I or texture II in the limit of vanishing $\alpha_{12, n}^{q}$ or $\alpha_{23, n}^{q}$, respectively ${ }^{12}$ However, the fit may come at the price of extreme fine-tuning. Indeed, in the context of each model, each fit point may translate to severely hierarchical superpotential parameters.

To quantify the hierarchies among model parameters, we take the total logarithmic amplitude $\mathrm{amp}_{\mathrm{tot}}=\mathrm{amp}_{u}+$ $\mathrm{amp}_{d}$, with

$$
\begin{equation*}
\operatorname{amp}_{q} \equiv \max \log _{10}\left|\alpha^{q}\right|-\min \log _{10}\left|\alpha^{q}\right| \tag{54}
\end{equation*}
$$

as a measure of model quality. By definition, $\mathrm{amp}_{q}$ is small when all superpotential constants are of the same order in that sector, e.g. if they are all $\mathcal{O}(1)$. A value of $\mathrm{amp}_{q}=1$ corresponds to $q$-sector constants differing at most by one order of magnitude among themselves.

In fitting textures I and II, the zero-weight (diagonal) entries alone already set a lower bound on the total amplitude, $\mathrm{amp}_{\text {tot }} \geq \mathrm{amp}_{\text {tot }, 0}$, for each fit point. Numerically, we find

$$
\begin{equation*}
\min \operatorname{amp}_{\mathrm{tot}, 0}^{\mathrm{I}} \simeq 1.99, \quad \min \mathrm{amp}_{\mathrm{tot}, 0}^{\mathrm{II}} \simeq 1.04 \tag{55}
\end{equation*}
$$

using $\sim 2 \times 10^{6}$ fit points for each texture. This means that type-I models in general carry a stronger hierarchy in superpotential parameters from the outset, when compared to type-II models.

To further analyse the minimal models, we randomly sample the parameter space of each fit. In particular, we select a manageable sample of 1000 fit points for each texture, appending also the points selected in Eq. (55). We then (exactly) fit each of the corresponding 231 minimal

[^9]models to each sample point. Not only is this second fit always possible; it is degenerate in some parameters, as the models possess more freedom than the textures. We thus vary model parameters, including $\tau$, in order to minimize the amplitude excess, defined as $\mathrm{amp}_{\text {tot }}-\mathrm{amp}_{\text {tot }, 0}$. For this procedure to be meaningful, the normalization of modular forms must be appropriately fixed. The list of normalized singlet modular forms we use, based on the proposal of Ref. [52, is explicitly given in Appendix B.

As a result of these scans, we identify the minimal models which can fit quark data with the lowest total amplitude:

- for texture I, the lowest-amplitude model is $(12,14)$, available for all $\Gamma_{N}^{\prime}$ and in the context of the full modular group, with $\mathrm{amp}_{\mathrm{tot}} \simeq 2.19$ $\left(\mathrm{amp}_{\text {tot }, 0} \simeq 1.99, \mathrm{amp}_{u} \simeq 0.18, \mathrm{amp}_{d} \simeq 2.01\right)$,
- for texture II, the lowest-amplitude model is the $A_{4}^{(\prime)}$-exclusive $\left(16^{\prime \prime}, 22\right)$, with $\mathrm{amp}_{\text {tot }} \simeq 2.34$ $\left(\mathrm{amp}_{\mathrm{tot}, 0} \simeq 1.49, \mathrm{amp}_{u} \simeq 0.05, \mathrm{amp}_{d} \simeq 2.29\right)$.

Across low-amplitude minimal models, one finds an overall tendency for hierarchies to originate mostly from the down-quark sector, with $\mathrm{amp}_{d} \sim 2$ vs. $\mathrm{amp}_{u}<1$.

One may also wonder if special values of $\tau$ are selected by minimizing the total amplitude, i.e. by asking for the smallest possible parametric hierarchies, in the context of each model. Note that an unambiguous selection of $\tau$ is not possible when the excess is zero. In Figs. 1 and 2 we show the selected values of the modulus VEV for type-I and type-II models with minimal $\mathrm{amp}_{\text {tot }}$ and non-zero excess. In the context of each model, any preference for either positive or negative values of $\operatorname{Re} \tau$ must be driven by the CPV phase $\delta$, which is the only observable sensitive to the sign of $\operatorname{Re} \tau$ (see e.g. 31]). Focusing on the models with relatively small total amplitude $\left(\mathrm{amp}_{\text {tot }}<3\right.$, blue dots in the figures), we find 20 type-I models selecting $\tau \simeq 0.24+1.80 i$ and no other apparent clusters. Instead, most low-amplitude type-II models cluster around $\tau \sim-0.45+1.8 i(58 \%), \tau \sim 0.1+1.9 i$ ( $29 \%$ ) or $\tau \sim 0.45+1.8 i(8 \%)$. To close, we present the details of a type-II model fit featuring a value of the modulus VEV within the central cluster.

## C. An $S_{4}$ benchmark

As a benchmark model, we consider the $S_{4}^{(1)}$-specific type-II minimal model denoted by $\left(10^{\prime}, 18\right)$. This modular model may arise as the result of, e.g., the following irrep and weight assignments under $\Gamma_{4}$ :

$$
\begin{align*}
Q & \sim(\mathbf{1}, 9) \oplus\left(\mathbf{1}^{\prime},-1\right) \oplus(\mathbf{1},-9), \\
u^{c}, d^{c} & \sim(\mathbf{1},-9) \oplus\left(\mathbf{1}^{\prime}, \mathbf{1}\right) \oplus(\mathbf{1}, 9), \tag{56}
\end{align*}
$$

resulting in $\left(k_{12}, \mathbf{1}_{12}\right)=\left(10, \mathbf{1}^{\prime}\right),\left(k_{13}, \mathbf{1}_{13}\right)=(18, \mathbf{1})$, and $\left(k_{23}, \mathbf{1}_{23}\right)=\left(8, \mathbf{1}^{\prime}\right)$. The fact that there is no $\mathbf{1}^{\prime}$ modular


FIG. 1. Values of $\tau$ in the fundamental domain selected by minimizing the total amplitude for each type-I minimal model. 52 zero-excess models are not shown. Points shown in blue (grey) correspond to $\mathrm{amp}_{\text {tot }}<3$ ( $\mathrm{amp}_{\text {tot }}>3$ ). The lowest-amplitude model is highlighted.


