On the importance of slow ions in the kinetic Bohm criterion

Alessandro Geraldini and Stephan Brunner
SPC-EPFL, Lausanne, Switzerland
(Dated: April 30, 2024)

Between a plasma and a solid target lies a positively charged 'sheath' of several Debye lengths λ_D in width, typically much smaller than the characteristic length scale L of the main plasma. This scale separation implies that the asymptotic limit $\epsilon = \lambda_D/L \to 0$ is useful to solve for the plasma-sheath system. In this limit, the Bohm criterion must be satisfied at the sheath entrance. A new derivation of the kinetic criterion, admitting a general ion velocity distribution, is presented. It is proven that for $\epsilon \to 0$ the distribution of the velocity component normal to the target, v_x, and its first derivative must vanish for $|v_x| \to 0$ at the sheath entrance. The kinetic Bohm criterion emerges by combining these two conditions with a third integral one upon integrating by parts twice. A subsequent interchange of the limits $\epsilon \to 0$ and $|v_x| \to 0$ is shown to be invalid, causing a divergence which underlies a common misconception that the criterion gives undue importance to slow ions.

For a quasineutral plasma to exist next to a wall or a solid target, the most mobile charged species, typically electrons due to their smaller mass, must be reflected by the wall to achieve no net charge loss (and thus preserve quasineutrality). The wall is thus negatively charged and the region where electron reflection occurs is called the 'sheath' [1–3]. The sheath is positively charged and shields the quasineutral plasma from the negative charge on the wall. Its characteristic length scale is the Debye length, defined as $\lambda_D = \sqrt{\epsilon_0 T_e/n_e e^2}$, where n_e is the electron density, e is the proton charge, T_e is the electron temperature (in units of energy) and ϵ_0 is the permittivity of free space. The characteristic length scale L of unmagnetised plasmas, or magnetised plasmas where the magnetic field is perpendicular to the target, is defined to be the smallest one between: the collisional mean free path of ions, the ionization mean free path of neutrals, and the target curvature [4]. It is usually much larger than the Debye length, such that the sheath is thin compared to the plasma

$$\epsilon = \frac{\lambda_D}{L} \ll 1. \tag{1}$$

In magnetised plasmas with an oblique magnetic field impinging on the target, L can also be the ion sound Larmor radius $\rho_S = \sqrt{m_i(ZT_e + T_i)}/(ZeB)$, where T_i is the ion temperature, B is the magnetic field strength, m_i is the ion mass and Z is the ionic charge state [5]. Once they reach the target, ions and electrons are absorbed, eventually recombine and are subsequently re-emitted as neutrals that penetrate back through the sheath all the way into the plasma before re-ionizing.

The physical principle treated in this paper states that the ions must reach the sheath at a minimum velocity, as first formalised by Bohm [6] in 1949 assuming mono-energetic (cold) ions:

$$-u_x \geqslant v_B \equiv \sqrt{ZT_e/m_i}. \tag{2}$$

Here $u_x = \vec{u}\cdot\vec{e}_x$, where $\vec{u}$ is the fluid velocity, $\hat{\vec{e}}_x$ is a unit vector normal to the target plane and pointing away from it, $v_B = \sqrt{ZT_e/m_i}$ is the Bohm speed (or cold-ion sound speed) and m_i is the ion mass. The negative sign in the velocity component is related to the choice of placing the target at $x = 0$ and the plasma at $x > 0$. The inequality (2) came to be known as the 'Bohm criterion'. It holds at the 'sheath entrance' (or 'sheath edge'), a position at this stage loosely defined as the distance from the wall below which collisions, ionisation, target curvature and curvature of ion Larmor orbits are negligible, and above which the space charge is negligible. It ensures an increasingly positive space charge in the sheath as the wall is approached, consistent with an increasing electric field directed towards the wall to reflect electrons.

Since mono-energetic ions are a significant idealisation, a *kinetic* formulation of the Bohm criterion, valid for arbitrary ion velocity distributions, is desirable. This was first given by Harrison & Thompson [7] in 1959,

$$v_B^2 \int_{-\infty}^{0} \frac{f(v_x)}{v_x^2} dv_x \leqslant \frac{n_e}{Z}. \tag{3}$$

