

# Approximation and FPT Algorithms for Finding DM-Irreducible Spanning Subgraphs

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## Abstract

Finding a minimum strongly connected spanning subgraph of a given directed graph generalizes the well-known strong connectivity augmentation problem, and it is NP-hard. For the weighted problem, a simple 2-approximation algorithm was proposed by Frederickson and Jájá (1981); surprisingly, it still achieves the best known approximation ratio in general. Also, the unweighted problem was shown to be FPT by Bang-Jensen and Yeo (2008), where the parameter is the difference from the trivial upper bound of the optimal value. In this paper, we consider a generalized problem related to the Dulmage–Mendelsohn decompositions of bipartite graphs instead of the strong connectivity of directed graphs, and extend the above approximation and FPT results to this setting.

**Keywords** Connectivity augmentation, Bipartite matching, Matroid intersection, Approximation algorithm, Fixed-parameter tractability.

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# 1 Introduction

The *Dulmage–Mendelsohn decomposition* (the *DM-decomposition*) [4, 5] of a bipartite graph is a partition of the vertex set that reflects the structure of all maximum matchings in that graph. A bipartite graph is *DM-irreducible* if its DM-decomposition consists of a single component. This property is closed under adding edges, like usual connectivity properties. In this paper, we consider a natural optimization problem, called the DM-irreducible spanning subgraph problem (DMISS), as follows: given a DM-irreducible bipartite graph (with edge weight), find a minimum (or minimum-weight) spanning subgraph that is DM-irreducible.

Bérczi, Iwata, Kato, and Yamaguchi [2] investigated the special cases of an augmentation setting, which is equivalent to DMISS in general. A bipartite graph  $G = (V^+, V^-; E)$  is said to be *balanced* if  $|V^+| = |V^-|$  and *unbalanced* otherwise. They proposed a polynomial-time algorithm for the case when the input bipartite graph is unbalanced with the aid of weighted matroid intersection algorithms. They also show that even when  $G$  is balanced, if  $G$  is a complete bipartite graph and the weight of each edge is 0 or 1, then it can be solved in polynomial time.

In general, DMISS for the balanced bipartite graphs includes the strongly connected spanning subgraph problem (SCSS) as follows: given a strongly connected directed graph (with edge weight), find a minimum (or minimum-weight) spanning subgraph that is strongly connected. This problem generalizes the strong connectivity augmentation problem and is known to be NP-hard [7]. For SCSS (with edge weight), Frederickson and Jájá [8] gave a 2-approximation algorithm; this is based on a very simple observation (see Section 3.1) and, surprisingly, it still achieves the best known approximation ratio in general. For the unweighted case, several algorithms with better approximation ratios [11, 12, 17, 18] have been proposed. Also, Bang-Jensen and Yeo [1] showed that the unweighted case is FPT if it is parameterized by the difference from the trivial upper bound of the optimal value (see Sections 2 and 4.1); here, the problem is called *FPT (fixed-parameter tractable)* if it can be solved in  $f(k) \cdot n^{O(1)}$  time, where  $k$  and  $f$  are prescribed parameter and computable function, respectively, and  $n$  is the input size.

In this paper, by extending the approaches to SCSS by Frederickson and Jájá [8] and by Bang-Jensen and Yeo [1], we propose a 2-approximation algorithm for DMISS in general and an FPT algorithm for the unweighted case of DMISS.

The rest of this paper is organized as follows. In Section 2, we describe necessary definitions, and formally state the problems and our results. In Section 3, we give a 2-approximation algorithm for DMISS. In Section 4, we prove the unweighted case of DMISS is FPT parameterized by the difference from the trivial upper bound of the optimal value.

## 2 Preliminaries

### 2.1 Definitions

We refer the readers to [16] for basic concepts and notation on graphs. In this paper, a *graph* means a directed graph or an undirected graph. We call a directed graph simply as a *digraph*.

For a graph  $G$ , we denote by  $V(G)$  and  $E(G)$  the vertex set and the edge set of  $G$ , respectively. A *subgraph* of  $G$  is a graph  $G'$  such that  $V(G') \subseteq V(G)$ ,  $E(G') \subseteq E(G)$ , and each edge  $e \in E(G')$  is between two vertices in  $V(G')$ . A subgraph  $G'$  of  $G$  is *induced by* a vertex set  $X \subseteq V(G)$ , denoted by  $G[X]$ , if  $V(G') = X$  and  $E(G')$  is the set of edges in  $E(G)$  whose end vertices are both in  $X$ .

We also define  $G - X := G[V \setminus X]$  for  $X \subseteq V$ . A subgraph  $G'$  of  $G$  is *spanning* if  $V(G') = V(G)$ . We often do not distinguish a spanning subgraph  $G'$  and its edge set  $E(G')$ .

A *stable set* in an undirected graph  $G$  is a vertex set  $X \subseteq V(G)$  such that no edge in  $E(G)$  has its both end vertices in  $X$ . An undirected graph  $G = (V, E)$  is *bipartite* if there exists a *bipartition*  $(V^+, V^-)$  of  $V$  such that  $V^+ \cap V^- = \emptyset$ ,  $V^+ \cup V^- = V$ , and both  $V^+$  and  $V^-$  are stable sets in  $G$  (possibly  $V^+$  or  $V^-$  is empty). To specify the bipartition, we denote by  $G = (V^+, V^-; E)$ . For a vertex set  $X$  on one side ( $V^+$  or  $V^-$ ), let  $\Gamma_G(X)$  denote the *neighbor* of  $X$ , i.e., the set of vertices adjacent to some vertex in  $X$ . A bipartite graph  $G = (V^+, V^-; E)$  is *balanced* if  $|V^+| = |V^-|$ .

For a digraph  $G = (V, E)$ , the *underlying graph* is an undirected graph  $\overline{G} = (V, \overline{E})$  defined as follows. For each directed edge  $e = (u, v) \in E$ , we denote by  $\overline{e} = \{u, v\}$  the corresponding undirected edge. We then define  $\overline{F} := \{\overline{e} \mid e \in F\}$  for each  $F \subseteq E$ .

For each edge  $e = \{u, v\}$  in a bipartite graph  $G = (V^+, V^-; E)$  such that  $u \in V^+$  and  $v \in V^-$ , we denote its orientations by  $\overrightarrow{e} = (u, v)$  and  $\overleftarrow{e} = (v, u)$ . Similarly, for each  $F \subseteq E$ , we define  $\overrightarrow{F}$  and  $\overleftarrow{F}$  as the sets of the orientations  $\overrightarrow{e}$  and  $\overleftarrow{e}$ , respectively, of the edges  $e \in F$ .

An undirected graph is said to be *connected* if for each pair of vertices, there exists a path between them. A digraph is said to be *strongly connected* if the same condition (where a path means a directed path) holds, and just *connected* if the underlying graph is connected.

For an undirected graph  $G$  and a vertex  $v \in V(G)$ , we denote by  $\deg_G(v)$  the *degree* of  $v$ , which is the number of edges incident to  $v$ . Similarly, for a directed graph  $G$  and a vertex  $v \in V(G)$ , we denote by  $\deg_G^{\text{in}}(v)$  and  $\deg_G^{\text{out}}(v)$  the *in-degree* and *out-degree* of  $v$ , respectively, which are the numbers of edges entering and leaving  $v$ . An *in-arborescence* (or *out-arborescence*) is a connected digraph in which exactly one vertex, called *the root*, has out-degree (or in-degree, respectively) 0 and all other vertices have out-degree (or in-degree, respectively) 1. An in- or out-arborescence having its root  $r$  is said to be *rooted at  $r$* .