FIG. 2. The same as in Fig. 1, but for type-II minimal models. All 231 models minimize the total amplitude with non-zero excess. Both the lowest-amplitude model and the benchmark model of Section V C are highlighted.
form of weight 8 guarantees the vanishing of the $(2,3)$ entries in both mass matrices $M_{q}$.

We present the found fit point for this model minimizing the total logarithmic amplitude, which was obtained for $\tau \simeq 0.06+1.69 i$, as indicated in Fig. 2 The values of superpotential parameters are (cf. Eq. (48b))

$$
\begin{align*}
& \alpha_{2}^{u} \simeq 1.00 \alpha_{1}^{u}, \quad \alpha_{3}^{u} \simeq 1.65 \alpha_{1}^{u}, \quad \alpha_{12}^{u} \simeq 1.27 \alpha_{1}^{u}, \\
& \alpha_{13,1}^{u} \simeq 1.47 \alpha_{1}^{u}, \quad \alpha_{13,2}^{u} \simeq 1.61 \alpha_{1}^{u}, \\
& \alpha_{2}^{d} \simeq 0.48 \alpha_{1}^{d}, \quad \alpha_{3}^{d} \simeq 0.05 \alpha_{1}^{d}, \quad \alpha_{12}^{d} \simeq 7.92 \alpha_{1}^{d},  \tag{57}\\
& \alpha_{13,1}^{d} \simeq-7.90 \alpha_{1}^{d}, \quad \alpha_{13,2}^{d} \simeq 0.05 \alpha_{1}^{d},
\end{align*}
$$

leading to the logarithmic amplitudes

$$
\begin{array}{ll}
\mathrm{amp}_{\mathrm{tot}} \simeq 2.46, & \mathrm{amp}_{\mathrm{tot}, 0} \simeq 1.56  \tag{58}\\
\mathrm{amp}_{u} \simeq 0.22, & \mathrm{amp}_{d} \simeq 2.24 .
\end{array}
$$

These values signal the fact that the hierarchy budget of the model is mostly associated with down sector constants, as can be checked from Eq. (57). The resulting values for the observables are

$$
\begin{align*}
& m_{u} / m_{c} \simeq 4.41 \times 10^{-3}, \quad m_{c} / m_{t} \simeq 2.67 \times 10^{-3} \\
& m_{d} / m_{s} \simeq 5.61 \times 10^{-2}, \quad m_{s} / m_{b} \simeq 1.45 \times 10^{-2}  \tag{59}\\
& \theta_{12}^{q} \simeq 13.2^{\circ}, \theta_{23}^{q} \simeq 2.10^{\circ}, \quad \theta_{13}^{q} \simeq 0.20^{\circ}, \delta \simeq 71.2^{\circ}
\end{align*}
$$

corresponding to a successful fit of GUT-scale quark data at the $2.9 \sigma$ level.

## VI. SUMMARY AND CONCLUSIONS

Modular symmetry can be used as a powerful generalization of traditional flavour symmetries. It may shed light on the observed patterns of fundamental fermion mixing and clarify the origins of fermion mass hierarchies and CP violation (CPV). Moreover, it has been shown that it may address the apparent absence of CPV in the strong sector, i.e. resolve the strong CP problem with high quality [20].

Here, we have identified the most general structures of modular weights (summarized in Eq. (18)) which allow for a large CKM phase $\delta$ while ensuring the vanishing of the QCD angle $\bar{\theta}$. The crucial condition of anomaly cancellation - implying zero modular weights for mass matrix determinants - together with the requirement of no massless quarks, severely restricts such structures. Note that this mechanism is, in essence, incompatible with the flavon-free mechanism of generating fermion mass hierarchies [13], as the latter requires non-zero weights for mass matrix determinants.

We have also generalized the existing approach, based on using the full modular group, to the finite modular groups $\Gamma_{N}^{\prime}$. In doing so, one naturally gains access to i) modular forms aside from trivial singlets and to ii) the
possibility of forbidding certain (zero-weight) entries of the mass matrices by appropriate choices of quark irreps.

A priori, one expects it may be possible to build viable models in this context involving quark irreps of dimension higher than one. We show this is not the case. Namely, the only potentially viable scenario of this kind turns out to be the one where all quark fields split into a doublet-plus-singlet configuration (see Section IV). This setup is amenable to analytic treatment, which however reveals an unavoidable conflict with quark data, either with the values of $\left|V_{u s}\right|$ and $\left|V_{u b}\right|$ or with $\left.\left|V_{c b}\right|\right|^{13}$ Hence, only scenarios where quarks furnish 1D irreps of the finite modular group have a chance to be phenomenologically viable.

In Section V, we classify the potentially viable (singletbased) quark mass structures, finding 5 inequivalent textures, see Eqs. 48 . We then focus on the models which feature a minimal or next-to-minimal number of real superpotential parameters, 12 and 14 respectively. These models are systematically identified and the corresponding landscape is presented in Table II. Finally, we numerically scan the minimal model landscape. While there is enough freedom to fit the data, this may require hierarchical superpotential parameters. Asking for the smallest possible hierarchies reveals some preferences for certain regions within the fundamental modular domain (see Figs. 11 and 2). An explicit $S_{4}$-based benchmark is presented in Section VC.

At present, finding restrictive but viable modularsymmetric models of quark masses and mixing which naturally address both the strong CP problem and the origin of quark mass hierarchies and CPV remains an interesting open challenge.

Note added: While the numerical scan was being finalized, 53 appeared on the arXiv, exploring in detail one of the two $A_{4}^{(\prime)}$-exclusive (type-III) next-to-minimal models denoted here by $\left(4,12^{\prime}\right)$, cf. Table $I$.

