Areas between cosines

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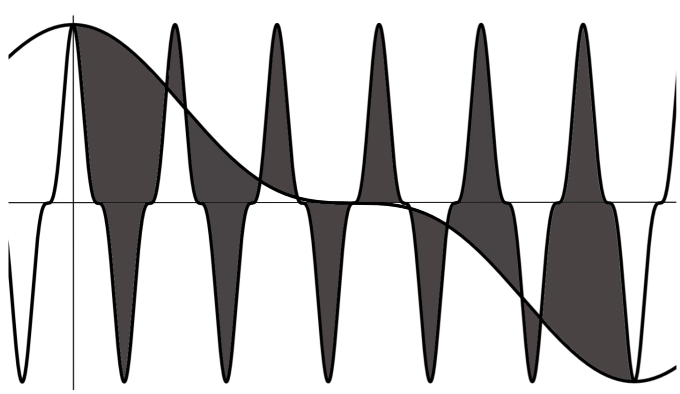
Abstract

We find the area between $\cos^n x$ and $\cos^n kx$ as k heads to infinity, and we establish connections between these limiting values, and coefficients from exponential generating functions involving $\arcsin x$.

1 Introduction

Shown here is the area between $\cos^3 x$ and $\cos^3 11x$ over the interval $[0, \pi]$.



It is not hard to calculate this area directly. We could write it as

$$\frac{6}{33}\left(\cot\frac{3\pi}{12}+9\cot\frac{\pi}{12}\right)-\frac{5}{33}\left(\cot\frac{3\pi}{10}+9\cot\frac{\pi}{10}\right)\approx 1.981887\ldots,$$

which has a pleasant symmetry, or we could write it as

$$\frac{1}{33}\left(144+54\sqrt{3}-5\left(19+9\sqrt{5}\right)\sqrt{5-2\sqrt{5}}\right)\approx 1.981887\ldots$$

which is not quite so nice. In this article, we are more interested in what happens when we replace the 11 in $\cos^3 11x$ with k (giving us $\cos^3 kx$) and then find the area between $\cos^3 x$ and $\cos^3 kx$ as k goes to infinity. In this case, the limiting area turns out to be

$$\frac{56}{9\pi}\approx 1.980594\ldots$$

which is rather surprising (and also fairly close numerically to the two expressions above).

Of course, there is no reason to restrict ourselves to just looking at the *third* power of cosine. With this in mind, we define A_n to be the limiting area (as $k\to\infty$) between $\cos^n x$ and $\cos^n kx$ over the interval $[0,\pi]$. In other words, we define

$$A_n=\lim_{k\to\infty}\int_0^\pi\left|\cos^n x-\cos^n kx\right|dx.$$

In what follows, we are able to find formulas for A_n involving sums with binomial coefficients (Theorems 1 and 2). We then find recursive formulas for A_n involving just A_{n-2} (Theorem 3). Finally, in a rather surprising result, we establish formulas for A_n that involve sums with double factorials, and we connect these formulas with two entries in the OEIS related to the exponential generating functions for $\arcsin(x)/(1-x)$ and $\arcsin^2(x)/(2(1-x))$ (Theorem 4).

2 Area formulas

We begin with a few values for the limiting area A_n for when n is odd.

n odd	A_n
1	$\frac{8}{\pi} \cdot \frac{1}{(1)^2}$
3	$\frac{8}{\pi} \cdot \frac{7}{(1 \cdot 3)^2}$
5	$\frac{8}{\pi} \cdot \frac{149}{(1 \cdot 3 \cdot 5)^2}$
7	$\frac{8}{\pi} \cdot \frac{6483}{(1 \cdot 3 \cdot 5 \cdot 7)^2}$

And what of these numbers $1, 7, 149, 6483$ that appear in the numerators of A_n for n odd? These are every other term in the sequence A296726 in the On-Line Encyclopedia of Integer Sequences (OEIS) [1], where we learn that they also appear as coefficients in the exponential generating function for $\arcsin x/(1-x)$. See Theorem 4 for details.

But first, we present the following theorem.

Theorem 1. *With A_n the limit, as $k \to \infty$, of the area between $\cos^n x$ and $\cos^n kx$ on the interval $[0, \pi]$, then*

$$\text{for } n \text{ odd}, \qquad A_n = \frac{8}{\pi} \cdot \frac{1}{2^{n-1}} \cdot \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2}. \tag{1}$$

Next, we present a few values for the limiting area A_n for when n is even. As it turns out, this case needs to be further subdivided depending on whether $n \equiv 2$ or $0 \bmod 4$.

$n \equiv 2$	A_n
2	$\frac{16}{\pi} \cdot \frac{1}{(2)^2} = \frac{4}{\pi}$
6	$\frac{16}{\pi} \cdot \frac{544}{(2 \cdot 4 \cdot 6)^2} = \frac{34}{9\pi}$
10	$\frac{16}{\pi} \cdot \frac{3096576}{(2 \cdot 4 \cdot 6 \cdot 8 \cdot 10)^2} = \frac{84}{25\pi}$

$n \equiv 0$	A_n
4	$\frac{16}{\pi} \cdot \frac{16}{(2 \cdot 4)^2} = \frac{4}{\pi}$
8	$\frac{16}{\pi} \cdot \frac{32768}{(2 \cdot 4 \cdot 6 \cdot 8)^2} = \frac{32}{9\pi}$
12	$\frac{16}{\pi} \cdot \frac{423493632}{(2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12)^2} = \frac{718}{225\pi}$

And what of these numbers $1, 16, 544, 32768, 3096576, \ldots$ that appear in the numerators of A_n for n even? These are every other term in the sequence A372324 in the OEIS, where we learn that they also appear as coefficients in the exponential generating function for $\arcsin^2 x/(2(1-x))$. See Theorem 4 for details.

For now, we have the following theorem.