A *matching* in an undirected graph  $G$  is an edge subset in which no two edges share a vertex. If a matching covers all the vertices in  $G$ , it is called a *perfect matching*. For a perfect matching  $M$  in a balanced bipartite graph  $G = (V^+, V^-; E)$ , the *auxiliary graph* of  $G$  with respect to  $M$  is a directed graph  $G_M := (V; \overrightarrow{E} \cup \overleftarrow{M})$ .

While the *DM-decomposition* [4, 5] of a bipartite graph is formally defined by the distributive lattice formed by the minimizers of a submodular function and a bipartite graph is called *DM-irreducible* if its DM-decomposition consists of a single component, we here employ its characterization as an alternative “definition” (cf. [2]).

**Lemma 2.1.** *For a balanced bipartite graph  $G$ , the following statements are equivalent.*

1.  $G$  is DM-irreducible.
2. For some perfect matching  $M$  in  $G$ ,  $G_M$  is strongly connected.
3.  $G$  has a perfect matching, and for any perfect matching  $M$  in  $G$ ,  $G_M$  is strongly connected.

Suppose that a graph  $G$  is associated with edge weight  $w: E \rightarrow \mathbb{R}$ . We define the *weight* of each edge set  $F \subseteq E$  as  $w(F) := \sum_{e \in F} w(e)$ .

## 2.2 Problems

We formally state the problems dealt with in this paper. Recall that DMIS for the unbalanced bipartite graphs can be solved in polynomial time [2], which is not considered in this paper.

**Problem** (STRONGLY CONNECTED SPANNING SUBGRAPH (SCSS)).

**Input:** A strongly connected digraph  $G = (V, E)$  with edge weight  $w: E \rightarrow \mathbb{R}_{\geq 0}$ .

**Goal:** Find a minimum-weight strongly connected spanning subgraph of  $G$ .

**Problem** (DM-IRREDUCIBLE SPANNING SUBGRAPH (DMISS)).

**Input:** A DM-irreducible balanced bipartite graph  $G = (V^+, V^-; E)$  with edge weight  $w: E \rightarrow \mathbb{R}_{\geq 0}$ .

**Goal:** Find a minimum-weight DM-irreducible spanning subgraph of  $G$ .

It is easy to observe that any inclusion-wise minimal strongly connected spanning subgraph of a digraph on  $n$  vertices consists of at most  $2n - 2$  edges (by Lemma 3.1 or 4.1). Similarly, it is easy to observe that any inclusion-wise minimal DM-irreducible spanning subgraph of a balanced bipartite graph on  $2n$  vertices ( $n$  vertices on each side) consists of at most  $3n - 2$  edges (by Lemma 3.2 or 4.2). The following problems are the unweighted version of the above two problems parameterized by the difference from these trivial upper bounds (note that it is obvious from the definition that  $|V|$  edges are necessary in either problem).

**Problem** (UNWEIGHTEDSCSS).

**Input:** A strongly connected digraph  $G = (V, E)$ .

**Parameter:** An integer  $k \leq |V| - 2$ .

**Goal:** Test whether  $G$  has a strongly connected spanning subgraph with at most  $2|V| - 2 - k$  edges or not.

**Problem** (UNWEIGHTEDDMISS).

**Input:** A DM-irreducible balanced bipartite graph  $G = (V^+, V^-; E)$ .

**Parameter:** An integer  $k \leq |V^+| - 2$ .

**Goal:** Test whether  $G$  has a DM-irreducible spanning subgraph with at most  $3|V^+| - 2 - k$  edges or not.

## 2.3 Results

The results shown in this paper are summarized as follows.

**Theorem 2.2.** *There exists a 2-approximation algorithm for DMISS that runs in  $O(n^3)$  time.*

**Theorem 2.3.** *There exists an FPT algorithm for UNWEIGHTEDDMISS.*

## 3 Approximation Algorithms for DMISS

In this section, we show Theorem 2.2 by presenting a 2-approximation algorithm for DMISS that runs in  $O(n^3)$  time.

In Section 3.1, as a base of our result, we describe a 2-approximation algorithm for SCSS by Frederickson and Jájá [8]. In Section 3.2, we show a 3-approximation algorithm for DMISS obtained by a naive extension of the result of Frederickson and Jájá. In Section 3.3, we improve it to a 2-approximation algorithm with the aid of matroid intersection.

### 3.1 2-approximation Algorithm for SCSS

The algorithm is based on the following characterization of strongly connected digraphs.

**Lemma 3.1.** *For a digraph  $G$ , the following statements are equivalent.*

1.  $G$  is strongly connected.
2. For some vertex  $r$ ,  $G$  has a spanning in-arborescence and out-arborescence rooted at  $r$ .
3. For any vertex  $r$ ,  $G$  has a spanning in-arborescence and out-arborescence rooted at  $r$ .

For the input  $(G = (V, E), w)$  of SCSS, the algorithm arbitrarily chooses a vertex  $r \in V$  and finds a minimum-weight spanning in-arborescence  $A_{\text{in}}$  and out-arborescence  $A_{\text{out}}$  rooted at  $r$ ; the existence of  $A_{\text{in}}$  and  $A_{\text{out}}$  follows from Lemma 3.1 (1  $\Rightarrow$  3). This can be done in  $O(m + n \log n)$  time [9], where  $n = |V|$  and  $m = |E|$ . By Lemma 3.1 (2  $\Rightarrow$  1), the union of  $A_{\text{in}}$  and  $A_{\text{out}}$  results in a strongly connected spanning subgraph of  $G$ , and the algorithm outputs this. Since any optimal solution has a spanning in-arborescence and out-arborescence rooted at  $r$  by Lemma 3.1 (1  $\Rightarrow$  3) and their weights are at least those of  $A_{\text{in}}$  and  $A_{\text{out}}$ , respectively, the weight of the output solution is at most twice of the optimal value.

### 3.2 3-approximation Algorithm for DMISS

A naive extension of the above result leads to a 3-approximation algorithm for DMISS. First, Lemma 3.1 is straightforwardly extended as follows (recall Lemma 2.1).

**Lemma 3.2.** *For a balanced bipartite graph  $G$ , the following statements are equivalent.*

1.  $G$  is DM-irreducible.
2. For some perfect matching  $M$  and some vertex  $r$ ,  $G_M$  has a spanning in-arborescence and out-arborescence rooted at  $r$ .
3.  $G$  has a perfect matching, and for any perfect matching  $M$  and any vertex  $r$ ,  $G_M$  has a spanning in-arborescence and out-arborescence rooted at  $r$ .

To achieve 3-approximation for DMISS, we simply divide the problem into two tasks: finding a minimum-weight perfect matching  $M$  in  $(G, w)$  and solving SCSS with the input  $(G_M, w_M)$ , where  $w_M: E(G_M) \rightarrow \mathbb{R}_{\geq 0}$  is defined as

$$w_M(e) := \begin{cases} w(\bar{e}) & (\bar{e} \notin M), \\ 0 & (\bar{e} \in M). \end{cases} \quad (1)$$

For the former part, polynomial-time exact algorithms initiated by the Hungarian method [15] are well-known. For the latter part, we apply the 2-approximation algorithm described in the previous section. We then output the union of the solutions (for the latter solution, we take the set of corresponding edges in  $G$ ). This intuitively leads to a 3-approximation algorithm for DMISS (as  $1 + 2 = 3$ ), which can be formally shown as follows.

