ON LIFTINGS OF MODULES OF FINITE PROJECTIVE DIMENSION

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Abstract. We introduce and study a notion of dimly liftable modules; these are modules that are liftable to the right dimension to a regular local ring. We establish special new cases of Serre's positivity conjecture over ramified regular local rings by proving it for dimly liftable modules. Furthermore, we show that the length of a nonzero finite length dimly liftable module is bounded below by the Hilbert-Samuel multiplicity of the local ring. This establishes special cases of the Length Conjecture of Iyengar-Ma-Walker.

Introduction

Suppose R is a noetherian local ring and Q is a regular local ring that surjects onto R. Classically, a finitely generated R-module M is said to *derived lift* to Q if there exists a Q-module L such that $L \otimes^{\mathsf{L}}_Q R \simeq M$. When this occurs, L is called a *lift* of M and the minimal R-free resolution of M lifts to the minimal Q-free resolution of L. In this situation, many properties of an R-module are dictated by the properties of its lift. Since over a regular ring, all modules have finite projective dimension, one expects that modules over an arbitrary local ring rarely lift. In this paper, we introduce a natural, more geometric notion of lifting R-modules that generalizes this derived lifting.

We begin by showing that if a finitely generated Q-module L is a lift of an R-module M, then we have the following equality of Krull dimensions; see Proposition 2.3:

$$\dim L + \dim R = \dim Q + \dim M.$$

Motivated by this result, we say a Q-module L is a *dim lift* of an R-module M if $L \otimes_Q R \cong M$ and the equality of dimension as above is achieved. When M admits such a lift, we say M is *dimly liftable*. While liftable modules must have finite projective dimension, dimly liftable modules have no such restriction. In fact, dimly liftable modules exist in abundance. Producing modules of finite projective dimension over a singular ring is harder but we show there exists a two dimensional hypersurface singularity that admits a module of finite length and finite projective dimension that is dimly liftable but unliftable; cf. Example 2.16.

Serre introduced the following fundamental notion of intersection multiplicity over regular local rings in [33]. Given finitely generated modules M and N over a regular local ring R whose supports intersect at the maximal ideal, we define

$$\chi(M, N) \coloneqq \chi(M \otimes^{\mathsf{L}}_R N) = \sum_{i \geq 0} (-1)^i \ell_R(\operatorname{Tor}_i(M, N)).$$

Serre proved that if R is an unramified regular ring and $\dim M + \dim N = \dim R$, then $\chi(M, N) > 0$. He conjectured that this holds in general but it remains wide open when R is a ramified regular local ring. In fact, the idea of lifting modules was initially motivated by the fact that Serre's positivity conjecture would hold [3] if one of the intersecting modules lifts to an unramified regular local ring. However, unliftable modules in this setting were subsequently found in [11]. Nevertheless, there exist unliftable modules that dimly lift to an unramified regular local ring; see Example 2.8. The following theorem establishes the positivity conjecture for such modules; see Theorem 2.9.

Theorem A. *Suppose R is a ramified regular local ring and M, N finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N = \dim R$. Fix a surjection $Q \twoheadrightarrow R = Q/f$ where Q is an unramified regular local ring. If M is dimly liftable to Q, then $\chi(M, N) > 0$.*

Interestingly, we have not been able to produce an example of a module over a ramified regular local ring that is not dimly liftable to an unramified regular local ring; cf. Question 2.11. Our initial interest in liftable and dimly liftable modules arose in trying to understand lengths of modules of finite projective dimension over local singularities.

One motivation for studying lengths of modules of finite projective dimension is to understand the behavior of the Hilbert Samuel multiplicity under local flat extensions. The long-standing Lech's conjecture [21] contends that if $R \to S$ is a faithfully flat extension of local rings, then $e(R) \leq e(S)$. The intuition is that under a faithfully flat extension, the singularity cannot get better. However, beyond dimension three [24] or the graded setting [25], this question remains unanswered. Consider then the following recent conjecture of Iyengar, Ma, and Walker.

Conjecture B. [14, Conjecture 1] *If M is a nonzero module of finite length and finite projective dimension over a noetherian local ring R, then $\ell_R(M) \geq e(R)$.*

Due to Ma, an affirmative answer to this conjecture settles Lech's conjecture for Cohen-Macaulay rings; see [23, Chapter V] for details. Unfortunately, Conjecture B is also only known in special cases. Iyengar, Ma, and Walker establish a stronger version of Conjecture B for standard graded k-algebras localized at the homogeneous maximal ideal where k is a perfect field of characteristic $p > 0$; cf. Theorem 3.10. On the other hand, without the graded structure, Conjecture B is not known in full generality even in dimension two. The following theorem is thus of interest; see Corollary 3.12.

Theorem C. *Suppose M is a Cohen-Macaulay module of finite projective dimension over a noetherian local ring R that is a quotient of a regular local ring Q. If M dimly lifts to a Q-module L where Q is an unramified regular local ring, then*

$$e(M) \geq e(L)e(R).$$

In particular, Theorem C establishes Conjecture B for the class of dimly liftable modules of finite projective dimension. It is well known that if I is a perfect ideal of a Cohen-Macaulay local ring R in the linkage class of a complete intersection, then R/I lifts, and hence dimly lifts, to an unramified regular ring; see Example 2.1. Hence, for such ideals Theorem C establishes that $e(R/I) \geq e(R)$. However, in general, there are obstructions to lifting modules of finite projective dimension to a regular local ring and we will give many such examples in Section 4. Notably, we report a class of perfect modules of projective dimension two over a Cohen Macaulay ring that cannot lift to any regular local ring; the main idea is due to

Heitmann. We also show there exist modules of finite length and finite projective dimension over singular rings that do not dimly lift; see Example 4.2.

Due to the New Intersection Theorem of Roberts [29], the only rings that admit a module of finite length and finite projective dimension are Cohen-Macaulay. To generalize Conjecture B to all local rings, we consider the so called short complexes. A *short complex* F over a local ring R is a complex

$$F : 0 \to F_d \to \cdots \to F_0 \to 0$$

such that $d = \dim R$, each F_i is a finite free R-module, and $\ell_R(\mathrm{H}(F)) < \infty$. For such a complex F with nonzero homology, Iyengar, Ma, and Walker also conjecture that $\chi_\infty(F) \geq e(R)$ where $\chi_\infty(-)$ is the *Dutta* multiplicity; see Conjecture 2. Validity of this conjecture implies Lech's conjecture in full generality [14, Proposition 8.3]. Motivated by the module case, we introduce the following notion of *dimly liftable* short complexes. A short complex F dimly lifts to a regular local ring Q if the finite length R-module $\mathrm{H}_0(F)$ dimly lifts to Q. On this topic, we prove the following; see Theorem 3.19.

Theorem D. *If F is a short complex over a local ring R of prime characteristic $p > 0$ with nonzero homology that dimly lifts to an unramified regular local ring, then*

$$\chi_\infty(F) \geq e(R).$$

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1. Preliminaries

In this section we establish some notation and give a quick introduction to the notion of intersection multiplicity over a local ring. Throughout this paper, all rings are noetherian local rings. We will say a local ring $(R, \mathfrak{m})$ is unramified if it is either of equal characteristic or it is of mixed characteristic $(0, p)$ for a prime integer p with $p \in \mathfrak{m} \setminus \mathfrak{m}^2$.

