# AN ORDER ANALYSIS OF HYPERFINITE BOREL EQUIVALENCE RELATIONS 

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#### Abstract

In this paper we first consider hyperfinite Borel equivalence relations with a pair of Borel $\mathbb{Z}$-orderings. We define a notion of compatibility between such pairs, and prove a dichotomy theorem which characterizes exactly when a pair of Borel $\mathbb{Z}$-orderings are compatible with each other. We show that, if a pair of Borel $\mathbb{Z}$-orderings are incompatible, then a canonical incompatible pair of Borel $\mathbb{Z}$-orderings of $E_{0}$ can be Borel embedded into the given pair. We then consider hyperfinite-over-finite equivalence relations, which are countable Borel equivalence relations admitting Borel $\mathbb{Z}^{2}$-orderings. We show that if a hyperfinite-over-hyperfinite equivalence relation $E$ admits a Borel $\mathbb{Z}^{2}$-ordering which is self-compatible, then $E$ is hyperfinite.


## 1. Introduction

This paper is a contribution to the study of hyperfinite Borel equivalence relations and, more generally, countable Borel equivalence relations which are conjectured to be hyperfinite. Hyperfinite Borel equivalence relations have been studied extensively by many researchers, first in the context of ergodic theory and operator algebras (see e.g. [2] and 44), and later in the context of descriptive set theory (see e.g. [19], [18] and [3). Despite an extensive literature, some problems about hyperfinite Borel equivalence relations are stubbornly open. One of the most wellknown open problems in this area is Weiss's question ([19]) of whether any orbit equivalence relation induced by a Borel action of a countable amenable group is hyperfinite.

By definition, an hyperfinite Borel equivalence relation is an increasing union of a sequence of Borel equivalence relations with finite equivalence classes. In [18, an equivalent formulation of hyperfiniteness is given: they are precisely those Borel equivalence relations for which there exists a Borel assignment of a linear ordering on each equivalence class so that the order type is a suborder of $\mathbb{Z}$.

These equivalent formulations of hyperfiniteness point to two different directions for generalizations. We call a Borel equivalence relation hyper-hyperfinite if it is a increasing union of a sequence of hyperfinite Borel equivalence relations. We call a Borel equivalence relation hyperfinite-over-hyperfinite if there is an Borel assignment of a linear ordering on each equivalence class so that the order type is a suborder of the lexicographic order of $\mathbb{Z}^{2}$.

It is unknown whether the two notions are equivalent to each other. The question whether any hyper-hyperfinite equivalence relation is hyperfinite is another

[^0]major open problem in the area, known as the Union Problem (see [3]). However, Kechris ( $[12]$ ) defined a notion of amenable equivalence relations (this is obviously motivated by and related to Weiss's question) and proved under CH that both hyper-hyperfinite equivalence relations and hyperfinite-over-hyperfinite equivalence relations are amenable.

What we study in this paper can be regarded as a small part of the more general study on structurable equivalence relations (see the recent paper by Chen and Kechris [1]. It is notable that Marks proved that any aperiodic countable Borel equivalence relation admits a Borel assignment of a linear ordering on each equivalence class so that the order type is exactly $\mathbb{Q}$ (see [1, Theorems 1.11 and 8.17]).

In this paper we study hyperfinite-over-hyperfinite equivalence relations and prove that, under certain conditions, they are hyperfinite. The condition is on the Borel $\mathbb{Z}^{2}$-ordering of the equivalence relation and is called self-compatible. Thus our main theorem is stated as follows.

Theorem 1.1. If $E$ is a hyperfinite-over-hyperfinite equivalence relation with $a$ Borel $\mathbb{Z}^{2}$-ordering which is self-compatible, then $E$ is hyperfinite.

More generally, the compatibility condition is between two Borel linear orderings on the equivalence classes. Before we prove our main theorem above, we give an analysis of hyperfinite Borel equivalence relations with two Borel $\mathbb{Z}$-orderings and characterize exactly when they are compatible with each other. We show that, if a pair of Borel $\mathbb{Z}$-orderings are incompatible, then there is a Borel embedding of a canonical pair of incompatible Borel $\mathbb{Z}$-orderings of $E_{0}$ into the given pair. Thus we obtain the following dichotomy theorem.

Theorem 1.2. There is a pair $\left(<_{0},<_{1}\right)$ of Borel $\mathbb{Z}$-orderings of $E_{0}$ such that, for any hyperfinite Borel equivalence relation $E$ on a standard Borel space $X$ and a pair $\left(<,<^{\prime}\right)$ of Borel $\mathbb{Z}$-orderings of $E$, exactly one of the following holds:
(I) $<$ and $<^{\prime}$ are compatible, or
(II) There is a Borel embedding $\theta: 2^{\omega} \rightarrow X$ witnessing $E_{0} \sqsubseteq_{B} E$ such that $\theta$ is order-preserving from $\left(<_{0},<_{1}\right)$ to $\left(<,<^{\prime}\right)$.

The theorem is proved using Gandy-Harrington forcing. As usual, the technical theorem is an effective version of the main dichotomy theorem in which all objects are $\Delta_{1}^{1}$. Some part of our proof is motivated by results of Kanovei [11], who defined the partial order $<_{0}$ and considered Borel reductions from $E_{0}$ to some $E$ which is order-preserving from $<_{0}$ to some $<$.

The rest of the paper is organized as follows. In section 2, we review some basic concepts and facts. In Section 3 we define the notion of compatible pairs of Borel $\mathbb{Z}$-orderings, state the main dichotomy theorem again, and prove some basic facts. In Section 4 we prove an effective version of the main dichotomy theorem (the technical theorem). In section 5 we turn to hyperfinite-over-hyperfinite equivalence relations, and formulate and prove Theorem 1.1.

## 2. Preliminaries

The standard notions of descriptive set theory we use in this paper can be found in, e.g., [16], [13] and [6].

A Polish space is a separable completely metrizable topological space. If $X$ is a Polish space then the collection of Borel sets on $X$ is the smallest $\sigma$-algebra of
subsets of $X$ containing the open sets. A standard Borel space is a pair $(X, \mathcal{B})$ where $X$ is a set and $\mathcal{B}$ is a $\sigma$-algebra of subsets of $X$ such that $\mathcal{B}$ is the collection of all Borel sets for some Polish topology on $X$.

Let $X$ be a standard Borel space. An equivalence relation $E$ on $X$ is Borel if $E$ is a Borel subset of $X^{2}$. Borel partial orders are similarly defined. Given a subset $A \subseteq X$ and an equivalence relation $E$ on $X$, we denote by $[A]_{E}=\{x: \exists y \in A(x E y)\}$ the $E$-saturation of $A . A$ is $E$-invariant if $A=[A]_{E}$. When $A=\{x\}$ is a singleton, $[A]_{E}$ is an $E$-equivalence class ( $E$-class for short), and we write $[x]_{E}$ for $[A]_{E}$. A subset $A \subseteq X$ is a complete section if it has nonempty intersection with every $E$ class. We say that $A$ is an infinite complete section if it has an infinite intersection with every $E$-class.

An equivalence relation $E$ on a standard Borel space $X$ is finite (or countable) if every $E$-class is finite (or countable, respectively). $E$ is hyperfinite if it is an increasing union of a sequence of finite Borel equivalence relations, i.e., $E=\bigcup_{n} E_{n}$, where each $E_{n}$ is a finite Borel equivalence relation, and $E_{n} \subseteq E_{n+1}$.

Following [12], we define a Borel structuring of a countable Borel equivalence relation $E$ on a standard Borel space $X$ as follows. Let $\mathcal{L}=\left\{R_{1}, \ldots, R_{n}\right\}$ be a finite relational language, with $k_{i}$ being the arity of $R_{i}$. Let $\mathcal{K}$ be a collection of countable $\mathcal{L}$-structures closed under isomorphism. An assignment $C \mapsto \mathcal{M}_{C}$, which for each $E$-class $C$ gives an $\mathcal{L}$-structure $\mathcal{M}_{C}=\left(C, R_{1}^{C}, \ldots, R_{n}^{C}\right)$ with universe $C$, is a Borel $\mathcal{K}$-structuring of $E$ if $\mathcal{M}_{C} \in \mathcal{K}$ for each $E$-class $C$, and the relations

$$
R_{i}\left(x, y_{1}, \ldots, y_{k_{i}}\right) \Longleftrightarrow y_{1}, \ldots, y_{k_{i}} \in[x]_{E} \text { and } R_{i}^{\mathcal{M}_{[x]_{E}}}\left(y_{1}, \ldots, y_{k_{i}}\right)
$$

are Borel subsets of $X^{k_{i}+1}$. In this paper we only consider two special cases, where $\mathcal{K}$ is either the collection of all suborders of $(\mathbb{Z},<)$ or the collection of all suborders of $\left(\mathbb{Z}^{2},<_{\text {lex }}\right)$, where $<_{\text {lex }}$ is the lexicographic order of $\mathbb{Z}^{2}$. We call them Borel $\mathbb{Z}$-orderings and Borel $\mathbb{Z}^{2}$-orderings respectively.

If $\mathcal{K}$ is a collection of countable linear orders and $C \mapsto \mathcal{M}_{C}$ is a Borel $\mathcal{K}$ structuring of $E$, then one can define a (strict) partial order $<_{X}$ on $X$ by

$$
x<_{X} y \Longleftrightarrow x<_{[x]_{E}} y
$$

Obviously $<_{X}$ is Borel. This motivates an equivalent but somewhat more intuitive concept as follows.

Definition 2.1. Let $X$ be a set and let $<$ be a (strict) partial order on $X$. We say that $<$ generates an equivalence relation $E$ on $X$ if for every pair $x, y \in X, x E y$ if and only if $x=y$ or there is a sequence $x=x_{0}, \ldots, x_{n}=y$ such that $x_{i}$ and $x_{i+1}$ are comparable in $<$ for each $i<n$.

Every partial order $<$ generates an equivalence relation $E_{<}$. We call each $E_{<^{-}}$ class a <-component.

Definition 2.2. Let $L$ be a countable linear order, $X$ be a set, $<$ be a partial order on $X$, and $E=E_{<}$. We say that $<$is class-wise $L$ if $<\upharpoonright[x]_{E}$ is isomorphic to a suborder of $L$ for every $x \in X$.

The following lemma without proof records the fact that the above two approaches are equivalent.

Lemma 2.3. Let $X$ be a standard Borel space and let $E$ be a Borel equivalence relation on $X$. Then the following are equivalent:
(i) There is a Borel $\mathbb{Z}$-ordering ( $\mathbb{Z}^{2}$-ordering) of $E$.
(ii) There is a Borel partial order $<$ on $X$ which is class-wise $\mathbb{Z}\left(\left(\mathbb{Z}^{2},<l_{\text {lex }}\right)\right.$, respectively; $\mathbb{Z}^{2}$, for short).

In the rest of the paper we work with Borel class-wise $\mathbb{Z}$-orders and Borel classwise $\mathbb{Z}^{2}$-orders.

Definition 2.4. Let $X$ be a Borel equivalence relation and let $E$ be a Borel equivalence relation on $X$.
(1) $E$ is hyper-hyperfinite if $E$ is the increasing union of a sequence of hyperfinite Borel equivalence relations, i.e., $E=\bigcup_{n} E_{n}$, where each $E_{n}$ is a hyperfinite Borel equivalence relation, and $E_{n} \subseteq E_{n+1}$.
(2) $E$ is hyperfinite-over-hyperfinite if $E$ admits a Borel $\mathbb{Z}^{2}$-ordering, or equivalently, there is a Borel class-wise $\mathbb{Z}^{2}$-order on $X$ for $E$.