**Lemma 3.3.** *There exists a polynomial-time 3-approximation algorithm for DMISS.*

*Proof.* Let  $M$  be a minimum-weight perfect matching in  $(G, w)$  (obtained in the former part), and  $A_{\text{in}}$  and  $A_{\text{out}}$  be a minimum-weight spanning in-arborescence and out-arborescence in  $(G_M, w_M)$  (obtained in the latter part), respectively. Also, let  $S_{\text{opt}}$  be an optimal solution and  $M_{\text{opt}}$  be a perfect matching in  $S_{\text{opt}}$ . We then have  $w(M) \leq w(M_{\text{opt}}) \leq w(S_{\text{opt}})$ .

Consider a subgraph  $G' = (V^+, V^-; S_{\text{opt}} \cup M)$  of  $G$ , which is DM-irreducible. By Lemma 3.2, the auxiliary graph  $G'_M$  has a spanning in-arborescence and out-arborescence rooted at any vertex. Let  $r$  be the root of  $A_{\text{in}}$  and  $A_{\text{out}}$ , and  $A'_{\text{in}}$  and  $A'_{\text{out}}$  be any spanning in-arborescence and out-arborescence of  $G'_M$  rooted at  $r$ , respectively. We then have

$$\begin{aligned}
w((\overline{A_{\text{in}}} \cup \overline{A_{\text{out}}}) \setminus M) &= w_M(A_{\text{in}} \cup A_{\text{out}}) && \text{(definition of } w_M) \\
&\leq w_M(A_{\text{in}}) + w_M(A_{\text{out}}) && (w_M \text{ is nonnegative}) \\
&\leq w_M(A'_{\text{in}}) + w_M(A'_{\text{out}}) && (G'_M \text{ is a subgraph of } G_M) \\
&= w(\overline{A'_{\text{in}}} \setminus M) + w(\overline{A'_{\text{out}}} \setminus M) && \text{(definition of } w_M) \\
&\leq 2w(S_{\text{opt}}) && (E(G') \setminus M \subseteq S_{\text{opt}}).
\end{aligned}$$

Thus, the weight of the output solution is

$$w(M \cup \overline{A_{\text{in}}} \cup \overline{A_{\text{out}}}) = w(M) + w((\overline{A_{\text{in}}} \cup \overline{A_{\text{out}}}) \setminus M) \leq 3w(S_{\text{opt}}). \quad \square$$

### 3.3 2-approximation Algorithm for DMISS

We improve the above 3-approximation algorithm with the aid of matroid intersection.

We first give an easy but important observation. In short, all the matching edges are included in any spanning in-arborescence in the auxiliary graph.

**Lemma 3.4.** *Let  $G = (V^+, V^-; E)$  be a DM-irreducible balanced bipartite graph,  $M$  be a perfect matching in  $G$ , and  $r \in V^+$ . Then, any spanning in-arborescence of  $G_M$  rooted at  $r$  contains all the edges in  $\overleftarrow{M}$ .*

*Proof.* Let  $A_{\text{in}}$  be a spanning in-arborescence of  $G_M$  rooted at  $r$ . Then, for any vertex  $v \in V^-$ , we have  $\deg_{A_{\text{in}}}^{\text{out}}(v) = 1$ . In  $G_M = (V^+, V^-; \overrightarrow{E} \cup \overleftarrow{M})$ , exactly one edge leaves each vertex in  $V^-$ , which is in  $\overleftarrow{M}$ . Thus,  $A_{\text{in}}$  must contain all the edges in  $\overleftarrow{M}$ .  $\square$

In the previous algorithm, we essentially solve three problems: finding a minimum-weight perfect matching, a minimum-weight spanning in-arborescence, and a minimum-weight spanning out-arborescence. Based on Lemma 3.4, we merge the first two into a single task as follows: finding a minimum-weight spanning tree  $T$  in  $(G, w)$  among those having a perfect matching  $M$  such that  $\overrightarrow{T} \setminus \overleftarrow{M} \cup \overleftarrow{M}$  is a spanning in-arborescence of  $G_M$ .

The main observation is the following characterization of such spanning trees.

**Lemma 3.5.** *For a spanning tree  $T$  of a balanced bipartite graph  $G$ , the following statements are equivalent.*

1.  $T$  has a perfect matching  $M$  in  $G$  and  $\overrightarrow{T} \setminus \overleftarrow{M} \cup \overleftarrow{M}$  is a spanning in-arborescence of  $G_M$ .
2. There exists a vertex  $r \in V^+$  such that  $\deg_T(r) = 1$  and  $\deg_T(v) = 2$  for every  $v \in V^+ \setminus \{r\}$ .

*Proof.* (1  $\Rightarrow$  2) Let  $A_{\text{in}} = \overrightarrow{T \setminus \overleftarrow{M}} \cup \overleftarrow{M}$ . Since  $M$  is a perfect matching in  $G$ ,  $\deg_{A_{\text{in}}}^{\text{in}}(v) = 1$  for every  $v \in V^+$  and  $\deg_{A_{\text{in}}}^{\text{out}}(v) = 1$  for every  $v \in V^-$ . Also, since  $A_{\text{in}}$  is a spanning in-arborescence,  $\deg_{A_{\text{in}}}^{\text{out}}(v) = 1$  for every  $v \in V^+ \setminus \{r\}$  and  $\deg_{A_{\text{in}}}^{\text{out}}(r) = 0$ , where  $r$  is the root of  $A_{\text{in}}$ . As  $\deg_T(v) = \deg_{A_{\text{in}}}^{\text{in}}(v) + \deg_{A_{\text{in}}}^{\text{out}}(v)$  for every vertex  $v$ , we are done.

(2  $\Rightarrow$  1) We first show that  $T$  has a perfect matching. By Hall's theorem [10], it suffices to show that for any vertex set  $X^+ \subseteq V^+$ , we have  $|X^+| \leq |\Gamma_T(X^+)|$ . Suppose to the contrary that there exists a vertex set  $X^+ \subseteq V^+$  such that  $|X^+| > |\Gamma_T(X^+)|$ . Then, the subgraph formed by the edges in  $T$  intersecting  $X^+$  has at most  $2|X^+| - 1$  vertices and at least  $2|X^+| - 1$  edges (due to the degree condition), which must contain a cycle, a contradiction.