1.1. For basic results on Hilbert-Samuel multiplicities, we refer to [2, 33]; alternatively, see [14, Section 2] as we use similar notation. Given a local ring $(R, \mathfrak{m})$ and a nonzero finitely generated R-module M, the Hilbert-Samuel multiplicity of M is

$$e(M) := (\dim M)! \lim_{n\to\infty} \frac{\ell_R(M/\mathfrak{m}^n M)}{n^{\dim M}}.$$

This limit exists and is strictly positive. When $M = R$ itself, $e(R)$ is the Hilbert-Samuel multiplicity of the local ring R.

1.2. Consider a local ring R and two finitely generated R-modules M and N such that their supports intersect at the maximal ideal of R. This is equivalent to the condition $\ell_R(M \otimes_R N) < \infty$. Simple examples indicate that the integer $\ell_R(M \otimes_R N)$ fails to correctly count the multiplicty of this intersection. Serre suggested that over regular rings one can extract the correct notion by considering the derived tensor product of M and N. Assuming $\operatorname{projdim}_R(M) < \infty$, Serre defined the following *intersection multiplicity* pairing

$$\chi^R(M, N) := \chi(M \otimes^{\mathsf{L}}_R N) = \sum_{i \geq 0} (-1)^i \ell_R(\operatorname{Tor}_i(M, N)).$$

When R is an unramified regular local ring, Serre proved several properties of this pairing that is expected of a good intersection theory. By [33, Ch.V, Thm 1], we have

(i) *Dimensional inequality*: $\dim M + \dim N \leq \dim R$,
(ii) *Non-negativity*: $\chi(M, N) \geq 0$,
(iii) *Vanishing:* If $\dim M + \dim N < \dim R$, then $\chi(M, N) = 0$,
(iv) *Positivity:* If $\dim M + \dim N = \dim R$, then $\chi(M, N) > 0$.

Dimensional inequality holds for all regular rings [33, Chapter V, Theorem 3]. Although positivity remains an open conjecture for ramified regular local rings, non-negativity (due to Gabber [32]) and the vanishing property (due to Gillet-Soulé [8] and Roberts [28] independently) have been proven for all regular rings. The main property of this invariant that we will utilize is the following theorem of Serre (in equal characteristic) and Skalit (in the unramified mixed characteristic).

Theorem 1.3. [34, Theorem A], [33, Ch.V, Thm 1 Complement] *Suppose R is an unramified regular local ring and M, N are finitely generated R-modules such that $\ell_R(M \otimes_R N) < \infty$. If $\chi(M, N) > 0$, then $\chi(M, N) \geq e(M)e(N)$.*

1.4. In prime characteristic $p > 0$ we will also utilize the Frobenius intersection multiplicities introduced by Dutta [5]. For the rest of this section, we consider a local ring R of characteristic $p > 0$. Let $f\colon R \to R$ denote the Frobenius endomorphism on R. We will assume that our rings of characteristic $p > 0$ are F-finite throughout this paper, i.e. $f : R \to R$ is a finite map of R-modules. For $e \in \mathbb{N}$, consider the eth iterate of the Frobenius map $f^e(r) = r^{p^e}$. Let ${}^{f^e}R$ denote R-R bimodule with left action via f^e and the canonical right action. The Frobenius functor on the category of R-modules is defined as $F^e(-) = - \otimes_R {}^{f^e}R$ where the R-module action is from the right factor. See [27, Chapter 1] for more details. Suppose R is a local ring of prime characteristic $p > 0$ and M, N two finitely generated R-modules such that $\ell_R(M \otimes N) < \infty$ and $\operatorname{projdim}_R M < \infty$. Dutta defined the following *Frobenius intersection multiplicity* pairing whenever the limit exists.

$$\chi_\infty(M, N) := \lim_{e \to \infty} \frac{\chi(F^e(M), N)}{p^{e \operatorname{codim} M}}.$$

If R is a local ring such that the vanishing property as above holds, then Dutta proved that $\chi_\infty(M, N) = \chi(M, N)$ [5, 1.3 Remark]. If M is an R-module of finite length and finite projective dimension, then the *Dutta multiplicity* of M is defined as $\chi_\infty(M) \coloneqq \chi_\infty(M, R)$. In general, $\chi_\infty(M) \neq \ell_R(M)$; see Example 4.3.

2. Dimly liftable modules

Throughout this section, we fix a surjection of local rings $\pi\colon Q \twoheadrightarrow R$. An R-complex is given the structure of Q-complex via restriction of scalars along π. All the modules considered are finitely generated.

2.1. **Liftability.** A finitely generated R-module M (derived) lifts to Q if there exists a finitely generated Q-module L such that $L \otimes^{\mathsf{L}}_Q R \simeq M$. We say L is a *(derived) lift* of M. This means that $L \otimes_Q R \cong M$ as modules and $\operatorname{Tor}_i^Q(L,R) = 0$ for $i > 0$. In other words, the minimal Q-free resolution of L tensors down to the minimal R-free resolution of M. If $\operatorname{projdim}_Q(L) < \infty$, we have $\operatorname{projdim}_R(M) < \infty$. This is a classical definition and is related to "Grothendieck's lifting problem"; see, for example, [3, 4, 11, 16, 17, 27]. It is now well-known that modules, even over a regular local rings, do not lift in general; see Example 2.8 for a particular one. In Section 4, we will report several examples and obstructions to liftability of modules over singular rings to regular rings. The following is one class of modules of finite projective dimension that lift to a regular ring.

Example 2.1. (Licci ideals) Suppose R is a Cohen-Macaulay local ring and fix a surjection $Q \twoheadrightarrow R$ where Q is an unramified regular local ring. Two perfect ideals I and J of R are said to be linked by a complete intersection, denoted $I \sim J$, if there exists a regular sequence $\boldsymbol{x} = x_1, \ldots, x_c$ such that $J = (\boldsymbol{x}) : I$ and $I = (\boldsymbol{x}) : J$. Jorgensen has shown that if $I \sim J$, R/I lifts to Q if and only if R/J also lifts to Q [17, Proposition 2.3]. Jorgensen assumes $R = Q/(x)$ and R is Gorenstein, but given our definition of linkage the proof applies at the level of generality stated.

We say I is in the linkage class of a complete intersection, or is *licci*, if there exists a finite sequence of links $I = I_0 \sim I_1 \sim \cdots \sim J$ where $J = (\boldsymbol{x})$ is an ideal generated by a regular sequence. Since the resolution of $R/(\boldsymbol{x})$ is the Koszul complex, it lifts to Q, whence any licci ideal lifts. It is well known that any perfect ideal of codimension two and any perfect, Gorenstein ideal of codimension three are licci. More recently, in equal characteristic zero, Gurrieri, Ni, and Weyman have shown that many classes of perfect ideals of codimension three are licci [9, Theorem 5.2].

2.2. **Dim liftability.** We now introduce the main objects of interest in this paper. We begin by recording a preliminary lemma.