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[^10]
## Appendix A: Modular transformation of determinants

Here we show explicitly that the determinant of a mass matrix $M$ in modular constructions is a singlet (i.e. a 1-dimensional) modular form of weight $k_{\mathrm{det}}=\sum_{i} k_{i}+k_{i}^{c}$. We start by noticing that, to guarantee modular invariance, the elements of the mass matrix $M$ - which are obtained from modular forms - must, in general, trans-
form in the LR convention we employ as (cf. Eq. (3)):

$$
\begin{align*}
M_{i j}(\tau) & \xrightarrow{\gamma} M_{i j}(\gamma \tau) \\
& =(c \tau+d)^{k_{i}+k_{j}^{c}} \rho_{i k}^{*}(\gamma) M_{k l}(\tau) \rho_{j l}^{c *}(\gamma) \tag{A1}
\end{align*}
$$

We can now look into how the determinant of the mass matrix changes under a modular transformation:

$$
\begin{align*}
& \operatorname{det} M=\frac{1}{n!} \varepsilon_{i_{1} \ldots i_{n}} \varepsilon_{j_{1} \ldots j_{n}} M_{i_{1} j_{1}} \ldots M_{i_{n} j_{n}} \\
& \xrightarrow{\gamma} \frac{1}{n!} \varepsilon_{i_{1} \ldots i_{n}} \varepsilon_{j_{1} \ldots j_{n}}(c \tau+d)^{k_{i_{1}}+k_{j_{1}}^{c}} \rho_{i_{1} k_{1}}^{*} \rho_{j_{1} l_{1}}^{c *} M_{k_{1} l_{1}} \\
& \ldots \\
&(c \tau+d)^{k_{i_{n}}+k_{j_{n}}^{c} \rho_{i_{n} k_{n}}^{*} \rho_{j_{n} l_{n}}^{c *} M_{k_{n} l_{n}}}  \tag{A2}\\
&=(c \tau+d)^{\sum_{i} k_{i}+k_{i}^{c}} \frac{1}{n!}\left(\varepsilon_{i_{1} \ldots i_{n}} \rho_{i_{1} k_{1}}^{*} \ldots \rho_{i_{n} k_{n}}^{*}\right)\left(\varepsilon_{j_{1} \ldots j_{n}} \rho_{j_{1} l_{1}}^{c *} \ldots \rho_{j_{n} l_{n}}^{c *}\right) M_{k_{1} l_{1}} \ldots M_{k_{n} l_{n}} \\
&=(c \tau+d)^{\sum_{i} k_{i}+k_{i}^{c}} \frac{1}{n!}\left(\operatorname{det} \rho^{*} \varepsilon_{k_{1} \ldots k_{n}}\right)\left(\operatorname{det} \rho^{c *} \varepsilon_{l_{1} \ldots l_{n}}\right) M_{k_{1} l_{1} \ldots M_{k_{n} l_{n}}} \\
&=(c \tau+d)^{k_{\operatorname{det}}} \underbrace{(\operatorname{det} \rho)^{*}\left(\operatorname{det} \rho^{c}\right)^{*}}_{\gamma \text {-dependent phase }} \operatorname{det} M,
\end{align*}
$$

where in the last line we have defined $k_{\text {det }} \equiv \sum_{i} k_{i}+k_{i}^{c}$. Alternatively, one can reach the same result by noticing that $M(\gamma \tau)$ can be cast in the form

$$
M(\gamma \tau)=\left(\begin{array}{lll}
(c \tau+d)^{k_{1}} & &  \tag{A3}\\
& \ddots & \\
& & (c \tau+d)^{k_{n}}
\end{array}\right) \rho^{*}(\gamma) M(\tau) \rho^{c \dagger}(\gamma)\left(\begin{array}{lll}
(c \tau+d)^{k_{1}^{c}} & & \\
& \ddots & \\
& & (c \tau+d)^{k_{n}^{c}}
\end{array}\right)
$$

and by taking a determinant. Since the determinant of a product of matrices equals the product of their determinants, one finds $\operatorname{det} M(\gamma \tau)=(c \tau+d)^{k_{\operatorname{det}}}(\operatorname{det} \rho)^{*}\left(\operatorname{det} \rho^{c}\right)^{*} \operatorname{det} M(\tau)$ directly.

Having found this transformation property, we verify that $\operatorname{det} M$ is a singlet modular form of weight $k_{\text {det }}$. Note that it need not be a trivial singlet of the finite modular group, given the presence of a $\gamma$-dependent phase factor. Indeed, consider as an example the mass matrix $M_{e}^{\dagger}$ of Ref. [32], obtained from level 4 modular forms, whose determinant is given in equation (E.4) therein. One can check then that, under $\tau \rightarrow \tau+1$ ( $T$ transformation), this determinant acquires a $-i$ factor, meaning it furnishes the (non-trivial) $\hat{\mathbf{1}}$ representation of $\Gamma_{4}^{\prime}$.

## Appendix B: Singlet modular forms

In this appendix we provide a list of all singlet (1D) modular forms available at weights up to 36 , for each level $N \leq 5$. The irreps available at each weight $w$ are summarized in Table III. Note that $A_{5}^{\prime}$ singlets coincide
with the trivial ones available for all $\Gamma_{N}^{\prime}$, as no other 1D irreps exist for this group. These "trivial singlet" forms are the well-known $S L(2, \mathbb{Z})$ modular forms generated as polynomials in $E_{4}$ and $E_{6}$, with

$$
\begin{equation*}
E_{2 n}(\tau)=\frac{1}{2 \zeta(2 n)} \sum_{\substack{m, m^{\prime} \in \mathbb{Z} \\\left(m, m^{\prime}\right) \neq(0,0)}}\left(m+m^{\prime} \tau\right)^{-2 n} \tag{B1}
\end{equation*}
$$

where $n>1$ is an integer. At a given weight $w>0$, the dimension of the linear space $\mathcal{M}_{w}(\Gamma)$ of these forms is given by

$$
\operatorname{dim} \mathcal{M}_{w}(\Gamma)= \begin{cases}\lfloor w / 12\rfloor, & \text { if } w \equiv 2 \bmod 12  \tag{B2}\\ \lfloor w / 12\rfloor+1, & \text { if } w \not \equiv 2 \bmod 12\end{cases}
$$

in agreement with the first data column of Table III.
Below we present the first few terms in the $q$ expansions, with $q \equiv \exp (2 \pi i \tau)$, of each of the singlet
forms $Y_{1^{*}}$ whose irreps have been collected in Table III. The listed terms are representative: they allow for the unambiguous identification of each form, but are in general insufficient to compute the forms to a good precision in all points of the fundamental domain $\mathcal{D}$.