Theorem 2. *For n even, and with A_n the limit, as $k \to \infty$, of the area between $\cos^n x$ and $\cos^n kx$ on the interval $[0, \pi]$, then*

$$\text{for } n \equiv 2 \text{ mod } 4, \qquad A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2 - 2j)^2}, \tag{2}$$

and

$$\text{for } n \equiv 0 \text{ mod } 4, \qquad A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-4)/4} \binom{n}{2j+1} \frac{1}{(n/2 - (2j+1))^2}. \tag{3}$$

The proofs of these two theorems are rather technical, and so we placed them at the end of this paper.

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3 Recursion formulas

Surprisingly, A_n has a fairly simple recursion formula.

Theorem 3. *For A_n the limit, as $k \to \infty$, of the area between $\cos^n x$ and $\cos^n kx$ on the interval $[0, \pi]$, then with $n \geq 3$ we have*

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{8}{\pi n^2} \quad \text{for } n \text{ odd, and}$$

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{16}{\pi n^2} \quad \text{for } n \text{ even.}$$

Proof. We start with the easily-verified statement that

$$\binom{n}{j} \cdot (n-j) \cdot j = \binom{n-2}{j-1} \cdot (n-1) \cdot n.$$

Next, we multiply both sides by 4 to get

$$\binom{n}{j} \cdot (2n-2j) \cdot 2j = 4 \cdot \binom{n-2}{j-1} \cdot (n-1) \cdot n.$$

We now divide both sides by $(n-2j)^2 = ((n-2)-2(j-1))^2$,to get

$$\binom{n}{j} \cdot \frac{(2n-2j)\cdot 2j}{(n-2j)^2} = 4 \cdot \binom{n-2}{j-1} \cdot \frac{(n-1)\cdot n}{((n-2)-2(j-1))^2}.$$

In the numerator on the left, we write $2n-2j$ as $n+n-2j$, and we write $2j$ as $n-(n-2j)$, giving us

$$\binom{n}{j} \cdot \frac{(n+(n-2j))\cdot(n-(n-2j))}{(n-2j)^2} = 4 \cdot \binom{n-2}{j-1} \cdot \frac{(n-1)\cdot n}{((n-2)-2(j-1))^2}.$$

We multiply out the numerator on the left to get

$$\binom{n}{j} \cdot \frac{n^2-(n-2j)^2}{(n-2j)^2} = 4 \cdot \binom{n-2}{j-1} \cdot \frac{(n-1)\cdot n}{((n-2)-2(j-1))^2},$$

and a further simplification on the left gives us

$$\binom{n}{j} \cdot \frac{n^2}{(n-2j)^2} - \binom{n}{j} = 4 \cdot \binom{n-2}{j-1} \cdot \frac{(n-1)\cdot n}{((n-2)-2(j-1))^2}.$$

We now divide both sides by n^2 and simplify a bit more to obtain

$$\binom{n}{j}\frac{1}{(n-2j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n}\binom{n-2}{j-1}\frac{1}{((n-2)-2(j-1))^2}. \tag{4}$$

At this point, we will consider three cases, for n odd, $n \equiv 2 \bmod 4$, and $n \equiv 0 \bmod 4$.

Assume, for now, that n is odd. We sum both sides of equation (4) from $j=1$ to $j=(n-1)/2$ to obtain

$$\sum_{j=1}^{(n-1)/2} \binom{n}{j}\frac{1}{(n-2j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n}\sum_{j=1}^{(n-1)/2} \binom{n-2}{j-1}\frac{1}{((n-2)-2(j-1))^2}.$$

On the left, we can start that sum at $j=0$ instead of $j=1$ without changing the value. On the right, we re-index the sum by using $j' = j-1$, so that j' starts at $j'=0$ and ends at $j'=(n-3)/2$. After distributing the sum on the left, this gives us

$$\sum_{j=0}^{(n-1)/2} \binom{n}{j}\frac{1}{(n-2j)^2} - \frac{1}{n^2}\sum_{j=0}^{(n-1)/2} \binom{n}{j} = \frac{4(n-1)}{n}\sum_{j'=0}^{(n-3)/2} \binom{n-2}{j'}\frac{1}{((n-2)-2j')^2}.$$

Thanks to our equation for A_n in equation (1) for n odd, we can re-write the above equation as

$$\pi 2^{n-4} \cdot A_n - \frac{1}{n^2} \sum_{j=0}^{(n-1)/2} \binom{n}{j} = \frac{4(n-1)}{n} \cdot \pi 2^{(n-2)-4} \cdot A_{n-2}.$$

Since n is odd, the sum on the left of the above equation is exactly half of the entire sum of the nth row of Pascal's triangle. The entire sum would be 2^n, so we have (after adjusting the powers of 2 on the right)

$$\pi 2^{n-4} \cdot A_n - \frac{1}{n^2} \cdot 2^{n-1} = \frac{n-1}{n} \cdot \pi 2^{n-4} \cdot A_{n-2}.$$

If we now divide by $\pi 2^{n-4}$ and re-arrange the terms, we obtain

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{8}{\pi n^2},$$

as desired (for n odd).

Next, we consider $n \equiv 2 \bmod 4$. Looking back at equation (4), we replace j with $2j$, giving us

$$\binom{n}{2j} \frac{1}{(n-4j)^2} - \frac{1}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{((n-2)-2(2j-1))^2}. \tag{5}$$

We factor out 2^2 from the $(n-4j)^2$ in the denominator on the left, and likewise from the denominator on the right, giving us

$$\binom{n}{2j} \frac{1}{4(n/2-2j)^2} - \frac{1}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{4((n-2)/2-(2j-1))^2}.$$

We multiply through by 4 to get

$$\binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \binom{n-2}{2j-1} \frac{1}{((n-2)/2-(2j-1))^2}. \tag{6}$$

We sum both sides of equation (6) from $j = 1$ to $j = (n-2)/4$ to obtain

$$\sum_{j=1}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2} \binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j=1}^{(n-2)/4} \binom{n-2}{2j-1} \frac{1}{((n-2)/2-(2j-1))^2}.$$

On the left, we can start that sum at $j = 0$ instead of $j = 1$ without changing the value. On the right, we re-index the sum by using $j' = j - 1$, so that j' starts at $j' = 0$ and ends at $j' = (n-6)/4$. This gives us

$$\sum_{j=0}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2} - \frac{4}{n^2}\binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j'=0}^{(n-6)/4} \binom{n-2}{2j'+1} \frac{1}{((n-2)/2-(2j'+1))^2}.$$