Lemma 2.2. *Suppose R is a local ring, M is an R-module of finite projective dimension and N is an R-complex with bounded homology. If $\ell(M \otimes_R \mathrm{H}_i(N)) < \infty$ for all i,*

$$\chi(M \otimes^{\mathsf{L}}_R N) = \sum_i (-1)^i \chi(M \otimes^{\mathsf{L}}_R \mathrm{H}_i(N)).$$

Proof. We induct on the number of nonzero homology modules of N. The base case is clear. Suppose $n = \max\{i : \mathrm{H}_i(N) \neq 0\}$. We have an exact triangle in $\mathsf{D}(R)$, the derived category of R-modules.

$$\Sigma^n \mathrm{H}_n(N) \xrightarrow{\iota} N \to C \to$$

Hence, by additivity of $\chi(M \otimes^{\mathsf{L}}_R -)$ on triangles in $\mathsf{D}(R)$, we get

$$\chi(M \otimes^{\mathsf{L}}_R N) = \chi(M \otimes^{\mathsf{L}}_R C) + \chi(M \otimes^{\mathsf{L}}_R \Sigma^n \mathrm{H}_n(N)).$$

Now by inductive hypothesis, we get

$$\chi(M \otimes^{\mathsf{L}}_R N) = \sum_{i\geq 0} \chi(M \otimes^{\mathsf{L}}_R \Sigma^i \mathrm{H}_i(N)).$$

since $\mathrm{H}_n(C) = 0$ and $\mathrm{H}_j(C) = \mathrm{H}_j(N)$ for $j \neq n$. The claimed equality follows as $\chi(M \otimes^{\mathsf{L}}_R \Sigma^i \mathrm{H}_i(N)) = (-1)^i \chi(M \otimes^{\mathsf{L}}_R \mathrm{H}_i(N))$. □

Proposition 2.3. *Suppose a finitely generated Q-module L is a lift of a finitely generated R-module M. If Q is regular, then*

$$\dim L + \dim R = \dim Q + \dim M.$$

Proof. Suppose $\dim M = r$. Choose $\boldsymbol{x} = x_1, \ldots, x_r$ in the maximal ideal of R which is a parameter sequence on both M and R. Since $L \otimes_Q R/(\boldsymbol{x}) \cong M/(\boldsymbol{x})M$ is of finite length, we have $\dim L + \dim R \leq \dim Q + \dim M$ by Serre's dimensional inequality; see 1.2. Let $K_R = K(\boldsymbol{x})$ be the Koszul complex on $\boldsymbol{x}$ over R. As L is a lift of M, we have the following quasi-isomorphism of complexes

$$L \otimes^{\mathsf{L}}_Q K_R \simeq M \otimes^{\mathsf{L}}_R K_R.$$

As $\mathrm{H}_i(K_R)$ is annihilated by $(\boldsymbol{x})$ as an R-module, $\mathrm{Supp}(\mathrm{H}_i(K_R)) \subseteq \mathrm{Supp}(R/(\boldsymbol{x}))$, whence $\dim \mathrm{H}_i(K_R) \leq \dim R/(\boldsymbol{x})$ and $L \otimes_Q \mathrm{H}_i(K_R)$ has finite length for all i. By Lemma 2.2, we have

$$\chi(L \otimes^{\mathsf{L}}_R K_R) = \sum_{i\geq 0} (-1)^i \chi(L \otimes^{\mathsf{L}}_R \mathrm{H}_i(K_R)).$$

By way of contradiction, assume $\dim L + \dim R < \dim Q + \dim M$. Equivalently, $\dim L + \dim R/(\boldsymbol{x}) < \dim Q$, whence $\dim L + \dim \mathrm{H}_i(K_R) < \dim Q$ for all i. The vanishing theorem of Roberts and Gillet-Soulé (see 1.2) implies $\chi(L \otimes^{\mathsf{L}}_Q K_R) = 0$. This is a contradiction as $\chi(L \otimes^{\mathsf{L}}_Q K_R) = \chi(M \otimes^{\mathsf{L}}_R K_R) = \chi(\boldsymbol{x}; M) = e(\boldsymbol{x}; M) > 0$ by Serre's theorem [33, Chapter IV, Theorem 1]. This completes our proof. □

Remark 2.4. The proof above actually generalizes the conclusion of Proposition 2.3 to the case when Q is a hypersurface and R is Cohen-Macaulay. Indeed, the dimensional inequality of Serre is known to hold over hypersurfaces by the work of Hochster [12]. Furthermore, since R is Cohen-Macaulay, $K_R \simeq R/(\boldsymbol{x})$ and as a Q-module $R/(\boldsymbol{x})$ has finite projective dimension. In fact, assuming R is Cohen-Macaulay, if Q is a local ring such that the dimensional inequality and vanishing conjectures hold, then the proof of Proposition 2.3 establishes the analogue of the derived depth formula for Krull dimensions. This raises the following question.

Question 2.5. Suppose $Q \twoheadrightarrow R$ is a surjection of noetherian local Cohen-Macaulay rings. Suppose a finitely generated R-module of finite projective dimension lifts to a Q-module L, i.e. $L \otimes^{\mathsf{L}}_Q R \simeq M$. Do we have $\dim L + \dim R = \dim Q + \dim M$?

2.6. Suppose a Q-module L is a lift of an R-module M, i.e. $L \otimes^{\mathsf{L}}_Q R \simeq M$. If $\mathrm{projdim}_R(M) < \infty$, by the derived depth formula [13, Corollary 2.2],

$$\mathrm{depth}\, L + \mathrm{depth}\, R = \mathrm{depth}\, Q + \mathrm{depth}\, M.$$

If Q is regular, by Proposition 2.3, $\dim L + \dim R = \dim Q + \dim M$ whence

$$(\dim L - \mathrm{depth}\, L) + (\dim R - \mathrm{depth}\, R) = (\dim M - \mathrm{depth}\, M).$$

Therefore, when Q is regular and R is Cohen-Macaulay, if L is a lift of M,

$$\dim L - \mathrm{depth}\, L = \dim M - \mathrm{depth}\, M.$$

In other words, M is Cohen-Macaulay if, and only if, so is its lift L.

Proposition 2.3 motivates the following weaker, more geometric notion of lifting.

Definition 2.7. A finitely generated Q-module L is a *dim lift* of a finitely generated R-module M if $L \otimes_Q R \cong M$ as modules and $\dim L + \dim R = \dim Q + \dim M$. If M admits such a lift, we say M is *dimly liftable* to Q.

2.3. **Dimly liftable modules over regular rings.** The motivation behind lifting modules over a regular ring comes from the "Grothendieck's lifting problem". Suppose Q is an unramified regular local ring and R is a ramified regular local ring. If every R-module M lifts to Q, then the still open Serre's positivity conjecture (see 1.2) would be resolved; see [3] for an argument. However, counterexamples were subsequently found in [11] (see also [4, 16]).

Example 2.8. (Hochster) [11, Example 2] Let $Q = \mathbb{Z}_{(2)}[[x, y, z, a, b, c]]$ and let $f = 2 + u$ where $u = x^2 + x^4 + y^4$. Consider the ramified regular local ring $R = Q/f$ and the surjection $Q \twoheadrightarrow R$. Let $I = (x^2, y^2, z^2, a^2, b^2, c^2, xa + yb + zc)$. Note that $M = R/(I)$ is an R-module of finite length. Hochster proved that M does not lift to Q. It is dimly liftable however; Q/I is a dim lift of M. Indeed, $I \subseteq \mathfrak{b}$ where $\mathfrak{b} = (x, y, z, a, b, c)$, so $V(I) \supseteq V(\mathfrak{b})$. Clearly, $V(\mathfrak{b})$ is 1-dimensional so, $\dim Q/I = \dim V(I) \geq \dim V(\mathfrak{b}) = 1$. Note that the dimension is also at most one, and therefore $\dim Q/I + \dim R = \dim Q$.