The fact that $q$-expansion coefficients are real is consistent with the requirement of real superpotential parameters in a gCP-invariant modular theory [17]. Each form is normalized according to the global (or integral) normalization based on the Petersson inner product, as advocated in Ref. [52]. Namely, one has $\left|\mathrm{N}\left[Y_{\mathbf{1}^{*}}\right]\right|=1$ for all these forms, with

$$
\begin{equation*}
\mathrm{N}\left[Y_{\mathbf{1}^{*}}^{(w)}(\tau)\right]^{2} \equiv \int_{\mathcal{D}}\left|Y_{\mathbf{1}^{*}}^{(w)}(x+i y)\right|^{2}(2 y)^{w} \frac{d x d y}{y^{2}} \tag{B3}
\end{equation*}
$$

with $x=\operatorname{Re} \tau$ and $y=\operatorname{Im} \tau$. Equation (B3) is valid for cusp (singlet) modular forms, i.e. forms vanishing in the limit $q \rightarrow 0$, in which case the integration is performed over the fundamental domain. For non-cusp (singlet) forms of weight $w>1$, it should be modified to

$$
\begin{align*}
& \overline{\mathrm{N}}\left[Y_{\mathbf{1}^{*}}^{(w)}(\tau)\right]^{2} \equiv \lim _{T \rightarrow \infty}\left(\int_{\mathcal{D}_{T}}\left|Y_{\mathbf{1}^{*}}^{(w)}\right|^{2}(2 y)^{w} \frac{d x d y}{y^{2}}\right.  \tag{B4}\\
&\left.-\left|a_{0}\right|^{2} \frac{2^{w}}{w-1} T^{w-1}\right)
\end{align*}
$$

where $a_{0}$ is the constant term in the $q$-expansion of $Y_{\mathbf{1}^{*}}$ and $\mathcal{D}_{T}$ is the truncated fundamental domain, with $y<T$ (see also [54]).

Finally, whenever more than one form is available for a given level, weight and irrep, one is free to choose the basis of the corresponding linear space. Particular basis choices are made below. However, we note that any such choice in general does not commute with the normalization procedure.

$$
\begin{aligned}
& 1 \mathrm{D} \text { trivial forms, available for all } \Gamma_{N}^{\prime} \\
& Y_{1}^{(4)} \simeq 0.489\left(1+240 q+2160 q^{2}+6720 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1}^{(6)} \simeq 1.04\left(1-504 q-16632 q^{2}-122976 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1}^{(8)} \simeq 0.333\left(1+480 q+61920 q^{2}+1050240 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1}^{(10)} \simeq 0.277\left(1-264 q-135432 q^{2}-5196576 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(12)} \simeq 15.4\left(q-24 q^{2}+252 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(12)} \simeq 0.0145\left(1-1008 q+220752 q^{2}+16519104 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1}^{(14)} \simeq 0.110\left(1-24 q-196632 q^{2}-38263776 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(16)} \simeq 2.65\left(q+216 q^{2}-3348 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(16)} \simeq 3.49 \times 10^{-3}\left(1-768 q-19008 q^{2}+67329024 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(18)} \simeq 0.911\left(q-528 q^{2}-4284 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(18)} \simeq 4.05 \times 10^{-3}\left(1+216 q-200232 q^{2}-85500576 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(20)} \simeq 0.340\left(q+456 q^{2}+50652 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(20)} \simeq 6.46 \times 10^{-4}\left(1-528 q-201168 q^{2}+61114944 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(22)} \simeq 0.109\left(q-288 q^{2}-128844 q^{3}+\mathcal{O}\left(q^{4}\right)\right),
\end{aligned}
$$

| $w$ | All $\Gamma_{N}^{\prime}$ | $S_{3}$ only | $A_{4}^{\prime}$ only | $S_{4}^{\prime}$ only |
| :---: | :---: | :---: | :---: | :---: |
| 1 | - | - | - | - |
| 2 | - | - | - | - |
| 3 | - | - | - | $\hat{1}^{\prime}$ |
| 4 | 1 | - | $1^{\prime}$ | - |
| 5 | - | - | - | - |
| 6 | 1 | $1^{\prime}$ | - | $1{ }^{\prime}$ |
| 7 | - | - | - | $\hat{1}^{\prime}$ |
| 8 | 1 | - | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | - |
| 9 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}$ |
| 10 | 1 | $1^{\prime}$ | $1^{\prime}$ | $1^{\prime}$ |
| 11 | - | - | - | $\hat{1}^{\prime}$ |
| 12 | 1, 1 | $1^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | $1{ }^{\prime}$ |
| 13 | - | - | - | $\hat{\mathbf{1}}, \hat{1}^{\prime}$ |
| 14 | 1 | $1^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | $1{ }^{\prime}$ |
| 15 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 16 | 1,1 | $1^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | $1{ }^{\prime}$ |
| 17 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}$ |
| 18 | 1,1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ |
| 19 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 20 | 1,1 | $1^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}$ |
| 21 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 22 | 1,1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}$ |
| 23 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 24 | 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}$ |
| 25 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 26 | 1,1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}$ |
| 27 | - | - |  | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 28 | 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}$ |
| 29 | - | - | - ${ }^{\prime}$ | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 30 | 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ |
| 31 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 32 | 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}$ |
| 33 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 34 | 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $1^{\prime}, 1^{\prime}, 1^{\prime}$ |
| 35 | - | - | - | $\hat{\mathbf{1}}, \hat{\mathbf{1}}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}, \hat{\mathbf{1}}^{\prime}$ |
| 36 | 1, 1, 1, 1 | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}, \mathbf{1}^{\prime \prime}$ | $\mathbf{1}^{\prime}, \mathbf{1}^{\prime}, \mathbf{1}^{\prime}$ |

TABLE III. Singlet modular form irreps available at a given weight $w \leq 36$ for the $\Gamma_{N}^{\prime}$ groups ( $N \leq 5$ ). Hatted (unhatted) irreps are in correspondence with odd (even) weights. For weights $w \geq 24(w \geq 36)$ one is sure to have at least two (three) independent modular forms furnishing each possible singlet irrep.