Here f is the ion distribution function of the velocity component normal to the target, $v_x = \vec{v}\cdot\hat{\vec{e}}_x$ with velocity $\vec{v}$, at the sheath entrance. By plasma quasineutrality at the sheath entrance, the electron density n_e satisfies

$$n_e = Z \int_{-\infty}^{0} f(v_x) dv_x. \tag{4}$$

The distribution f is, in practice, the full three-dimensional distribution function which has been integrated over the velocity components tangential to the target. Note that the fluid criterion (2) is *always* fulfilled by the flow velocity u_x defined by

$$u_x \int_{-\infty}^{0} f(v_x) dv_x = \int_{-\infty}^{0} f(v_x) v_x dv_x, \tag{5}$$

provided that f satisfies the kinetic criterion (3), as can be shown by two applications of Schwarz's inequality [4, 7]. In deriving (3), it has been assumed that no ions travel away from the target at the sheath entrance, such that $f(v_x) = 0$ for $v_x > 0$.

Although Harrison & Thompson's kinetic criterion (3) stood the test of time, Hall [8] promptly criticised their derivation for implicitly assuming the conditions $f(0^-) \equiv \lim_{v_x \to 0^-} f(v_x) = 0$ and $f'(0^-) = 0$ without justification. The derivative of any function $G(\zeta)$ of a single variable ζ is denoted $G'(\zeta) \equiv dG(\zeta)/d\zeta$. Performing a calculation that will be shown here to be flawed, Hall refutes $f'(0^-) = 0$ and concludes that (3) ascribes "undue importance" to the slow ions with $v_x \approx 0$.

In parallel, Caruso and Cavaliere [9] recognised the asymptotic framework of the plasma-sheath transition as a singular perturbation theory [10, 11], following ideas in reference [12]. In the limit $\epsilon \to 0$ the equations on the plasma and sheath scales are distinct, and thus the two regions can be treated separately. The sheath entrance is defined in this limit by $x/L \to 0$ and $x/\lambda_{\mathrm{D}} \to \infty$ [13]. In practice, any point $x_\epsilon = \epsilon^q L$ on an intermediate length scale with $q \in (0,1)$ satisfies $\lim_{\epsilon\to 0} x_\epsilon/\lambda_{\mathrm{D}} \to \infty$ and $\lim_{\epsilon \to 0} x_\epsilon / L \to 0$ and is thus a valid reference choice for the sheath entrance [14]. The actual distribution function at this point is denoted $f_\epsilon(v_x)$, and the asymptotic sheath entrance distribution function is defined by

$$f(v_x) \equiv \lim_{\epsilon \to 0} f_\epsilon(v_x). \tag{6}$$

The Bohm criterion at the sheath entrance arises as a necessary condition for an electron-repelling *sheath* solution to exist in the limit $\epsilon \to 0$.

In section 3.2 of his seminal 1991 review [4], Riemann derived the kinetic Bohm criterion for $\epsilon \to 0$, including ion reflection and re-emission from the wall. For comparison with this work, his derivation is considered in the limit of a perfectly absorbing wall. There, it is proven in an intermediate step that $f(0^-) = 0$, while the condition $f'(0^-) = 0$ emerges from the assumption that f can be represented by a Taylor expansion in kinetic energy $v_x^2/2$ near $v_x = 0$. This assumption was criticised [15–18] for restricting the class of possible ion distribution functions at the sheath entrance. These criticisms emphasise the divergence of the left hand side of (3) when applied to the actual distribution function,

$$\lim_{\epsilon \to 0} v_{\mathrm{B}}^2 \int_{-\infty}^{0} \frac{f_\epsilon(v_x)}{v_x^2} dv_x = \lim_{\epsilon \to 0} \infty, \tag{7}$$

caused by $f_\epsilon(0) \neq 0$ and $f'_\epsilon(0) \neq 0$ for finite ϵ. Echoing Hall [8] around 50 years later, Baalrud & Hegna [16, 17] claim that the kinetic criterion places "undue importance" to slow ions. While this assertion appears reasonable in light of (7), it is motivated by the misguided expectation that (3) should still apply *as it is* to f_ϵ for small but finite ϵ [19].

