A SIMPLE PROOF OF WERNER SCHULTE'S CONJECTURE

DJAMEL HIMANE

Abstract: Lately, Werner Schulte has conjectured that for all positive $n > 1$, n divides $\frac{(n-2)!\,(n-1)!}{2^{n-3}} + 4$ if and only if n is prime. In this paper, We use elementary methods, to give a simple proof of this conjecture.

Keywords: Legendre's formula, p-adic valuation, Wilson Theorem, Fermat's Little Theorem.

MSC2020: 11A41, 11A51.

1 Introduction and statement of results

Taking

$$A_n \;=\; 1\,(1+2)\,(1+2+3)\,...\,(1+2+3+...+(n-1)) \qquad (1)$$

the question is: does this sequence of positive integers significant?
In an attempt to find an explanation for this sequence. The triangular number is $T_s = 1 + 2 + \cdots + s = \dfrac{s(s+1)}{2}$, as shown

$$A_n \;=\; \prod_{j=1}^{n-1} T_j \;=\; \prod_{j=1}^{n-1} \frac{j(j+1)}{2} \;=\; \frac{(n-1)!\,n!}{2^{n-1}} \qquad (2)$$

The first terms of this sequence are $1, 1, 3, 18, 180, 2700, 56700, ...$ (sequence A006472 in the OEIS [1]), also (sequence M3052 [1]). In the OEIS, we found a conjecture attributed to W. Schulte, concerning the previous sequence. for which we will present a simple proof.

The main result we seek to prove is

Main Theorem (*Werner Schulte's conjecture*)**.**
For all positive $n > 1$, n divides $\frac{(n-2)!\,(n-1)!}{2^{n-3}} + 4$ if and only if n is prime.

[1] A006472, The On-Line Encyclopedia of Integer Sequences.

2 Preliminaries

For all prime numbers p and non-negative integer n, let $v_p(n)$ is the exponent of the greatest power of p dividing n (i.e. the p-adic valuation of n). Then

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor, \tag{3}$$

where $\lfloor x \rfloor$ is the floor function. The formula of Legendre (3) gives an expression to the greatest power of a prime p the exponent of which divide the factorial $n!$ [6]. Whereas the right-hand formula is an infinite sum, for every particular value of n and p, there are only a few nonzero terms: for each i sufficiently big as $p^i > n$, we have $\left\lfloor \frac{n}{p^i} \right\rfloor = 0$.
We also have the recurrence relation:

$$v_p(n!) = \lfloor n/p \rfloor + v_p(\lfloor n/p \rfloor !) \tag{4}$$

Legendre's formula can also be reformulated in terms of a base-p expansion of n. Let $s_p(n)$ denotes the sum of the digits in the base-p expansion of n, then

$$v_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{5}$$

You can easily check the veracity of the following relationships. For any positive integers a, b and any prime p [7].

- $v_p\,(a\,b) = v_p(a) + v_p(b)$.
- $v_p\left(\dfrac{a}{b}\right) = v_p(a) - v_p(b)$.
- if $a \mid b$, *then* $\quad v_p(a) \leq v_p(b)$, in particularly $v_p(a) < a$.

The proof of the following lemma is known and easy, but is included here for completeness as the proof is short.

Lemma 1. *For all positive integers $m > 4$, we have* $\ m < 2^{m/2}$.

Proof. For the induction step, note that instead of using that the left side increases by 1, we will use the increase in multiplication (since the right side increases by a multiplicative factor) to get

$$m < 2^{m/2}$$

equivalent to

$$m\left(\frac{m+1}{m}\right) \;<\; 2^{m/2}\left(\frac{m+1}{m}\right)$$

equivalent to

$$m+1 \;<\; 2^{m/2}\,\left(\frac{m+1}{m}\right). \tag{6}$$

Next, let

$$\mathrm{g}(w) = \sqrt{2} - \frac{w+1}{w} = (\sqrt{2}-1) - \frac{1}{w}.$$

This gives $g(5) = (\sqrt{2}-1) - \frac{1}{5} \approx 0.2142$. Also, $g'(w) = \frac{1}{w^2} > 0$, which means $g(w)$ is a strictly increasing function for $w \geq 5$, so it's always positive then,

$$\sqrt{2} - \frac{w+1}{w} > 0 \qquad \textit{implies} \qquad \sqrt{2} > \frac{w+1}{w}$$

Thus, for $m \geq 5$,

$$\frac{m+1}{m} \quad < \quad 2^{1/2} \qquad \textit{gives} \qquad 2^{m/2}\,(\frac{m+1}{m}) \quad < \quad 2^{(m+1)/2}$$

Using this in (6) gives

$$m+1 \quad < \quad 2^{(m+1)/2}$$

which completes the induction step. As a result for $m > 4$, we get

$$m \; < \; 2^{m/2} \qquad \textit{implies} \qquad v_2(m) \; < \; \frac{m}{2}.$$

$\square$

Lemma 2 (Wilson Theorem)**.** *If p is prime then $(p-1)!+1$ is divisible by p.*

This theorem was affirmed by Alhazen (c. 965-1000 A.D.), [2] and, in the eighteenth century, from J. Wilson [3]. Proof of this came from J. L. Lagrange in 1771 [4]. An integer primality test derives from the Wilson theorem: a non-negative integer $n > 1$ is prime, If and only if

$$(n-1)! + 1 \equiv 0 \pmod{n}. \tag{7}$$

Lemma 3 (Fermat's Little Theorem)**.** *For any positive integers a which cannot be divided by a prime number p, we have*

$$a^{p-1} \equiv 1 \pmod{p}. \tag{8}$$

Among one of the basic results of the theory of elemental numbers Little theorem of Fermat, This is the basis of Fermat's primality test. this theorem is called from Pierre de Fermat, who declared during 1640. It is referred to as the: ≪Little theorem≫ to set it apart from Last Theorem of Fermat. (see page 514, [5]) .