The following are some examples of Borel equivalence relations relevant to our study in this paper.
(a) $E_{0}$ is the equivalence relation defined on $2^{\omega}$ by

$$
x E_{0} y \Longleftrightarrow \exists n<\omega \forall m>n x(m)=y(m) .
$$

(b) $E_{t}$ is the equivalence relation defined on $2^{\omega}$ by

$$
x E_{t} y \Longleftrightarrow \exists n, m<\omega \forall k x(n+k)=y(m+k) .
$$

(c) If $X$ is an uncountable standard Borel space, then $E_{t}(X)$ is the equivalence relation defined on $X^{\omega}$ by

$$
x E_{t}(X) y \Longleftrightarrow \exists n, m<\omega \forall k x(n+k)=y(m+k) .
$$

(d) $E_{\mathcal{S}}$ is the equivalence relation defined on $2^{\mathbb{Z}}$ by

$$
x E_{\mathcal{S}} y \Longleftrightarrow \exists n \in \mathbb{Z} \forall k \in \mathbb{Z} x(n+k)=y(k) .
$$

(e) If $E$ is an equivalence relation over $X$, then $E^{\omega}$ is defined on $X^{\omega}$ by

$$
x E^{\omega} y \Longleftrightarrow \forall n x(n) E y(n) .
$$

The equivalence relations $E_{0}, E_{t}, E_{\mathcal{S}}$ are all hyperfinite. Hyperfiniteness can also be characterized in the language of Borel reducibility, as follows.

For Borel equivalence relations $E$ and $F$ on standard Borel spaces $X$ and $Y$ respectively, we say that $E$ is Borel reducible to $F$, and write $E \leq_{B} F$, if there is a Borel map $f: X \rightarrow Y$ such that

$$
x_{1} E x_{2} \Longleftrightarrow f\left(x_{1}\right) F f\left(x_{2}\right)
$$

for all $x_{1}, x_{2} \in X . f$ is called a Borel reduction from $E$ to $F$. Moreover, if $f$ can be taken to be injective, then we say that $E$ is Borel embeddable into $F$, and write $E \sqsubseteq_{B} F$.

The following are some basic but nontrivial results about hyperfiniteness.
Theorem 2.5. Let $E$ be a Borel equivalence relation on a standard Borel space. Then the following hold:
(i) (Dougherty-Jackson-Kechris [3) $E$ is hyperfinite if and only if $E \leq{ }_{B} E_{0}$.
(ii) (Hjorth-Kechris [9) If $E$ is countable, then $E$ is hyperfinite if and only if $E \leq_{B} E_{0}^{\omega}$.
(iii) (Kechris-Louveau [14, Dougherty-Jackson-Kechris [3]) If $E$ is countable, then for any standard Borel space $X, E$ is hyperfinite if and only if $E \leq_{B}$ $E_{t}(X)$.

We construct more Borel equivalence relations in the following.
Definition 2.6. Let $E$ be a Borel equivalence relation on a standard Borel space $X$.
(f) $E_{0}(E)$ is the equivalence relation on $X^{\omega}$ defined by

$$
E_{0}(E)=\left\{\left(\left(x_{n}\right)_{n<\omega},\left(y_{n}\right)_{n<\omega}\right): \exists N<\omega \forall n>N\left(x_{n} E y_{n}\right)\right\}
$$

(g) $E_{\mathcal{S}}(E)$ is the equivalence relation on $X^{\mathbb{Z}}$ defined by

$$
E_{\mathcal{S}}(E)=\left\{\left(\left(x_{n}\right)_{n \in \mathbb{Z}},\left(y_{n}\right)_{n \in \mathbb{Z}}\right): \exists n \in \mathbb{Z} \forall k \in \mathbb{Z}\left(x_{n+k} E y_{k}\right)\right\}
$$

It turns out that hyper-hyperfiniteness and hyperfinite-over-hyperfiniteness can both be characterized by Borel reducibility, similarly to the above mentioned result of Hjorth-Kechris.

Proposition 2.7. A countable Borel equivalence relation $E$ on a standard Borel space $X$ is hyper-hyperfinite if and only if $E \leq_{B} E_{0}\left(E_{0}\right)$.
Proof. Suppose first $E \leq_{B} E_{0}\left(E_{0}\right)$. Without loss of generality, we may assume that $X \subseteq\left(2^{\omega}\right)^{\omega}, E=E_{0}\left(E_{0}\right) \upharpoonright X$, and $E$ is countable. By the theorem of HjorthKechris (Theorem [2.5 (ii)), we have

$$
F_{n}=\left\{(x, y) \in X^{2}: \forall k>n\left(x_{k} E_{0} y_{k}\right)\right\}
$$

is hyperfinite on $X$. Then note that $F_{n} \subseteq F_{n+1}$ and $E=\bigcup F_{n}$.
Conversely, suppose $E=\bigcup F_{n}$ is an increasing union of hyperfinite Borel equivalence relations $F_{n}$. Let $\phi_{n}$ be a Borel reduction from $F_{n}$ to $E_{0}$. Then $\left(\phi_{n}(x)\right)_{n<\omega}$ reduces $E$ to $E_{0}\left(E_{0}\right)$.

Proposition 2.8. A countable Borel equivalence relation $E$ on a standard Borel space $X$ is hyperfinite-over-hyperfinite if and only if $E \leq_{B} E_{\mathcal{S}}\left(E_{0}\right)$.

Proof. Suppose first $E \leq_{B} E_{\mathcal{S}}\left(E_{0}\right)$. We construct a Borel partial order < which generates $E$ and is class-wise $\mathbb{Z}^{2}$. Without loss of generality, we can assume that $X \subseteq\left(2^{\omega}\right)^{\mathbb{Z}}, E=E_{\mathcal{S}}\left(E_{0}\right) \upharpoonright X$, and $E$ is countable. Consider the Borel $E$-invariant set

$$
A=\left\{x \in X: \exists n \neq 0 \forall k \in \mathbb{Z}\left(x_{n+k} E_{0} x_{k}\right)\right\} .
$$

Then for each $x \in A,[x]_{E}$ can be characterized by finitely many $E_{0}$-classes. Hence $E \upharpoonright A$ is hyperfinite, and we can define a Borel partial order $<$ on $A$ which is class-wise $\mathbb{Z}$; in particular it is class-wise $\mathbb{Z}^{2}$.

It remains to define a Borel class-wise $\mathbb{Z}^{2}$-order on $X \backslash A$ for $E$. Let $F$ be the equivalence relation on $X$ defined by

$$
x F y \Longleftrightarrow \forall k \in \mathbb{Z}\left(x_{k} E_{0} y_{k}\right)
$$

Clearly $F \subseteq E$, hence $F$ is also countable. By the theorem of Hjorth-Kechris (Theorem [2.5 (ii)), $F$ is hyperfinite, and we can define Borel partial order $<_{*}$ on $X$ which is class-wise $\mathbb{Z}$ and generates $F$. Now if $x, y \in X \backslash A$ are from the same $E$-class but different $F$-classes, put

$$
x<^{*} y \Longleftrightarrow \exists n>0 \forall k \in \mathbb{Z}\left(x_{n+k} E_{0} y_{k}\right)
$$

Clearly $<^{*}$ is a well-defined Borel partial order on $X \backslash A$ that is $F$-invariant, i.e., if $x F x^{\prime}, y F y^{\prime}$ and $x<^{*} y$, then $x^{\prime}<^{*} y^{\prime}$. Also clear is that $<^{*}$ linearly orders $F$-classes inside a single $E$-class into a $\mathbb{Z}$-order. Now define a Borel partial order $<$ on $X \backslash A$ by

$$
x<y \Longleftrightarrow\left(x F y \text { and } x<_{*} y\right) \text { or }\left(x E y \text { and } \neg x F y \text { and } x<^{*} y\right)
$$

Then $<$ is class-wise $\mathbb{Z}^{2}$ and generates $E$.
Conversely, fix a Borel partial order $<$ witnessing that $E$ is hyperfinite-overhyperfinite. For $x E y$ and $x<y$, define $x F y$ iff $y F x$ iff there is a finite sequence $x=x_{0}<x_{1}<\cdots<x_{k}=y$ which is maximal of this form. Then $F$ is a Borel equivalence relation, and each $F$ class is order-embeddable into $\mathbb{Z}$, thus $F$ is hyperfinite. Notice that < orders $F$-classes in a single $E$-class into an order which is order-embeddable into $\mathbb{Z}$. Without loss of generality we can assume this order is isomorphic to $\mathbb{Z}$, since in all other cases we may in a Borel way choose a single $F$-class from the $E$-class it lies in, which implies that on the set of such points $E$ is hyperfinite.

For $x<y$ that are not $F$-equivalent, if for any $z$ such that $x<z<y$ either $x F z$ or $z F y$, we say that $x$ is just below $y$, or $y$ is just above $x$; we denote this relation as $B(x, y)$. Clearly $B$ is $F$-invariant. Use countable uniformization on $B$ twice, we obtain a Borel partial injection $\gamma$ such that $\operatorname{both} \operatorname{dom}(\gamma)$ and range $(\gamma)$ are $F$-complete sections, and $B(\gamma(x), x)$ for $x \in \operatorname{dom}(\gamma)$. Now for every $x \in X$ let $p(x) \in \operatorname{dom}(\gamma) \cap[x]_{F}$ be the closest element to $x$ in the $<$-order (which is a $\mathbb{Z}$-order on $[x]_{F}$ ), if a unique such element exists; otherwise there are two such elements, and noting that they are ordered by $<$, so we can let $p(x)$ be the smaller one in the $<$-order. Let $\phi(x)=\gamma(p(x))$. Similarly, define $q(x) \in$ range $(\gamma) \cap[x]_{F}$ to be a closest element to $x$ in the <-order, and let $\psi(x)=\gamma^{-1}(q(x))$. Then the map $x \mapsto\left(\ldots, \psi^{2}(x), \psi(x), x, \phi(x), \phi^{2}(x), \ldots\right)$ reduces $E$ to $E_{\mathcal{S}}\left(E_{0}\right)$.

As we stated in the introduction, the following problems are open.
Problem 2.9 (The Union Problem). Is every hyper-hyperfinite equivalence relation hyperfinite?

Problem 2.10 (The Hyperfinite-over-Hyperfinite Problem). Is every hyperfinite-over-hyperfinite equivalence relation hyperfinite?

The Union Problem is stated in [3] and better known. The Hyperfinite-overHyperfinite Problem has been in the folklore.

## 3. The main dichotomy theorem

We define the notion of compatibility for two Borel class-wise $\mathbb{Z}$-orders for a hyperfinite equivalence relation.

Definition 3.1. Let $E$ be a hyperfinite Borel equivalence relation on a standard Borel space $X$. Let $(X,<)$ and $\left(X,<^{\prime}\right)$ be two Borel class-wise $\mathbb{Z}$-orders generating $E$. We say that $<$ and $<$ are compatible if there is a Borel complete section $X^{\prime} \subseteq X$ such that on each $E$-class restricted to $X^{\prime}$, either $<=<^{\prime}$ or $<=>^{\prime}$. Such an $X^{\prime}$ is called a E-monotonic subset for $\left(<,<^{\prime}\right)$, or just $E$-monotonic, if $<$ and $<^{\prime}$ are clear from the context. If $<$ and $<^{\prime}$ are not compatible, we say they are incompatible.

Let us first look at an example of an incompatible pair.

