

LOWER BOUNDS ON LOEWY LENGTHS OF MODULES OF FINITE PROJECTIVE DIMENSION

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ABSTRACT. In this article we study nonzero modules of finite length and finite projective dimension over a local ring. We show the Loewy length of such a module is larger than the regularity of the ring whenever the ring is strict Cohen-Macaulay, extending work of Avramov–Buchweitz–Iyengar–Miller beyond the Gorenstein setting. Applications include establishing a conjecture of Corso–Huneke–Polini–Ulrich and verifying a Lech-like conjecture, comparing generalized Loewy length along flat local extensions, for strict Cohen-Macaulay rings. We also give significant improvements on known lower bounds for Loewy lengths of modules of finite projective dimension without any assumption on the associated graded ring. The strongest general bounds are achieved over complete intersection rings.

INTRODUCTION

This work is concerned with modules of finite length and finite projective dimension over a local ring. Such modules have received a great deal of attention and encode the singularity of the ring; for instance, a consequence of Roberts’ New Intersection Theorem [34] is that if a ring admits a nonzero module of finite length and finite projective dimension, then the ring must be Cohen-Macaulay. There are a number of interesting questions that remain open regarding these modules. In this article we revisit one on a uniform lower bound for their Loewy lengths.

Fix a local ring R with maximal ideal \mathfrak{m} . Recall that the Loewy length of an R -module M is $\ell_R(M) = \inf\{i \geq 0 \mid \mathfrak{m}^i M = 0\}$. We are interested in understanding

$$\inf\{\ell_R(M) \mid M \neq 0 \text{ finitely generated and } \text{projdim}_R(M) < \infty\},$$

which (by the New Intersection Theorem) is finite if and only if R is Cohen-Macaulay; recall that if R is Cohen-Macaulay then $R/(\mathbf{x})$ is a finite length module of finite projective dimension for any system of parameters \mathbf{x} on R . These observations suggest the following, conjectured by Corso, Huneke, Polini and Ulrich [10]; we refer to this as the *Loewy Length Conjecture*.

Conjecture A. *For a local ring R and a nonzero R -module M with finite projective dimension, the following inequality holds:*

$$\ell_R(M) \geq \min\{\ell_R(R/(\mathbf{x})) \mid \mathbf{x} \text{ is a system of parameters on } R\}.$$

The proposed uniform lower bound in the conjecture is called the generalized Loewy length of R , denoted $g\ell_R(R)$; this invariant is a Loewy length analog to

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the Hilbert-Samuel multiplicity of R , and it has been studied in a number of works; see, for example, [5, 8, 10, 12, 13, 19, 28, 33].

Previously, the most significant progress on the Loewy Length Conjecture was established by Avramov–Buchweitz–Iyengar–Miller [5, Theorem 1.2]: *If R is a Gorenstein ring with Cohen-Macaulay associated graded ring $R^{\mathfrak{g}}$ and infinite residue field, then Conjecture A holds.* Without any assumption on the residue field, they also provide a bound in terms of the (Castelnuovo-Mumford) regularity of $R^{\mathfrak{g}}$, denoted $\text{reg}(R^{\mathfrak{g}})$. The main result of this article extends this result in the following.

Theorem B. *If R is a local with Cohen-Macaulay associated graded ring, then any nonzero finitely generated R -module M of finite projective dimension satisfies:*

$$\ell_R(M) \geq \text{reg}(R^{\mathfrak{g}}) + 1.$$

Furthermore, if R has an infinite residue field, then $\ell_R(M) \geq g\ell_R(R)$.

The proof of Theorem B can be found at the end of Section 2. An essential ingredient is Lemma 2.6, where we leverage the hypothesis that $R^{\mathfrak{g}}$ is Cohen-Macaulay (that is, R is *strict* Cohen-Macaulay) to produce the minimal free resolution of a certain artinian quotient of R ; this resolution has the property that entries in its differentials are in sufficiently high powers of the maximal ideal. Theorem B then follows from the calculation in Lemma 2.7 which shows that upon tensoring this resolution with modules having ‘small enough’ Loewy length there must be homology in arbitrarily high degrees.

There are several consequences of Theorem B to strict Cohen-Macaulay rings. An immediate one is that the generalized Loewy length of a nonzero Cohen-Macaulay module with finite projective dimension is bounded below by the generalized Loewy length of the ring; see Theorem 2.1. Another is that

$$g\ell_R(R) = \ell_R(R/(\mathbf{x}))$$

for all maximal superficial sequences \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$, where \mathfrak{m} denotes the maximal ideal of R ; cf. Corollary 2.2. Furthermore, we introduce a version of the long-standing conjecture of Lech [29], where Hilbert-Samuel multiplicity is replaced with generalized Loewy length (see Conjecture 3.1), and we provide the following evidence:

Corollary C. *Suppose $R \rightarrow S$ is a map of Cohen-Macaulay local rings with infinite residue fields. If $\text{flatdim}_R(S) < \infty$ and R is strict Cohen-Macaulay, then*

$$g\ell_R(R) \leq g\ell_S(S).$$

This corollary establishes interesting ring theoretic properties along flat local extensions whose base is strict Cohen-Macaulay; cf. Corollary 3.3. In particular, we verify a conjecture of Hanes [18, Conjecture 3.1] on reduction numbers along flat local extensions when the base is assumed to be strict Cohen-Macaulay.