Let  $M$  be the perfect matching in  $T$  (note that it is unique). We next show that  $A_{\text{in}} = \overrightarrow{T \setminus \overleftarrow{M}} \cup \overleftarrow{M}$  is an in-arborescence. As  $T$  is a tree, it suffices to show that  $\deg_{A_{\text{in}}}^{\text{out}}(v) = 1$  for any vertex  $v \in V \setminus \{r\}$ , where  $r \in V^+$  is the vertex in Statement 2. By definition, every vertex  $v \in V^-$  has a unique outgoing edge, which is in  $\overleftarrow{M}$ , and hence  $\deg_{A_{\text{in}}}^{\text{out}}(v) = 1$ . Also, every vertex  $v \in V^+$  has a unique incoming edge, which is in  $\overleftarrow{M}$ , and hence  $\deg_{A_{\text{in}}}^{\text{in}}(v) = 1$ . Thus, for any vertex  $v \in V^+ \setminus \{r\}$ , we have  $\deg_{A_{\text{in}}}^{\text{out}}(v) = \deg_T(v) - \deg_{A_{\text{in}}}^{\text{in}}(v) = 2 - 1 = 1$ , and we are done.  $\square$

Let us call a spanning tree satisfying Statement 2 in Lemma 3.5 *strongly balanced*, and say that the vertex  $r$  is the *root* and the tree is *rooted at  $r$* . Any DM-irreducible graph has a strongly balanced spanning tree by Lemma 3.2, and then we formulate the first task as follows.

**Problem** (STRONGLY BALANCED SPANNING TREE (SBST)).

**Input:** A DM-irreducible bipartite graph  $G = (V^+, V^-; E)$  with edge weight  $w: E \rightarrow \mathbb{R}_{\geq 0}$ .

**Goal:** Find a minimum-weight strongly balanced spanning tree of  $G$ .

For the sake of simplicity of the following discussion, we consider the following problem, which is polynomially equivalent and actually enough for our purpose.

**Problem** (STRONGLY BALANCED SPANNING TREE WITH ROOT (SBSTR)).

**Input:** A DM-irreducible bipartite graph  $G = (V^+, V^-; E)$  with edge weight  $w: E \rightarrow \mathbb{R}_{\geq 0}$ , and  $r \in V^+$ .

**Goal:** Find a minimum-weight strongly balanced spanning tree of  $G$  rooted at  $r$ .

Our proposed algorithm for DMISS is described as follows.

**Step 1.** Choose an arbitrary vertex  $r \in V^+$  as a root, and solve SBSTR with the input  $(G, w, r)$ . Let  $T$  be the obtained solution, and  $M$  be the perfect matching in  $T$ .

**Step 2.** Find a minimum-weight spanning out-arborescence  $A_{\text{out}}$  in  $(G_M, w_M)$  rooted at  $r$  (recall that  $w_M$  is defined as (1)).

**Step 3.** Output  $T \cup \overrightarrow{A_{\text{out}}}$ .

The output is indeed a DM-irreducible spanning subgraph (by Lemma 3.2). To complete the proof of Theorem 2.2, we analyze the computational time and the approximation ratio.

## Computational Time

As we already mentioned, the latter part (finding a minimum-weight spanning arborescence) can be done in  $O(m + n \log n)$  time [9]. The following observation leads to an  $O(n^3)$ -time algorithm for the former part (SBSTR), which completes the analysis. We omit the basics on matroids, for which we refer the readers to [16].

**Lemma 3.6.** *SBST and SBSTR are special cases of the weighted matroid intersection problem, i.e., the set of feasible solutions can be represented as the set of common bases of two matroids.*

*Proof.* Let  $\mathbf{M}_1 = (E, \mathcal{I}_1)$  be the cycle matroid of  $G$ , i.e.,

$$\mathcal{I}_1 = \{F \subseteq E \mid F \text{ forms a forest}\},$$

and  $\mathbf{M}_2 = (E, \mathcal{I}_2)$  be a partition matroid defined by

$$\mathcal{I}_2 = \{F \subseteq E \mid \deg_F(v) \leq 2 \ (\forall v \in V^+)\}.$$

Note that  $V^+$  is a stable set in  $G$  by definition.

In SBST, the set of feasible solutions is represented as the set of common bases of  $\mathbf{M}_1$  and the truncation of  $\mathbf{M}_2$  by  $2|V^+| - 1$ , whose independent set family is  $\{F \in \mathcal{I}_2 \mid |F| \leq 2|V^+| - 1\}$ . In SBSTR, the set of feasible solutions is represented as the set of common bases of  $\mathbf{M}_1$  and a slightly modified partition matroid  $\mathbf{M}'_2 = (E, \mathcal{I}'_2)$  such that

$$\mathcal{I}'_2 = \{F \subseteq E \mid \deg_F(r) \leq 1 \text{ and } \deg_F(v) \leq 2 \ (\forall v \in V^+ \setminus \{r\})\}. \quad \square$$

Lemma 3.6 implies that SBST and SBSTR can be solved in polynomial time with the aid of matroid intersection algorithms. In particular, Brezovec, Cornuéjols, and Glover [3] proposed an  $O(nm + kn^2)$ -time algorithm for the special case when the two matroids are a cycle matroid of a graph and a partition matroid whose partition classes are induced by a stable set of the graph (i.e., each partition class is the set of edges incident to a vertex in the stable set), where  $n$  and  $m$  are the numbers of vertices and of edges in the input graph, respectively, and  $k$  is the size of the stable set inducing the partition classes. By the proof of Lemma 3.6, SBSTR completely fits this situation, and hence we can exactly solve it in  $O(n^3)$  time, where we have  $k = |V^+| = \frac{n}{2}$ .

*Remark 3.7.* It seems not easy to essentially improve the computational time. Both SBST and SBSTR actually include two fundamental special cases of the matroid intersection problem: finding a minimum-weight perfect matching in a bipartite graph, and finding a minimum-weight spanning arborescence in a digraph. In particular, for the former problem, one of the best known bound in general is  $O(n(m + n \log n))$  [6], which is not better than  $O(n^3)$  when the graph is dense.

Let us see the inclusion. For a balanced bipartite graph  $G = (V^+, V^-; E)$ , add two vertices  $s^+, s^-$  and  $|V^+| + 1$  edges  $\{v, s^-\}$  ( $v \in V^+ \cup \{s^+\}$ ) of weight 0, and add a sufficiently large value to the weights of all the edges in  $E$ . Then, a minimum-weight perfect matching in  $G$  corresponds to a minimum-weight strongly balanced spanning tree of the resulting graph, which is rooted at  $s^+$ .

Also, for a digraph  $G = (V, E)$ , split each vertex  $v \in V$  into two vertices  $v^+$  and  $v^-$  with an edge  $\{v^+, v^-\}$  of weight 0, and replace each edge  $(u, v)$  with  $\{u^+, v^-\}$  of the same weight plus a sufficiently large value. Then, a minimum-weight spanning in-arborescence in  $G$  rooted at  $r \in V$  corresponds to a minimum-weight strongly balanced spanning tree of the resulting graph (which is indeed a balanced bipartite graph) rooted at  $r^+$ .

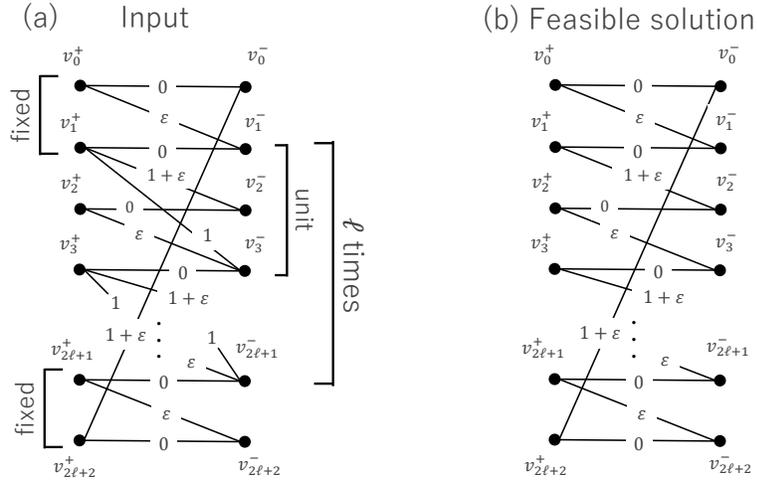


Figure 1: A tight example of our 2-approximation algorithm for DMISS.