Theorem 2.9. *Suppose R is a ramified regular local ring and M, N two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N = \dim R$. Fix a surjection $\pi\colon Q \twoheadrightarrow R = Q/f$ where Q is an unramified regular local ring. Suppose M is dimly liftable to Q. Then, $\chi^R(M, N) > 0$.*

Proof. Suppose a Q-module L is a dim lift of M. Let $G \xrightarrow{\simeq} L$ denote the minimal Q-free resolution of L and set $\overline{G} \coloneqq G \otimes_Q R$. As $\mathrm{H}_i(\overline{G}) = \mathrm{Tor}_i^Q(L, R)$, we have that $\overline{G}$ is a perfect complex over R with $\mathrm{H}_i(\overline{G}) = 0$ for $i \geq 2$ and $\mathrm{H}_0(\overline{G}) \cong M$. Furthermore, $\mathrm{Supp}(\mathrm{H}_1(\overline{G})) \cap \mathrm{Supp}(N) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \{\mathfrak{m}\}$ where $\mathfrak{m}$ is the maximal ideal of R. Thus we may apply Lemma 2.2 to get

$$\chi(\overline{G} \otimes_R^{\mathsf{L}} N) = \chi(M \otimes_R^{\mathsf{L}} N) - \chi(\mathrm{H}_1(\overline{G}) \otimes_R^{\mathsf{L}} N).$$

Since L is a dim lift of M, $\dim L + \dim N = \dim Q$ so by Serre's positivity theorem over unramified regular local rings (see 1.2),

$$\chi^Q(L, N) = \chi(\overline{G} \otimes_R^{\mathsf{L}} N) > 0.$$

On the other hand, by Gabber's nonnegativity theorem [32] over ramified regular local rings, $\chi(\mathrm{H}_1(\overline{G}) \otimes_R^{\mathsf{L}} N) \geq 0$. Therefore, $\chi^R(M, N) > 0$ as claimed. $\square$

Remark 2.10. As in Example 2.8, there exist unliftable modules over ramified regular local rings that dimly lift to an unramified regular local ring. Hence Theorem 2.9 establishes Serre's positivity conjecture in new cases. The obstructions to dim liftability we have discovered so far are only applicable for modules over singular rings, and so we have not been able to find an example of a module over a ramified regular local ring that does not dimly lift in the setting of Theorem 2.9. This raises the following question.

Question 2.11. Fix a surjection $Q \twoheadrightarrow Q/f = R$ where Q and R are unramified and ramified regular local rings respectively. Does there exist a prime ideal $\mathfrak{p}$ of R such that $R/\mathfrak{p}$ does not dimly lift to Q?

2.4. **Dimly liftable modules over local singularities.** We now focus our attention on dimly liftable modules over local singularities. As we alluded to earlier, an R-module M that lifts to a regular ring Q must have finite projective dimension and this is a strong obstruction when R is not regular. However, dimly liftable modules do not have this restriction, so they exist in abundance.

Example 2.12. In fact, it is easy to produce examples of dimly liftable modules. Suppose $(Q, \mathfrak{n})$ is a regular ring and L is some Q-module of dimension r. Choose a sequence of elements $\boldsymbol{x} = x_1, \ldots, x_r \in \mathfrak{n}$ that is a parameter sequence on L. Set $M \coloneqq L/(\boldsymbol{x})L$. Then M is a module of finite length over $R = Q/(\boldsymbol{x})$ that is dimly liftable to Q.

Example 2.13. There exist dimly liftable but not liftable finite length R-modules. Suppose R is a non-Cohen-Macaulay ring such that $\dim R = d$. Given some system of parameters $\boldsymbol{x} = x_1, \ldots, x_d$ on R, we have that $\operatorname{projdim}_R(R/(\boldsymbol{x})) = \infty$ by the New Intersection theorem; see 3.2. Therefore, $M = R/(\boldsymbol{x})$ is an R-module of finite length that cannot lift to any regular ring. However, it dimly lifts to any regular ring Q such that $\pi \colon Q \twoheadrightarrow R$. Indeed, we may choose arbitrary $\widetilde{x}_i \in Q$ such that $\pi(\widetilde{x}_i) = x_i$. Let $\widetilde{\boldsymbol{x}} = \widetilde{x}_1, \ldots, \widetilde{x}_d$ and let $L = Q/(\widetilde{\boldsymbol{x}})$. As $L \otimes_Q R = M$, we have by Serre's dimensional inequality, $\dim L + \dim R \leq \dim Q$. We also have $\dim L \geq \dim Q - d$. That is, $\dim L + \dim R = \dim Q$. Hence L is a dim lift of M.

We now show that there exist a dimly liftable but unliftable module of finite length and finite projective dimension over a local singularity.

2.14. We begin with outlining an obstruction to liftability of modules over complete intersection rings via the notion of cohomological operators due to Gulliksen [10] and Eisenbud [7]. Suppose R is a local complete intersection ring and assume R is complete. Then $R \cong Q/(f_1, \ldots, f_c)$ where Q is a regular ring and $f_1, \ldots, f_c$ a regular sequence on Q. Consider a finitely generated R-module M and its minimal free resolution $F \xrightarrow{\simeq} M$ over R.

Following [7], we naively lift the complex F to Q, denoted $\widetilde{F}$. The lift of differentials $\widetilde{d}$ square to zero modulo $(f_1, \ldots, f_c)$. That is, $\widetilde{d}^2 = \sum_{i=1}^{c} t_i f_i$ where t_i denote the maps $t_i : \widetilde{F} \to \Sigma^2 \widetilde{F}$. The maps $\chi_i = t_i \otimes_Q R$ induce a well-defined action on $\operatorname{Ext}^*_R(M, N)$ for N some finitely generated R-module and are independent of the choice of the lift $\widetilde{F}$ to Q [7, Section 1]. The ring of cohomological operators $E = R[\chi_1, \ldots, \chi_c]$ acts on $\operatorname{Ext}^*_R(M, N)$.

Proposition 2.15. *With setup as above, if M is an R-module that lifts to Q, then $(\chi_1, \ldots, \chi_c)\operatorname{Ext}^*_R(M, N) = 0$ for any finitely generated R-module N.*

Proof. Since M lifts, its minimal R-free resolution $F \simeq M$ lifts to the minimal Q-free resolution $G \simeq L$ where L is the lift of M. That is, E acts on $\operatorname{Ext}^*_R(M, N)$ trivially, whence the assertion follows. $\square$

Example 2.16. Let $(R, \mathfrak{m})$ be the two dimensional local hypersurface

$$k[[x, y, z]]/(z^2 - xy)$$

defined over some field k. Let M be the module with the following presentation:

$$d_1 \colon R^4 \xrightarrow{\begin{pmatrix} x & 0 & z & y \\ y & x & y & 0 \end{pmatrix}} R^2.$$

The 0th Fitting ideal of M is $F_0(M) = I_2(d_1) = \big(z^2,\, y\,z,\, x\,z,\, y^2,\, x^2\big)$ and hence, $\operatorname{Supp}(M) = \{\mathfrak{m}\}$. This module also has finite projective dimension. Indeed,

$$0 \to R^2 \xrightarrow{\begin{pmatrix} -y-z & -y \\ -y & 0 \\ x+y+z & y \\ -z & x-z \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x & 0 & z & y \\ y & x & y & 0 \end{pmatrix}} R^2.$$

is an acyclic complex: it is easy to check that this is indeed an R-complex and, using the Buchsbaum-Eisenbud acyclicity criterion, one can show that this is acyclic. Therefore $\operatorname{projdim}_R(M) = 2$.