Thanks to our equation for A_n in equation (2) for $n \equiv 2 \bmod 4$, we recognize that we can re-write the sum of the first expression on the left above as $\pi 2^{n-4} \cdot A_n$. When we do so (after distributing that sum on the left) it give us

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2} \sum_{j=0}^{(n-2)/4} \binom{n}{2j} = \frac{4(n-1)}{n} \sum_{j'=0}^{(n-6)/4} \binom{n-2}{2j'+1} \frac{1}{((n-2)/2-(2j'+1))^2}.$$

Likewise, since $n \equiv 2 \bmod 4$, then $n-2 \equiv 0 \bmod 4$, and so if we use equation (3) for A_{n-2} then we recognize that the sum on the right is equal to $\pi 2^{(n-2)-4} A_{n-2}$. This means we can re-write the above equation as

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2} \sum_{j=0}^{(n-2)/4} \binom{n}{2j} = \frac{4(n-1)}{n} \cdot \pi 2^{n-6} \cdot A_{n-2}$$

Since $n \equiv 2 \bmod 4$, then the sum on the left of the above equation is exactly one quarter of the entire sum of the nth row of Pascal's triangle. The entire sum would be 2^n, so we have (after adjusting the powers of 2 on the right)

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2} \cdot 2^{n-2} = \frac{n-1}{n} \cdot \pi 2^{n-4} \cdot A_{n-2}.$$

If we now divide by $\pi 2^{n-4}$ and re-arrange the terms, we obtain

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{16}{\pi n^2},$$

as desired (for $n \equiv 2 \bmod 4$).

Finally, we consider $n \equiv 0 \bmod 4$. Looking back once more at equation (4), we first factor out 2^2 from the $(n-2j)^2$ in the denominator on the left, and likewise from the denominator in the right. We also replace $n-2$ with q in the expression on the right, leaving us with

$$\binom{n}{j} \frac{1}{4(n/2-j)^2} - \frac{1}{n^2}\binom{n}{j} = \frac{4(n-1)}{n} \binom{q}{j-1} \frac{1}{4(q/2-(j-1))^2}.$$

We now multiply through by 4, and replace j with $2j+1$, giving us

$$\binom{n}{2j+1}\frac{1}{(n/2-(2j+1))^2}-\frac{4}{n^2}\binom{n}{2j+1}=\frac{4(n-1)}{n}\binom{q}{2j}\frac{1}{(q/2-2j)^2}. \tag{7}$$

We sum both sides of equation (7) from $j=0$ to $j=(n-4)/4$ to obtain

$$\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}\frac{1}{(n/2-(2j+1))^2}-\frac{4}{n^2}\binom{n}{2j+1}=\frac{4(n-1)}{n}\sum_{j=0}^{(n-4)/4}\binom{q}{2j}\frac{1}{(q/2-2j)^2}.$$

Thanks to our equation for A_n in equation (3) for $n \equiv 0 \bmod 4$, we recognize that we can re-write the sum of the first expression on the left above as $\pi 2^{n-4} \cdot A_n$. When we do so (after distributing that sum on the left, and after also replacing $n-4$ with $q-2$ in the upper bound of the sum on the right) it give us

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2}\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}=\frac{4(n-1)}{n}\sum_{j=0}^{(q-2)/4}\binom{q}{2j}\frac{1}{(q/2-2j)^2}.$$

Since $n \equiv 0 \bmod 4$ and since $q=n-2$, then $q \equiv 2 \bmod 4$, and so if we use equation (2) for $A_q = A_{n-2}$ then we recognize that the sum on the right is equal to $\pi 2^{(n-2)-4}A_{n-2}$. This means we can re-write the above equation as

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2}\sum_{j=0}^{(n-4)/4}\binom{n}{2j+1}=\frac{4(n-1)}{n}\cdot \pi 2^{n-6}\cdot A_{n-2}$$

Since $n \equiv 0 \bmod 4$, then the sum on the left of the above equation is exactly one quarter of the entire sum of the nth row of Pascal's triangle. That entire sum would be 2^n, so we have (after adjusting the powers of 2 on the right)

$$\pi 2^{n-4} \cdot A_n - \frac{4}{n^2}\cdot 2^{n-2} = \frac{n-1}{n}\cdot \pi 2^{n-4}\cdot A_{n-2}.$$

If we now divide by $\pi 2^{n-4}$ and re-arrange the terms, we obtain

$$A_n = \frac{n-1}{n}A_{n-2}+\frac{16}{\pi n^2},$$

as desired (for $n \equiv 0 \bmod 4$).

Having covered all the cases for n, this completes the proof. $\square$

4 Connections to the OEIS

As we mentioned earlier, the numbers that appear in Theorem 1 are related to the sequence A296726. Here is the connection.

Theorem 4. *For A_n as defined above, then*

$$\text{for } n \text{ odd,} \qquad A_n = \frac{8}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}, \tag{8}$$

$$\text{and for } n \text{ even,} \qquad A_n = \frac{16}{\pi} \cdot \frac{n!}{(n!!)^2} \cdot \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2}. \tag{9}$$

Furthermore, the numbers

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \qquad \text{for } n \text{ odd,} \tag{10}$$

from equation (8) above, appear as every other entry in A296726, the terms from the exponential generating function for $\arcsin(x)/(1-x)$*. Likewise, the numbers*

$$n! \sum_{j=0}^{(n-2)/2} \frac{(2j)!!}{(2j+1)!!} \frac{1}{2j+2} \qquad \text{for } n \text{ even,} \tag{11}$$

from equation (9) above, appear as every other entry in A372324, the terms from exponential generating function for $\arcsin^2(x)/(2(1-x))$*.*

We recall that the notation $n!$ refers to the usual factorial function, and the notation $n!!$ is the less-familiar *double factorial* function [3]. Really, it's more like an "every other factorial", and here is the definition:

$$(2j)!! = (2j)(2j-2)(2j-4)\cdots 6 \cdot 4 \cdot 2,$$
$$(2j+1)!! = (2j+1)(2j-1)(2j-3)\cdots 5 \cdot 3 \cdot 1.$$

We also agree that $0!! = (-1)!! = 1$.