In the remainder of this article, the derivation of the kinetic Bohm criterion (3) will be generalised to remove the assumption, and prove the physical legitimacy of, $f'(0^-) = 0$. Condition (3) is shown to emerge from

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} f_\epsilon(v_x) \equiv f(0^-) = 0, \tag{8}$$

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} f'_\epsilon(v_x) \equiv f'(0^-) = 0, \tag{9}$$

$$\lim_{v_x \to 0^-} \lim_{\epsilon \to 0} v_{\mathrm{B}}^2 \int_{-\infty}^{v_x} f''_\epsilon(v'_x) \ln(-1/v'_x) dv'_x$$
$$\equiv v_{\mathrm{B}}^2 \int_{-\infty}^{0} f''(v_x) \ln(-1/v_x) dv_x \leqslant \frac{n_{\mathrm{e}}}{Z} \equiv \lim_{\epsilon \to 0} \frac{n_{\mathrm{e},\epsilon}}{Z}. \tag{10}$$

by integrating (10) by parts and imposing (8)-(9). In (10), the electron density at x_ϵ was defined as $n_{\mathrm{e},\epsilon}$. To conclude, (8)-(10) are combined to obtain the reformulation of (3) in terms of f_ϵ. It is explained that the divergence (7) follows from an incautious interchange of the limits $\epsilon \to 0$ and $v_x \to 0^-$ after integration by parts.

The following dimensionless variables are introduced: the sheath-scale position $\xi = x/\lambda_{\mathrm{D}}$, the (negative of the) electrostatic potential relative to the sheath entrance $\chi = \lim_{\epsilon \to 0} e(\phi(x_\epsilon) - \phi(x))/T_{\mathrm{e}}$, the ion velocity component directed towards the wall $w = -v_x/v_{\mathrm{B}}$, and the ion distribution function $g(w) = Z f(v_x) v_{\mathrm{B}}/n_{\mathrm{e}}$ satisfying $\int_0^\infty g(w)\, dw = 1$ from (4).

For $\epsilon \to 0$, in the sheath $x \leqslant x_\epsilon$, corresponding to $\xi \in [0, \infty]$, the ion distribution function $g_\xi(\xi, w)$ satisfies the kinetic equation

$$w \frac{\partial g_\xi}{\partial \xi} - \chi' \frac{\partial g_\xi}{\partial w} = 0. \tag{11}$$

Equation (11) results from the normalised full kinetic equation which has been integrated in the other two velocity components, after all the terms small in ϵ have vanished in the limit $\epsilon \to 0$. The ϵ-dependent terms are related to the target curvature, collisions and ionisation in the plasma, the magnetic force on the ions, and explicit time dependence on the plasma scale $v_{\mathrm{B}}/L = \epsilon v_{\mathrm{B}}/\lambda_{\mathrm{D}}$. By imposing the boundary condition at the sheath entrance $g_\xi(\infty, w) = g(w)$, assuming a perfectly absorbing wall, $g_\xi(0, w < 0) = 0$, and assuming no ion reflection within the sheath, $g_\xi(\xi, w < 0) = 0$, one obtains $g_\xi(\xi, w) = g(\sqrt{w^2 - 2\chi})$ from equation (11). In order for no ions to be reflected, $\chi(\xi) \geqslant 0$ is required. For the electrons, a Boltzmann distribution is assumed, which is justified when the sheath reflects most electrons back into the bulk such that $e^{-\chi(0)} \ll 1$. Hence, Poisson's equation in the sheath is

$$\chi''(\xi) = \int_{\sqrt{2\chi}}^{\infty} g\left(\sqrt{w^2 - 2\chi}\right) dw - e^{-\chi}. \tag{12}$$

In order to solve (12) for $\chi(\xi)$ locally near the sheath entrance (where $\chi = 0$), the electron density may be expanded as a Taylor series in $\chi \ll 1$, $e^{-\chi} = 1 - \chi + \frac{1}{2}\chi^2 + O(\chi^3)$. The ion density integral for $\chi \ll 1$ is calculated via a matched asymptotic expansion [10] hinging on the observation that the function $g(\sqrt{w^2 - 2\chi})$ can be represented using two different approximations with a common range of validity. For $\bar{w} = \sqrt{w^2 - 2\chi} \ll 1$ (slow ions), the function $g(\bar{w})$ is expanded as a power series close to $\bar{w}=0$,

$$g(\bar{w}) = g_p \bar{w}^p + O(\bar{w}^{p_2}) \text{ for } \bar{w} \ll 1, \tag{13}$$

where $p > -1$ (the density should remain finite) is defined to be the smallest power in the expansion of $g(\bar{w})$ near $\bar{w}=0$, $g_p > 0$ is the corresponding positive constant coefficient ($g(w) > 0$), and $p_2 > p$ is the second smallest power in the expansion. Note that the form of the distribution function $g(w)$ near $w = 0$ assumed here is much more general than in previous derivations of the kinetic Bohm criterion. For $\bar{w} \sim w \gg \sqrt{2\chi}$, $g(\sqrt{w^2-2\chi})$ can be Taylor expanded near $g(w)$,

$$g\left(\sqrt{w^2-2\chi}\right) = g(w) - \chi\frac{g'(w)}{w} - \frac{1}{2}\chi^2\frac{g'(w)}{w^3} + \frac{1}{2}\chi^2\frac{g''(w)}{w^2} + O\left(\frac{\chi^3 g'(w)}{w^5}, \frac{\chi^3 g''(w)}{w^4}, \frac{\chi^3 g'''(w)}{w^3}\right). \tag{14}$$