Proposition 2.17. *In the notation of Example 2.16, M is dimly liftable to $S = k[[x,y,z]]$ but it does not lift to any regular ring.*

Proof. We claim that L with the following presentation is a dim lift of M:

$$\widetilde{d_1}\colon S^4 \xrightarrow{\begin{pmatrix} x & 0 & z & y \\ y & x & y & 0 \end{pmatrix}} S^2.$$

Clearly $L \otimes_S R \cong M$. Computing the support via the 0th Fitting ideal, we get $\dim L = \dim \mathrm{V}(\big(y\,z,\, x\,z,\, y^2,\, x\,y,\, x^2\big)) = \dim \mathrm{V}(x,y) = 1$. So $\dim L = 1$. Hence $\dim L + \dim R = 1 + 2 = 3 = \dim S + \dim M$.

Now we show M does not lift to any regular ring. If M is liftable to some regular ring Q along a surjection $\pi : Q \twoheadrightarrow R$, then it also lifts to the completion of Q at its maximal ideal, denoted $\widehat{Q}$. Furthermore, since we can factor the map $\widehat{Q} \twoheadrightarrow R$ to $\widehat{Q} \twoheadrightarrow k[[x,y,z]] \twoheadrightarrow R$, we may assume M lifts to the regular local ring $S = k[[x,y,z]]$. Let $E = k[\chi]$ be the ring of Eisenbud operator that acts on $\operatorname{Ext}^*_R(M,k)$. The action of χ on $\operatorname{Ext}^*_R(M,k)$ is independent of the choice of lift of the minimal free resolution F of M to S [7, Proposition 1.2]. Naively lifting the R-free resolution in Example 2.16 above to S, we can compute that

$$\widetilde{d}^2 = \begin{pmatrix} z^2 - xy & 0 \\ 0 & 0 \end{pmatrix} = (z^2 - xy)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, χ acts on $\operatorname{Ext}^*_R(M,k)$ via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, therefore does not annihilate it. Therefore, by Proposition 2.15, M cannot lift to any regular ring. $\square$

3. Lengths of dimly liftable modules.

In this section, we will show that lengths of dimly liftable modules over a local ring R is bounded below by the Hilbert-Samuel multiplicity of R. This is closely related to two recent conjectures of Iyengar, Ma, and Walker. We begin by introducing these conjectures and surveying some known results.

3.1. Length Conjectures.

Conjecture 1. [14, Conjecture 1] (Length Conjecture) If M is a nonzero module of finite length and finite projective dimension over a local noetherian ring R, then $\ell_R(M) \geq e(R)$.

Remark 3.1. This question has been around for a long time although not formally stated as a conjecture until recently. In their 1993 paper [1], Avramov and Buchweitz remarked that it is "not known whether over a local or graded ring R there can be a non-zero module M of finite projective dimension whose multiplicity is smaller than the multiplicity of R." In the same paper, they show that for a graded module M with $\operatorname{projdim}_R(M) < \infty$, we have an equality $e(M) = ce(R)$ for c some positive constant depending on M. Hence the conjecture holds in the graded setting.

3.2. The New Intersection Theorem [31, Theorem 13.4.1] implies that a local ring R admits a nonzero module of finite length and finite projective dimension if and only if R is Cohen-Macaulay. Over an arbitrary local ring R we consider "short complexes" defined as follows: A *short complex* F over R is a complex

$$F : 0 \to F_d \to \cdots \to F_0 \to 0$$

such that $d = \dim R$, each F_i is a finite free R-module, and $\ell_R(\mathrm{H}(F)) < \infty$. When R is Cohen-Macaulay, any short complex is a finite free resolution of a finite length R-module by the acyclicity lemma of Peskine-Szpiro [27, Lemma 1.8]. In this sense, these objects naturally generalize the notion of a finite, free resolution of a finite length module to all local rings. Roberts has shown the existence of a short complex F over a local ring with $\chi(F) < 0$ [30]. To study the analogue of Conjecture 1 for all local rings, the *Dutta multiplicity* is a more suitable invariant.

3.3. We will restrict ourselves to the case of prime characteristic $p > 0$ but, due to Kurano [18], one may define χ_∞ in a characteristic free way that generalizes the definition given below in prime characteristic $p > 0$; see [18, 19, 20] for more details. We plan to address the mixed characteristic case in future work. Let R be an F-finite local ring of prime characteristic $p > 0$ and $\dim R = d$. If F is a short complex over R, we define

$$\chi_\infty(F) = \lim_{e\to\infty} \frac{\chi(F^e(F))}{p^{de}}.$$

This limit exists [31, Theorem 7.3.3]. Furthermore, a fundamental result of Roberts and Kurano-Roberts [20] states that if R is an equicharacteristic complete local domain and F is a short complex over R with $\mathrm{H}_0(F) \neq 0$, then $\chi_\infty(F) > 0$. This positivity property plays a key role in the proof of the New Intersection Theorem in mixed characteristic [31, Theorem 13.4.1] and Walker's proof of Total Rank Conjecture in characteristic $p > 0$ [38, Theorem 2]. However, it is an open conjecture of Kurano [18, Conjecture 3.1] that this also holds in mixed characteristic.

On the other hand, Iyengar, Ma, and Walker have recently conjectured an even tighter lower bound. An affirmative answer to Conjecture 2 implies Lech's conjecture holds in full generality [14, Proposition 8.3].

Conjecture 2. [14, Conjecture 2] If F is a short complex over a local ring R with nonzero homology, then $\chi_\infty(F) \geq e(R)$.

Example 3.4. Suppose R is a local ring of characteristic $p > 0$ and $\boldsymbol{x} = x_1, \ldots, x_d$ system of parameters on R. Note that $K = \operatorname{Kos}(\boldsymbol{x})$ is a short complex over R. Furthermore,

$$\chi_\infty(K) = \lim_{e\to\infty} \frac{\chi(\operatorname{Kos}(x_1^{p^e}, \ldots, x_d^{p^e})}{p^{de}} = \chi(K)$$

The first equality follows by definition and the second by [31, Proposition 5.2.3]. By a theorem of Serre [33, Ch. IV, Thm 1], $\chi(K) \geq e(R)$.

3.5. We now briefly discuss how Conjectures 1 and 2 are related to one another. This leads us to the class of *numerically Roberts* rings. We have noted that Conjecture 1 only concerns Cohen-Macaulay rings; see 3.2. Even if R is Cohen-Macaulay, in general, $\ell_R(M) \neq \chi_\infty(M)$ and there is no inequality that relates these two invariants. So, as far as we know, one conjecture does not imply the other in the Cohen-Macaualy setting where both are sensible. However, there is a class of rings, called *numerically Roberts rings* [19] over which $\chi(F) = \chi_\infty(F)$ for any short complex F. In his thesis, Ma proved the following [23, Proposition V.45]: Suppose R is a numerically Roberts ring of prime characteristic $p > 0$ and of dimension at most three. If $\operatorname{projdim}_R(R/I) < \infty$, then $\ell_R(R/I) \geq e(R)$. Since, for example, complete intersections, Cohen-Macaulay rings of dimension at most two, and Gorenstein rings of dimension at most three are known to be numerically Roberts, Ma's result establishes the cyclic case of Conjecture 1 and 2 for these class of rings of dimension at most three in prime characteristic $p > 0$. Recall that we define for a local ring of prime characteristic $p > 0$ of dimension d with maximal ideal $\mathfrak{m}$, the Hilbert-Kunz multiplicity is defined to be the following limit:

$$e_{HK}(R) \coloneqq \lim_{e\to\infty} \frac{\ell_R(R/\mathfrak{m}^{[p^{ed}]})}{p^{ed}}.$$