Example 3.2. Consider the canonical hyperfinite equivalence relation $E_{0}$ on $2^{\omega}$. For $x, y \in 2^{\omega}$, define
$x<_{0} y \quad \Longleftrightarrow x(n)<y(n)$ for the largest $n$ such that $x(n) \neq y(n)$,
$x<_{1} y \Longleftrightarrow$ for the largest $n$ such that $x(n) \neq y(n)$,
$x(n)<y(n)$ if $n$ is odd and $y(n)<x(n)$ if $n$ is even.
Then $<_{0}$ and $<_{1}$ are both Borel class-wise $\mathbb{Z}$-orders for $E_{0}$, and $<_{0}$ and $<_{1}$ are incompatible.

Here is an argument for the incompatibility of the pair $\left(<_{0},<_{1}\right)$. As the space $2^{\omega}$ can be covered by countably many homeomorphic images of any complete section, any complete section cannot be meager. Hence it suffices to show that any nonmeager subset of $2^{\omega}$ with the Baire property cannot be $E_{0}$-monotonic for $\left(<_{0},<_{1}\right)$. Let $S$ be a non-meager subset of $2^{\omega}$ with the Baire property. Let $t \in 2^{<\omega}$ be of even length such that $S$ is comeager in $N_{t}=\left\{x \in 2^{\omega}: t \subseteq x\right\}$. It follows that

$$
\begin{aligned}
& \left\{x \in 2^{\omega}: t^{\wedge} 00^{\wedge} x \in S\right\} \\
& \left\{x \in 2^{\omega}: t^{\wedge} 10^{\wedge} x \in S\right\} \text { and } \\
& \left\{x \in 2^{\omega}: t^{\wedge} 11^{\wedge} x \in S\right\}
\end{aligned}
$$

are all comeager in $2^{\omega}$ and thus has nonempty intersection. Take an $x$ from their intersection. Then $t^{\wedge} 00^{\wedge} x<_{0} t^{\wedge} 10^{\wedge} x$ and $t^{\wedge} 00^{\wedge} x<_{1} t^{\wedge} 10^{\wedge} x$, but $t^{\wedge} 10^{\wedge} x<_{0}$ $t^{\wedge} 11^{\wedge} x$ while $t^{\wedge} 11^{\wedge} x<_{1} t^{\wedge} 10^{\wedge} x$. Thus $<_{0}$ and $<_{1}$ agree on some pairs of points in $S$, while disagree on other pairs of points from the same $E_{0}$-class in $S$. Therefore, $S$ is not $E$-monotonic.

Our main dichotomy theorem states that $\left(<_{0},<_{1}\right)$ is a canonical obstruction to compatibility, in the following sense.

Theorem 3.3. Let $E$ be a hyperfinite Borel equivalence relation on a standard Borel space $X$. Let $<$ and $<^{\prime}$ be Borel class-wise $\mathbb{Z}$-orders generating E. Then exactly one of the following holds:
(I) $<$ and $<^{\prime}$ are compatible;
(II) There is an injective Borel map $\theta: 2^{\omega} \rightarrow X$ such that $\theta$ reduces $E_{0}$ to $E$ and $\theta$ is order-preserving from $\left(<_{0},<_{1}\right)$ to $\left(<,<^{\prime}\right)$, i.e., for any $x, y \in 2^{\omega}$, $x<_{0} y \Longleftrightarrow x<y$ and $x<_{1} y \Longleftrightarrow x<^{\prime} y$.

The following proposition gives some equivalent characterizations of (I) in the above theorem. It will imply that (I) and (II) in the above theorem are mutually exclusive.

Proposition 3.4. Let $E$ be a hyperfinite Borel equivalence relation on a standard Borel space $X$. Let $<$ and $<^{\prime}$ be Borel class-wise $\mathbb{Z}$-orders generating $E$. Then the following are equivalent:
(1) There is a Borel E-monotonic complete section.
(2) There is a Borel E-monotonic complete section $A \subseteq X$ such that for any $x \in X$, if $[x]_{E}$ is infinite, then either $<\upharpoonright[x]_{E} \cap A$ is order-isomorphic to $<\upharpoonright[x]_{E}$ or $<^{\prime} \uparrow[x]_{E} \cap A$ is order-isomorphic to $<^{\prime} \uparrow[x]_{E}$ or both.
(3) For every Borel complete section $A \subseteq X$, there is a Borel E-monotonic complete section $C \subseteq A$.
(4) For every Borel complete section $A \subseteq X$, there is a Borel E-monotonic complete section $C \subseteq A$ such that for any $x \in X$, if $[x]_{E}$ is infinite, then
either $<\upharpoonright[x]_{E} \cap C$ is order-isomorphic to $<\upharpoonright[x]_{E}$ or $<^{\prime} \uparrow[x]_{E} \cap C$ is orderisomorphic to $<\uparrow[x]_{E}$ or both.
Proof. Clearly $(4) \Rightarrow(1)$. We show $(1) \Rightarrow(2) \Rightarrow(3) \Rightarrow(4)$.
For $(1) \Rightarrow(2)$, let $Y \subseteq X$ be a Borel $E$-monotonic complete section. Let $Y^{\prime}$ be the set of all $y \in Y$ such that $[y]_{E}$ is infinite but neither $<\upharpoonright[y]_{E} \cap Y$ is order-isomorphic to $<\upharpoonright[y]_{E}$ nor $<^{\prime} \uparrow[y]_{E} \cap Y$ is order-isomorphic to $<^{\prime} \uparrow[y]_{E}$. Then for $y \in Y, y \in Y^{\prime}$ if and only if $[y]_{E}$ is infinite and exactly one of the following holds:
(i) $[y]_{E} \cap Y$ is finite, and at least one of the following happens:

- there is either a <-least element or a <-largest element, but not both, of $[y]_{E}$;
- there is either a $<^{\prime}$-least element or a $<^{\prime}$-largest element, but not both, of $[y]_{E}$;
(ii) both $<\upharpoonright[y]_{E}$ and $<^{\prime} \uparrow[y]_{E}$ are isomorphic to $\mathbb{Z}$, and there is either a <-least element or a <-largest element, or both, of $[y]_{E} \cap Y$.
$Y^{\prime}$ is Borel. If $Y^{\prime}=\varnothing$ then there is nothing to prove. Thus we assume $Y^{\prime} \neq \varnothing$. Let $X^{\prime}=\left[Y^{\prime}\right]_{E}$. Then $X^{\prime}$ is a standard Borel space, every $E \upharpoonright X^{\prime}$-class is infinite, and $Y^{\prime}$ is a Borel complete section of $E \upharpoonright X^{\prime}$. To prove (2) it suffices to find a Borel $E \upharpoonright X^{\prime}$-monotonic complete section $A \subseteq X^{\prime}$ such that for any $x \in X^{\prime}$, either $<\upharpoonright[x]_{E} \cap A$ is order-isomorphic to $<\upharpoonright[x]_{E}$ or ${<^{\prime}}^{\prime}[x]_{E} \cap A$ is order-isomorphic to $<\upharpoonright[x]_{E}$ or both. For notational simplicity, assume $Y^{\prime}=Y$ and $X^{\prime}=X$. Our assumption implies that there is a Borel selector $\sigma: X \rightarrow X$ for $E$, i.e., a Borel function $\sigma$ such that for all $x \in X, \sigma(x) E x$ and if $x, y \in X, \sigma(x)=\sigma(y)$.

For every $x \in X$, define a 2 -coloring

$$
c(x)= \begin{cases}0, & \text { if either }\left(x<\sigma(x) \text { and } x<^{\prime} \sigma(x)\right) \text { or }\left(\sigma(x)<x \text { and } \sigma(x)<^{\prime} x\right), \\ 1, & \text { otherwise. }\end{cases}
$$

Then $c$ is Borel. For any Borel infinite complete section $C \subseteq X$, by the pigeonhole principle, there is a Borel infinite complete section $C^{\prime} \subseteq C$ such that for any $x \in X$, $c$ is constant on $[x]_{E} \cap C^{\prime} . C^{\prime}$ is an $E$-monotonic set.

Let $X_{1}$ be the set of all $x \in X$ such that either $<\upharpoonright[x]_{E}$ or $<\upharpoonright[x]_{E}$ is not orderisomorphic to $\mathbb{Z}$. Then $X_{1}$ is an $E$-invariant Borel subset of $X$. Consider $c \upharpoonright X_{1}$. There is a Borel infinite complete section $A_{1} \subseteq X_{1}$ such that for any $x \in X_{1}, c$ is constant on $[x]_{E} \cap A_{1}$. $A_{1}$ is an $E$-monotonic complete section of $X_{1}$ such that for any $x \in X_{1}$, either $<\upharpoonright[x]_{E} \cap A_{1}$ is order-isomorphic to $<\upharpoonright[x]_{E}$ or $<^{\prime} \upharpoonright[x]_{E} \cap A_{1}$ is order-isomorphic to $<^{\prime} \uparrow[x]_{E}$.

Let $X_{2}=X \backslash X_{1}$. Then $x \in X_{2}$ if and only if both $<\uparrow[x]_{E}$ and $<^{\prime} \uparrow[x]_{E}$ are order-isomorphic to $\mathbb{Z}$. We note that, for any $x \in X_{2}$, at least one of the following holds:
(a) there are infinitely many $y \in[x]_{E}, y>\sigma(x)$, such that $c(y)=0$, and there are infinitely many $y \in[x]_{E}, y<\sigma(x)$, such that $c(y)=0$;
(b) there are infinitely many $y \in[x]_{E}, y>\sigma(x)$, such that $c(y)=1$, and there are infinitely many $y \in[x]_{E}, y<\sigma(x)$, such that $c(y)=1$.
Toward a contradiction, assume neither (a) nor (b) holds for some $x \in X_{2}$. Then there are $x_{0}, x_{1} \in[x]_{E}, x_{0} \leq \sigma(x) \leq x_{1}$ such that $c(y)$ is constant for all $y>x_{1}$ and $c(z)$ is constant for all $z<x_{0}$, but $c(y) \neq c(z)$ for any $z<x_{0} \leq x_{1}<y$. For definiteness, assume $c(y)=0$ for $y>x_{1}$ and $c(z)=1$ for $z<x_{0}$. Then each of the sets $\left\{y \in[x]_{E}: y<x_{0}\right\},\left\{y \in[x]_{E}: x_{0} \leq y \leq x_{1}\right\}$, and $\left\{y \in[x]_{E}: y>x_{1}\right\}$ has a $<^{\prime}$-least element, which implies that $[x]_{E}$ has a $<^{\prime}$-least element, a contradiction.

Now with (a) and (b), we obtain a Borel $E$-monotonic infinite complete section $A_{2}$ of $X_{2}$ such that for each $x \in X_{2}$, both $<\upharpoonright[x]_{E} \cap A_{2}$ and $<^{\prime} \uparrow[x]_{E} \cap A_{2}$ are order-isomorphic to $\mathbb{Z}$.

Let $A=A_{1} \cup A_{2}$. Then $A$ is as required in (2).
Next we prove $(2) \Rightarrow(3)$. For this, fix a Borel complete section $Y \subseteq X$ as in (2), as well as another arbitrary Borel complete section $A \subseteq X$. We construct $C$.

Let $X^{\prime}$ be the set of all $x \in X$ such that the orders $<\upharpoonright[x]_{E} \cap Y,<^{\prime} \uparrow[x]_{E} \cap Y$, $<\upharpoonright[x]_{E} \cap A$ and $<^{\prime} \uparrow[x]_{E} \cap A$ are all order-isomorphic to $\mathbb{Z} . \quad X^{\prime}$ is a Borel $E$ invariant subset of $X$. If $X \neq X^{\prime}$ then there is a Borel selector $\sigma$ for $X \backslash X^{\prime}$, and we may assume $\sigma: X \backslash X^{\prime} \rightarrow A$. Then the set $C_{0}=\left\{\sigma(x): x \in X \backslash X^{\prime}\right\} \subseteq A$ is a Borel $E$-monotonic complete section of $X \backslash X^{\prime}$. For notational simplicity, we assume $X=X^{\prime}$ for the rest of the proof.