$$
\begin{aligned}
& Y_{1,2}^{(22)} \simeq 2.38 \times 10^{-4}\left(1+456 q-146232 q^{2}-133082976 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(24)} \simeq 145\left(q^{2}-48 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(24)} \simeq 0.0316\left(q+696 q^{2}+162252 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,3}^{(24)} \simeq 1.08 \times 10^{-4}\left(1-288 q-325728 q^{2}+11700864 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(26)} \simeq 8.51 \times 10^{-3}\left(q-48 q^{2}-195804 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(26)} \simeq 1.22 \times 10^{-5}\left(1+696 q-34632 q^{2}-167186976 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(28)} \simeq 29.3\left(q^{2}+192 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,2}^{(28)} \simeq 2.09 \times 10^{-3}\left(q+936 q^{2}+331452 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,3}^{(28)} \simeq 3.77 \times 10^{-5}\left(1-48 q-392688 q^{2}-67089216 q^{3}+\mathcal{O}\left(q^{4}\right)\right), \\
& Y_{1,1}^{(30)} \simeq 10.2\left(q^{2}-552 q^{3}+\mathcal{O}\left(q^{4}\right)\right),
\end{aligned}
$$

```
\(Y_{1,2}^{(30)} \simeq 4.79 \times 10^{-4}\left(q+192 q^{2}-205164 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,3}^{(30)} \simeq 5.12 \times 10^{-7}\left(1+936 q+134568 q^{2}-173988576 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,1}^{(32)} \simeq 4.83\left(q^{2}+432 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,2}^{(32)} \simeq 1.02 \times 10^{-4}\left(q+1176 q^{2}+558252 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,3}^{(32)} \simeq 5.31 \times 10^{-7}\left(1+192 q-402048 q^{2}-161431296 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,1}^{(34)} \simeq 1.88\left(q^{2}-312 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,2}^{(34)} \simeq 2.03 \times 10^{-5}\left(q+432 q^{2}-156924 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,3}^{(34)} \simeq 1.73 \times 10^{-8}\left(1+1176 q+361368 q^{2}-139663776 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,1}^{(36)} \simeq 1290\left(q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,2}^{(36)} \simeq 0.683\left(q^{2}+672 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,3}^{(36)} \simeq 3.82 \times 10^{-6}\left(q+1416 q^{2}+842652 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\),
\(Y_{1,4}^{(36)} \simeq 8.85 \times 10^{-9}\left(1+432 q-353808 q^{2}-257501376 q^{3}+\mathcal{O}\left(q^{4}\right)\right)\).
```

1D forms exclusive to $\Gamma_{2} \simeq S_{3}$

$$
\begin{aligned}
& Y_{1^{\prime}}^{(6)} \simeq 4.72\left(\sqrt{q}-12 q^{3 / 2}+54 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}}^{(10)} \simeq 0.661\left(\sqrt{q}+228 q^{3 / 2}-666 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{1^{\prime}}^{(12)} \simeq 0.199\left(\sqrt{q}-516 q^{3 / 2}-10530 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}}^{(14)} \simeq 0.0551\left(\sqrt{q}+468 q^{3 / 2}+56214 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}}^{(16)} \simeq 0.0128\left(\sqrt{q}-276 q^{3 / 2}-132210 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(18)} \simeq 2.59 \times 10^{-3}\left(\sqrt{q}+708 q^{3 / 2}+170694 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(18)} \simeq 2.59 \times 10^{-3}\left(\sqrt{q}-1020 q^{3 / 2}+232902 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}}^{(20)} \simeq 4.66 \times 10^{-4}\left(\sqrt{q}-36 q^{3 / 2}-196290 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(22)} \simeq 7.52 \times 10^{-5}\left(\sqrt{q}+948 q^{3 / 2}+342774 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(22)} \simeq 7.52 \times 10^{-5}\left(\sqrt{q}-780 q^{3 / 2}-9738 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(24)} \simeq 1.09 \times 10^{-5}\left(\sqrt{q}-1524 q^{3 / 2}+730350 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{1^{\prime}, 2}^{(24)} \simeq 1.09 \times 10^{-5}\left(\sqrt{q}+204 q^{3 / 2}-202770 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(26)} \simeq 1.47 \times 10^{-6}\left(\sqrt{q}+1188 q^{3 / 2}+572454 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(26)} \simeq 1.47 \times 10^{-6}\left(\sqrt{q}-540 q^{3 / 2}-194778 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(28)} \simeq 1.81 \times 10^{-7}\left(\sqrt{q}-1284 q^{3 / 2}+366750 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(28)} \simeq 1.81 \times 10^{-7}\left(\sqrt{q}+444 q^{3 / 2}-151650 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(30)} \simeq 2.07 \times 10^{-8}\left(\sqrt{q}+1428 q^{3 / 2}+859734 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{1^{\prime}, 2}^{(30)} \simeq 2.07 \times 10^{-8}\left(\sqrt{q}-2028 q^{3 / 2}+1481814 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 3}^{(30)} \simeq 2.07 \times 10^{-8}\left(\sqrt{q}-300 q^{3 / 2}-322218 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(32)} \simeq 2.20 \times 10^{-9}\left(\sqrt{q}-1044 q^{3 / 2}+60750 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(32)} \simeq 2.20 \times 10^{-9}\left(\sqrt{q}+684 q^{3 / 2}-42930 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(34)} \simeq 2.19 \times 10^{-10}\left(\sqrt{q}+1668 q^{3 / 2}+1204614 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(34)} \simeq 2.19 \times 10^{-10}\left(\sqrt{q}-60 q^{3 / 2}-392058 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 3}^{(34)} \simeq 2.19 \times 10^{-10}\left(\sqrt{q}-1788 q^{3 / 2}+997254 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 1}^{(36)} \simeq 2.06 \times 10^{-11}\left(\sqrt{q}-804 q^{3 / 2}-187650 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 2}^{(36)} \simeq 2.06 \times 10^{-11}\left(\sqrt{q}-2532 q^{3 / 2}+2487294 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{1^{\prime}, 3}^{(36)} \simeq 2.06 \times 10^{-11}\left(\sqrt{q}+924 q^{3 / 2}+123390 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) .
\end{aligned}
$$