The two approximations (13)-(14) have a common range of validity $\sqrt{2\chi} \ll w \ll 1$. Therefore, (12) can be re-expressed by choosing a cutoff velocity parameter w_c satisfying $\sqrt{2\chi} \ll w_\mathrm{c} \ll 1$, and using the approximation (13) for $w \leqslant w_\mathrm{c}$ and (14) for $w \geqslant w_\mathrm{c}$. This results in

$$\chi'' = Q_0 + Q_1 + Q_2 + \bar{Q}_{(p+1)/2} + O(\chi^3 w_\mathrm{c}^{p-5}, \chi^3, \chi^{(p_2+1)/2}, \chi w_\mathrm{c}^{p_2-1}, \chi^2 w_\mathrm{c}^{p_2-3}), \tag{15}$$

with the zeroth order, first order, second order, and slow ion charge densities, respectively, defined by:

$$Q_0 = \int_0^\infty g(w)\, dw - 1 = 0, \tag{16}$$

$$Q_1 = \chi\left[1 - \int_{w_\mathrm{c}}^\infty \frac{g'(w)}{w} dw\right], \tag{17}$$

$$Q_2 = \frac{1}{2}\chi^2\left[\int_{w_\mathrm{c}}^\infty \frac{g''(w)}{w^2} dw - \int_{w_\mathrm{c}}^\infty \frac{g'(w)}{w^3} dw - 1\right], \tag{18}$$

$$\begin{aligned}\bar{Q}_{(p+1)/2} &= g_p\left[\int_{\sqrt{2\chi}}^{w_\mathrm{c}} \left(w^2-2\chi\right)^{p/2} dw - \int_0^{w_\mathrm{c}} w^p dw\right] \\ &= g_p\left[-\frac{(2\chi)^{(p+1)/2}}{p+1} \right.\\ &\left. + \sum_{n=1}^\infty \frac{(2\chi)^n}{n!} \prod_{m=0}^{n-1}\left(m-\frac{p}{2}\right) \int_{\sqrt{2\chi}}^{w_\mathrm{c}} \frac{dw}{w^{2n-p}}\right].\end{aligned} \tag{19}$$

The quasineutrality condition at the sheath entrance (where $\chi = 0 = \chi''$) was used in (16). The errors $O(\chi^3 w_\mathrm{c}^{p-5}, \chi^3)$ in (15) result from integrating the errors in (14) in the interval $w \in [w_\mathrm{c}, \infty]$ and from another $O(\chi^3)$ term in the expansion of the electron density. The errors $O(\chi^{(p_2+1)/2}, \chi w_\mathrm{c}^{p_2-1}, \chi^2 w_\mathrm{c}^{p_2-3})$ come from the term $\bar{Q}_{(p_2+1)/2}$ which has been neglected in (15). Hall [8] takes $w_\mathrm{c} = \sqrt{2\chi}$ and uses a Taylor expansion instead of a power expansion in (13), thus considering $p = 0$ (and $p_2 = 1$) or $p = 1$, while keeping terms up to order $O(\chi)$. Yet, even the expansion (15) up to the higher order $O(\chi^2)$ is invalid for $w_\mathrm{c} = \sqrt{2\chi}$ since some of the neglected terms are of order $O(\chi^{(p+1)/2})$, which is $O(\chi^{1/2})$ for $p = 0$ and $O(\chi)$ for $p = 1$.

The assumption of an electron-repelling sheath, $\chi(\xi) \geqslant \chi(\infty) = 0$, requires that the charge density χ'' in equation (15) become positive in the sheath. The sign of χ'' is established in the following cases sequentially: $-1 < p < 1$, $p = 1$, $1 < p < 3$, $p = 3$, and $p > 3$, thus covering all $p > -1$ in (13). It will be convenient to first evaluate the density contributions (17)-(19) for specific values or intervals of p in (13). Equation (17) for Q_1 leads to:

$$Q_1 = O(\chi w_\mathrm{c}^{p-1}) \text{ for } -1 < p < 1, \tag{20}$$

$$Q_1 = \chi\left[1 - \int_0^\infty g''(w)\ln(1/w)dw - g_1\ln(1/w_\mathrm{c})\right] + O(\chi w_\mathrm{c}^{p_2-1}) \text{ for } p = 1, \tag{21}$$

$$Q_1 = \chi\left[1 - \int_0^\infty g''(w)\ln(1/w)dw\right] + \frac{p g_p \chi w_\mathrm{c}^{p-1}}{p-1} + O\left(\chi w_\mathrm{c}^{p_2-1}\right) \text{ for } p > 1. \tag{22}$$