Proposition 3.6. *Suppose R is a local ring of prime characteristic $p > 0$ and $\dim R = d$. If F is a short complex with nonzero homology over R, then*

$$\chi_\infty(F) \geq \beta_0(F) e_{HK}(R) \geq \frac{\beta_0(F)}{d!} e(R).$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. We have

$$\chi_\infty(F) = \lim_{e\to\infty} \frac{\ell_R(F^e \operatorname{H}_0(F))}{p^{ed}} \geq \beta_0(F) \lim_{e\to\infty} \frac{\ell_R(F^e(R/\mathfrak{m}))}{p^{ed}}$$

where the first equality is due to Roberts [31, Theorem 7.3.5] and the second inequality holds by the fact $F^e \operatorname{H}_0(F) \twoheadrightarrow F^e(R/\mathfrak{m})$ for all $e \geq 0$. We recall that the limit on the right hand side above is the Hilbert Kunz multiplicity $e_{HK}(R)$ of the maximal ideal $\mathfrak{m}$. Therefore, $\chi_\infty(F) \geq \beta_0(F) e_{HK}(R)$. The final inequality follows because $e_{HK}(R) \geq e(R)/d!$ by definition. □

Corollary 3.7. *(Ma) Conjectures 1 and 2 hold for two dimensional Cohen-Macaulay rings of prime characteristic $p > 0$.*

Proof. Since R is Cohen-Macaulay of dimension two, it is Numerically Roberts. So, for a nonzero R-module of finite length and finite projective dimension, $\chi_\infty(M) = \ell_R(M)$. Now note that the case $\beta_0(M) = 1$ is the result of Ma, so we may assume $\beta_0(M) \geq 2$. Suppose $F \to M$ is the minimal R-free resolution. By the proposition above,

$$\ell_R(M) = \chi_\infty(F) \geq \beta_0(M) e(R)/2! \geq e(R).$$ □

3.8. More recently, Iyengar, Ma, and Walker prove these length conjectures for the following class of rings. In fact, their results say more.

Theorem 3.9. [14, Theorem 3.8] *Suppose R is a strict complete intersection of $\dim R = d$. If M is an R-module of finite length and finite projective dimension, then for each i,*

$$\binom{d}{i}\ell_R(M) \geq \beta_i(M)e(R).$$

Theorem 3.10. [14, Corollary 7.2] *Suppose A is a standard graded k-algebra of $\dim A = d$ where k is a perfect field of characteristic $p > 0$ and $\mathfrak{m} = \bigoplus_{i>0} A_i$ is the irrelevant maximal ideal. If $R = A_{\mathfrak{m}}$ and F is a short complex over R with nonzero homology, then for each i,*

$$\binom{d}{i}\chi_\infty(F) \geq \beta_i(F)e(R).$$

These results follow from the existence of the Ulrich modules or more generally lim Ulrich sequence of modules with special properties; see [14] for details. Recently, Iyengar, Ma, Walker, and Zhuang have shown that even local complete intersections of codimension two do not admit any Ulrich module [15].

3.2. **Lower bounds on lengths.** We will now show that modules, not necessarily of finite projective dimension, that are dimly liftable to an unramified regular local ring must have length at least the Hilbert-Samuel multiplicity of the ring. Below, whenever we say an R-module M dimly lifts to a regular ring Q, we assume that there is some surjection $Q \twoheadrightarrow R$ as in Section 2.

Theorem 3.11. *Suppose a nonzero R-module M of finite length dimly lifts to a Q-module L where Q is an unramified regular local ring. Then $\ell_R(M) \geq e(L)e(R)$.*

Proof. We have $\ell(L \otimes_Q R) = \ell(M) < \infty$ and $\dim L + \dim R = \dim Q$ by the dim lift assumption. Hence, by Serre's positivity theorem, $\chi(L \otimes^{\mathsf{L}}_Q R) > 0$. Furthermore,

$$\chi(L \otimes^{\mathsf{L}}_Q R) = \ell_R(M) - \chi_1(L \otimes^{\mathsf{L}}_Q R).$$

We have $\chi_1(L \otimes^{\mathsf{L}}_Q R) \geq 0$ due to Lichtenbaum [22, Theorem 2]. Therefore,

$$\ell_R(M) \geq \chi(L \otimes^{\mathsf{L}}_Q R) \geq e(L)e(R)$$

where the final inequality follows by Theorem 1.3. □

Corollary 3.12. *Suppose M is a nonzero Cohen-Macaulay module of finite projective dimension over a local ring R. If M dimly lifts to a Q-module L where Q is an unramified regular local ring, then $e(M) \geq e(L)e(R)$.*

Proof. We claim that R must be Cohen-Macaulay if it admits a nonzero Cohen-Macaulay module of finite projective dimension. Recall that by Auslander's zero-divisor theorem [31, Theorem 6.2.3]: a regular element on a module of finite projective dimension is a regular element on R. Choose $\boldsymbol{x} = x_1, \ldots, x_r$ a maximal regular sequence on M. Then $M/(\boldsymbol{x})M$ is a module of finite length and finite projective dimension over $R/(\boldsymbol{x})$. Therefore, by the New Intersection theorem, $R/(\boldsymbol{x})$ is Cohen-Macaulay. Since $\boldsymbol{x}$ is a regular sequence on R, this implies R is also Cohen-Macaulay. Furthermore, we may assume R has an infinite residue field.

We have a surjection $Q \twoheadrightarrow R$. Let $\mathfrak{m}$ denote the maximal ideal of Q and, by slight abuse of notation, also of R. We have the following commutative square.

$$\begin{tikzcd} Q \arrow[r] \arrow[d, two heads] & Q[t]_{\mathfrak{m}Q[t]} \arrow[d, two heads] \\ R \arrow[r] & R[t]_{\mathfrak{m}R[t]} \end{tikzcd}$$

Note that $Q \to Q' = Q[t]_{\mathfrak{m}Q[t]}$ and $R \to R' = R[t]_{\mathfrak{m}R[t]}$ are faithfully flat maps with fields as their closed fibers. Hence, Q' is an unramified regular local ring, $L \otimes_Q Q'$ is a dim lift of the module $M \otimes_R R'$, and all the invariants of interest remain unchanged. Thus, we may now choose a sufficiently general sequence of elements $\boldsymbol{x} = x_1, \ldots, x_r \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $e(M) = \ell_R(M/(\boldsymbol{x})M)$ and $e(R) = e(R/(\boldsymbol{x})R)$ [2, Corollary 4.6.10]. We have $L \otimes_Q R/(\boldsymbol{x}) \cong M/(\boldsymbol{x})M$ and $\dim L + \dim R/(\boldsymbol{x}) = \dim Q$. It now remains to apply Theorem 3.11. □

Corollary 3.13. *Suppose R is a Cohen-Macaulay ring and I is a perfect ideal in the linkage class of a complete intersection. Then $e(R/I) \geq e(R)$.*

Proof. Indeed, by Example 2.1, R/I is liftable so we may apply Corollary 3.12. □

Example 3.14. Theorem 3.11 provides an obstruction to dim liftability of finite length R-modules (and for perfect R-modules by Corollary 3.12) to an unramified regular ring. For example, if R is a local ring such that $e(R) > 1$, then the residue field k of R does not dimly lift to an unramified regular ring.