3 Proof of the Main Theorem

Proof. **First step.** It is shown that if n is prime, then n divides

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4. \tag{9}$$

We assume $n = p$ is odd prime, thus from Wilson's theorem 2, we get p divided both $(p-1)!+1$ and $(p-2)!-1$. Hence p divided $(p-1)!(p-2)!+1$. Instead, from little theorem of Fermat 3, we get p divided $2^{p-1}-1$, hence from (2) we get

$$2 \times A_{p-1} \;=\; \frac{4\,(p-2)!\,(p-1)!}{2^{p-1}} \equiv 4\;(p-2)!\,(p-1)! \;\equiv -4 \pmod{p}.$$

So p divides the number

$$\frac{(p-2)!\,(p-1)!}{2^{p-3}}+4.$$

Finally, for $n=2$, the validity of the property can be easily checked.

Second step. We show that if n is a composite, then

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4$$

It is not divisible by n.

We assume $n=pq$ is a composite, where $1<p\leq q<n$ three sub-cases arise :

- **First case.** if $n=4$, then

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4\;=\;10,$$

It is not divisible by n.

- **Second case.** if n odd $(n>4)$, then $n \mid (n-2)!(n-1)!$ and

$$\gcd(n,2^{n-3})=1$$

from Euclid's lemma, we have

$$n \mid \frac{(n-2)!\,(n-1)!}{2^{n-3}},$$

Hence

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4\;\equiv\;4\pmod{n},$$

So

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4$$

it is not divisible by n.

- **Third case.** if n even $(n=2^{v_2(n)}\times k>4)$ where k odd, then $n\mid(n-2)!(n-1)!$ from Euclid's lemma, we get $k\mid 2\times A_{n-1}$, hence $n\mid 2\times A_{n-1}$ if and only if

$$v_2(n)\leq v_2(2\times A_{n-1}).$$

Through the above properties of p-adic valuation, we get

$$v_2(2\times A_{n-1})=v_2((n-1)!)+v_2((n-2)!)-v_2(2^{n-3})$$

We also have the recurrence relation:

$$v_2((n-1)!)=v_2(n-1)+v_2((n-2)!)=v_2((n-2)!),$$

Because $n-1$ odd, and

$$v_2((n-2)!)=\lfloor\frac{n-2}{2}\rfloor+\lfloor\frac{n-2}{4}\rfloor+v_2(\lfloor\frac{n-2}{4}\rfloor!)\geq\frac{n-2}{2}+\frac{n-2}{4}$$

Therefore

$$v_2(2\times A_{n-1}) \geq \left(\frac{n-2}{2}+\frac{n-2}{4}\right)+\left(\frac{n-2}{2}+\frac{n-2}{4}\right)-(n-3)$$

As a result

$$v_2(2\times A_{n-1}) \;\geq\; \frac{n}{2}.$$

From Lemma 1, we obtain

$$\frac{n}{2} \;>\; v_2(n).$$

Then

$$v_2(2\times A_{n-1}) \;\geq\; v_2(n).$$

Hence

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4 \;\equiv\; 4 \pmod{n},$$

So

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4$$

is not divisible by n.

In summary, we have obtained to all positive integer $n>1$, that n divides

$$\frac{(n-2)!\,(n-1)!}{2^{n-3}}+4$$

if and only if n is prime. □

References

[1] N. J. A. Sloane, S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, (1995), ISBN 9780125586306 (includes this sequence, page 319).
[2] O'Connor, John J., Robertson, Edmund F., ≪*Abu Ali al-Hasan ibn al-Haytham*≫, MacTutor History of Mathematics archive, University of St Andrews.
[3] Edward Waring, *Meditationes Algebraicae*, In the third edition of Waring's (1782), Wilson's theorem appears as problem 5 on page 380 (in Latin).
[4] J. L. Lagrange, ≪*Démonstration d'un théorème nouveau concernant les nombres premiers*≫ (Proof of a new theorem concerning prime numbers), Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres (Berlin), vol. 2, pages 125-137 (1771).
[5] Burton, David M., *The History of Mathematics : An Introduction* (7th ed, 2011), McGraw-Hill, ISBN 9780073383156, page 514 .
[6] A. M. Legendre, *Théorie des nombres*, Paris, (1830), page 10.
[7] J.W.S. Cassels, A. Fröhlich, *Algebraic Number Theory* Academic Press,(1967), page 9.

Djamel Himane
LA3C Laboratory,
Faculty of Mathematics, USTHB,
Po. Box 32, El Alia, 16111,
Bab Ezzouar, Algiers, Algeria.
Email address: dhimane@usthb.dz