Equation (18) for Q_2 gives:

$$Q_2 = O(\chi^2 w_\mathrm{c}^{p-3}) \text{ for } -1 < p < 3, \tag{23}$$

$$Q_2 = \frac{1}{4}\chi^2\left[3g_3 + \int_0^\infty g''''(w)\ln(1/w)dw + 6g_3\ln(1/w_\mathrm{c}) - 2\right] + O\left(\chi^2 w_\mathrm{c}^{p_2-3}\right) \text{ for } p = 3, \tag{24}$$

$$Q_2 = \frac{1}{2}\chi^2\left[3\int_0^\infty \frac{g(w)}{w^4}dw - 1\right] - \frac{p(p-2)}{2(p-3)} g_p \chi^2 w_\mathrm{c}^{p-3} + O\left(\chi^2 w_\mathrm{c}^{p_2-3}\right) \text{ for } p > 3. \tag{25}$$

Equation (19) for $\bar{Q}_{(p+1)/2}$ becomes:

$$\bar{Q}_{(p+1)/2} = A_{(p+1)/2}\chi^{(p+1)/2} - \frac{p g_p \chi w_\mathrm{c}^{p-1}}{p-1} + \frac{p(p-2)}{2(p-3)} g_p \chi^2 w_\mathrm{c}^{p-3} + O(\chi^3 w_\mathrm{c}^{p-5}) \text{ for } p > -1, p \neq 2n-1 \text{ with } n \in \mathbb{N}, \tag{26}$$

$$\bar{Q}_1 = -g_1\chi\left[1 + \frac{1}{2}\sum_{n=2}^\infty \frac{\prod_{m=1}^{n-1}\left(m-\frac{1}{2}\right)}{n!(n-1)} + \ln\left(\frac{w_\mathrm{c}}{\sqrt{2\chi}}\right)\right] + O(\chi^2 w_\mathrm{c}^{-2}) \text{ for } p = 1, \tag{27}$$

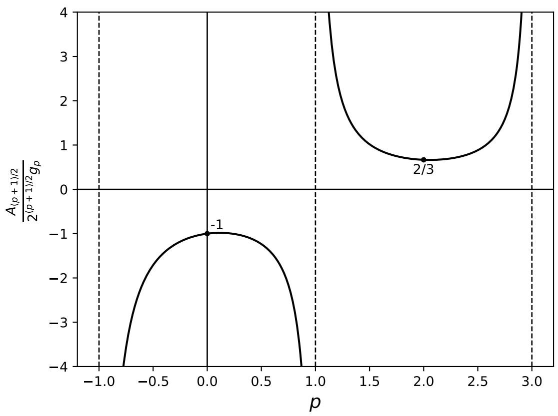

FIG. 1. The quantity $A_{p+1/2}/(2^{(p+1)/2}g_p)$ is plotted as a function of p in the intervals $p \in (-1,1)$ and $p \in (1,3)$. The asymptotes at $p = -1$, 1 and 3 are shown as dashed lines. The value of the curve at $p = 0$ and 2 is marked.

$$\bar{Q}_2 = g_3\chi^2 \left[2 + 3\sum_{n=3}^{\infty} \frac{\prod_{m=2}^{n-1}\left(m - \frac{3}{2}\right)}{n!(2n-4)} + \frac{3}{2}\ln\left(\frac{w_c}{\sqrt{2\chi}}\right)\right] - \frac{3}{2}g_3\chi w_c^2 + O(\chi^3 w_c^{-2}) \text{ for } p = 3, \tag{28}$$

where $A_{(p+1)/2}$ in (26) is defined by

$$\frac{A_{(p+1)/2}}{2^{(p+1)/2}} = g_p\left[\sum_{n=1}^{\infty} \frac{\prod_{m=0}^{n-1}\left(m - \frac{p}{2}\right)}{n!(2n-p-1)} - \frac{1}{p+1}\right] \text{for } p > -1, p \neq 2n-1 \text{ with } n \in \mathbb{N}. \tag{29}$$

Gauss' test can be used to show that the series in (27)-(29) are convergent for the given values of p.