### Approximation Ratio

Let  $S_{\text{opt}}$  be an optimal solution, and we show  $w(T \cup \overline{A_{\text{out}}}) = w(T) + w_M(A_{\text{out}}) \leq 2w(S_{\text{opt}})$  (recall that  $w_M: E(G_M) \rightarrow \mathbb{R}_{\geq 0}$  is defined by (1)). The following two claims conclude this.

**Claim 3.8.**  $w(T) \leq w(S_{\text{opt}})$ .

*Proof.* By Lemmas 3.2 and 3.4,  $S_{\text{opt}}$  has a perfect matching  $M_{\text{opt}}$ , and there exists a spanning in-arborescence  $A'_{\text{in}}$  of  $G_{M_{\text{opt}}}$  rooted at  $r$  such that  $\overleftarrow{M_{\text{opt}}} \subseteq A'_{\text{in}} \subseteq \overrightarrow{S_{\text{opt}}} \cup \overleftarrow{M_{\text{opt}}}$ . Since  $T$  is optimal and  $\overline{A'_{\text{in}}}$  is feasible in SBSTR with the input  $(G, w, r)$ , we have  $w(T) \leq w(\overline{A'_{\text{in}}}) \leq w(S_{\text{opt}})$ .  $\square$

**Claim 3.9.**  $w_M(A_{\text{out}}) \leq w(S_{\text{opt}})$ .

*Proof.* As with the proof of Lemma 3.3, consider a subgraph  $G' = (V^+, V^-; S_{\text{opt}} \cup M)$  of  $G$ . Then, by Lemma 3.2,  $G'_M$  has a spanning out-arborescence  $A'_{\text{out}}$  rooted at  $r$ , and hence

$$w_M(A_{\text{out}}) \leq w_M(A'_{\text{out}}) \leq w(\overline{A'_{\text{out}}} \setminus M) \leq w(S_{\text{opt}}). \quad \square$$

*Remark 3.10.* The above analysis is tight. Consider the input  $(G, w)$  shown in Figure 1, consisting of

- $2(2\ell + 3)$  vertices  $v_i^+, v_i^-$  ( $i = 0, 1, \dots, 2\ell + 2$ ),
- $2\ell + 3$  edges  $\{v_i^+, v_i^-\}$  ( $i = 0, 1, \dots, 2\ell + 2$ ) of weight 0,
- $\ell$  edges  $\{v_{2i-1}^+, v_{2i}^-\}$  ( $i = 1, 2, \dots, \ell$ ) of weight  $1 + \epsilon$ ,
- $\ell$  edges  $\{v_{2i}^+, v_{2i+1}^-\}$  ( $i = 1, 2, \dots, \ell$ ) of weight  $\epsilon$ ,
- $\ell$  edges  $\{v_{2i-1}^+, v_{2i+1}^-\}$  ( $i = 1, 2, \dots, \ell$ ) of weight 1,
- two edges  $\{v_0^+, v_1^-\}$  and  $\{v_{2\ell+1}^+, v_{2\ell+2}^-\}$  of weight  $\epsilon$ , and

- an edge  $\{v_{2\ell+2}^+, v_0^-\}$  of weight  $1 + \varepsilon$ ,

where  $\ell$  is a positive integer and  $\varepsilon := \frac{1}{\ell} > 0$ .

On the one hand, a Hamiltonian cycle  $(v_0^-, v_0^+, v_1^-, v_1^+, \dots, v_{2\ell+2}^-, v_{2\ell+2}^+, v_0^-)$  illustrated as in (b) is a feasible solution. On the other hand, the output of our algorithm is the whole graph as follows. First, the edges of weight at most 1 form a spanning tree of  $G$ , which is of minimum weight (consider Kruskal's algorithm [14]). This minimum-weight spanning tree is strongly balanced with root  $r = v_{2\ell+2}^+$ , and hence it is eligible for  $T$  in Step 1. The edges of weight 0 form a perfect matching, say  $M$ , and any spanning out-arborescence of  $G_M$  (candidate of  $A_{\text{out}}$  in Step 2) must contain all the remaining edges, which are of weight  $1 + \varepsilon$ .

Thus, the approximation ratio of our algorithm is at least

$$\frac{w(T \cup \overline{A_{\text{out}}})}{w(S_{\text{opt}})} \geq \frac{\ell(2 + 2\varepsilon) + 2\varepsilon + 1 + \varepsilon}{\ell(1 + 2\varepsilon) + 2\varepsilon + 1 + \varepsilon} = \frac{2\ell^2 + 3\ell + 3}{\ell^2 + 3\ell + 3} \rightarrow 2 \quad (\ell \rightarrow \infty).$$

## 4 FPT Algorithm for Unweighted DMISS

In this section, we show Theorem 2.3 by presenting an FPT algorithm for UNWEIGHTEDDMISS.

In Section 4.1, as a base of our result, we sketch the idea of an FPT algorithm for UNWEIGHTEDSCSS proposed by Bang-Jensen and Yeo [1]. In Section 4.2, by extending it, we propose an FPT algorithm for UNWEIGHTEDDMISS.

### 4.1 FPT Algorithm for Unweighted SCSS

The algorithm utilizes another well-known characterization of strongly connected digraphs. For a digraph  $G = (V, E)$ , an *ear decomposition* of  $G$  is a sequence  $\mathcal{E} = (P_0, P_1, \dots, P_f)$  of edge-disjoint paths and cycles in  $G$  with  $E = \bigcup_{i=0}^f E(P_i)$  such that

- $P_0$  is a cycle, and
- $P_i$  is either a path between two different vertices in  $\bigcup_{j=0}^{i-1} V(P_j)$  whose inner vertices are in  $V \setminus \bigcup_{j=0}^{i-1} V(P_j)$ , or a cycle intersecting exactly one vertex in  $\bigcup_{j=0}^{i-1} V(P_j)$ .

Each path or cycle  $P_i$  is called an *ear*.

**Lemma 4.1.** *A digraph is strongly connected if and only if it has an ear decomposition.*

The idea of the FPT algorithm for UNWEIGHTEDSCSS is sketched as follows. It first computes an ear decomposition  $\mathcal{E} = (P_0, P_1, \dots, P_f)$  of the input graph  $G$  so that the set of long ears is inclusion-wise maximal (the number is not necessarily maximized), where an ear is *long* if it is of length at least 3. If the number of long ears is large enough (depending on  $k$ ), then the ear decomposition  $\mathcal{E}$  gives a strongly connected spanning subgraph that is small enough by removing all the ears of length 1, and the answer is yes. Otherwise, let  $X$  be the set of vertices intersecting a long ear in  $\mathcal{E}$ . Under the assumption that  $\mathcal{E}$  does not immediately give a solution, we can show that  $|X| = O(k)$ . Also, by the maximality of long ears in  $\mathcal{E}$ , the vertices in  $V \setminus X$  are added by ears of length 2, and this is basically true for some minimum strongly connected spanning subgraph. Thus, we can test the existence of a solution by exhaustive search on the subgraphs of  $G$  on the vertex set  $X$  (plus a small subset), where we have only  $2^{O(k^2)}$  candidates.