Proof of Theorem 4. We begin with n odd. We define A'_n to be the right-hand side of equation (8), so that

$$A'_n = \frac{8}{\pi} \cdot \frac{p!}{(n!!)^2} \cdot \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}. \tag{12}$$

We know from the first statement in Theorem 3 that

$$A_n = \frac{n-1}{n} A_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2}$$

We now seek to prove that

$$A'_n = \frac{n-1}{n} A'_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2}. \tag{13}$$

This, along with the fact that $A_1 = A'_1 = 8/\pi$, is all we will need.

We begin with the easily-verified statement

$$\frac{n!}{(n!!)^2} \cdot \frac{(n-2)!!}{(n-1)!!} \cdot \frac{1}{n} = \frac{1}{n^2}. \tag{14}$$

Now, starting with the right-hand side of equation (12), we use the definition of A'_n to write

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2} = \frac{8}{\pi} \left(\frac{1}{n^2} + \frac{n-1}{n} \cdot \frac{(n-2)!}{((n-2)!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right).$$

Since

$$\frac{n-1}{n} \cdot \frac{(n-2)!}{((n-2)!!)^2} = \frac{n(n-1)}{n^2} \cdot \frac{(n-2)!}{((n-2)!!)^2} = \frac{n!}{(n!!)^2},$$

we can re-write the previous equation as

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2} = \frac{8}{\pi} \left(\frac{1}{n^2} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right).$$

We now use the expression for $1/n^2$ in equation (14) to re-write the above equation as

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2} = \frac{8}{\pi} \left(\frac{n!}{(n!!)^2} \cdot \frac{(n-2)!!}{(n-1)!!} \cdot \frac{1}{n} + \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-3)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right),$$

and we add that first term on the right into the sum (as the $j = (n-1)/2$ term) to give us

$$\frac{n-1}{n} A'_{n-2} + \frac{8}{\pi} \cdot \frac{1}{n^2} = \frac{8}{\pi} \frac{n!}{(n!!)^2} \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} = A'_n,$$

as desired.

Hence, both A_n and A'_n satisfy the same recursion from equation (13), and since they also start at the same value of $A_1 = A'_1 = 1$, then they are indeed identical, thus giving us the desired equality in equation (8) in the statement of our theorem.

Next, we will show that the numbers

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1}$$

from equation (10) really are the same as every other entry in A296726, which is the list of coefficients for the exponential generating function for $\arcsin(x)/(1-x)$. To show this, we begin with the series for $1/(1-x)$ which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots ,$$

and for $\arcsin x$ which is

$$\arcsin x = x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + \frac{225}{7!}x^7 + \frac{11025}{9!}x^9 + \cdots ,$$

thanks to A177145. And furthermore, thanks to A001818, we can re-write those numerators as follows:

$$\arcsin x = \frac{((-1)!!)^2}{1!}x + \frac{(1!!)^2}{3!}x^3 + \frac{(3!!)^2}{5!}x^5 + \frac{(5!!)^2}{7!}x^7 + \frac{(7!!)^2}{9!}x^9 + \cdots .$$

Hence, since the generating function for $\arcsin(x)/(1-x)$ will be the convolution of the generating functions for $\arcsin x$ and $1/(1-x)$, then the nth term in the *exponential* generating function for $\arcsin(x)/(1-x)$, for n odd, will be

$$n! \left(\frac{((-1)!!)^2}{1!} + \frac{(1!!)^2}{3!} + \frac{(3!!)^2}{5!} + \cdots + \frac{((n-2)!!)^2}{n!} \right),$$

which we can write as

$$n! \sum_{j=0}^{(n-1)/2} \frac{((2j-1)!!)^2}{(2j+1)!}.$$

Now, since $(2j+1)! = (2j-1)!!(2j)!!(2j+1)$, then the above expression becomes

$$n! \sum_{j=0}^{(n-1)/2} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1},$$

as seen in equation (10).

The cases for n even are quite similar, and we leave the details to the reader. $\square$

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5 Technical Results

Before we can begin the proof, we will need some preliminary results.

Lemma 5. *Let N and q be positive integers. Then,*

$$\text{for } N \text{ even,} \qquad \sum_{\ell=1}^{N/2} \sin \frac{q\ell 2\pi}{N} = \begin{cases} 0 & \text{for } q \text{ even} \\ \cot \dfrac{q\pi}{N} & \text{for } q \text{ odd.} \end{cases} \tag{15}$$

Proof. We call upon Lagrange's Trigonometric Identity [2], which states that

$$\sum_{\ell=0}^{m} \sin \ell\theta = \frac{\cos \theta/2 - \cos(m+1/2)\theta}{2 \sin \theta/2}. \tag{16}$$

Since we are assuming that N is even, we replace m with $N/2$ and we replace θ with $q2\pi/N$ in equation (16) to get

$$\sum_{\ell=0}^{N/2} \sin \frac{q\ell 2\pi}{N} = \frac{\cos \dfrac{q\pi}{N} - \cos \dfrac{(N+1)q\pi}{N}}{2 \sin \dfrac{q\pi}{N}}. \tag{17}$$

Now, $\cos \frac{(N+1)q\pi}{N}$ can be written as $\cos\left(q\pi + \frac{q\pi}{N}\right)$, and for q even then $q\pi$ is an even multiple of π and so $\cos\left(q\pi + \frac{q\pi}{N}\right)$ equals $\cos\frac{q\pi}{N}$. However, for q odd then $q\pi$ is an odd multiple of π and so $\cos\left(q\pi + \frac{q\pi}{N}\right)$ equals $-\cos\frac{q\pi}{N}$. When we plug these simplifications into the numerator of equation (17), we get either 0 or $2\cos\frac{q\pi}{N}$ in the numerator depending on whether q is even or odd, respectively, and this gives us our desired formula. $\square$