If $-1 < p < 1$, (20), (23) and (26) are inserted into (15) to obtain

$$\chi''(\xi) = A_{(p+1)/2}\chi^{(p+1)/2} + O(\chi w_c^{p-1}, \chi^{(p_2+1)/2}). \tag{30}$$

Figure 1 is a plot of the numerical evaluation of $A_{(p+1)/2}$ as a function of $p \in (-1,1) \cup (1,3)$, illustrating that $A_{(p+1)/2} < 0$ for $p \in (-1,1)$. This assumed interval for p has therefore led to a negative charge density, $\chi'' \leqslant 0$, in contradiction with an electron-repelling sheath. Hence, the form (13) requires $p \geqslant 1$: the distribution function satisfies $g(0^+) = 0$, corresponding to (8), and has a non-divergent (zero or finite) first derivative $g'(0^+)$.

If $p = 1$, (21), (23) and (27) are inserted into (15) to obtain (the $O(\chi w_c^{p_2-1})$ errors in (15) and (21) cancel)

$$\chi'' = B_1\chi\ln(1/\chi) + A_1\chi + O(\chi w_c \ln w_c, \chi^2 w_c^{-2}, \chi^{(p_2+1)/2}), \tag{31}$$

where A_1 and B_1 are defined by

$$A_1 = g_1\left[\frac{1}{2}\ln 2 - \frac{1}{2}\sum_{n=2}^{\infty}\frac{\prod_{m=1}^{n-1}\left(m-\frac{1}{2}\right)}{n!(n-1)} - 1\right] + 1 - \int_0^{\infty} g''(w)\ln(1/w)dw, \tag{32}$$

$$B_1 = -g_1/2 < 0. \tag{33}$$

Equation (31), which follows from the assumption $p = 1$, is also in contradiction with an electron-repelling sheath, since $B_1 < 0$ from (33) and $\ln(1/\chi) \gg 1$ for $\chi \ll 1$ imply that $\chi'' \leqslant 0$. Hence, $p > 1$ is required in (13), corresponding to $g'(0^+) = 0$. This conclusively proves that condition (9) must be satisfied at the sheath entrance.

Considering $p > 1$ and inserting (22), either of (23)-(25), and (26) in (15) gives (the $O(\chi w_c^{p_2-1})$ errors in (15) and (22) cancel)

$$\chi''(x) = \chi\left[1 - \int_0^{\infty} g''(w)\ln(1/w)dw\right] + O(\chi^{(p+1)/2}, \chi^2 w_c^{p-3}, \chi^2). \tag{34}$$

For (34) to be compatible with an electron-repelling sheath, the condition $\int_0^{\infty}\ln(1/w)g''(w)\,dw < 1$ is required, which is the dimensionless form of (10) without the equality. In order to prove (10), and consequently (3), it must still be verified that if the equality holds,

$$\int_0^{\infty} g''(w)\ln(1/w)dw = \int_0^{\infty}\frac{g(w)}{w^2}dw = 1, \tag{35}$$

the higher-order terms in equation (34) are consistent with an electron-repelling sheath solution with $\chi(\xi) \geqslant 0$.

If $1 < p < 3$, inserting (22), (23), (26) and (35) in (15) leads to

$$\chi'' = A_{(p+1)/2}\chi^{(p+1)/2} + O(\chi^2 w_c^{p-3}, \chi^{(p_2+1)/2}). \tag{36}$$

From figure 1 it is seen that $A_{(p+1)/2} > 0$ for $1 < p < 3$, which makes (36) consistent with a positive space charge, $\chi'' \geqslant 0$. If $p = 3$, inserting (22), (24), (28) and (35) into (15) gives (the $O(\chi^2 w_c^{p_2-3})$ errors in (15) and (24) cancel)

$$\chi'' = B_2\chi^2\ln(1/\chi) + A_2\chi^2 + O\left(\chi^2 w_c\ln w_c, \chi^3 w_c^{-2}, \chi^{(p_2+1)/2}\right), \tag{37}$$

with

$$A_2 = g_3\left[\frac{11}{4} + 3\sum_{n=3}^{\infty}\frac{\prod_{m=2}^{n-1}\left(m-\frac{3}{2}\right)}{n!(2n-4)} - \frac{3}{4}\ln 2\right] + \frac{1}{4}\int_0^{\infty} g''''(w)\ln(1/w)dw - \frac{1}{2}, \tag{38}$$

$$B_2 = 3g_3/4 > 0. \tag{39}$$

Equation (37) is again consistent with a positive space charge, since $B_2 > 0$ from (39). If $p > 3$, inserting (22), (25), (26) and (35) into (15) gives (the $O(\chi^2 w_c^{p_2-3})$ errors in (15) and (25) cancel)

$$\chi'' = A_2\chi^2 + O\left(\chi^3 w_c^{p-5}, \chi^3, \chi^{(p+1)/2}\right), \tag{40}$$

$$A_2 = \frac{3}{2}\int_0^\infty \frac{g(w)}{w^4}dw - \frac{1}{2} > 0. \tag{41}$$