## 4.2 FPT Algorithm for Unweighted DMISS

First, we give an alternative characterization of DM-irreducible balanced bipartite graphs by ear decompositions. An ear decomposition of a bipartite graph  $G = (V^+, V^-; E)$  is defined in the same way as above, and we call it *odd proper* if

- $P_0$  is a cycle of length at least 4, and
- each  $P_i$  ( $i = 1, 2, \dots, f$ ) is a path of odd length, which implies that one end vertex of  $P_i$  is in  $V^+$  and the other is in  $V^-$ .

**Lemma 4.2.** *A bipartite graph on at least four vertices is balanced and DM-irreducible if and only if it has an odd proper ear decomposition.*

*Proof.* Suppose that a bipartite graph  $G = (V^+, V^-; E)$  with  $|V| \geq 4$  is balanced and DM-irreducible. By Lemma 2.1, there exists a perfect matching  $M$  in  $G$  and  $G_M$  is strongly connected. Since  $|V| \geq 4$  and  $G_M$  is connected, there exists an edge  $e_0 = \{u_0, v_0\} \in E \setminus M$ , where  $u_0 \in V^+$  and  $v_0 \in V^-$ . As  $G_M$  is strongly connected, it contains a path from  $v_0$  to  $u_0$ , which forms a cycle of length at least 4 together with the edge  $\vec{e}_0 = (u_0, v_0)$ . Let  $(P_0, P_1, \dots, P_f)$  be an arbitrary ear decomposition of  $G_M$  such that  $P_0$  is this cycle (the existence follows from Lemma 4.1 with shrinking  $P_0$  into a single vertex). We then see by induction on  $i = 0, 1, \dots, f - 1$  that

- for every  $e = \{u, v\} \in M$ , either  $\{u, v\} \cap \bigcup_{j=0}^i V(P_j) = \emptyset$  or  $\{u, v\} \subseteq \bigcup_{j=0}^i V(P_j)$  and  $\overleftarrow{e} = (v, u) \in \bigcup_{j=0}^i E(P_j)$ , and
- $P_{i+1}$  starts and ends with edges in  $\vec{E}$ , which implies that it is of odd length.

Thus, we are done by taking the corresponding ear decomposition of the underlying graph (with removing each ear consisting of a single edge  $\vec{e} \in \vec{M}$ , which is duplicated).

Suppose that a bipartite graph  $G$  with  $|V(G)| \geq 4$  has an odd proper ear decomposition  $(P_0, P_1, \dots, P_f)$ . We then see by induction on  $i = 0, 1, \dots, f$  that the subgraph  $G_i$  of  $G$  defined by  $V(G_i) := \bigcup_{j=0}^i V(P_j)$  and  $E(G_i) := \bigcup_{j=0}^i E(P_j)$  is balanced and DM-irreducible (it is straightforward by Lemma 2.1), which concludes that  $G$  itself is balanced and DM-irreducible.  $\square$

An ear is called *trivial* if it consists of a single edge, and *nontrivial* otherwise. Also, an ear is called *long* if it is of length at least 5, and *short* otherwise.

We start to describe our FPT algorithm for UNWEIGHTEDDMISS. Let  $G = (V^+, V^-; E)$  be the input DM-irreducible balanced bipartite graph, where  $n = |V^+|$  and  $m = |E|$  and assume  $n \geq 2$ . We first find an odd proper ear decomposition  $\mathcal{E} = (P_0, P_1, \dots, P_f)$  such that

- $\{P_1, P_2, \dots, P_s\}$  is the set of long ears in  $\mathcal{E}$ , which is inclusion-wise maximal (the number  $s$  is not necessarily maximized), and
- $\{P_1, P_2, \dots, P_r\}$  is the set of nontrivial ears in  $\mathcal{E}$  (note that  $s \leq r \leq f$ ).

This can be done in polynomial time (e.g., by starting with an arbitrary odd proper ear decomposition of  $G$  and applying Claim 4.4; if the condition does not hold, add a long ear constructed in the proof).

For each  $i = 0, 1, \dots, f$ , we define a subgraph  $G_i$  of  $G$  by  $V(G_i) := \bigcup_{j=0}^i V(P_j)$  and  $E(G_i) := \bigcup_{j=0}^i E(P_j)$ . Let  $X = V(G_s)$  and  $Y = V \setminus X$ . We then observe the following two claims.

**Claim 4.3.**  $G_r$  is a DM-irreducible spanning subgraph of  $G$ , which consists of  $(3n-2) - \left(\frac{|X|}{2} - s - 2\right)$  edges.

*Proof.* Since  $V(G_r) = V(G)$ , by Lemma 4.2,  $G_r$  is a DM-irreducible spanning subgraph of  $G$ . By definition, we have

$$\begin{aligned} |V(G_r)| &= |V(P_0)| + \sum_{i=1}^r (|V(P_i)| - 2) = |V(P_0)| + \sum_{i=1}^r (|E(P_i)| - 1), \\ |E(G_r)| &= |E(P_0)| + \sum_{i=1}^r |E(P_i)| = |V(P_0)| + \sum_{i=1}^r |E(P_i)|, \end{aligned}$$

and hence  $|E(G_r)| = |V(G_r)| + r = 2n + r = (3n - 2) - (n - r - 2)$ . We also have

$$|X| = 2n - |Y| = 2n - \sum_{i=s+1}^r (|E(P_i)| - 1) = 2n - 2(r - s),$$

and we are done.  $\square$

By Claim 4.3, if  $k \leq \frac{|X|}{2} - s - 2$ , then  $(G, k)$  is a yes-instance. We also have  $|X| \geq 4s + 4$  by definition. In what follows, we assume  $k \geq \frac{|X|}{2} - s - 1 \geq \frac{|X|}{4}$ , i.e.,  $|X| \leq 4k$ .

**Claim 4.4.**  $E(G[Y])$  is a perfect matching.

*Proof.* By definition,  $P_{s+1}, P_{s+2}, \dots, P_r$  are paths of length 3 and the two inner vertices of each path are in  $Y$ . Thus,  $E(G[Y])$  indeed has a perfect matching

$$\{\{y_i^+, y_i^-\} \mid \{y_i^+, y_i^-\} \text{ is the middle edge of the ear } P_i \ (s+1 \leq i \leq r)\}.$$

By the maximality of long ears, we can see that  $E(G[Y])$  contains no other edge as follows.