For $x, y \in X$ with $x E y$, define

$$
d(x, y)= \begin{cases}0, & \text { if } x=y \\ \mid\{z \in X: x<z<y \text { or } y<z<x\} \mid+1, & \text { otherwise } .\end{cases}
$$

Then $d$ is Borel and is a metric on every $[x]_{E}$.
For each $x \in A$, let $\sigma(x)=\sup _{<}\{y \in Y: y \leq x\}$ and $\sigma^{\prime}(x)=\sup _{<^{\prime}}\left\{y \in Y: y \leq^{\prime}\right.$ $x\}$. Let $h(x)=\min \left\{d\left(\sigma(y), \sigma^{\prime}(y)\right): y E x\right\}$. Shrink $A$ to

$$
A_{0}=\left\{x \in A: d\left(\sigma(x), \sigma^{\prime}(x)\right)=h(x)\right\}
$$

which is clearly still a Borel complete section.
Define a binary relation $R \subseteq E \upharpoonright A$ on $A$ by

$$
R(x, y) \Longleftrightarrow \begin{cases}x<y \text { and } y<^{\prime} x, & \text { if }<=<^{\prime} \text { on }[x]_{E} \cap Y, \\ x<y \text { and } x<^{\prime} y, & \text { if }<=>^{\prime} \text { on }[x]_{E} \cap Y .\end{cases}
$$

Extend $R$ to be symmetric. We claim that $R$ is locally finite on $A_{0}$, i.e., for each $x \in A_{0}$ there are only finitely many $y \in A_{0}$ with $R(x, y)$ or $R(y, x)$. To see this, let $(x, y) \in E \upharpoonright A_{0}$. Then $h(x)=h(y)=d\left(\sigma(x), \sigma^{\prime}(x)\right)=d\left(\sigma(y), \sigma^{\prime}(y)\right)$. For definiteness, suppose $<=<^{\prime}$ on $[x]_{E} \cap Y$ and $R(x, y)$. Since $x<y$, we have $\sigma(x) \leq$ $\sigma(y)$. Since $y<^{\prime} x$, we have $\sigma^{\prime}(y) \leq^{\prime} \sigma^{\prime}(x)$. Since $\sigma^{\prime}(y), \sigma(x) \in[x]_{E} \cap Y$ and $<=<^{\prime}$ on $[x]_{E} \cap Y$, we have $\sigma^{\prime}(y) \leq \sigma^{\prime}(x)$. Considering the points $x, \sigma(x), \sigma^{\prime}(x), \sigma(y), \sigma^{\prime}(y)$ in the <-order, we conclude that

$$
d(x, \sigma(y)) \leq d(\sigma(x), x)+2 h(x)
$$

Now for a fixed $x$, there are finitely many $z$ such that $d(x, z) \leq d(\sigma(x), x)+2 h(x)$, and for each $z \in Y$ there are finitely many $y$ such that $\sigma(y)=z$. Therefore $R$ is locally finite. The cases where $R(y, x)$ holds or $<=>^{\prime}$ are similar.

Now we can obtain a Borel maximal $R$-anticlique $C \subseteq A_{0}$ (see [10, Lemma 1.17]), which is a Borel $E$-monotonic complete section.

The proof of $(3) \Rightarrow(4)$ is identical to the proof of $(1) \Rightarrow(2)$.

With this proposition in mind, we notice that if both (I) and (II) in Theorem 3.3 hold, then by $(1) \Rightarrow(3)$ of the above proposition we can construct a Borel $E$-monotonic complete section in the image of $\theta$ and pull it back to $2^{\omega}$, resulting in a contradiction.

## 4. The technical theorem

Our strategy to prove the main dichotomy theorem is to prove the following effective version of the main dichotomy theorem which we call the technical theorem. We state this theorem below in the non-relativized form but from the proof it will be clear that this theorem can be relativized.

Theorem 4.1 (The technical theorem). Let $X$ be a recursively presented Polish space, let $E$ be a $\Delta_{1}^{1}$ equivalence relation on $X$ which is generated by $\Delta_{1}^{1}$ class-wise $\mathbb{Z}$-orders $<$ and $<^{\prime}$. Then exactly one of the following holds:
(I) For every $x \in X$ there is an $\Delta_{1}^{1} E$-monotonic subset $S \subseteq X$ so that $x \in S$;
(II) There is an injective continuous map that $\theta: 2^{\omega} \rightarrow X$ such that $\theta$ reduces $E_{0}$ to $E$ and $\theta$ is order-preserving from $\left(<_{0},<_{1}\right)$ to $\left(<,<^{\prime}\right)$.
We then obtain the following corollary, from which the main dichotomy theorem follows immediately because all Polish spaces of the same cardinality are isomorphic as standard Borel spaces.

Corollary 4.2. Let $X$ be a recursively presented Polish space, let $E$ be a hyperfinite Borel equivalence relation on $X$, and let $<$ and $<^{\prime}$ be Borel class-wise $\mathbb{Z}$-orders on $X$ generating $E$. Then exactly one of following holds:
(I) $<$ and $<^{\prime}$ are compatible;
(II) There is an injective continuous map $\theta: 2^{\omega} \rightarrow X$ such that $\theta$ reduces $E_{0}$ to $E$ and $\theta$ is order-preserving from $\left(<_{0},<_{1}\right)$ to $\left(<,<^{\prime}\right)$.
Proof. By relativization, we may assume without loss of generality that $E,<$ and $<^{\prime}$ are $\Delta_{1}^{1}$. Note that the second alternate is the same as in Theorem 4.1. So we only need to show that (I) of Theorem 4.1 implies the first alternate of this corollary.

Suppose (I) of Theorem 4.1 holds. Let $\mathcal{F}=\left\{S \in \Delta_{1}^{1}: S \subseteq X\right.$ is $E$-monotonic $\}$. Since there are only countably many $\Delta_{1}^{1}$ subsets, we can enumerate the elements of $\mathcal{F}$ as $S_{0}, S_{1}, \ldots$. For every $x \in X$ there is $n \in \omega$ so that $x \in S_{n}$.

To construct a Borel $E$-monotonic complete section, we inductively define

$$
\begin{aligned}
& A_{0}=S_{0} \\
& A_{n+1}=S_{n} \backslash\left[A_{n}\right]_{E}
\end{aligned}
$$

Then each $A_{n}$ is $E$-monotonic and Borel. Additionally, $\left[A_{n}\right]_{E}$ are Borel, $E$-invariant and pairwise disjoint. Therefore $A=\bigcup_{n<\omega} A_{n}$ is $E$-monotonic and Borel. Since every $x$ is contained in some $S_{n}$, it must be that $x \in[S]_{E}$, thus $S$ is a complete section.

In our treatment of $\Delta_{1}^{1}$ hyperfinite equivalence relations, we are going to frequently and tacitly use the fact that quantifiers bounded by $E$-classes $\left(\forall x \in[y]_{E}\right.$ and $\left.\exists x \in[y]_{E}\right)$ are in fact number quantifiers. In the classical setting this is true for any countable Borel equivalence relation, which is a consequence of the FeldmanMoore theorem ([5, Theorem 1]), or in the case of hyperfinite Borel equivalence relations $E$, by direct computations using the equivalent characterization that $E$ is generated by a single Borel automorphism ([18]; also see [3, Theorem 5.1(4)]). In the effective setting, this can be seen by applying e.g. [17, Theorem 4.5].

The Gandy-Harrington forcing will be the main tool in our proof of the technical theorem. Detailed introductions of Gandy-Harrington forcing can be found in [7]
and [11. Here we briefly review some basic notions and prove a few facts to be used in our proof.

For the rest of this section, let $X$ be a fixed recursively represented Polish space. For any natural number $n \geq 1$, the Gandy-Harrington forcing notion on $X^{n}$ is the poset

$$
\mathrm{P}_{n}=\left\{A \subseteq X^{n}: A \in \Sigma_{1}^{1}, A \text { is uncountable }\right\}
$$

ordered by inclusion. The following is a basic fact about the Gandy-Harrington forcing.
Lemma 4.3. Let $n \geq 1$ and let $M$ be a model of sufficiently many axioms of ZFC with $\mathrm{P}_{n} \in M$. If $\mathcal{G}$ is $\mathrm{P}_{n}$-generic over $M$, then $\bigcap \mathcal{G}$ is a singleton $\left\{x_{\mathcal{G}}\right\}$ with $x_{\mathcal{G}} \in X^{n} \cap M[\mathcal{G}]$.

Note that $P_{n}$ is a different forcing notion from the product $P_{1}^{n}$, but projection maps $\pi$ onto a specific coordinate are open maps for both $\mathrm{P}_{n}$ and $\mathrm{P}_{1}^{n}$.

Consider the full binary tree $2^{<\omega}=\bigcup_{n<\omega} 2^{n}$. An element $t \in 2^{n}$ is a 0,1 sequence of length $n$. We denote the length of $t$ by $|t|$. If $t=\left(t_{1}, \ldots, t_{n}\right) \in 2^{n}$ and $m \leq n$, then $t \upharpoonright m=\left(t_{1}, \ldots, t_{m}\right)$ denotes the initial segment of $t$ of length $m$. If $t \in 2^{n}, s \in 2^{m}$ and $m \leq n$, then we say $t$ extends $s$, and write $s \subseteq t$ or $t \supseteq s$, if $t \upharpoonright m=s$. If $t=\left(t_{1}, \ldots, t_{n}\right) \in 2^{n}$ and $s=\left(s_{1}, \ldots, s_{m}\right) \in 2^{m}$, then the concatenation of $t$ and $s$ is $t^{\wedge} s=\left(t_{1}, \ldots, t_{n}, s_{1}, \ldots, s_{n}\right) \in 2^{n+m}$. When $|s|=1$, instead of writing $t^{\wedge}(i)$ for $i=0,1$, we write $t^{\wedge} i$. Concatenations can be composed.

Now for each $n<\omega$, let $\mathrm{P}_{2^{n}}$ be the Gandy-Harrington forcing on $X^{2^{n}}$. In this point of view, each $t \in 2^{n}$ is a coordinate of a point in $X^{2^{n}}$. We let $\mathrm{P}=\bigcup_{n<\omega} \mathrm{P}_{2^{n}}$ be the disjoint union of $\mathrm{P}_{2^{n}}$ (in the same sense as $2^{<\omega}$ being a disjoint union of $2^{n}$ ). $\mathrm{P}_{2^{n}}$ is called the level $n$ of P . For each $p \in \mathrm{P}$, let $\operatorname{dim}(p)$ be the unique $n$ such that $p \in \mathrm{P}_{2^{n}}$.