In (41), Schwartz's inequality and the equalities (16) and (35) constrain $\int_0^\infty dw g(w)/w^4 \geqslant 1$ [4], such that $A_2 > 0$. Equation (40) is compatible with an electron-repelling sheath, and exhausts all possible remaining values of p. This completes the proof that an electron-repelling sheath solution (with adiabatic electrons) exists if (8)-(10), which can be combined into (3), are satisfied.

The physical interpretation of the kinetic Bohm criterion (3) for $\epsilon \to 0$ is as follows. The sheath must be positively charged to be consistent with an electron-repelling field. The cold-ion Bohm criterion (2) imposes a minimum ion flow velocity at the sheath entrance because slower ions undergo a much larger relative velocity increment than faster ions for a given potential drop, and by continuity a larger drop in density. Similarly, with an arbitrary velocity distribution instead of a mono-energetic one, conditions (8)-(9) limit the number of slow ions with $v_x \approx 0$ as they would enhance the ion density drop and make it exceed the electron density drop. Correspondingly, slow ions have a larger weight in condition (3) and force the bulk ions to compensate by moving faster towards the wall, leading to the over-satisfaction of (2).

For small but finite ϵ, ions moving away from the wall ($v_x > 0$) may be generated in the small region $x \in (0, x_\epsilon)$, causing the presence of a small additional number of slow ions with $v_x \approx 0$ at x_ϵ. This is consistent with conditions (8)-(9), which imply $f_\epsilon(0^-) \sim \epsilon^{r_1(q)} n_e/Zv_B$ and $f'_\epsilon(0^-) \sim \epsilon^{r_2(q)} n_e/Zv_B^2$, where $r_1(q) > 0$ and $r_2(q) > 0$ depend on q in $x_\epsilon = \epsilon^q L$ [20]. Condition (3) results from integrating (10) by parts twice, which in terms of f_ϵ gives

$$\lim_{v_x\to 0^-}\lim_{\epsilon\to 0}\left[\int_{-\infty}^{v_x}\frac{f_\epsilon(v'_x)}{v_x'^2}dv'_x + f'_\epsilon(v_x)\ln(-1/v_x) + \frac{f_\epsilon(v_x)}{v_x}\right] \leqslant \lim_{\epsilon\to 0}\frac{n_{e,\epsilon}}{Zv_B^2}. \tag{42}$$

If the limits on the left hand side of (42) are interchanged, all terms diverge, as exemplified by (7) for the first term. Without the interchange, the second and third terms vanish by (8) and (9), and (3) is recovered. The limits in (42) can be combined, without interchange, by taking $v_x = -v_\epsilon$, with $v_\epsilon = \epsilon^r v_B$ and $r \in (0, r_1(q))$, such that $\lim_{\epsilon\to 0} f_\epsilon(-v_\epsilon)/v_\epsilon = 0$ and $\lim_{\epsilon\to 0} f'_\epsilon(-v_\epsilon)\ln(1/v_\epsilon) = 0$ [21]. Then, (42) becomes the reformulation of the kinetic Bohm criterion (3) in terms of f_ϵ:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{-v_\epsilon}\frac{f_\epsilon(v_x)}{v_x^2}dv_x \leqslant \lim_{\epsilon\to 0}\frac{n_{e,\epsilon}}{Zv_B^2}. \tag{43}$$

The contribution to the ion density drop of only a small number (vanishing for $\epsilon \to 0$) of slow ions with $|v_x| < v_\epsilon$ is ignored, as it would be sharply over-estimated due to neglecting their generation within the sheath.

This work has been carried out within the framework of the EUROfusion Consortium, via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion) and funded by the Swiss State Secretariat for Education, Research and Innovation (SERI). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union, the European Commission, or SERI. Neither the European Union nor the European Commission nor SERI can be held responsible for them.