In the final part of the paper, Section 4, we turn our attention to lower bounds for the Loewy lengths of nonzero modules of finite projective dimension without making assumptions on the structure of $R^{\mathfrak{g}}$. Building on [5], it was shown in [33] that when R is Gorenstein then for such a module M one has $\ell_R(M) \geq \text{ord}(R)$; here $\text{ord}(R)$ denotes the minimal order of a defining relation of R in one of its minimal Cohen presentations. We improve this bound in a rather drastic way; below, the complexity of an R -module M , denoted $\text{cx}_R(M)$, measures the polynomial rate of growth of the Betti sequence for M .

Theorem D. *Assume R is a local ring, and fix a minimal Cohen presentation $\widehat{R} \cong Q/(f_1, \dots, f_c)$ and a nonzero R -module M .*

- (1) *If $\ell_R(M) < \text{ord}(R)$, then $\text{cx}_R(M) \geq c$.*
- (2) *If M has finite projective dimension, then*

$$\ell_R(M) \geq \max\{\text{ord}(f_i) : 1 \leq i \leq c\}.$$

- (3) *If M has finite projective dimension and R is a complete intersection ring, then*

$$\ell_R(M) \geq \sum_{i=1}^c \text{ord}(f_i) - c + 1.$$

The first two parts of the theorem can be found in Theorem 4.3, and their proofs make use of a dg algebra structure on the minimal free resolution of the residue field. Part (3) of Theorem D, see Theorem 4.6, was first suggested to the authors by Mark Walker. We present two proofs: the first is similar to the proof of Theorem B, making use of the explicit free resolution of the socle of an artinian complete intersection, while the second was sketched to the authors by Walker and makes use of a construction from [21].

Finally, the central problem in this article (Conjecture A) parallels the *Length Conjecture* studied in [26]; cf. Remark 3.4. As Loewy length and length are subtle invariants in their own way, it is perhaps unsurprising we employ seemingly different techniques in the present article to establish our uniform lower bounds for Loewy lengths compared to the methods used by Iyengar–Ma–Walker for length (with the exception of the second proof of Theorem D). It is worth remarking that the existence of Ulrich modules, which played a pivotal role in [26], are now known to not exist over rings where the Loewy Length Conjecture was settled in the positive here; for example, [27] provides examples of complete intersection rings that are strict Cohen-Macaulay where the method from [26] cannot be applied.

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1. LOEWY LENGTH AND THE ASSOCIATED GRADED RING

This section recounts the necessary background on (generalized) Loewy length. Throughout, R is a commutative noetherian local ring with maximal ideal \mathfrak{m} .

1.1. The *Loewy length* of an R -module M , denoted $\ell_R(M)$, is the least nonnegative integer i such that $\mathfrak{m}^i M = 0$; if no such integer exists set $\ell_R(M) = \infty$. When M is a finitely generated R -module, its *generalized Loewy length* is

$$g\ell_R(M) = \inf\{\ell_R(M/\mathbf{x}M) \mid \mathbf{x} \text{ is a system of parameters on } M\}.$$

The invariant $g\ell_R(R)$ can be regarded as a measure of the singularity of R , as it equals one if and only if R is regular.

Remark 1.2. Generalized Loewy length is to Loewy length as Hilbert-Samuel multiplicity is to length. Whereas there is much known on the Hilbert-Samuel multiplicity of modules over a local ring, generalized Loewy length is a more subtle and difficult to understand invariant. For example, the Hilbert-Samuel multiplicity of a Cohen-Macaulay local ring R is the length of $R/(\mathbf{x})$ where \mathbf{x} is any reduction of the maximal ideal. Whether this holds for generalized Loewy length is a question of DeStefani over Gorenstein rings having infinite residue fields in [12, Question 4.5]. This is also the content of a conjecture in [10].