We define a collection of projections from $\mathrm{P}_{2^{n}}$ to $\mathrm{P}_{2^{m}}$ for $n>m$ as follows.
Definition 4.4. For natural numbers $m<n$ and $t \in 2^{n-m}$, the projection map $\pi_{m, t}: \mathrm{P}_{2^{n}} \rightarrow \mathrm{P}_{2^{m}}$, which we call the projection from level $n$ to level $m$ along $t$, is defined by

$$
\pi_{m, t}(p)=\left\{\left(x_{s}\right)_{s \in 2^{m}}: \exists\left(y_{r}\right)_{r \in 2^{n}} \in p \forall s \in 2^{m} y_{s \sim t}=x_{s}\right\}
$$

for $p \in \mathrm{P}_{2^{n}}$.
The following basic facts are easy to verify. We state them without proof.
Lemma 4.5. Suppose $k<m<n, s \in 2^{m-k}$ and $t \in 2^{n-m}$. Then the following hold:
(i) $\pi_{k, s^{\circ} t}=\pi_{k, s} \circ \pi_{m, t}$.
(ii) If $D \subseteq \mathrm{P}_{2^{m}}$ is open dense, then so is

$$
\pi_{m, t}^{-1}(D)=\left\{p \in \mathrm{P}_{2^{n}}: \pi_{m, t}(p) \in D\right\}
$$

Next we define a partial order $\leq_{P}$ on $P$ to turn it into a poset.
Definition 4.6. Define a partial order $\leq_{\mathrm{P}}$ on P by letting $p \leq_{\mathrm{P}} q$ iff either $p \subseteq q$ or there are $m<n$ such that $p \in \mathrm{P}_{2^{n}}, q \in \mathrm{P}_{2^{m}}$ and for every $t \in 2^{n-m}, \pi_{m, t}(p) \subseteq q$.

In this paper, we do not need the full genericity for $P$. Instead, we use $P$ to construct objects that are, in the sense of the above projections, simultaneously generic for all $P_{2^{n}}$. The sense of sufficient genericity for $P$ is formulated in the following proposition.

Proposition 4.7. Let $M$ be a countable model of sufficiently many axioms of ZFC with $\mathrm{P} \in M$. Then there is a sequence of subsets $\left\{D_{n}\right\}_{n<\omega}$ of P in $M$ such that:
(i) Each $D_{n} \subseteq \mathrm{P}_{2^{n}}$ is open dense in $\mathrm{P}_{2^{n}}$;
(ii) If a filter $\mathcal{G} \subseteq \mathrm{P}$ intersects each $D_{n}$, then for every $b \in 2^{\omega}$ and $n \in \omega$,

$$
\left\{\pi_{n, b \upharpoonright k}(p): \exists k \in \omega\left(p \in \mathrm{P}_{2^{n+k}} \cap \mathcal{G}\right)\right\}
$$

is $\mathrm{P}_{2^{n}-\text { generic over }} M$.
Proof. For each $n \in \omega$, enumerate all open dense subsets of $\mathrm{P}_{2^{n}}$ in $M$ as $\left\{U_{k}^{n}\right\}_{k<\omega}$. Let $V_{k}^{n}=\bigcap_{i \leq k} U_{i}^{n}$. Since this is a finite intersection, each $V_{k}^{n} \in M$ and is still open dense. Moreover, a filter is $\mathrm{P}_{n}$-generic over $M$ if and only if it has nonempty intersection with every $V_{k}^{n}$ (or, just a tail of $\left\{V_{k}^{n}\right\}_{k}$, since it is a decreasing family).

Next, we inductively shrink each $V_{k}^{n}$ to an open dense $W_{k}^{n}$. Let $W_{k}^{0}=V_{k}^{0}$ for all $k<\omega$. Suppose we have already defined $W_{k}^{n}$ for a fixed $n$ and all $k<\omega$. Define $W_{k}^{n+1}$ for all $k<\omega$ by induction on $k$ :

$$
\begin{aligned}
& W_{0}^{n+1}=V_{0}^{n+1} \cap \pi_{n, 0}^{-1}\left(W_{0}^{n}\right) \cap \pi_{n, 1}^{-1}\left(W_{0}^{n}\right), \\
& W_{k}^{n+1}=V_{k}^{n+1} \cap \pi_{n, 0}^{-1}\left(W_{k}^{n}\right) \cap \pi_{n, 1}^{-1}\left(W_{k}^{n}\right) \cap W_{k-1}^{n+1}, \text { for } k>0 .
\end{aligned}
$$

By Lemma 4.5 (ii), all $W_{k}^{n}$ are still open dense. By Lemma 4.5 (i), the twoparameter family $\left\{W_{k}^{n}\right\}_{n, k}$ satisfies that for every $n>m$ and every $t \in 2^{n-m}$, $\pi_{m, t}^{-1}\left(W_{k}^{n}\right) \subseteq W_{k}^{m}$.

Finally, take $D_{n}=W_{n}^{n}$. Then $\left\{D_{n}\right\}_{n}$ is as required.
The rest of this section is devoted to a proof of Theorem 4.1.
Assume that (I) fails. Then

$$
Y=\left\{x \in X: \forall S \in \Delta_{1}^{1}(S \text { is } E \text {-monotonic } \rightarrow x \notin S)\right\}
$$

is nonempty. $Y$ is $\Sigma_{1}^{1}$. Since singletons are $E$-monotonic, $Y$ does not contain any $\Delta_{1}^{1}$ member. By the effective perfect set theorem (see, e.g., [16, 4F.1]), $Y$ is uncountable. Thus $Y \in \mathrm{P}_{1}$. We note that $Y$ does not contain any nonempty $\Sigma_{1}^{1}$ $E$-monotonic subset of $X$. In fact, by counting quantifiers, we can see that being $E$-monotonic is $\Pi_{1}^{1}$ on $\Sigma_{1}^{1}$ sets. By the first reflection theorem (see, e.g., [8, Lemma 1.2]), every nonempty $\Sigma_{1}^{1} E$-monotonic subset of $X$ is included in a nonempty $\Delta_{1}^{1}$ $E$-monotonic subset, and is therefore disjoint from $Y$ by definition.

Recall that $<_{0}$ and $<_{1}$ are two $\Delta_{1}^{1}$ class-wise $\mathbb{Z}$-orders on $2^{\omega}$ generating $E_{0}$. Now we extend them to each level of the tree $2^{<\omega}$. For $n<\omega, t, s \in 2^{n}$ and $i=0,1$, define

$$
t<_{i} s \Longleftrightarrow \exists x \in 2^{\omega}\left(t^{\frown} x<_{i} s^{\wedge} x\right) .
$$

Note that $t<_{i} s$ if and only if for every $x \in 2^{\omega}, t^{\wedge} x<_{i} s^{\wedge} x$. Thus $<_{i}$ is $\Delta_{1}^{1}$. The following facts are easy to verify. We state them without proof.

Lemma 4.8. For all $n<\omega$ and $t, s \in 2^{n}$, the following hold:
(i) For $i=0,1$ and $j=0,1, t<_{i} s$ if and only if $t^{\wedge} j<_{i} s^{\wedge} j$;
(ii) $t^{\curvearrowleft} 0<{ }_{0} s^{\wedge} 1$;
(iii) $t^{\curvearrowright} 0<_{1} s^{\wedge} 1$ if $n$ is even, and $t^{\wedge} 1<_{1} s^{\wedge} 0$ if $n$ is odd.

Let $M$ be a countable model of sufficiently many axioms of ZFC with $\mathrm{P} \in M$. Let $\left\{D_{n}\right\}_{n<\omega}$ be as in Proposition 4.7. For each $\Sigma_{1}^{1}$ set $q \in \mathrm{P}_{2^{n}}$, write $D_{n}(q)=$ $\left\{p \cap q: p \in D_{n}\right\}$. Clearly these are nonempty sets that are downward closed by the open denseness of $D_{n}$.

Let $\tau$ be the topology generated by $\mathrm{P}_{1}$ on $X$. Let $\bar{E}$ be the closure of $E$ in the $\tau \times \tau$ topology (which corresponds to the product forcing $\mathrm{P}_{1} \times \mathrm{P}_{1}$ ). By [7, Lemma 5.2], $E$ is $G_{\delta}$ in $\tau \times \tau$. Also, any partial transversal for $E$ is automatically $E$-monotonic, so $Y$ is included in the set

$$
\left\{x \in X:[x]_{E} \neq[x]_{\bar{E}}\right\} .
$$

By [7, Lemma 5.3], $E$ is both dense and meager in the relative $\tau \times \tau$ topology on $\bar{E} \cap(Y \times Y)$. Let $\left\{F_{n}\right\}_{n<\omega}$ enumerate the open dense subsets of $\mathrm{P}_{1} \times \mathrm{P}_{1}$ restricted to $\bar{E} \cap(Y \times Y)$ in $M$.

Let $<$ and $<^{\prime}$ be $\Delta_{1}^{1}$ class-wise $\mathbb{Z}$-orders on $X$. Next we define $p_{n} \in \mathrm{P}_{2^{n}}$ with the following properties:
(1) $p_{n} \in D_{n}\left(Y^{2^{n}}\right)$;
(2) $\pi_{n, i}\left(p_{n+1}\right) \subseteq p_{n}$ for $i=0,1$;
(3) For any $x \in p_{n}$ and $t, s \in 2^{n}, x(t)<x(s) \Longleftrightarrow t<_{0} s$ and $x(t)<^{\prime} x(s)$ $\Longleftrightarrow t<_{1} s$
(4) For every pair $t, s$ such that $|t|=|s|$ and $t(|t|) \neq s(|s|), \pi_{0, t}\left(p_{n}\right) \times \pi_{0, s}\left(p_{n}\right) \in$ $F_{n}$.
To simplify our argument, define
$u_{n}=\left\{\left(x_{t}\right)_{t \in 2^{n}} \in Y^{2^{n}}: \forall t, s \in 2^{n}\left(x_{t}<x_{s} \Longleftrightarrow t<_{0} s\right.\right.$ and $\left.\left.x_{t}<^{\prime} x_{s} \Longleftrightarrow t<_{1} s\right)\right\}$.
Then $u_{n} \in \mathrm{P}_{2^{n}}$, and properties (1) and (3) can together be written as $p_{n} \in D_{n}\left(u_{n}\right)$.
Granting the existence of such $p_{n}$, we show that (II) holds. In fact, define $\theta: 2^{\omega} \rightarrow X$ by

$$
\{\theta(b)\}=\bigcap_{n<\omega} \pi_{0, b \upharpoonright n}\left(p_{n}\right)
$$

for $b \in 2^{\omega}$. To see that this makes sense, note that by properties (1) and (2) and Proposition 4.7 the sequence $\pi_{0, b \upharpoonright n}\left(p_{n}\right)$ is $\mathrm{P}_{1}$-generic over $M$, and therefore, by Lemma 4.3, the set on the right hand side above is a singleton. Thus $\theta$ is well defined.

To see that $\theta$ is continuous, let $\rho$ be a compatible metric on $X$. Then for any rational $\epsilon>0$, the set $A=\left\{p \in \mathrm{P}_{1}: \operatorname{diam}(p)<\epsilon\right\}$ is open dense in $\mathrm{P}_{1}$. Thus for any rational $\epsilon>0$, and for any $b \in 2^{\omega}$, there is $n<\omega$ such that $\pi_{0, b \upharpoonright n}\left(p_{n}\right) \in A$; now if $b^{\prime} \upharpoonright n=b \upharpoonright n$, then $\rho\left(\theta\left(b^{\prime}\right), \theta(b)\right)<\epsilon$. Thus $\theta$ is continuous.

To see that $\theta$ is injective, note that, by property (3), for any $n<\omega, x \in p_{n}$, and distinct $t, s \in 2^{n}, x(t) \neq x(s)$. It follows that the set

$$
\left\{p \in \mathrm{P}_{n}: \forall t, s \in 2^{n}\left(t \neq s \rightarrow \pi_{0, t}(p) \cap \pi_{0, s}(p)=\varnothing\right)\right\}
$$

is open dense below $p_{n} \in \mathrm{P}_{2^{n}}$. Thus if $b, b^{\prime} \in 2^{\omega}$ and $b \neq b^{\prime}$, then there is $n<\omega$ such that $b \upharpoonright n \neq b^{\prime} \upharpoonright n$ and $\pi_{0, b \upharpoonright n}\left(p_{n}\right) \cap \pi_{0, b^{\prime} \upharpoonright n}\left(p_{n}\right)=\varnothing$; this implies $\theta(b) \neq \theta\left(b^{\prime}\right)$.