[1] I. Langmuir, Positive ion currents from the positive column of mercury arcs, Science **58**, 290 (1923).

[2] I. Langmuir, The interaction of electron and positive ion space charges in cathode sheaths, Physical Review **33**, 954 (1929).

[3] L. Tonks and I. Langmuir, A general theory of the plasma of an arc, Physical Review **34**, 876 (1929).

[4] K.-U. Riemann, The Bohm criterion and sheath formation, Journal of Physics D: Applied Physics **24**, 493 (1991).

[5] R. Chodura, Plasma–wall transition in an oblique magnetic field, Physics of Fluids (1958-1988) **25**, 1628 (1982).

[6] D. Bohm, *The Characteristics of Electrical Discharges in Magnetic fields, Chapter 3, ed A Guthrie and R K Wakerling (New York: McGraw–Hill)* (1949).

[7] E. Harrison and W. Thompson, The low pressure plane symmetric discharge, Proceedings of the Physical Society **74**, 145 (1959).

[8] L. Hall, Harrison-Thompson generalization of Bohm's sheath condition, in *Proc. Phys. Soc.(London)*, Vol. 80 (Univ. of California, Livermore, 1962).

[9] A. Caruso and A. Cavaliere, The structure of the collisionless plasma-sheath transition, Il Nuovo Cimento (1955-1965) **26**, 1389 (1962).

[10] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*, Vol. 1 (Springer Science & Business Media, 1999).

[11] M. Van Dyke, Nineteenth-century roots of the boundary-layer idea, Siam Review **36**, 415 (1994).

[12] B. Bertotti, Theory of an electrostatic probe in a strong magnetic field, The Physics of Fluids **4**, 1047 (1961).

[13] B. Bertotti and A. Cavaliere, Boundary conditions for plasma ion waves, Il Nuovo Cimento (1955-1965) **35**, 1244 (1965).

[14] For small but finite ϵ, the plasma and sheath solutions can be asymptotically matched to the solution obtained in a transition region on a suitably-chosen intermediate

length scale between plasma and sheath [22–27]; one may therefore argue that this region is the most accurate manifestation of the concept of a sheath entrance.

[15] R. Fernsler, S. Slinker, and G. Joyce, Quasineutral plasma models, Physical Review E **71**, 026401 (2005).

[16] S. Baalrud and C. Hegna, Kinetic theory of the presheath and the Bohm criterion, Plasma Sources Science and Technology **20**, 025013 (2011).

[17] S. Baalrud and C. Hegna, Reply to comment on 'Kinetic theory of the presheath and the Bohm criterion', Plasma Sources Science and Technology **21**, 068002 (2012).

[18] S. Baalrud, B. Scheiner, B. Yee, M. Hopkins, and E. Barnat, Extensions and applications of the Bohm criterion, Plasma Physics and Controlled Fusion **57**, 044003 (2015).

[19] K. Riemann, Comment on 'Kinetic theory of the presheath and the bohm criterion', Plasma Sources Science and Technology **21**, 068001 (2012).

[20] It is presumed here that f_ϵ is differentiable, like f.

[21] In the unlikely case that $f_\epsilon(v_x) \sim \epsilon^{r_3} v_x^{p_\epsilon}$ for $v_x \ll v_{\mathrm{B}}$, with $r_3 > 0$ and $p_\epsilon \in (-1,0) \cup (0,1)$, the limits are non-interchangeable in (8) (if $p_\epsilon \in (-1,0)$), (9) and (10), and $r < r_3/(1-p_\epsilon)$ is required.

[22] R. Franklin and J. Ockendon, Asymptotic matching of plasma and sheath in an active low pressure discharge, Journal of Plasma Physics **4**, 371 (1970).

[23] M. Slemrod, Monotone increasing solutions of the Painlevé 1 equation $y'' = y^2 + x$ and their role in the stability of the plasma-sheath transition, European Journal of Applied Mathematics **13**, 663 (2002).

[24] K. Riemann, Kinetic analysis of the collisional plasma–sheath transition, Journal of Physics D: Applied Physics **36**, 2811 (2003).

[25] K.-U. Riemann, Plasma-sheath transition in the kinetic Tonks-Langmuir model, Physics of Plasmas **13**, 063508 (2006).

[26] K.-U. Riemann, The influence of collisions on the plasma sheath transition, Physics of Plasmas **4**, 4158 (1997).

[27] K. U. Riemann, J. Seebacher, D. Tskhakaya, and S. Kuhn, The plasma–sheath matching problem, Plasma physics and controlled fusion **47**, 1949 (2005).