Lemma 6. *Let N and q be positive integers. Then,*

$$\text{for } N, q \text{ even,} \qquad \sum_{\ell=1}^{N/2} \sin \frac{q\ell\pi}{N} = \begin{cases} 0 & \text{for } q \equiv 0 \bmod 4, \\ \cot \dfrac{q\pi}{2N} & \text{for } q \equiv 2 \bmod 4. \end{cases} \tag{18}$$

Proof. We call once more upon Lagrange's Trigonometric Identity (16). Since N is again even, we will replace m with $N/2$ and we replace θ with $q\pi/N$ in equation (16) to get

$$\sum_{\ell=0}^{N/2} \sin\frac{q\ell\pi}{N} = \frac{\cos\dfrac{q\pi}{2N} - \cos\dfrac{(N+1)q\pi}{2N}}{2\sin\dfrac{q\pi}{2N}}. \tag{19}$$

Now, $\cos\frac{(N+1)q\pi}{2N}$ can be written as $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$, and for $q \equiv 0 \bmod 4$ then $\frac{q\pi}{2}$ is an even multiple of π and so $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$ simplifies to $\cos\frac{q\pi}{2N}$. However, for $q \equiv 2 \bmod 4$ then $\frac{q\pi}{2}$ is an odd multiple of π is odd and so $\cos\left(\frac{q\pi}{2}+\frac{q\pi}{2N}\right)$ simplifies to $-\cos\frac{q\pi}{2N}$ When we plug these simplifications into the right-hand side of equation (19), we get either 0 or $2\cos\frac{q\pi}{2N}$ in the numerator depending on whether q is equivalent to 0 or 2 mod 4, respectively, and this gives us our desired formula. □

Lemma 7. *For x any real number,*

$$\lim_{k\to\infty} \left(\frac{1}{k}-1\right)\cot\frac{x}{k-1} + \left(\frac{1}{k}+1\right)\cot\frac{x}{k+1} \;=\; \frac{4}{x}. \tag{20}$$

Proof. We begin with the Taylor expansion for the cotangent, which gives us

$$\cot\theta \;=\; \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} + \cdots \;=\; \frac{1}{\theta} + \mathcal{O}(\theta).$$

If we apply this to our limit, we get

$$\left(\frac{1}{k}-1\right)\left(\frac{k-1}{x} + \mathcal{O}\left(\frac{x}{k-1}\right)\right) \;+\; \left(\frac{1}{k}+1\right)\left(\frac{k+1}{x} + \mathcal{O}\left(\frac{x}{k+1}\right)\right).$$

Since x is fixed, we can remove it from inside the $\mathcal{O}$. After expanding the above expression, we get

$$\left(\frac{1-k}{k}\right)\left(\frac{k-1}{x}\right) + \left(\frac{1-k}{k}\right)\cdot\mathcal{O}\left(\frac{1}{k-1}\right) \;+\; \left(\frac{1+k}{k}\right)\left(\frac{k+1}{x}\right) + \left(\frac{1+k}{k}\right)\cdot\mathcal{O}\left(\frac{1}{k+1}\right).$$

This simplifies nicely to

$$\left(\frac{(1-k)(k-1)+(1+k)(k+1)}{kx}\right) + \mathcal{O}\left(\frac{1}{k}\right).$$

We reduce this to get

$$\left(\frac{(1+k)^2-(1-k)^2}{kx}\right)+\mathcal{O}\left(\frac{1}{k}\right) \;=\; \left(\frac{4k}{kx}\right)+\mathcal{O}\left(\frac{1}{k}\right) \;=\; \frac{4}{x}+\mathcal{O}\left(\frac{1}{k}\right),$$

which, as $k\to\infty$, gives us our desired $4/x$. □

Lemma 8. *Let k and q be odd numbers. Then, if we define*

$$B_q=\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)}\Big(\cos qkx-\cos qx\Big)\,dx, \tag{21}$$

we have that

$$B_q=\frac{4}{q^2\pi}.$$

Proof. First, we integrate the right-hand side of equation (21) to get

$$B_q=\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\frac{1}{kq}\sin qkx-\frac{1}{q}\sin qx\,\Bigg|_{x=\ell 2\pi/(k+1)}^{x=\ell 2\pi/(k-1)}. \tag{22}$$

Taking out the $1/q$ and plugging in the endpoints, we get

$$B_q=\frac{1}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(\frac{1}{k}\sin\frac{qk\ell 2\pi}{k-1}-\sin\frac{q\ell 2\pi}{k-1}\right)-\left(\frac{1}{k}\sin\frac{qk\ell 2\pi}{k+1}-\sin\frac{q\ell 2\pi}{k+1}\right) \tag{23}$$

Now, if we write

$$\frac{qk\ell 2\pi}{k-1}=\frac{q(k-1+1)\ell 2\pi}{k-1}=q\ell 2\pi+\frac{q\ell 2\pi}{k-1}$$

then we see that

$$\sin\frac{qk\ell 2\pi}{k-1}=\sin\frac{q\ell 2\pi}{k-1}. \tag{24}$$

Likewise, if we write

$$\frac{qk\ell 2\pi}{k+1}=\frac{q(k+1-1)\ell 2\pi}{k-1}=q\ell 2\pi-\frac{q\ell 2\pi}{k-1}$$

then we see that

$$\sin\frac{qk\ell 2\pi}{k+1}=-\sin\frac{q\ell 2\pi}{k-1}. \tag{25}$$

By substituting equations (24) and (25) into the right-hand side of equation (23), we have that

$$B_q = \frac{1}{q} \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \left(\frac{1}{k} - 1\right) \sin\frac{q\ell 2\pi}{k-1} + \left(\frac{1}{k} + 1\right) \sin\frac{q\ell 2\pi}{k+1}. \tag{26}$$

We now distribute the sum, and change the upper limit of the second summation from $(k-1)/2$ to $(k+1)/2$, which fortunately does not change the value of the sum, to get

$$B_q = \frac{1}{q} \lim_{k\to\infty} \left(\frac{1}{k} - 1\right) \sum_{\ell=1}^{(k-1)/2} \sin\frac{q\ell 2\pi}{k-1} + \left(\frac{1}{k} + 1\right) \sum_{\ell=1}^{(k+1)/2} \sin\frac{q\ell 2\pi}{k+1}. \tag{27}$$