To see that $\theta$ is order-preserving from $\left(<_{0},<_{1}\right)$ to $\left(<,<^{\prime}\right)$, consider an arbitrary pair $b, b^{\prime} \in 2^{\omega}$ with $b E_{0} b^{\prime}$ and $b<_{0} b^{\prime}$. Let $k \in \omega$ be the largest so that $b(k) \neq b^{\prime}(k)$. Let $n=k+1$ and $c \in 2^{\omega}$ be such that $b=(b \upharpoonright n)^{\wedge} c$. Then $b \upharpoonright n<_{0} b^{\prime} \upharpoonright n$. By property (3), we have that for any $x \in p_{n}, x(b \upharpoonright n)<x\left(b^{\prime} \upharpoonright n\right)$. By property (2), we have that for all $m>n$ and $x \in p_{m}, x(b \upharpoonright m)<x\left(b^{\prime} \upharpoonright m\right)$. By property (1) and Proposition 4.7 the sequence

$$
q_{m}=\pi_{n, c \upharpoonright(m-n)}\left(p_{m}\right), m>n
$$

is $\mathrm{P}_{n}$-generic over $M$. By Lemma 4.3, $\bigcap_{m>n} q_{m}$ is a singleton, whose only element we denote as $z$. By the definition of $\theta$, we have that $z(b \upharpoonright n)=\theta(b)$ and $z\left(b^{\prime} \upharpoonright n\right)=$
$\theta\left(b^{\prime}\right)$. By property (3), we have $\theta(b)<\theta\left(b^{\prime}\right)$. The proof for $\theta$ being order-preserving from $<_{1}$ to $<^{\prime}$ is similar.

Finally, to see that $\theta$ is a reduction from $E_{0}$ to $E$, note that property (3) implies that for any $b, b^{\prime} \in 2^{\omega}$, if $b E_{0} b$ then $\theta(b) E \theta\left(b^{\prime}\right)$. By property (4), $\left(\theta(b), \theta\left(b^{\prime}\right)\right)$ is $\mathrm{P}_{1} \times \mathrm{P}_{1}$-generic if $\left(b, b^{\prime}\right) \notin E_{0}$ (c.f. [11, Section 5]), thus the pair cannot lie in any meager set of the $\mathrm{P}_{1} \times \mathrm{P}_{1}$ forcing, in particular $\left(\theta(b), \theta\left(b^{\prime}\right)\right) \notin E$. We have thus established (II).

Now, let us turn to the construction of $p_{n}$. For $n=0$ we simply put $p_{0}=Y$. Inductively, we assume that we have already defined $p_{n}$ to satisfy properties (1)(4). By property (3) we have that for any $x \in p_{n}$ and $t, s \in 2^{n}, x(t) E x(s)$. We proceed to defining $p_{n+1}$. We assume that $n+1$ is odd. The even case is similar. Our strategy is to construct $p_{n+1} \subseteq u_{n+1}$ to satisfy properties (3) and (4), and then extend it further to satisfy property (1). Property (2) will be clear from our construction.

We first work with property (4). We enumerate all pairs mentioned in (4) as $\left\{\left(t_{i}, s_{i}\right)\right\}_{i<k}$. Let $p_{n, 0,0}=p_{n, 1,0}=p_{n}$. At each step $i<k$, Let $A=\pi_{0, t_{i}}\left(p_{n, 0, i}\right)$ and $B=\pi_{0, s_{i}}\left(p_{n, 1, i}\right)$. Note that $(A \times B) \cap \bar{E}$ is nonempty for $i=0$, and we promise that this will be the case for each step $i<k$. Since $A \times B$ is $\tau \times \tau$ open and $F_{n+1}$ is open dense in the relative $\tau \times \tau$ on $\bar{E}$, we are able to find $A^{\prime} \subseteq A$ and $B^{\prime} \subseteq B$ so that $A^{\prime} \times B^{\prime} \in F_{n+1}$.

Let

$$
\begin{aligned}
& p_{n, 0, i+1}=\left\{x \in p_{n, 0, i}: x\left(t_{i}\right) \in A^{\prime}\right\}, \\
& p_{n, 1, i+1}=\left\{x \in p_{n, 1, i}: x\left(s_{i}\right) \in B^{\prime}\right\} .
\end{aligned}
$$

Since $E$ is dense in $\bar{E},\left(A^{\prime} \times B^{\prime}\right) \cap E \neq \varnothing$. Let $p_{n, 0}=p_{n, 0, k}$ and $p_{n, 1, k}$. We write $p_{n, 0} \otimes p_{n, 1}$ for the set

$$
\left\{z \in X^{2^{n+1}}: \exists x \in p_{n, 0} \exists y \in p_{n, 1} \forall t \in 2^{n}\left(z\left(t^{\wedge} 0\right)=x(t) \text { and } z\left(t^{\wedge} 1\right)=y(t)\right)\right\}
$$

Clearly, $p_{n, 0} \otimes p_{n, 1}$, and any of its extensions in $P_{2^{n+1}}$, satisfies property (2) and (4).

Then we turn to (3). We claim that there exists a pair $x \in p_{n, 0}$ and $y \in p_{n, 1}$ such that both $x(1 \ldots 1)<y(0 \ldots 0)$ and $x(1010 \ldots 10)<^{\prime} y(0101 \ldots 01)$. Granting this claim, we set $z\left(t^{\wedge} 0\right)=x(t)$ and $z\left(t^{\wedge} 1\right)=y(t)$ for all $t \in 2^{n}$. By the transitivity of $<$ and $<^{\prime}$ it must be that $z \in u_{n+1}$. In particular $\left(p_{n, 0} \otimes p_{n, 1}\right) \cap u_{n+1}$ is nonempty. Taking this intersection and extending it to an element of $D_{n+1}$ will give us the required $p_{n+1}$.

Toward a contradiction, assume the claim fails. This means that
$\left(^{*}\right)$ for any pair $x \in p_{n, 0}$ and $y \in p_{n, 1}$, we have that

$$
x(11 \ldots 1)<y(00 \ldots 0) \Longleftrightarrow x(0101 \ldots 01)<^{\prime} y(1010 \ldots 10)
$$

Since $\left(^{*}\right)$ is $\Pi_{1}^{1}$ on $\Sigma_{1}^{1}$, using the first reflection theorem twice, we can extend $p_{n, 0}$ and $p_{n, 1}$ to $\Delta_{1}^{1}$ sets $p_{n, 0}^{\prime} \supseteq p_{n, 0}$ and $p_{n, 1}^{\prime} \supseteq p_{n, 1}$, respectively, while keeping $\left({ }^{*}\right)$ for $p_{n, 0}^{\prime}$ and $p_{n, 1}^{\prime}$. In addition, we can assure that for $i=0,1$, for any $x \in p_{n, i}$ and $s, t \in 2^{n}, x(s) E x(t)$. Next we shrink $p_{n, 0}^{\prime}$ and $p_{n, 1}^{\prime}$.

For distinct $x, y \in X$ with $x E y$, define

$$
d(x, y)=\mid\{z \in X: x<z<y \text { or } y<z<x\} \mid .
$$

$d$ is $\Delta_{1}^{1}$ and is a metric on each $E$-class. Now, for each $x=\left(x_{t}\right)_{t \in 2^{n}} \in p_{n}$ let

$$
\operatorname{diam}(x)=\max \left\{d\left(x_{t}, x_{s}\right): t, s \in 2^{n}\right\}
$$

This is well defined since $x_{t} E x_{s}$ for all $t, s \in 2^{n}$. Now let

$$
a=\min \left\{\operatorname{diam}(x): x \in p_{n, 0}^{\prime}\right\}
$$

and define

$$
q_{n, 0}=\left\{x \in p_{n, 0}^{\prime}: \operatorname{diam}(x)=a\right\}
$$

Define a binary relation $R$ on $q_{n, 0}$ by

$$
\left(\left(x_{t}\right)_{t \in 2^{n}},\left(y_{t}\right)_{t \in 2^{n}}\right) \in R \Longleftrightarrow \exists t, s, r \in 2^{n}\left(x_{t}<y_{s}<x_{r} \text { or } y_{t}<x_{s}<y_{r}\right)
$$

By the definition of $q_{n, 0}, R$ is locally finite on $q_{n, 0} . R$ is clearly $\Delta_{1}^{1}$, so by [10, Lemma 1.17] we are able to find a $\Delta_{1}^{1}$ maximal $R$-anticlique $q_{n, 0}^{\prime} \subseteq q_{n, 0}$. We also shrink $p_{n, 1}^{\prime}$ in the same manner to obtain $q_{n, 1}^{\prime}$.

Let $q_{n}$ be the set of all $y \in q_{n, 1}^{\prime}$ such that there is $x \in q_{n, 0}^{\prime}$ with $x(11 \ldots 1)<$ $y(00 \ldots 0)$ but there is no $z \in q_{n, 1}^{\prime}$ such that $x(11 \ldots 1)<z(00 \ldots 0)$ and

$$
x(0101 \ldots 01)<^{\prime} z(1010 \ldots 10)<^{\prime} y(1010 \ldots 10)
$$

All the quantifiers in the definition of $q_{n}$ are first-order. Hence $q_{n}$ is still $\Delta_{1}^{1} . q_{n}$ is nonempty since on each $E$-class $<^{\prime}$ is order-isomorphic to $\mathbb{Z}$.

Now we show that $\pi_{0,1010 \ldots 10}\left(q_{n}\right)$ is a $E$-monotonic subset of $Y$, which contradicts the definition of $Y$. For this we show that for any $y, z \in q_{n}$,

$$
y(1010 \ldots 10)<z(1010 \ldots 10) \Longleftrightarrow y(1010 \ldots 10)<^{\prime} z(1010 \ldots 10)
$$

Assume this fails. Let $y, z \in q_{n}$ satisfy

$$
y(1010 \ldots 10)<z(1010 \ldots 10) \text { and } z(1010 \ldots 10)<^{\prime} y(1010 \ldots 10)
$$

Since $q_{n} \subseteq q_{n, 1}^{\prime}$ and $q_{n, 1}^{\prime}$ is an $R$-anticlique, we must have $y(t)<z(s)$ for any $t, s \in 2^{n}$. Let $x \in q_{n, 0}^{\prime}$ be a witness for $y \in q_{n}$. Then

$$
x(11 \ldots 1)<y(00 \ldots 0)<z(00 \ldots 0)
$$

By $\left(^{*}\right)$ we have $x(0101 \ldots 01)<^{\prime} z(1010 \ldots 10)$. Thus

$$
x(0101 \ldots 01)<^{\prime} z(1010 \ldots 10)<^{\prime} y(1010 \ldots 10)
$$

This contradicts the definition of $y \in q_{n}$.
The proof of the technical theorem is thus complete.

## 5. Hyperfinite-over-hyperfinite equivalence relations

In this final section we consider a special class of hyperfinite-over-hyperfinite Borel equivalence relations and show that they are indeed hyperfinite. To define the class, we consider a hyperfinite-over-hyperfinite equivalence relation and from it define a hyperfinite Borel equivalence relation with two Borel class-wise $\mathbb{Z}$-orders. We then compare the two orders and see if they are compatible. The details are as follows.