Notation 1.3. The associated graded ring of R is

$$R^{\mathfrak{g}} := \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{j=0}^{\infty} \frac{\mathfrak{m}^j}{\mathfrak{m}^{j+1}}.$$

For a sequence of elements \mathbf{x} in R , write \mathbf{x}^* for its corresponding sequence of initial forms in $R^{\mathfrak{g}}$. Also, for a minimal R -complex C , its linear part is the associated graded complex

$$C^{\mathfrak{g}} := \bigoplus_{j \in \mathbb{Z}} \frac{\mathfrak{m}^j C}{\mathfrak{m}^{j+1} C}(-j);$$

this is a complex of graded $R^{\mathfrak{g}}$ -modules where $(-j)$ is shifting the internal grading (that is, the non-homological one). In particular $C^{\mathfrak{g}}$ is bigraded with

$$C_{i,j}^{\mathfrak{g}} = \frac{\mathfrak{m}^{j-i} C_i}{\mathfrak{m}^{j-i+1} C_i}.$$

1.4. A sequence \mathbf{x} in R is *superficial* if its sequence of initial forms in $R^{\mathfrak{g}}$ is part of a system of parameters for $R^{\mathfrak{g}}$; such a sequence is part of a system of parameters for R . Moreover, if R has an infinite residue field, then there exists a superficial system of parameters in $\mathfrak{m} \setminus \mathfrak{m}^2$.

1.5. A local ring R is *strict Cohen-Macaulay* if its associated graded ring $R^{\mathfrak{g}}$ is Cohen-Macaulay. Strict Cohen-Macaulay rings are, in particular, Cohen-Macaulay; see, for example, [1] (or [16]). Hypersurface rings are strict Cohen-Macaulay, however complete intersection rings of higher codimension need not be strict Cohen-Macaulay (see, for example, Remark 4.7).

1.6. For a standard graded algebra S over a field S_0 , write S_+ for its homogeneous maximal ideal. For a homogeneous ideal I of S , let $H_I(-)$ denote the local cohomology functor with respect to I . The *regularity* of S is

$$\mathrm{reg}(S) := \max\{i + j \mid H_{S_+}^i(S)_j \neq 0 \text{ for some } i\}.$$

Following [22], the *regularity* of R is $\mathrm{reg}(R) := \mathrm{reg}(R^{\mathfrak{g}})$.

Lemma 1.7. *If R is a local ring with a superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$, and set $A = \mathrm{Kos}^R(\mathbf{x})$, then*

$$\ell_R(R/(\mathbf{x})) - 1 \leq \mathrm{reg}(R) = \max\{j \geq 0 \mid H_i(A^{\mathfrak{g}})_{i+j} \neq 0 \text{ for some } i\}.$$

Proof. Set $d = \dim(R)$, and fix a Noether normalization $S = k[t_1, \dots, t_d] \hookrightarrow R^{\mathfrak{g}}$ with each t_i mapping to x_i^* , the initial form of x_i in $R^{\mathfrak{g}}$. By the assumptions on \mathbf{x} , we have

$$H_{R^{\mathfrak{g}}}^i(R^{\mathfrak{g}}) \cong H_{(\mathbf{t})}^i(R^{\mathfrak{g}})$$

and hence it follows from a theorem of Eisenbud–Goto (see, for example, [9, Theorem 4.3.1]) that

$$\operatorname{reg}(R) = \max\{j \geq 0 \mid H_i(\operatorname{Kos}^S(\mathbf{t}; R^{\mathfrak{g}}))_{i+j} \neq 0 \text{ for some } i\}.$$

Finally, it remains to observe that $\operatorname{Kos}^S(\mathbf{t}; R^{\mathfrak{g}}) \cong A^{\mathfrak{g}}$ to justify the desired equality.

For the inequality, it follows from [24, Proposition 8.2.4] that

$$\ell\ell_R(R/(\mathbf{x})) \leq \min\{j \mid (R^{\mathfrak{g}}/(\mathbf{x}^*))_{\geq j} = 0\}.$$

Now from the already established equality, using that $H_0(A^{\mathfrak{g}}) = R^{\mathfrak{g}}/(\mathbf{x}^*)$, we obtain the desired inequality. \square

Lemma 1.8. *If R is strict Cohen-Macaulay, then $\operatorname{reg}(R) = \ell\ell_R(R/(\mathbf{x})) - 1$ for any superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$.*

Proof. By [35, Lemma 0.1], since $R^{\mathfrak{g}}$ is Cohen-Macaulay, for any superficial sequence \mathbf{x} we have

$$(R/(\mathbf{x}))^{\mathfrak{g}} \cong R^{\mathfrak{g}}/(\mathbf{x}^*).$$

Also, when \mathbf{x} is a system of parameters for R in $\mathfrak{m} \setminus \mathfrak{m}^2$ it follows from [19, Lemma 6.2] that

$$\ell\ell_R(R/(\mathbf{x})) = \min\{j \mid (R^{\mathfrak{g}}/(\mathbf{x}^*))_{\geq j} = 0\}.$$

Now since $R^{\mathfrak{g}}$ is Cohen-Macaulay, applying Lemma 1.7 yields

$$\ell\ell_R(R/(\mathbf{x})) = \operatorname{reg}(R) + 1. \quad \square$$

2. MAIN RESULT

This section is devoted to proving Theorem B from the introduction. In fact, we prove the following stronger statement, affirming a conjecture of Corso–Huneke–Polini–Ulrich [10] for strict Cohen-Macaulay rings.