1D forms exclusive to $\Gamma_{3}^{\prime} \simeq A_{4}^{\prime}$
$Y_{\mathbf{1}^{\prime}}^{(4)} \simeq 3.10\left(q^{1 / 3}-8 q^{4 / 3}+20 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}}^{(8)} \simeq 0.372\left(q^{1 / 3}+232 q^{4 / 3}+260 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime \prime}}^{(8)} \simeq 7.05\left(q^{2 / 3}-16 q^{5 / 3}+104 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}}^{(10)} \simeq 0.0976\left(q^{1 / 3}-512 q^{4 / 3}-12580 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}}^{(12)} \simeq 0.0217\left(q^{1 / 3}+472 q^{4 / 3}+58100 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}}^{(12)} \simeq 1.09\left(q^{2 / 3}+224 q^{5 / 3}-1576 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}}^{(14)} \simeq 3.94 \times 10^{-3}\left(q^{1 / 3}-272 q^{4 / 3}-133300 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime \prime}}^{(14)} \simeq 0.351\left(q^{2 / 3}-520 q^{5 / 3}-8464 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(16)} \simeq 32.8\left(q^{4 / 3}-32 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(16)} \simeq 6.12 \times 10^{-4}\left(q^{1 / 3}+712 q^{4 / 3}+173540 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime \prime}}^{(16)} \simeq 0.111\left(q^{2 / 3}+464 q^{5 / 3}+54344 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}}^{(18)} \simeq 8.28 \times 10^{-5}\left(q^{1 / 3}-32 q^{4 / 3}-196420 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}}^{(18)} \simeq 0.0300\left(q^{2 / 3}-280 q^{5 / 3}-131104 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 1}^{(20)} \simeq 6.07\left(q^{4 / 3}+208 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 2}^{(20)} \simeq 9.91 \times 10^{-6}\left(q^{1 / 3}+952 q^{4 / 3}+346580 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(20)} \simeq 69.1\left(q^{5 / 3}-40 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(20)} \simeq 7.16 \times 10^{-3}\left(q^{2 / 3}+704 q^{5 / 3}+167864 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(22)} \simeq 2.13\left(q^{4 / 3}-536 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(22)} \simeq 1.06 \times 10^{-6}\left(q^{1 / 3}+208 q^{4 / 3}-201940 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime \prime}}^{(22)} \simeq 1.54 \times 10^{-3}\left(q^{2 / 3}-40 q^{5 / 3}-196144 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 1}^{(24)} \simeq 0.879\left(q^{4 / 3}+448 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(24)} \simeq 1.03 \times 10^{-7}\left(q^{1 / 3}+1192 q^{4 / 3}+577220 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(24)} \simeq 13.5\left(q^{5 / 3}+200 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(24)} \simeq 3.00 \times 10^{-4}\left(q^{2 / 3}+944 q^{5 / 3}+338984 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(26)} \simeq 0.310\left(q^{4 / 3}-296 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(26)} \simeq 9.25 \times 10^{-9}\left(q^{1 / 3}+448 q^{4 / 3}-149860 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(26)} \simeq 4.73\left(q^{5 / 3}-544 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(26)} \simeq 5.36 \times 10^{-5}\left(q^{2 / 3}+200 q^{5 / 3}-203584 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(28)} \simeq 301\left(q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(28)} \simeq 0.100\left(q^{4 / 3}+688 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 3}^{(28)} \simeq 7.60 \times 10^{-10}\left(q^{1 / 3}+1432 q^{4 / 3}+865460 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(28)} \simeq 2.10\left(q^{5 / 3}+440 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(28)} \simeq 8.80 \times 10^{-6}\left(q^{2 / 3}+1184 q^{5 / 3}+567704 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(30)} \simeq 0.0311\left(q^{4 / 3}-56 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(30)} \simeq 5.79 \times 10^{-11}\left(q^{1 / 3}+688 q^{4 / 3}-40180 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(30)} \simeq 0.788\left(q^{5 / 3}-304 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(30)} \simeq 1.34 \times 10^{-6}\left(q^{2 / 3}+440 q^{5 / 3}-153424 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(32)} \simeq 63.0\left(q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 2}^{(32)} \simeq 8.75 \times 10^{-3}\left(q^{4 / 3}+928 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 3}^{(32)} \simeq 4.11 \times 10^{-12}\left(q^{1 / 3}-56 q^{4 / 3}-392284 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(32)} \simeq 623\left(q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(32)} \simeq 0.273\left(q^{5 / 3}+680 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 3}^{(32)} \simeq 1.90 \times 10^{-7}\left(q^{2 / 3}+1424 q^{5 / 3}+854024 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 1}^{(34)} \simeq 21.7\left(q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(34)} \simeq 2.34 \times 10^{-3}\left(q^{4 / 3}+184 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{1^{\prime}, 3}}^{(34)} \simeq 2.73 \times 10^{-13}\left(q^{1 / 3}+928 q^{4 / 3}+127100 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(34)} \simeq 0.0933\left(q^{5 / 3}-64 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(34)} \simeq 2.53 \times 10^{-8}\left(q^{2 / 3}+680 q^{5 / 3}-45664 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 1}^{(36)} \simeq 10.8\left(q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 2}^{(36)} \simeq 5.86 \times 10^{-4}\left(q^{4 / 3}+1168 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime}, 3}^{(36)} \simeq 1.70 \times 10^{-14}\left(q^{1 / 3}+184 q^{4 / 3}-403564 q^{7 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 1}^{(36)} \simeq 134\left(q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 2}^{(36)} \simeq 0.0287\left(q^{5 / 3}-808 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$,
$Y_{1^{\prime \prime}, 3}^{(36)} \simeq 3.16 \times 10^{-9}\left(q^{2 / 3}+1664 q^{5 / 3}+1197944 q^{8 / 3}+\mathcal{O}\left(q^{10 / 3}\right)\right)$.
1D forms exclusive to $\Gamma_{4}^{\prime} \simeq S_{4}^{\prime}$