At this point, since k is odd, then both $k-1$ and $k+1$ are even and so we can apply Lemma 5 (with q odd) to rewrite the above equation as

$$B_q = \frac{1}{q} \lim_{k\to\infty} \left(\frac{1}{k} - 1\right) \cot\frac{q\pi}{k-1} + \left(\frac{1}{k} + 1\right) \cot\frac{q\pi}{k+1}. \tag{28}$$

We can now apply Lemma 7 with $x = q\pi$ to the above equation to get that

$$B_q = \frac{1}{q}\frac{4}{q\pi} = \frac{4}{q^2\pi},$$

as desired. $\square$

Lemma 9. *For k odd and q even, if we define*

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_{q,k}(x)\,dx - \int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} f_{q,k}(x)\,dx \tag{29}$$

with

$$f_{q,k}(x) = \cos qx - \cos qkx,$$

then we have that

$$C_q = \begin{cases} 0 & \text{for } q \equiv 0 \mod 4, \\ \dfrac{16}{q^2\pi} & \text{for } q \equiv 2 \mod 4. \end{cases}$$

Proof. If we let $F_{q,k}(x)$ be the anti-derivative of $f_{q,k}(x) = \cos qx - \cos qkx$, then equation (29) becomes

$$C_q = \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} F_{q,k}(x)\Big|_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} + F_{q,k}(x)\Big|_{\ell\pi/(k-1)}^{\ell\pi/(k+1)} \tag{30}$$

where we replaced $-F_{q,k}$ with $F_{q,k}$ and reversed the limits in the second integral. We note that almost every term in the above expression for C_q will appear twice when we plug in the endpoints and write out the sum, with the exception of $F_{q,k}(0)$ and $F_{q,k}(\pi/2)$ which will each appear once. However, since an easy calculation gives us that

$$F_{q,k}(x) = \frac{1}{q}\sin qx - \frac{1}{qk}\sin qkx, \tag{31}$$

then $F_{q,k}(0) = 0$ and since q is even then $F_{q,k}(\pi/2) = 0$ as well.

So, if we plug in the endpoints, write out the sum, and replace the $F_{q,k}(0)$ term with $F_{q,k}(\pi/2)$, them equation (30) becomes

$$C_q = 2\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2} F_{q,k}\left(\frac{\ell\pi}{k+1}\right) - F_{q,k}\left(\frac{\ell\pi}{k-1}\right). \tag{32}$$

Replacing $F_{q,k}$ with the expression in equation (31) and taking out the $1/q$ gives us

$$C_q = \frac{2}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(\sin\frac{q\ell\pi}{k+1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k+1}\right) - \left(\sin\frac{q\ell\pi}{k-1} - \frac{1}{k}\sin\frac{qk\ell\pi}{k-1}\right). \tag{33}$$

Now, if we write

$$\frac{qk\ell\pi}{k+1} = \frac{q(k+1-1)\ell\pi}{k+1} = q\ell\pi - \frac{q\ell\pi}{j+1}$$

and if we remember that q is even, then we see that

$$\sin\frac{qk\ell\pi}{k+1} = -\sin\frac{q\ell\pi}{k+1}. \tag{34}$$

Likewise, if we write

$$\frac{qk\ell\pi}{k-1} = \frac{q(k-1+1)\ell\pi}{k-1} = q\ell\pi + \frac{q\ell\pi}{k-1}$$

and again recall that q is even, then we see that

$$\sin\frac{qk\ell\pi}{k-1} = \sin\frac{\ell 2\pi}{k-1}. \tag{35}$$

By substituting equations (34) and (35) into the right-hand side of equation (33), we have that

$$C_q = \frac{2}{q}\lim_{k\to\infty}\sum_{\ell=1}^{(k-1)/2}\left(1+\frac{1}{k}\right)\sin\frac{q\ell\pi}{k+1} - \left(1-\frac{1}{k}\right)\sin\frac{q\ell\pi}{k-1}. \tag{36}$$

We now distribute the sum, factor through the negative in the second expression, and change the upper limit of the first summation from $(k-1)/2$ to $(k+1)/2$, which fortunately does not change the value of the sum, to get

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\frac{1}{k}+1\right) \sum_{\ell=1}^{(k+1)/2} \sin\frac{q\ell\pi}{k+1} + \left(\frac{1}{k}-1\right) \sum_{\ell=1}^{(k-1)/2} \sin\frac{q\ell\pi}{k-1}. \tag{37}$$

At this point, since k is odd, then both $k+1$ and $k-1$ are even and so we can apply Lemma 6 (with q even). If $q/2$ is even, then Lemma 6 tells us that both the above sums are zero and so $C_q = 0$ in this case. If $q/2$ is odd, we apply Lemma 6 to tell us that

$$C_q = \frac{2}{q} \lim_{k\to\infty} \left(\frac{1}{k}+1\right) \cot\frac{q\pi}{2(k+1)} + \left(\frac{1}{k}-1\right) \cot\frac{q\pi}{2(k-1)} \qquad \text{for } q/2 \text{ odd}. \tag{38}$$

We can now apply Lemma 7 with $x = q\pi/2$ to the above equation to get that

$$C_q = \frac{2}{q}\frac{4}{q\pi/2} = \frac{16}{q^2\pi} \qquad \text{for } q/2 \text{ odd},$$

as desired. ☐

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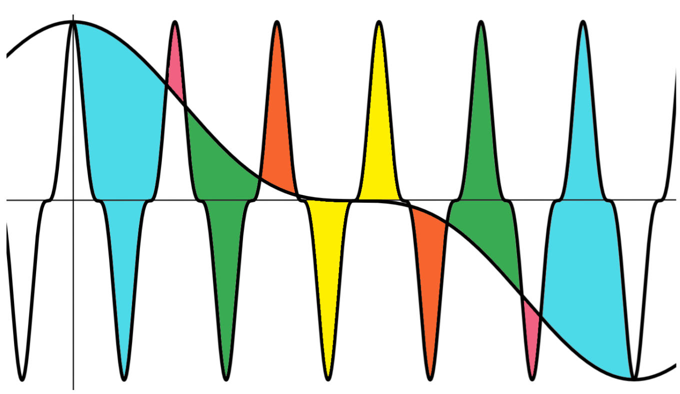
6 Proof of Theorem 1

Proof of Theorem 1. We begin with the area between $\cos^n x$ and $\cos^n kx$ for n odd. As seen in this picture with $n = 3$ and $k = 11$, there is odd symmetry across the midpoint $x = \pi/2$ and so each region "below" $\cos^3 x$ (in color) has an equivalent area "above" $\cos^3 x$ (in a matching color).