Let $E$ be a hyperfinite-over-hyperfinite equivalence relation on a standard Borel space $X$. Suppose $E$ is generated by a Borel class-wise $\mathbb{Z}^{2}$-order $\prec$. Following the proof of Proposition 2.8, define an equivalence relation $F \subseteq E$ by $x F y$ iff $x E y$ and there are only finitely many elements in between $x$ and $y$ in the $\prec$-order. $\prec$ linearly orders the $F$-classes within a single $E$-class into a suborder of $\mathbb{Z}$. We say that an $x \in X$ is full if the $\prec$-order of $\left\{[y]_{F}: y E x\right\}$ is order-isomorphic to $\mathbb{Z}$. The set of all full points of $X$ is called the full part of $X$, and denoted full $(X)$; its complement is called the non-full part. The full part is an $E$-invariant Borel subset. $E$ is hyperfinite on the non-full part.

As in the proof of Proposition 2.8, we can define a partial Borel injection $\gamma$ on full $(X)$ so that for any $x \in \operatorname{full}(X),[\gamma(x)]_{F}$ is the immediate predecessor of $[x]_{F}$ in the $\prec$-order of $F$-classes. Thus dom $(\gamma)$ is a Borel complete section of full $(X)$ for $F$. Moreover, we may require that for each $x \in \operatorname{full}(X), \prec \upharpoonright\left(\operatorname{dom}(\gamma) \cap[x]_{F}\right)$ is order-isomorphic to $\prec\left\lceil[x]_{F}\right.$.

For distinct $x, y \in X$ with $x F y$, let $d(x, y)$ be the distance between $x$ and $y$ in the $\prec$-order. If $A \subseteq \operatorname{full}(X)$ is a Borel complete section, then there is a unique Borel function $\eta_{A}:$ full $(X) \rightarrow A$ such that for each $x \in \operatorname{full}(X), d\left(\eta_{A}(x), x\right)=$ $\min \{d(y, x): y \in A\}$, and for any $y \in A$ with $d(y, x)=d\left(\eta_{A}(x), x\right), \eta_{A}(x) \leq$ $y$. Intuitively, $\eta_{A}(x)$ is the $d$-closest point to $x$ in $A$, and in the case when $x$ is equidistant to two points of $A, \eta_{A}(x)$ is the $\prec$-smaller one. When $A=\operatorname{dom}(\gamma)$ where $\gamma$ is the partial Borel injection above, $\eta_{A}$ is finite-to-one.

Now we define two Borel partial orders $<$ and $<^{\prime}$ on full $(X)$ so that they both generate $F$. For $x, y \in \operatorname{full}(X)$, let

$$
x<y \Longleftrightarrow x F y \text { and } x \prec y
$$

and

$$
\begin{aligned}
x<^{\prime} y \Longleftrightarrow & x F y \text { and either } \\
& \left(\eta_{A}(x)=\eta_{A}(y) \text { and } x \prec y\right) \text { or } \\
& {\left[\eta_{A}(x) \neq \eta_{A}(y) \text { and } \gamma\left(\eta_{A}(x)\right) \prec \gamma\left(\eta_{A}(y)\right)\right], }
\end{aligned}
$$

where $A=\operatorname{dom}(\gamma)$. Both $<$ and $<^{\prime}$ are Borel class-wise $\mathbb{Z}$-orders on full $(X)$ for $F$.
Definition 5.1. We say that the order $\prec$ is self-compatible if $<$ and $<^{\prime}$ are compatible.

It may look like the definition depends on the choice of the partial Borel injection $\gamma$. The following proposition shows that this is not the case.

Proposition 5.2. Denote the Borel partial order $<^{\prime}$ as $<_{\gamma}^{\prime}$. For any $\gamma_{0}$ and $\gamma_{1},<$ and $<_{\gamma_{0}}^{\prime}$ are compatible if and only if $<$ and $<_{\gamma_{1}}^{\prime}$ are compatible.

Proof. Suppose $<$ and $<_{\gamma_{0}}^{\prime}$ are compatible. Let $A_{0}=\operatorname{dom}\left(\gamma_{0}\right), A_{1}=\operatorname{dom}\left(\gamma_{1}\right)$, $\eta_{0}=\eta_{A_{0}}$ and $\eta_{1}=\eta_{A_{1}}$. Let $B \subseteq \operatorname{full}(X)$ be a Borel $F$-monotonic complete section for $<$ and $<_{\gamma_{0}}^{\prime}$. By Proposition 3.4 (4), we may assume that $B \subseteq \operatorname{dom}\left(\gamma_{0}\right)$ and that for any $x \in$ full $(X)$, if $[x]_{F}$ is infinite, then either $<\uparrow\left([x]_{F} \cap B\right)$ is order-isomorphic to $<\left\lceil[x]_{F}\right.$ or $<_{\gamma_{0}}^{\prime} \upharpoonright\left([x]_{F} \cap B\right)$ is order-isomorphic to $<_{\gamma_{0}}^{\prime} \upharpoonright[x]_{F}$ or both.

Let $X_{0}$ be the set of all $x \in$ full $(X)$ such that either $<\left\lceil\left([x]_{F} \cap B\right)\right.$ or $<_{\gamma_{0}}^{\prime} \upharpoonright\left([x]_{F} \cap\right.$ $B)$ is not order-isomorphic to $\mathbb{Z} . X_{0}$ is a Borel $F$-invariant subset of full $(X)$. On $X_{0}$ there is a Borel selector $\sigma$, from which we get a Borel $F$-monotonic complete section for $<$ and $<_{\gamma_{1}}^{\prime}$. For each $x \in \operatorname{full}(X) \backslash X_{0}$, both $<\upharpoonright\left([x]_{F} \cap B\right)$ and $<_{\gamma_{0}}^{\prime} \upharpoonright\left([x]_{F} \cap B\right)$ are order-isomorphic to $\mathbb{Z}$.

Let

$$
\begin{aligned}
& X_{+}=\left\{x \in \operatorname{full}(X) \backslash X_{0}: \forall a, b \in[x]_{F} \cap B\left(a<b \Longleftrightarrow a<_{\gamma_{0}}^{\prime} b\right)\right\} \\
& X_{-}=\left\{x \in \operatorname{full}(X) \backslash X_{0}: \forall a, b \in[x]_{F} \cap B\left(a<b \Longleftrightarrow b<_{\gamma_{0}}^{\prime} a\right)\right\}
\end{aligned}
$$

Then $X_{+}$and $X_{-}$are both $F$-invariant Borel sets.
For any $x \in X_{+}$and $y \in\left[\gamma_{0}\left(\eta_{0}(x)\right)\right]_{F}$, define

$$
\begin{aligned}
& I_{x, y}^{+}=\left\{z \in[x]_{F} \cap B: x<z \text { and } \gamma_{0}\left(\eta_{0}(z)\right)<y\right\} \\
& I_{x, y}^{-}=\left\{z \in[x]_{F} \cap B: z<x \text { and } y<\gamma_{0}\left(\eta_{0}(z)\right)\right\} .
\end{aligned}
$$

Since $x \in X_{+}, \gamma_{0}$ is increasing on $[x]_{F} \cap B$, and thus both $I_{x, y}^{+}$and $I_{x, y}^{-}$are finite, and at least one of them is empty. For $x \in X_{+}$, let

$$
n(x)=\left|I_{x, \gamma_{1}\left(\eta_{1}(x)\right)}^{+} \cup I_{x, \gamma_{1}\left(\eta_{1}(x)\right)}^{-}\right|
$$

Let $a=\min \{n(y): y F x\}$ and

$$
X_{+}^{\prime}=\left\{x \in X_{+}: n(x)=a\right\}
$$

Then $X_{+}^{\prime}$ is a Borel complete section of $X_{+}$for $F$. Define a binary relation $R$ on $X_{+}^{\prime}$ by

$$
\begin{aligned}
R(x, y) \quad \Longleftrightarrow \quad\left(x<y \text { and } \gamma_{1}\left(\eta_{1}(y)\right)<\gamma_{1}\left(\eta_{1}(x)\right)\right) \text { or } \\
\left(y<x \text { and } \gamma_{1}\left(\eta_{1}(x)\right)<\gamma_{1}\left(\eta_{1}(y)\right)\right) .
\end{aligned}
$$

$R$ is a Borel graph on $X_{+}^{\prime}$. We claim that $R$ is locally finite. To see this, consider $x, y \in X_{+}^{\prime}$ with $R(x, y)$. Without loss of generality assume that $x<y$ and $\gamma_{1}\left(\eta_{1}(y)\right)<\gamma_{1}\left(\eta_{1}(x)\right)$. We first show that there are only finitely many $z \in B$ such that $x<z<y$. Consider any such $z \in B$. Note that if $\gamma_{0}\left(\eta_{0}(z)\right)<\gamma_{1}\left(\eta_{1}(y)\right)$, then $\gamma_{0}\left(\eta_{0}(z)\right)<\gamma_{1}\left(\eta_{1}(x)\right)$, and thus $z \in I_{x, \gamma_{1}\left(\eta_{1}(x)\right)}^{+}$. Similarly if $\gamma_{1}\left(\eta_{1}(x)\right)<\gamma_{0}\left(\eta_{0}(z)\right)$ then $\gamma_{1}\left(\eta_{1}(y)\right)<\gamma_{0}\left(\eta_{0}(z)\right)$ and $z \in I_{y, \gamma_{1}\left(\eta_{1}(y)\right)}^{-}$. Therefore there are at most $2 a$ such $z$. This in turn implies that there are only finitely many $y$ satisfying our assumption. Thus $R$ is locally finite as claimed.

Let $C_{+}$be a Borel maximal $R$-anticlique. Then $C_{+}$is Borel $F$-monotonic complete section of $X_{+}$for $<$and $<_{\gamma_{1}}^{\prime}$.

A similar construction can be done on $X_{-}$to obtain a Borel $F$-monotonic complete section $C_{-}$of $X_{-}$for $<$and $<_{\gamma_{1}}^{\prime}$.

We have thus shown that $<$ and $<_{\gamma_{1}}^{\prime}$ are compatible.
Now we are ready for the main theorem of this section.
Theorem 5.3. If $E$ is a hyperfinite-over-hyperfinite equivalence relation on a standard Borel space $X$ and $E$ is generated by a Borel class-wise $\mathbb{Z}^{2}$-order which is self-compatible, then $E$ is hyperfinite.

Proof. Because $E$ is hyperfinite on the non-full part of $X$, it suffices to show that $E$ is hyperfinite on full $(X)$. We continue to use the notation developed in the above discussions, in particular the equivalence relation $F$, the partial Borel injection $\gamma$, and the Borel class-wise $\mathbb{Z}$-orders $<$ and $<^{\prime}$ on full $(X)$ which generate $F$. Let $A=\operatorname{dom}(\gamma)$. Let $B \subseteq A$ be a Borel $F$-monotonic complete section of full $(X)$ such that for all $x \in \operatorname{full}(X)$, if $[x]_{F}$ is infinite, then either $<\upharpoonright\left([x]_{F} \cap B\right)$ is orderisomorphic to $<\upharpoonright[x]_{F}$ or $<^{\prime} \uparrow\left([x]_{F} \cap B\right)$ is order-isomorphic to $<^{\prime} \uparrow[x]_{F}$ or both. Now if we consider $\gamma^{\prime}=\gamma \upharpoonright B$, then $<_{\gamma^{\prime}}^{\prime}=<^{\prime}$ on $B$ and thus $B$ is still a Borel $F$ monotonic complete section of full $(X)$ with the stated properties with $<_{\gamma^{\prime}}^{\prime}$ replacing $<^{\prime}$. Thus we may assume $A=B$ and $<_{\gamma^{\prime}}^{\prime}=<^{\prime}$. Let $\eta=\eta_{A}=\eta_{B}$.