$$
\begin{aligned}
& Y_{\hat{1}^{\prime}}^{(3)} \simeq 2.49\left(q^{1 / 4}-6 q^{5 / 4}+9 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{1^{\prime}}^{(6)} \simeq 4.72\left(\sqrt{q}-12 q^{3 / 2}+54 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{\hat{1}^{\prime}}^{(7)} \simeq 0.261\left(q^{1 / 4}+234 q^{5 / 4}+729 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{\hat{1}}^{(9)} \simeq 8.59\left(q^{3 / 4}-18 q^{7 / 4}+135 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{\hat{1}^{\prime}}^{(9)} \simeq 0.0605\left(q^{1 / 4}-510 q^{5 / 4}-13599 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{1^{\prime}}^{(10)} \simeq 0.661\left(\sqrt{q}+228 q^{3 / 2}-666 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{\hat{1}^{\prime}}^{(11)} \simeq 0.0112\left(q^{1 / 4}+474 q^{5 / 4}+59049 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{1^{\prime}}^{(12)} \simeq 0.199\left(\sqrt{q}-516 q^{3 / 2}-10530 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{\hat{\mathbf{1}}}^{(13)} \simeq 1.37\left(q^{3 / 4}+222 q^{7 / 4}-2025 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{\hat{\mathbf{1}}^{\prime}}^{(13)} \simeq 1.68 \times 10^{-3}\left(q^{1 / 4}-270 q^{5 / 4}-133839 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{1^{\prime}}^{(14)} \simeq 0.0551\left(\sqrt{q}+468 q^{3 / 2}+56214 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{\hat{1}}^{(15)} \simeq 0.453\left(q^{3 / 4}-522 q^{7 / 4}-7425 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{\hat{1}^{\prime}, 1}^{(15)} \simeq 27.1\left(q^{5 / 4}-30 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right), \\
& Y_{\hat{1}^{\prime}, 2}^{(15)} \simeq 2.11 \times 10^{-4}\left(q^{1 / 4}-1014 q^{5 / 4}+226809 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{1^{\prime}}^{(16)} \simeq 0.0128\left(\sqrt{q}-276 q^{3 / 2}-132210 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{\hat{1}}^{(17)} \simeq 0.151\left(q^{3 / 4}+462 q^{7 / 4}+53415 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{\hat{1}^{\prime}}^{(17)} \simeq 2.29 \times 10^{-5}\left(q^{1 / 4}-30 q^{5 / 4}-196479 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{1^{\prime}, 1}^{(18)} \simeq 47.6\left(q^{3 / 2}-36 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{1^{\prime}, 2}^{(18)} \simeq 2.59 \times 10^{-3}\left(\sqrt{q}-1020 q^{3 / 2}+232902 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right) \text {, } \\
& Y_{\hat{1}}^{(19)} \simeq 0.0430\left(q^{3 / 4}-282 q^{7 / 4}-130545 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(19)} \simeq 4.96\left(q^{5 / 4}+210 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{\hat{1}^{\prime}, 2}^{(19)} \simeq 2.18 \times 10^{-6}\left(q^{1 / 4}-774 q^{5 / 4}-14391 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, } \\
& Y_{1^{\prime}}^{(20)} \simeq 4.66 \times 10^{-4}\left(\sqrt{q}-36 q^{3 / 2}-196290 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right), \\
& Y_{\hat{\mathbf{i}}, 1}^{(21)} \simeq 0.0109\left(q^{3 / 4}+702 q^{7 / 4}+166455 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right) \text {, }
\end{aligned}
$$

$Y_{\hat{\mathbf{1}}, 2}^{(21)} \simeq 0.0108\left(q^{3 / 4}-1026 q^{7 / 4}+239031 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(21)} \simeq 1.73\left(q^{5 / 4}-534 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}^{\prime}, 2}^{(21)} \simeq 1.85 \times 10^{-7}\left(q^{1 / 4}+210 q^{5 / 4}-201519 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(22)} \simeq 9.07\left(q^{3 / 2}+204 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(22)} \simeq 7.52 \times 10^{-5}\left(\sqrt{q}-780 q^{3 / 2}-9738 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{\mathbf{1}}}^{(23)} \simeq 2.51 \times 10^{-3}\left(q^{3 / 4}-42 q^{7 / 4}-196065 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(23)} \simeq 0.699\left(q^{5 / 4}+450 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 2}^{(23)} \simeq 1.42 \times 10^{-8}\left(q^{1 / 4}-534 q^{5 / 4}-197991 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(24)} \simeq 3.19\left(q^{3 / 2}-540 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(24)} \simeq 1.09 \times 10^{-5}\left(\sqrt{q}+204 q^{3 / 2}-202770 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{\mathbf{1}}, 1}^{(25)} \simeq 16.4\left(q^{7 / 4}+198 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}, 2}^{(25)} \simeq 5.27 \times 10^{-4}\left(q^{3 / 4}+942 q^{7 / 4}+337095 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}^{\prime}, 1}^{(25)} \simeq 0.242\left(q^{5 / 4}-294 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}^{\prime}, 2}^{(25)} \simeq 9.92 \times 10^{-10}\left(q^{1 / 4}+450 q^{5 / 4}-148959 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(26)} \simeq 1.37\left(q^{3 / 2}+444 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(26)} \simeq 1.47 \times 10^{-6}\left(\sqrt{q}-540 q^{3 / 2}-194778 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{\mathbf{1}}, 1}^{(27)} \simeq 5.75\left(q^{7 / 4}-546 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{i}}, 2}^{(27)} \simeq 1.01 \times 10^{-4}\left(q^{3 / 4}+198 q^{7 / 4}-203985 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\mathbf{1}^{\prime}, 1}^{(27)} \simeq 251\left(q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 2}^{(27)} \simeq 0.0766\left(q^{5 / 4}+690 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 3}^{(27)} \simeq 6.36 \times 10^{-11}\left(q^{1 / 4}-294 q^{5 / 4}-323991 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(28)} \simeq 0.499\left(q^{3 / 2}-300 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(28)} \simeq 1.81 \times 10^{-7}\left(\sqrt{q}+444 q^{3 / 2}-151650 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{\mathbf{1}}, 1}^{(29)} \simeq 2.60\left(q^{7 / 4}+438 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}, 2}^{(29)} \simeq 1.80 \times 10^{-5}\left(q^{3 / 4}-546 q^{7 / 4}-191529 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(29)} \simeq 0.0230\left(q^{5 / 4}-54 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 2}^{(29)} \simeq 3.77 \times 10^{-12}\left(q^{1 / 4}+690 q^{5 / 4}-38799 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(30)} \simeq 433\left(q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(30)} \simeq 0.168\left(q^{3 / 2}+684 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 3}^{(30)} \simeq 2.07 \times 10^{-8}\left(\sqrt{q}-300 q^{3 / 2}-322218 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{1}, 1}^{(31)} \simeq 0.984\left(q^{7 / 4}-306 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}, 2}^{(31)} \simeq 2.98 \times 10^{-6}\left(q^{3 / 4}+438 q^{7 / 4}-154305 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(31)} \simeq 52.1\left(q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 2}^{(31)} \simeq 6.31 \times 10^{-3}\left(q^{5 / 4}-798 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{\mathbf{1}}^{\prime}, 3}^{(31)} \simeq 2.07 \times 10^{-13}\left(q^{1 / 4}-54 q^{5 / 4}-392391 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{1^{\prime}, 1}^{(32)} \simeq 0.0548\left(q^{3 / 2}-60 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{1^{\prime}, 2}^{(32)} \simeq 2.20 \times 10^{-9}\left(\sqrt{q}+684 q^{3 / 2}-42930 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)$,
$Y_{\hat{\mathbf{1}}, 1}^{(33)} \simeq 747\left(q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\mathbf{1}, 2}^{(33)} \simeq 0.346\left(q^{7 / 4}+678 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,
$Y_{\hat{1}, 3}^{(33)} \simeq 4.61 \times 10^{-7}\left(q^{3 / 4}-306 q^{7 / 4}-320409 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)$,