In other words, we can just find the areas "above" $\cos^n x$ on the interval $[0, \pi]$ and then double them. To do so, we first need to find the intersection points. Since n is odd, then to find the the solutions to $\cos^n x = \cos^n kx$ we take the nth root of both sides and rewrite it to get $\cos x - \cos kx = 0$, and we then use a trig identity to write that as

$$\sin \frac{(k+1)x}{2} \cdot \sin \frac{(k-1)x}{2} \;=\; 0.$$

This has solutions $x = \ell \cdot 2\pi/(k+1)$ and $x = \ell \cdot 2\pi/(k-1)$ for ℓ any integer, and we note that we can order these as follows:

$$\begin{aligned} 0 \quad < \quad \frac{1 \cdot 2\pi}{k+1} < \frac{1 \cdot 2\pi}{k-1} \quad < \quad \frac{2 \cdot 2\pi}{k+1} < \frac{2 \cdot 2\pi}{k-1} \quad < \quad \cdots \\ \cdots \quad < \quad \frac{\ell \cdot 2\pi}{k+1} < \frac{\ell \cdot 2\pi}{k-1} \quad < \quad \frac{(\ell+1) \cdot 2\pi}{k+1} < \frac{(\ell+1) \cdot 2\pi}{k-1} \quad < \quad \cdots \\ \cdots \quad < \quad \frac{(k-1)/2 \cdot 2\pi}{k+1} < \frac{(k-1)/2 \cdot 2\pi}{k-1} = \pi, \end{aligned}$$

and in particular we have that

$$\frac{\ell \cdot 2\pi}{k-1} \quad < \quad \frac{(\ell+1) \cdot 2\pi}{k+1} \qquad \text{so long as } \ell < (k-1)/2.$$

With these intersection points, we have the following formula for the total area which takes just the "upper" regions and doubles them:

$$2 \sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \Big(\cos^n kx - \cos^n x \Big)\, dx. \tag{39}$$

We now use the power-reduction formula for cosine to an odd power n,

$$\cos^n \theta = \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \cos(n-2j)\theta,$$

and when we substitute this into equation (39), twice, we get the following expression for the area:

$$2 \sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \frac{2}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \Big(\cos(n-2j)kx - \cos(n-2j)x \Big)\, dx. \tag{40}$$

Of course, we want the limit of the expression in (40) as k goes to infinity, so when we do this, and re-arrange the sums and integrals and such, we get

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \lim_{k\to\infty} \sum_{\ell=1}^{(k-1)/2} \int_{\ell 2\pi/(k+1)}^{\ell 2\pi/(k-1)} \Big(\cos(n-2j)kx - \cos(n-2j)x \Big)\, dx. \tag{41}$$

We now recognize the limit in the right-hand side of equation (41) as being the same as in Lemma 8. In other words, we have that

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} B_{n-2j}, \tag{42}$$

and thanks to Lemma 8, this becomes

$$A_n = \frac{4}{2^n} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{4}{(n-2j)^2\pi} \tag{43}$$

$$= \frac{8}{\pi} \cdot \frac{1}{2^{n-1}} \cdot \sum_{j=0}^{(n-1)/2} \binom{n}{j} \frac{1}{(n-2j)^2}, \tag{44}$$

as desired.

The area between $\cos^n x$ and $\cos^n kx$ for k even is quite similar and leads to the same formula as seen in equation (44); we leave the details to the reader. □

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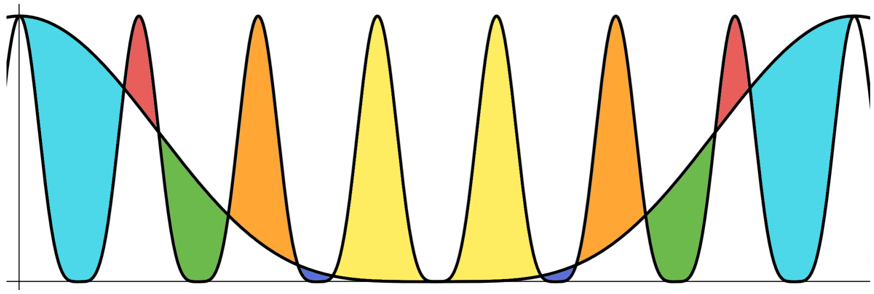
7 Proof of Theorem 2

Proof of Theorem 2. We begin with the area between $\cos^n x$ and $\cos^n kx$ for k odd. As seen in this picture with $n = 4$ and $k = 7$, there is even symmetry across the midpoint $x = \pi/2$ and so each region on the left of $x = \pi/2$ (in color) has an equivalent area on the right of $w = \pi/2$ (in a matching color).