Let $X_{0}$ be the set of all $x \in$ full $(X)$ such that either $<\uparrow\left([x]_{F} \cap B\right)$ or $<^{\prime} \uparrow\left([x]_{F} \cap B\right)$ is not order-isomorphic to $\mathbb{Z}$. Let $Y=\left[X_{0}\right]_{E}$. Then $X_{0}$ is a Borel $F$-invariant subset of full $(X)$ and $Y$ is a Borel $E$-invariant subset of full $(X)$. We claim that $E \upharpoonright Y$ is hyperfinite. To see this, let $\sigma: X_{0} \rightarrow X_{0}$ be a Borel selector for $F$. By the FeldmanMoore theorem, $\sigma$ can be extended to $Y$, which we still denote as $\sigma$. Then the set $C=\{x \in Y: \sigma(x)=x\}$ is a Borel complete section of $E$ on $Y$, and $\prec$ on $C$ is a Borel class-wise $\mathbb{Z}$-order. Thus $E \upharpoonright C$ is hyperfinite, and by [3, Proposition 5.2(4)], $E \upharpoonright Y$ is hyperfinite.

Now let $X_{1}=\operatorname{full}(X) \backslash Y$. Then for any $x \in X_{1}$, both $<\uparrow\left([x]_{F} \cap B\right)$ and $<^{\prime} \uparrow\left([x]_{F} \cap B\right)$ are order-isomorphic to $\mathbb{Z}$. Define $\phi: X_{1} \rightarrow X_{1}^{\omega}$ by

$$
\phi(x)=\left(x,(\gamma \circ \eta)(x),(\gamma \circ \eta)^{2}(x), \cdots\right) .
$$

Consider the tail equivalence relation $E_{t}\left(X_{1}\right)$. We split $X_{1}$ into two parts:

$$
\begin{aligned}
& X_{2}=\left\{x \in X_{1}: \forall y \in[x]_{E}\left(\phi(x) E_{t}\left(X_{1}\right) \phi(y)\right)\right\}, \\
& X_{3}=X_{1} \backslash X_{2} .
\end{aligned}
$$

In other words, $X_{2}$ is precisely the part of $X_{1}$ on which $\phi$ reduces $E$ to $E_{t}\left(X_{1}\right)$. By the theorems of Kechris-Louveau and Dougherty-Jackson-Kechris (Proposition 2.5 (iii)), $E \upharpoonright X_{2}$ is hyperfinite.

Now, we focus on $X_{3}$, which is a Borel $E$-invariant subset of $X_{1}$. Let $E^{\prime}=$ $(\phi \times \phi)^{-1}\left(E_{t}\left(X_{1}\right)\right)$ be the pullback of $E_{t}\left(X_{1}\right)$. Then $E^{\prime}$ is hyperfinite. Note that for any $x, y, z \in X_{3}$, if $x<y<z$ and $x E^{\prime} z$ then $x E^{\prime} y E^{\prime} z$. Thus the $E^{\prime}$-classes within an $F$-class consist of intervals in the <-order. Let $Z$ be the set of $x \in X_{3}$ such that there is an infinite $E^{\prime}$-class within $[x]_{F}$. Then $Z$ is a Borel $F$-invariant subset of $X_{3}$, and there is a Borel selector for $F$ on $Z$. A similar argument as the above for $E \upharpoonright Y$ shows that $E \upharpoonright Z$ is hyperfinite. Thus we assume without loss of generality that $Z=\varnothing$, i.e., all $S^{\prime}$-classes within an $F$-class in $X_{3}$ are finite.

Let

$$
S=\left\{x \in X_{3}: \forall y<x(y, x) \notin E^{\prime}\right\} .
$$

Then $S$ is a Borel complete section of $E$ on $X_{3}$ such that for any $x \in X_{3},<\uparrow$ $\left([x]_{F} \cap S\right.$ ) is order-isomorphic to $\mathbb{Z}$. By [3, Proposition 5.2(4)], we only need to prove that $E \upharpoonright S$ is hyperfinite.

Note that for any $x \in X_{3},[x]_{F} \cap[x]_{E^{\prime}} \cap S$ is a singleton. Therefore, for every $x \in S$ there is a unique $y \in S$ so that $(\gamma \circ \eta)(x) E^{\prime} y$. Let $\alpha: S \rightarrow S$ denote this map $x \mapsto y$. Then $\alpha$ is Borel. We note that $\alpha$ is an injection. In fact, if $x, y \in S$ such that $\alpha(x)=\alpha(y)$, then $x F y$ and $x E^{\prime} y$, hence $x=y$. For every $x \in S$, we define $\alpha^{-\infty}(x)$ to be, if it exists, the unique $z \in S$ so that $z \notin \operatorname{range}(\alpha)$ and there is $n \geq 0$ with $\alpha^{n}(z)=x$. If such $z$ does not exist we leave $\alpha^{-\infty}(x)$ undefined. $\alpha^{-\infty}$ is a partial Borel function.

Consider the case where $\alpha$ is a bijection on a Borel complete section $S^{\prime} \subseteq S$ for $E \upharpoonright S$, i.e., $\alpha: S^{\prime} \rightarrow S^{\prime}$ is a bijection from $S^{\prime}$ onto $S^{\prime}$. Again, we may assume that for any $x \in S^{\prime},<\upharpoonright\left([x]_{F} \cap S^{\prime}\right)$ is order-isomorphic to $\mathbb{Z}$, since otherwise there is a Borel selector for its $F$-class and we deal with such points by a similar argument as that for $E \upharpoonright Y$. Now we can define a Borel action of the additive group $\mathbb{Z}^{2}$ on $S^{\prime}$ by letting $(1,0) \cdot x=\alpha(x)$ and $(0,1) \cdot x=\min _{<}\left\{y \in S^{\prime}: y>x\right\}$. One readily checks that this is indeed a $\mathbb{Z}^{2}$-action that generates $E \upharpoonright S^{\prime}$. By a theorem of Weiss (3), $E \upharpoonright S^{\prime}$ is hyperfinite, and it follows from [3 Proposition 5.2(4)] that $E \upharpoonright S$ as well as $E \upharpoonright X_{3}$ is hyperfinite.

So we further focus on $E$-classes in which $\alpha$ is not a bijection on any Borel complete section for $E \upharpoonright S$. Note that for any $x \in S$, if there is $y \in[x]_{E}$ such that $\alpha^{-\infty}(y)$ is undefined, then we can in a Borel way produce a subset of $[x]_{E}$ on which $\alpha$ is a bijection. In other words, we consider the part

$$
S^{\prime}=\left\{x \in S: \forall y \in[x]_{E}\left(y \in S \rightarrow y \in \operatorname{dom}\left(\alpha^{-\infty}\right)\right)\right\}
$$

For every $x \in S^{\prime}$ define

$$
\psi(x)= \begin{cases}\alpha^{-1}(x), & \text { if } x \in \operatorname{range}(\alpha), \\ \alpha^{-1}\left(\max _{\prec}\{y \in \operatorname{range}(\alpha): y<x\}\right), & \text { otherwise } .\end{cases}
$$

Then $\psi(x)$ is well defined for every $x \in S^{\prime}$ and is Borel.
To finish the proof in the case that $\gamma$ is increasing on every $F$-class, we claim that for every pair $x, y \in S^{\prime}$ with $x E y$, we have that

$$
\left(x, \psi(x), \ldots, \psi^{n}(x), \ldots\right) E_{t}\left(S^{\prime}\right)\left(y, \psi(y), \ldots, \psi^{m}(y), \ldots\right)
$$

To see this, fix such a pair $x, y$. Either there is $n \geq 0$ so that $\alpha^{n}(x) F y$ or there is $n \geq 0$ so that $\alpha^{n}(y) F x$. Without loss of generality we assume that $\alpha^{n}(x) F y$ for some $n \in \mathbb{N}$. Also without loss of generality assume that $\alpha^{n}(x)<y$. Then for each $k \geq 0, \psi^{k}\left(\alpha^{n}(x)\right)$ and $\psi^{k}(y)$ are in the same $F$-class and $\psi^{k}\left(\alpha^{n}(x)\right) \leq \psi^{k}(y)$. For each $k \geq 0$, let $N_{k}$ denote the number of elements of $z \in S^{\prime}$ so that $\psi^{k}\left(\alpha^{n}(x)\right)<$ $z \leq \psi^{k}(y)$. When $N_{k}=0$ we have that $\psi^{k}\left(\alpha^{n}(x)\right)=\psi^{k}(y)$. Now observe that
(a) if $\psi^{k}(y) \in \operatorname{range}(\alpha)$, then $N_{k+1} \leq N_{k}$;
(b) if $\psi^{k}(y) \notin$ range $(\alpha)$, then $N_{k+1}<N_{k}$ if $N_{k}>0$.

Since $x, y \in S^{\prime}$ we conclude that case (b) must happen as $k$ increases, and therefore for some large enough $k, N_{k}=0$.

Now we extend this result to the general case where $\gamma$ is monotonic on every $F$-class. Consider an equivalence relation $F^{\prime}$ on $X_{3} \times\{0,1\}$ defined as

$$
(x, i) F^{\prime}(y, j) \Longleftrightarrow x F y \text { and } i=j
$$

Define a Borel partial order $\triangleleft$ on $X_{3} \times\{0,1\}$ by

$$
\begin{aligned}
(x, i) \triangleleft(y, j) \Longleftrightarrow & (x \prec y \text { and }(x, y) \notin F) \text { or } \\
& (x F y \text { and } i<j) \text { or } \\
& (x F y \text { and } x<y \text { and } i=0) \text { or } \\
& (x F y \text { and } y<x \text { and } i=1) .
\end{aligned}
$$

Then $\triangleleft$ is a Borel class-wise $\mathbb{Z}^{2}$-order on $X_{3} \times\{0,1\}$. Let $E_{\triangleleft}$ be the equivalence relation generated by $\triangleleft$.

Define $\gamma^{\prime}$ on $X_{3} \times\{0,1\}$ by

$$
\gamma^{\prime}(x, i)= \begin{cases}(\gamma(x), i), & \text { if } \gamma \text { is increasing on }[x]_{E}, \\ (\gamma(x), 1-i), & \text { otherwise }\end{cases}
$$

Let $E_{\gamma^{\prime}}$ be the equivalence relation generated by $F^{\prime}$ together with the map $\gamma^{\prime}$, i.e., $E_{\gamma^{\prime}}$ is the symmetric and transitive closure of the union of $F^{\prime}$ and the graph of $\gamma^{\prime}$. Then $\triangleleft \cap E_{\gamma^{\prime}}$ is still a Borel class-wise $\mathbb{Z}^{2}$-order on $X_{3} \times\{0,1\}$ for $E_{\gamma^{\prime}}$. Let $\ll$ and $<^{\prime}$ be the induced class-wise $\mathbb{Z}$-order on $X_{3} \times\{0,1\}$ for $F^{\prime}$. Then $\gamma^{\prime}$ is increasing with respect to $\ll$ and $<^{\prime}$ on every $F^{\prime}$-class. By the above argument for the increasing case, $E_{\gamma^{\prime}}$ is hyperfinite.

Now note that $E_{\gamma^{\prime}} \subseteq E_{\triangleleft}$ and each $E_{\triangleleft}$ contains exactly two $E_{\gamma^{\prime}}$-classes. Hence by [3, Proposition $1.3\left(\right.$ vii)], $E_{\triangleleft}$ is hyperfinite. Now $x \mapsto(x, 0)$ is a natural Borel embedding of $X_{3}$ into $X_{3} \times\{0,1\}$ which is a reduction of $E$ to $E_{\triangleleft}$. Thus $E$ is hyperfinite.

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