Suppose to the contrary that  $E(G[Y])$  contains another edge  $e = \{y_i^+, y_j^-\}$  with  $i \neq j$ . We pick such an edge minimizing  $i + j$ , and assume  $i < j$  by symmetry. Then,  $G_j$  has a path  $Q_i$  of length 2 between  $y_i^+$  and some vertex  $x^+ \in X \cap V^+$  (a part of the ear  $P_i$ ) and a path  $Q_j$  of length at least 2 between  $y_j^-$  and some vertex  $x^- \in X \cap V^-$  starting with the edge  $\{y_j^+, y_j^-\}$  whose inner vertices are in  $Y$ . If  $Q_i$  and  $Q_j$  are disjoint, then  $Q_i$ ,  $Q_j$ , and  $e$  form a long ear that can be added just after  $P_s$  in  $\mathcal{E}$ , which contradicts the maximality of long ears. Otherwise,  $Q_j$  start with the edges  $\{y_j^+, y_j^-\}$  and  $\{y_j^+, y_i^-\}$  (otherwise, there exists another edge  $e' = \{y_{i'}^+, y_{j'}^-\}$  with  $i' \neq j'$  and  $i' + j' < i + j$ , a contradiction). In this case, the first two edges of  $Q_j$ ,  $e$ , and the two end edges in the ear  $P_i$  form a long ear that can be added just after  $P_s$  in  $\mathcal{E}$ , which contradicts the maximality of long ears.  $\square$

We consider a bipartite graph  $\tilde{H}$  with bipartition  $(\tilde{Z}, \tilde{Y})$  defined as follows:

$$\begin{aligned} \tilde{Y} &:= \{\tilde{y} := \{y^+, y^-\} \mid y^+, y^- \in Y, \{y^+, y^-\} \in E(G)\}, \\ \tilde{Z} &:= \{\tilde{z} := \{z^+, z^-\} \mid \tilde{y} \in \tilde{Y}, \{z^+, y^-\}, \{z^-, y^+\} \in E(G)\}, \\ E(\tilde{H}) &:= \{\{\tilde{z}, \tilde{y}\} \mid \tilde{z} \in \tilde{Z}, \tilde{y} \in \tilde{Y}, \{z^+, y^-\}, \{z^-, y^+\} \in E(G)\}. \end{aligned}$$

Let  $\tilde{Z}_{\text{out}} \subseteq \tilde{Z}$  be a vertex set  $\tilde{Z}'$  maximizing  $|\tilde{Z}'|$  subject to  $2|\tilde{Z}'| > |\Gamma_{\tilde{H}}(\tilde{Z}')|$ , and let  $\tilde{Z}_{\text{in}} := \tilde{Z} \setminus \tilde{Z}_{\text{out}}$ . Then, by the maximality of  $\tilde{Z}_{\text{out}}$ , we have  $2|\tilde{Z}'| \leq |\Gamma_{\tilde{H}}(\tilde{Z}')|$  for any  $\tilde{Z}' \subseteq \tilde{Z}_{\text{in}}$ .

Let  $\tilde{Y}_{\text{out}} := \Gamma_{\tilde{H}}(\tilde{Z}_{\text{out}}) \setminus \Gamma_{\tilde{H}}(\tilde{Z}_{\text{in}})$ ,  $\tilde{Y}_{\text{in}} := \tilde{Y} \setminus \tilde{Y}_{\text{out}}$ , and  $\tilde{H}_{\text{in}} := \tilde{H}[\tilde{Z}_{\text{in}} \cup \tilde{Y}_{\text{in}}]$ . Then, as  $\Gamma_{\tilde{H}}(\tilde{Z}_{\text{in}}) \subseteq \tilde{Y}_{\text{in}}$ , we also have  $2|\tilde{Z}'| \leq |\Gamma_{\tilde{H}_{\text{in}}}(\tilde{Z}')|$  for any  $\tilde{Z}' \subseteq \tilde{Z}_{\text{in}}$ . By Hall's theorem [10],  $\tilde{H}_{\text{in}}$  has two disjoint matchings such that each of them covers  $\tilde{Z}_{\text{in}}$  and no vertex in  $\tilde{Y}_{\text{in}}$  is covered by both of them. In other words,  $\tilde{H}_{\text{in}}$  contains a subgraph  $\tilde{H}'$  such that  $\deg_{\tilde{H}'}(\tilde{z}) = 2$  for every  $\tilde{z} \in \tilde{Z}_{\text{in}}$  and  $\deg_{\tilde{H}'}(\tilde{y}) \leq 1$  for every  $\tilde{y} \in \tilde{Y}_{\text{in}}$ . Let  $\tilde{M} := E(\tilde{H}')$ , and  $\tilde{Z}(\tilde{M})$  and  $\tilde{Y}(\tilde{M})$  be the sets of end vertices of edges in  $\tilde{M}$  that are in  $\tilde{Z}$  and in  $\tilde{Y}$ , respectively. We define also  $\tilde{Y}' := \tilde{Y}_{\text{in}} \setminus \tilde{Y}(\tilde{M})$ . For a set  $\tilde{B}$  of vertex pairs, let  $B$  denote the set of corresponding vertices, i.e.,  $B := \bigcup_{\tilde{b} \in \tilde{B}} \tilde{b}$ .

If  $\tilde{Z}_{\text{in}} = \emptyset$ , then  $\tilde{Y}_{\text{out}} = \tilde{Y}$ . Hence,  $|\tilde{Y}| = |\tilde{Y}_{\text{out}}| < 2|\tilde{Z}_{\text{out}}| \leq 2|\tilde{Z}| \leq 2(2k)^2 = 8k^2$ , which implies  $|V| = |X| + |Y| \leq 4k(4k + 1)$ . In this case, the input size is bounded by a parameter, and hence we can solve the problem by exhaustive search in FPT time.

Otherwise, i.e., if  $\tilde{Z}_{\text{in}} \neq \emptyset$ , the following claim holds.

**Claim 4.5.** *There exists a minimum DM-irreducible spanning subgraph  $G^*$  of  $G$  satisfying the following two conditions.*

- $G^* - Y'$  is a DM-irreducible spanning subgraph of  $G - Y'$ .
- For any  $\tilde{y} = \{y^+, y^-\} \in \tilde{Y}'$ , there exists  $\tilde{z} = \{z^+, z^-\} \in \tilde{Z}$  with  $\{\tilde{z}, \tilde{y}\} \in E(\tilde{H})$  such that the edges in  $E(G^*)$  incident to  $y^+$  or  $y^-$  are  $\{z^+, y^-\}$ ,  $\{y^+, y^-\}$ , and  $\{y^+, z^-\}$ .

Before proving Claim 4.5, we complete our FPT algorithm for UNWEIGHTEDDMISS. By Claim 4.5, the answer is yes if and only if there exists a DM-irreducible spanning subgraph of  $G - Y'$  consisting of at most  $3n - 2 - k - 3|\tilde{Y}'|$  edges. We can check all possible spanning subgraphs of  $G - Y'$  by exhaustive search in FPT time, since

$$|V(G - Y')| = |X| + 2|\tilde{Y}(\tilde{M})| + 2|\tilde{Y}_{\text{out}}| < |X| + 4|\tilde{Z}(\tilde{M})| + 4|\tilde{Z}_{\text{out}}| = |X| + 4|\tilde{Z}| \leq 4k(4k + 1).$$

*Proof of Claim 4.5.* Let  $G'$  be a minimum DM-irreducible spanning subgraph of  $G$ . If  $G' - Y'$  is DM-irreducible, then we are done as follows. By Claim 4.4, any DM-irreducible spanning subgraph of  $G$  (including  $G'$ ) contains, for each pair  $\tilde{y} = \{y^+, y^-\} \in \tilde{Y}$ , either

- a path of length 3 between two vertices in  $V(G' - Y)$  whose middle edge is  $\{y^+, y^-\}$ , or
- at least two edges between  $y^+$  and  $V(G' - Y)$  and between  $y^-$  and  $V(G' - Y)$  (at least four in total).