In other words, we can just find the areas from 0 to $\pi/2$ and double them. To do so, we first need to find the intersection points. If we set

$$\cos^n x = \cos^n kx \tag{45}$$

and take the nth root of both sides, then since n is even we will get

$$\cos x = \pm \cos kx,$$

which becomes two equations,

$$\cos x - \cos kx = 0 \qquad \text{and} \qquad \cos x + \cos kx = 0.$$

Using two familiar trig identities, these become

$$\sin \frac{(k+1)x}{2} \sin \frac{(k-1)x}{2} = 0 \qquad \text{and} \qquad \cos \frac{(k+1)x}{2} \cos \frac{(k-1)x}{2} = 0$$

The first equation has solutions $x = 0$, and also $x = 2\pi/(k+1)$ and $x = 2\pi/(k-1)$, and also $x = 4\pi/(k+1)$ and $x = 4\pi/(k-1)$, and so on. The second equation has solutions $x = \pi/(k+1)$ and $x = \pi/(k-1)$, and also $x = 3\pi/(k+1)$ and $x = 3\pi/(k-1)$, and so on. Hence, the complete list of solutions to equation (45) in the interval $[0, \pi/2]$, written out in order, is

$$0 < \frac{\pi}{k+1} < \frac{\pi}{k-1} < \frac{2\pi}{k+1} < \frac{2\pi}{k-1} < \frac{3\pi}{k+1} < \frac{3\pi}{k-1} < \cdots$$
$$\cdots < \frac{(k-1)/2 \cdot \pi}{k+1} < \frac{(k-1)/2 \cdot \pi}{k-1} = \frac{\pi}{2}.$$

With these intersection points, we have the following formula for the total area (for k any fixed odd number) which takes just the regions on the right of $x = \pi/2$ and doubles them:

$$2\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)} f_n(x)\,dx+\int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)} -f_n(x)\,dx, \tag{46}$$

where $f_n(x)=\cos^n x-\cos^n kx$.

We now use the power-reduction formula for cosine to an even power n,

$$\cos^n\theta=\frac{1}{2^n}\binom{n}{n/2}+\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\cos(n-2j)\theta,$$

to give us that

$$f_n(x)=\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x-\cos(n-2j)kx\Big).$$

When we substitute this into equation (46) twice, and distribute the outer sum, we get the following expression for the area:

$$\begin{aligned}2\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)}\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x-\cos(n-2j)kx\Big)\,dx\\
-2\sum_{\ell=1}^{(k-1)/2}\int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)}\frac{2}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\Big(\cos(n-2j)x-\cos(n-2j)kx\Big)\,dx.\end{aligned} \tag{47}$$

Of course, we want the limit of the expression in (47) as k goes to infinity, so when we do this, and re-arrange the sums and integrals and such, we get

$$\begin{aligned}A_n=\frac{4}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}\lim_{k\to\infty}\Bigg(\sum_{\ell=1}^{(k-1)/2}\int_{(\ell-1)\pi/(k-1)}^{\ell\pi/(k+1)}\Big(\cos(n-2j)x-\cos(n-2j)kx\Big)\,dx\\
-\sum_{\ell=1}^{(k-1)/2}\int_{\ell\pi/(k+1)}^{\ell\pi/(k-1)}\Big(\cos(n-2j)x-\cos(n-2j)kx\Big)\,dx\Bigg)\end{aligned} \tag{48}$$

We now recognize the limit in the right-hand side of equation (48) as being the same as in Lemma 9. In other words, we now have that

$$A_n=\frac{4}{2^n}\sum_{j=0}^{(n/2)-1}\binom{n}{j}C_{n-2j} \tag{49}$$

where C_{n-2j} from Lemma (9) is defined as

$$C_{n-2j} = \begin{cases} 0 & \text{for } n-2j \equiv 0 \bmod 4, \\ \dfrac{16}{(n-2j)^2\pi} & \text{for } n-2j \equiv 2 \bmod 4. \end{cases}$$

We now consider the case when $n \equiv 2 \bmod 4$. In this case, if we write out the terms in equation (49) and use our definition of C_{n-2j} from above, we have only the terms with j even (as that is when $n - 2j \equiv 2 \bmod 4$), giving us

$$A_n = \frac{4}{2^n}\left(\binom{n}{0}\frac{16}{(n)^2\pi} + \binom{n}{2}\frac{16}{(n-4)^2\pi} + \binom{n}{4}\frac{16}{(n-8)^2\pi} + \cdots + \binom{n}{(n/2)-1}\frac{16}{(2)^2\pi}\right)$$

We now factor out $16/(2^2\pi)$ from each term, giving us

$$A_n = \frac{4}{2^n}\frac{16}{2^2\pi}\left(\binom{n}{0}\frac{1}{(n/2)^2} + \binom{n}{2}\frac{1}{(n/2-2)^2} + \binom{n}{4}\frac{1}{(n/2-4)^2} + \cdots + \binom{n}{(n/2)-1}\frac{1}{(1)^2}\right)$$

We re-index the above sum, and simplify the coefficients on the left, to get

$$A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-2)/4} \binom{n}{2j} \frac{1}{(n/2-2j)^2},$$

as desired (for $n \equiv 2 \bmod 4$).

Finally, for $n \equiv 0 \bmod 4$, we again write out the terms in equation (49) and use our definition of C_{n-2j} from above. This time, the only non-zero contributions come from j odd (as this is when $n - 2j \equiv 2 \bmod 4$), giving us

$$A_n = \frac{4}{2^n}\left(\binom{n}{1}\frac{16}{(n-2)^2\pi} + \binom{n}{3}\frac{16}{(n-6)^2\pi} + \binom{n}{5}\frac{16}{(n-10)^2\pi} + \cdots + \binom{n}{(n/2)-1}\frac{16}{(2)^2\pi}\right)$$

We again factor out $16/(2^2\pi)$ from each term, giving us

$$A_n = \frac{4}{2^n}\frac{16}{2^2\pi}\left(\binom{n}{1}\frac{1}{(n/2-1)^2} + \binom{n}{3}\frac{1}{(n/2-3)^2} + \binom{n}{5}\frac{1}{(n/2-5)^2} + \cdots + \binom{n}{(n/2)-1}\frac{1}{(1)^2\pi}\right)$$

We re-index the above sum, and simplify the coefficients on the left, to get

$$A_n = \frac{16}{\pi} \cdot \frac{1}{2^n} \cdot \sum_{j=0}^{(n-4)/4} \binom{n}{2j+1} \frac{1}{(n/2-(2j+1))^2},$$

as desired (for $n \equiv 0 \bmod 4$).

The area between $\cos^n x$ and $\cos^n kx$ for k even is quite similar and leads to the same formulas as seen above; we leave the details to the reader. $\square$

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