3.15. The notion of lifting modules of finite length and finite projective dimension is restricted to Cohen Macaulay rings; see 3.2. For all local rings, we will now consider the lifting of short complexes. Let us fix a surjection $\pi : Q \twoheadrightarrow R$, where Q is a regular local ring. A finite free R-complex F *lifts* to Q if there exists a finite free Q-complex G such that $G \otimes_Q R \cong F$. We say F is *liftable* to Q and that G is a lift of F. For example, a Koszul complex on a sequence of elements of R is liftable. We utilize the following result of Kurano [19, Theorem 1.2]: Suppose F is a short complex over a local ring R that lifts to a finite free Q-complex G over a regular local ring Q. Then $\mathrm{H}_i(G) = 0$ for $i > 0$ and $\dim \mathrm{H}_0(G) + \dim R = \dim Q$.

Remark 3.16. Suppose R is Cohen Macaulay and M is an R-module of finite length and finite projective dimension. If $F \xrightarrow{\cong} M$ is the minimal free resolution of M, then the complex F lifts to Q if and only if the module M lifts to Q. Indeed, the forward direction follows by Kurano's result and the converse is clear. This also highlights the fact that the main obstruction to lifting an R-module of finite length and finite projective dimension is lifting the complex structure of its free resolution. The acylicity comes for free.

Proposition 3.17. *Suppose F is a short complex over R that lifts to Q. Then the R-module $\mathrm{H}_0(F)$ dimly lifts to Q.*

Proof. Let G be the lift of F. By Kurano's result, $\mathrm{H}_i(G) = 0$ for $i > 0$ and $\dim \mathrm{H}_0(G) + \dim R = \dim Q$. Now since $\mathrm{H}_0(G) \otimes_Q R = \mathrm{H}_0(F)$ by the right exactness of tensor product, $\mathrm{H}_0(G)$ is a dim lift of $\mathrm{H}_0(F)$. □

Definition 3.18. A short complex F over a local ring R *dimly lifts* to Q if the R-module $\mathrm{H}_0(F)$ dimly lifts to Q.

Theorem 3.19. *Suppose R is a local ring of prime characteristic $p > 0$ and F is a short complex over R with nonzero homology. If F is dimly liftable to an unramified regular local ring Q, then*

$$\chi_\infty(F) \geq e(R).$$

Proof. Let the Q-module L denote the dim lift of $H_0(F)$ and let $G \xrightarrow{\simeq} L$ denote its minimal Q-free resolution. For $e \geq 0$, $\chi(F^e G \otimes_Q R) = \ell(\mathrm{H}_0(F^e F)) - \chi_1(F^e G \otimes_Q R)$. We have $\operatorname{codim} L = \dim R = d$ and since Q is regular $\chi_\infty(L, R) = \chi(L, R)$ [5, 1.3 Remark]. Taking limits,

$$\chi(L,R) = \chi_\infty(L,R) = \lim_{e\to\infty} \frac{\ell(\mathrm{H}_0(F^e F))}{p^{ed}} - \lim_{e\to\infty} \frac{\chi_1(F^e G \otimes_Q R)}{p^{ed}}.$$

Due to Roberts (proof of [31, Theorem 7.3.5]), as F is a short complex, the first term on the left hand side is equal to $\chi_\infty(F)$ and due to Lichtenbaum [22, Theorem 2] we know $\chi_1(F^e G \otimes_Q R) \geq 0$ and therefore the second limit is at least zero. This shows that $\chi_\infty(F) \geq \chi(L, R) > 0$. The proof is then complete by Theorem 1.3. □

Remark 3.20. Suppose R is a local ring of characteristic $p > 0$ and dimension d. If M is a finite length R-module, the following limit exists.

$$\ell_\infty(M) = \lim_{e\to\infty} \frac{\ell_R(F^e(M))}{p^{ed}}.$$

The proof of Theorem 3.19 shows that if M is dimly liftable to an unramified regular local ring, then $\ell_\infty(M) \geq e(R)$.

3.3. **Extremal case.** What modules of finite length and finite projective dimension (resp. short complexes) have length (resp. χ_∞) that equals $e(R)$? We will provide a new proof to the following result of Ma [23, Theorem V.28].

Proposition 3.21. *Suppose R is a strict complete intersection of dimension d. If M is an R-module of finite projective dimension with $\ell_R(M) = e(R)$, then for some system of parameters $\boldsymbol{x} = x_1, \ldots, x_d$ on R, we have $M \cong R/(\boldsymbol{x})$.*

Proof. Given Theorem 3.9, if we sum up both sides in the inequalities, we get $2^d \ell_R(M) \geq \sum_{i=1}^d \beta_i(M) e(R)$. By cancelling $e(R)$, we get $2^d \geq \sum_{i=1}^d \beta_i(M)$. If the characteristic of the residue field is not two, Walker's total rank theorem [38, Theorem 2] implies $2^d = \sum_{i=1}^d \beta_i(M)$. Again by Walker's theorem, this condition forces $M \simeq \operatorname{Kos}(x_1, \ldots, x_d)$ for some system of parameters $x_1, \ldots, x_d$. If the characteristic is two, the same conclusion follows due to [37, Theorem 1.12]. □

An analogous argument can be carried out for short complexes over standard graded algebras as in Theorem 3.10 but we do not know whether for a short complex F, the equality $\sum_{i=1}^d \beta_i(F) = 2^d$ forces $F \cong \operatorname{Kos}(x_1, \ldots, x_d)$ for some system of parameters. This is formulated as a conjecture by VandeBogert and Walker [37, Conjecutre 1.11].

Proposition 3.22. *Suppose R is a local ring and M is a cyclic R-module of finite length that is liftable to an unramified regular local ring Q. Then, if $\ell_R(M) = e(R)$, then $M \cong R/(\boldsymbol{x})$ for some system of parameters $\boldsymbol{x} = x_1, \ldots, x_d$ on R.*

Proof. Due to Theorem 3.11, we have $e(R) = \ell_R(M) \geq e(L)e(R)$ where L is the lift of M. Therefore, $e(L) = 1$. Since M is cyclic, we may assume L is also cyclic. Assume $L \cong Q/J$ for some ideal J. Since Q/J is a Cohen Macaulay algebra (see

2.6) such that $e(Q/J) = 1$, it follows that Q/J must be regular. So J is genereated by a part of a regular system of parameters on Q. Therefore, $M \cong R/(\boldsymbol{x})$ where $\boldsymbol{x} = x_1, \dots, x_d$ is the image of J on R and $d = \dim R$. $\square$

We do not know of any R-module of finite projective dimension with $\ell_R(M) = e(R)$ which is not isomorphic to $R/(\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}$ on R.

4. Unliftable modules

In general, there are many obstructions for a module of finite projective dimension over a singular ring to lift to a regular ring. We report several such examples. Notably, we will also show there exists a module of finite projective dimension over a singular ring that cannot dimly lift to a regular ring.

Example 4.1. Suppose R is a local ring of dimension d and M is an R-module of finite length and finite projective dimension. If for some R-module N such that $\dim N < \dim R$, $\chi(M, N) \neq 0$, then M cannot lift to any regular ring Q. Indeed, if L is a lift of M, then $\chi^Q(L, N) = \chi^R(M, N) \neq 0$. But this contradicts the vanishing theorem over regular local rings (see 1.2) as $\dim L + \dim N < \dim Q$.