Thus, a spanning subgraph of  $G$  obtained from  $G' - Y'$  as follows has as few edges as  $G'$ , and is a desired graph: for each pair  $\tilde{y} = \{y^+, y^-\} \in \tilde{Y}'$ , choose an arbitrary pair  $\tilde{z} = \{z^+, z^-\} \in \tilde{Z}$  with  $\{\tilde{z}, \tilde{y}\} \in E(\tilde{H})$  and add three edges  $\{z^+, y^-\}$ ,  $\{y^+, y^-\}$ , and  $\{y^+, z^-\}$ .

Otherwise, i.e., if  $G' - Y'$  is not DM-irreducible, we transform  $G'$  as follows: remove all the edges incident to  $Y(\tilde{M})$ , and for each  $\{\tilde{z}, \tilde{y}\} \in \tilde{M}$ , add three edges  $\{z^+, y^-\}$ ,  $\{y^+, y^-\}$ , and  $\{y^+, z^-\}$ . Let  $G^\sharp$  be the resulting graph. By the discussion in the previous paragraph, we have  $|E(G^\sharp)| \leq |E(G')|$  and it suffices to show that  $G^\sharp - Y'$  is DM-irreducible.

Let  $M'$  be a perfect matching in  $G'$ . As  $G'$  is DM-irreducible,  $G'_{M'}$  is strongly connected. We first show that  $G^\sharp - Y'$  has a perfect matching, which we construct from  $M'$  minus the edges incident to  $Y_{\text{in}}$  as follows. For each removed edge  $\{y^+, y^-\} \in E(G[Y(\tilde{M})])$ , we just add it again. The other removed edges are paired as  $\{y^+, z^-\}$  and  $\{z^+, y^-\}$  such that  $\tilde{y} = \{y^+, y^-\} \in \tilde{Y}_{\text{in}}$ ,  $\tilde{z} = \{z^+, z^-\} \in \tilde{Z}_{\text{in}}$ , and  $\{\tilde{z}, \tilde{y}\} \in E(\tilde{H}_{\text{in}})$ . By definition of  $\tilde{M}$ , there exists a pair  $\tilde{y}' = \{y'^+, y'^-\} \in \tilde{Y}(\tilde{M})$  with  $\{\tilde{z}, \tilde{y}'\} \in \tilde{M} \subseteq E(\tilde{H}_{\text{in}})$ , which means that  $\{z^+, y'^-\}$ ,  $\{y'^+, y'^-\}$ ,  $\{y'^+, z^-\} \in E(G^\sharp)$ . By definition

of  $\tilde{M}$  again, such  $\tilde{y}'$  is distinct for each  $\tilde{z} \in \tilde{Z}_{\text{in}} = \tilde{Z}(\tilde{M})$ , and hence we can add  $\{z^+, y'^-\}$  and  $\{y'^+, z^-\}$  instead of each pair of  $\{y^+, z^-\}$  and  $\{z^+, y^-\}$ . Finally, for each pair  $\{y^+, y^-\} \in \tilde{Y}(\tilde{M})$ , if at least one of  $y^+$  and  $y^-$  is not matched, then both  $y^+$  and  $y^-$  are not matched due to the above procedure, and add an edge  $\{y^+, y^-\}$ . Let  $M^\sharp$  be the obtained perfect matching in  $G^\sharp$ .

The remaining task is to show that  $G_{M^\sharp}^\sharp - Y'$  is strongly connected. By the above construction, it suffices to show that for each pair of two vertices  $s, t \in V \setminus Y_{\text{in}}$ , we have  $t$  is reachable from  $s$  in  $G_{M^\sharp}^\sharp - Y'$ . Suppose to the contrary that there exists a pair  $(s, t)$  such that  $t$  is not reachable from  $s$  in  $G_{M^\sharp}^\sharp - Y'$ . As  $G_{M'}^\sharp$  is strongly connected, there exists a path  $P$  from  $s$  to  $t$  in  $G_{M'}^\sharp$ . Take such  $(s, t)$  and  $P$  so that the length of  $P$  is minimized. Then, all the inner vertices of  $P$  are in  $Y_{\text{in}}$  (otherwise, we can take an inner vertex  $v \notin Y_{\text{in}}$  such that  $(s, v)$  and the prefix of  $P$  or  $(v, t)$  and the suffix of  $P$  is eligible, a contradiction), and hence  $s, t \in Z_{\text{in}}$ .

If  $\{s, t\} \in \tilde{Z}_{\text{in}} = \tilde{Z}(\tilde{M})$ , then by the above construction, there exists a path from  $s$  to  $t$  through a pair  $\tilde{y} \in \tilde{Y}(\tilde{M})$  in  $G_{M^\sharp}^\sharp - Y'$ , a contradiction. Otherwise,  $P$  consists of two edges  $(s, v)$  and  $(v, t)$  such that  $v \in Y_{\text{in}}$  and exactly one of  $\{s, v\}$  and  $\{t, v\}$  is in  $M'$ . By symmetry, assume that  $s, t \in V^+$ ,  $v \in V^-$ , and  $\{t, v\} \in M'$ . Let  $\tilde{y} = \{y^+, y^-\} \in \tilde{Y}_{\text{in}}$  be the pair with  $y^- = v$ . Then, there exist two pairs  $\tilde{z}_1 = \{z_1^+, z_1^-\}$  and  $\tilde{z}_2 = \{z_2^+, z_2^-\}$  in  $\tilde{Z}_{\text{in}} = \tilde{Z}(\tilde{M})$  such that  $z_1^+ = s$ ,  $z_2^+ = t$ ,  $z_1^- = z_2^-$  and  $\{y^+, z_1^-\} \in M'$ . By the above construction, there exist two pairs  $\tilde{y}_1 = \{y_1^+, y_1^-\}$  and  $\tilde{y}_2 = \{y_2^+, y_2^-\}$  in  $\tilde{Y}(\tilde{M})$  such that  $\{\tilde{z}_1, \tilde{y}_1\}, \{\tilde{z}_2, \tilde{y}_2\} \in \tilde{M}$  and  $\{y_1^+, y_1^-\}, \{z_2^+, y_2^-\}, \{y_2^+, z_2^-\} \in M^\sharp$ . This concludes that  $G_{M^\sharp}^\sharp - Y'$  has a path from  $s = z_1^+$  to  $t = z_2^+$  that consists of six edges  $(s, y_1^-)$ ,  $(y_1^-, y_1^+)$ ,  $(y_1^+, z_1^-)$ ,  $(z_1^-, y_2^+)$ ,  $(y_2^+, y_2^-)$ , and  $(y_2^-, t)$ , a contradiction. Thus, we are done.  $\square$

*Remark 4.6.* Another possible parameterized approach is, as a connectivity augmentation problem (like [2]), taking the parameter  $k$  as the number of additional edges to achieve DM-irreducibility. For strong connectivity augmentation, Klinkby, Misra, and Saurabh [13] recently showed the first FPT algorithm in this direction. How to extend their result to the DM-irreducibility problem is highly nontrivial because there are too many possible choices of perfect matchings in the resulting augmented graphs. This may be possible future work.

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