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\(Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(33)} \simeq 18.0\left(q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{1}^{\prime}, 2}^{(33)} \simeq 1.63 \times 10^{-3}\left(q^{5 / 4}+186 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{1}^{\prime}, 3}^{(33)} \simeq 1.07 \times 10^{-14}\left(q^{1 / 4}+930 q^{5 / 4}+128961 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{1^{\prime}, 1}^{(34)} \simeq 92.1\left(q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\),
\(Y_{1^{\prime}, 2}^{(34)} \simeq 0.0162\left(q^{3 / 2}-804 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\),
\(Y_{1^{\prime}, 3}^{(34)} \simeq 2.19 \times 10^{-10}\left(\sqrt{q}-60 q^{3 / 2}-392058 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\),
\(Y_{\hat{\mathbf{1}}, 1}^{(35)} \simeq 0.120\left(q^{7 / 4}-66 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{\mathbf{i}}, 2}^{(35)} \simeq 6.68 \times 10^{-8}\left(q^{3 / 4}+678 q^{7 / 4}-47025 q^{11 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{\mathbf{1}}^{\prime}, 1}^{(35)} \simeq 8.85\left(q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{1}^{\prime}, 2}^{(35)} \simeq 3.94 \times 10^{-4}\left(q^{5 / 4}+1170 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{\hat{1}^{\prime}, 3}^{(35)} \simeq 5.17 \times 10^{-16}\left(q^{1 / 4}+186 q^{5 / 4}-403191 q^{9 / 4}+\mathcal{O}\left(q^{13 / 4}\right)\right)\),
\(Y_{1^{\prime}, 1}^{(36)} \simeq 31.6\left(q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\),
\(Y_{1^{\prime}, 2}^{(36)} \simeq 4.61 \times 10^{-3}\left(q^{3 / 2}+180 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\),
\(Y_{1^{\prime}, 3}^{(36)} \simeq 2.06 \times 10^{-11}\left(\sqrt{q}+924 q^{3 / 2}+123390 q^{5 / 2}+\mathcal{O}\left(q^{7 / 2}\right)\right)\).
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    ${ }^{1}$ See also [26] for a possible modular origin of the axion.

[^1]:    ${ }^{2}$ A CP-conserving mechanism of SUSY breaking is also assumed, resulting in a real gluino mass so that $\bar{\theta}=0$ is preserved.

[^2]:    ${ }^{3}$ In the presence of a massless quark, an additional mechanism might be invoked to generate the up- or down-quark masses, such as SUSY-breaking effects or higher-dimensional operators, along the lines of what is described in Refs. [12, 44]. The latter possibility may require non-trivial weights for $H_{u, d}$. We do not consider such possibilities here.
    ${ }^{4}$ Eq. 4. holds even when the Higgs doublets carry non-zero modular weights $k_{H_{q}}$, which would imply $k_{\text {det }}^{q}=3 k_{H_{q}}+\sum_{i} k_{i}+k_{i}^{c}$. We thank M. Parriciatu for this observation.

[^3]:    5 This is true even if we are dealing with representations of dimension higher than 1 , in which case some weights are forced to be equal and the action of permutations must simply keep the block structure of the mass matrix.

[^4]:    ${ }^{6}$ If a viable structure is found, then its transpose may also be viable a priori.

[^5]:    ${ }^{7}$ Here and in what follows, $\mathbf{r}^{*}$ denotes some unspecified $\Gamma_{N}^{(\prime)}$ irrep of a given dimension. In the case of $\Gamma_{4} \simeq S_{4}$, for example, $\mathbf{1}^{*} \in\left\{\mathbf{1}, \mathbf{1}^{\prime}\right\}, \mathbf{2}^{*}=\mathbf{2}$ (the only possibility), and $\mathbf{3}^{*} \in\left\{\mathbf{3}, \mathbf{3}^{\prime}\right\}$.

[^6]:    ${ }^{8}$ Even though textures I and II resemble those of Eq. 22 in an appropriate basis, they are not automatically excluded (cf. Section IV as they allow different values along the diagonal.

[^7]:    9 This is also why one cannot consistently allow only one non-zero positive-weight entry in texture IV (similarly for V).
    10 Note that the VEV of $\tau$ may be shared with the lepton sector or fixed at a particular, motivated value in extended constructions.

[^8]:    11 Only the next-to-minimal pairs $(4,8),(6,6),(6,10)$, and $(8,8)$, featuring 7 parameters per sector, and a model with 8 parameters per sector were considered in Ref. 20], whose selection is based on minimal field weights.

[^9]:    12 The detailed analysis of non-minimal models is beyond the scope of this work.

[^10]:    13 These results are expected to hold even in extensions based on vector-valued modular forms.