Dutta, Hochster, and McLaughlin produced the following example [6]. Let $R = k[[x, y, z, w]]/(xy - zw)$ and $\mathfrak{p} = (x, w)R$. There exists a module M such that $\ell_R(M) = 15$, $\operatorname{projdim}_R(M) = 3$ such that $\chi^R(M, N) = -1$. Since $\dim N = 2$, M cannot lift to a regular ring by the discussion above. Building on this example, Smoke shows there exists a cyclic R-module R/I of finite length and finite projective dimension such that $\chi^R(R/I, R/\mathfrak{p}) = \chi^R(M, R/\mathfrak{p}) = -1$ [35, Section 6]. Therefore, this cyclic R-module of projective dimension three does not lift either.

Example 4.2. In this example, we show there exists a module of finite length and finite projective dimension that is not dimly liftable to any regular local ring. Let R be as in the example above. Suppose an R-module M dimly lifts to some regular ring Q. Since R is complete and of equal characteristic, we may assume so is Q. Since R is a complete, complete intersection, we have $R = Q/(g_1, \dots, g_c)$. As R is a hypersurface, we can assume $g_1, \dots, g_{c-1} \in \mathfrak{m} \setminus \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of Q. Let L be the Q-module that is a dim lift of M. Choosing a sufficiently general sequence of elements in $g'_1, \dots, g'_{c-1} \in (g_1, \dots, g_{c-1}) \setminus \mathfrak{m}^2$, we can consider the regular ring $Q' = Q/(g'_1, \dots, g'_{c-1})$. Then the Q'-module $L' = L \otimes_Q Q'$ has dimension $\dim L - (c - 1)$ and $L' \otimes_{Q'} R = M$. Therefore, if M dimly lifts to some regular ring Q, we may assume $R = Q/(f)$.

Now consider M and $\mathfrak{p}$ as in Example 4.1 above. Assume L is a dim lift of M. Consider the exact sequence

$$0 \to \operatorname{ann}_L(f) \to L \xrightarrow{f} L \to M \to 0.$$

Note $M_1 \coloneqq \operatorname{ann}_L(f)$ is a nonzero R-module as M is not liftable. Using the standard change of ring long exact sequence of Tors, we have $\operatorname{projdim}_R(M_1) < \infty$. On the other hand, since $\dim L = 1$, $\chi(f; L) > 0$ whence $\ell_R(M) > \ell_R(M_1)$. Furthermore,

$$0 = \chi(L, R/\mathfrak{p}) = \chi(M, R/\mathfrak{p}) - \chi(M_1, R/\mathfrak{p})$$

where the first equality follows by the vanishing theorem over regular rings and the second by Lemma 2.2. Therefore, $\chi(M_1, R/\mathfrak{p}) = -1$ and $\ell_R(M_1) < \ell_R(M)$. In particular, M_1 is not liftable by the argument in Example 4.1.

If M_1 is not dimly liftable, we have found our desired example. Otherwise, we run the same argument to deduce an existence of an unliftable nonzero module M_2 of finite projective dimension such that $\ell_R(M_2) < \ell_R(M_1)$ and $\chi_R(M_2, R/\mathfrak{p}) = -1$. Clearly, this process must terminate at some point giving us the desired example.

Example 4.3. Due to Kurano [19, Theorem 1.2], if R is a local ring and F is a short complex over R that is liftable to a regular ring, then $\chi_\infty(F) = \chi(F)$. Miller-Singh construct a five dimensional Gorenstein ring and module of finite length and finite projective dimension such that $\chi(F) \neq \chi_\infty(F)$ where F is the module's minimal free resolution [26]. Hence, this module is also unliftable.

Example 4.4. The following example is due to Hietmann and communicated to us by Walker. We will construct a class of Cohen Macaulay rings of dimension two and a module of finite length and finite projective dimension with betti numbers $(2, 4, 2)$ such that they do not lift to a regular ring. Consider the "generic" complex with betti numbers $(2, 4, 2)$.

$$0 \to S^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \end{pmatrix}} S^2 \to 0 \tag{1}$$

where $S = \mathbb{Z}[X; Y]$ with x_i, y_j considered indeterminates. We want to introduce relations on S to make (1) a complex. First, let $I = I_1(YX)$, i.e. the ideal generated by entries of the product YX. Modding out S by I makes (1) a complex. Now let $P \subseteq \{1, 2, 3, 4\}$ be an ordered subset of two elements and let P^c denote its complement. Let Δ_P^Y denote the two by two minor of Y with columns indexed by P and let $\Delta_{P^c}^X$ denote the two by two minor of X with with rows indexed by P^c. Let $(P \mid P^c)$ denote the permutation defined by the sets P and P^c, and let $\sigma_P = \sigma(P \mid P^c)$ denote the sign of this permutation. We let J denote the ideal generated by elements $\Delta_P^Y - \sigma_P \Delta_{P^c}^X$ as P ranges over all ordered subsets of $\{1, 2, 3, 4\}$ consisting of two elements. From the theory of generic resolutions of length two, over $R = S/(I + J)$, we have that (1) is an acyclic complex [36, Section 2]. Furthermore, R is Cohen Macaulay. Let M denote the zeroth homology module of the acyclic complex above. By the symmetry of the construction, the R-linear dual of (1) is also acyclic. Hence, M is a Cohen Macaulay module since it is perfect module over a Cohen Macaulay ring.

We localize R appropriately, say at (p, X, Y) for a mixed characteristic example. Then we may choose a maximal regular sequence $r_1, \ldots, r_{\dim M}$ on M. Taking powers, we may assume $r_1, \ldots, r_{\dim M} \subseteq \mathfrak{m}_R^2$. Now let $\overline{R} = R/(r_1, \ldots, r_{\dim M})$ and $\overline{M} = M/(r_1, \ldots, r_{\dim M})M$. We have $\overline{R}$ is Cohen Macaulay and $\dim \overline{R} = \dim R - \dim M = \operatorname{projdim}_R(M) = 2$ and $\overline{M}$ is a finite length $\overline{R}$-module of finite projective dimension. We claim that this module cannot lift to a regular ring.

Proof. For contradiction, assume it does lift to some regular ring Q. Then we may assume $Q \twoheadrightarrow \overline{R}$ minimally, i.e. $\mathfrak{m}_Q/\mathfrak{m}_Q^2 = \mathfrak{m}_{\bar{R}}/\mathfrak{m}_{\bar{R}}^2$. We have $\bar{x_1}, \ldots, \bar{x_8}, \bar{y_1}, \ldots, \bar{y_8}$ minimally generates the maximal ideal $\mathfrak{m}_{\bar{R}}$. Their lifts to Q minimally generate the maximal ideal of Q, but as Q is a regular local ring, they must form a regular system of parameters on Q. If we can lift the resolution of $\bar{M}$, we have that

$$0 \to Q^2 \xrightarrow{\widetilde{X}} Q^4 \xrightarrow{\widetilde{Y}} Q^2 \to 0$$

is in particular a Q-complex. But $\widetilde{Y}\widetilde{X} = 0$ implies there are non-Koszul relations amongst lifts of $\bar{x_1}, \ldots, \bar{x_8}, \bar{y_1}, \ldots, \bar{y_8}$ to Q. This is a contradiction as these lifts form a regular sequence on Q. □

One may generalize this example to produce many non-cyclic modules of finite length and finite projective dimension over a two dimensional Cohen Macaulay ring that do not lift to a regular ring. The length conjecture for such modules remains open in mixed characteristic.

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