Theorem 2.1. *Suppose a local ring R is strict Cohen-Macaulay. If M is a nonzero Cohen-Macaulay R -module of finite projective dimension, then*

$$g\ell\ell_R(M) \geq \operatorname{reg}(R) + 1.$$

Furthermore, if R has an infinite residue field, then $g\ell\ell_R(M) \geq g\ell\ell_R(R)$.

Before starting on the proof, we present the following corollary. This establishes another conjecture from [10] (again, for strict Cohen-Macaulay rings).

Corollary 2.2. *For a strict Cohen-Macaulay ring (R, \mathfrak{m}) with infinite residue field,*

$$g\ell\ell_R(R) = \ell\ell_R(R/(\mathbf{x})) = \operatorname{reg}(R) + 1$$

where \mathbf{x} is a sufficiently general system of parameters for R in $\mathfrak{m} \setminus \mathfrak{m}^2$.

Proof. As the residue field is infinite, a sufficiently general system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$ is, in particular, superficial; see [24, Theorem 8.6.6]. Since R is strict Cohen-Macaulay, from Lemma 1.8 we have $\ell\ell_R(R/(\mathbf{x})) = \operatorname{reg}(R) + 1$ and from the inequality in Theorem 2.1 (applied for $M = R$), we obtain

$$g\ell\ell_R(R) = \ell\ell_R(R/(\mathbf{x})) = \operatorname{reg}(R) + 1. \quad \square$$

The proof of Theorem 2.1 is given at the end of the section; it takes a bit of setup and is an immediate consequence of the more technical result in Lemma 2.7.

Notation 2.3. For the remainder of the section we assume R is a commutative noetherian local ring with maximal ideal \mathfrak{m} and infinite residue field k ; the fact that Loewy length decreases and regularity is unchanged among passage to an infinite residue field is due to [19, Proposition 6.3] and that local cohomology is invariant under flat base change, respectively. We can also assume that R is Cohen-Macaulay because of the New Intersection Theorem [34].

The next lemma abstracts a discussion in [11, Section 2], building on work in [22]. It is also related to the main theorem of [38], as well as [7, Section 3].

Lemma 2.4. *Suppose (R, \mathfrak{m}) is a local ring, \mathbf{x} is a superficial system of parameters in $\mathfrak{m} \setminus \mathfrak{m}^2$, and set $A = \text{Kos}^R(\mathbf{x})$. Then for $i \geq 0$ and $j \geq \text{reg}(R)$ we have*

$$\partial(A_{i+1}) \cap \mathfrak{m}^{j+1}A_i = \partial(\mathfrak{m}^j A_{i+1}).$$

Proof. There is no harm in completing R to show the desired equality. As A is a minimal complex, it is clear that $\partial(\mathfrak{m}^j A_{i+1}) \subseteq \partial(A_{i+1}) \cap \mathfrak{m}^{j+1}A_i$.

For the reverse containment, consider $\partial x \in \partial(A_{i+1}) \cap \mathfrak{m}^{j+1}A_i$. Its image in $A^{\mathfrak{g}}$ is a cycle in $A_{i,i+j+1}^{\mathfrak{g}}$, and so it defines the following homology class

$$[\partial x] \in H_i(A^{\mathfrak{g}})_{i+j+1}.$$

Since $j \geq \text{reg}(R)$, by Lemma 1.7, the homology class $[\partial x]$ is zero. That is to say, there exists $u_0 \in \mathfrak{m}^j A_{i+1}$ such that

$$\partial^A(x) - \partial^A(u_0) = \partial^A(x - u_0) \in \mathfrak{m}^{j+2}A_i.$$

Repeating the argument above produces $u_n \in \mathfrak{m}^{j+n}A_{i+1}$ with

$$\partial^A(x) - \partial^A\left(\sum_{\ell=0}^n u_\ell\right) = \partial^A\left(x - \sum_{\ell=0}^n u_\ell\right) \in \mathfrak{m}^{j+n+2}A_i.$$

Since R is complete, we can let $u = \sum_{\ell=0}^{\infty} u_\ell$, and by construction $\partial^A(x) = \partial^A(u)$ with $u \in \mathfrak{m}^j A_{i+1}$. \square

2.5. Assuming the setup in Notation 2.3, there exists a superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$; moreover, as R is Cohen-Macaulay, \mathbf{x} is a maximal regular sequence of R . Set $\overline{R} = R/(\mathbf{x})$ and let A be the Koszul complex on \mathbf{x} over R . Note that A is the minimal free resolution of \overline{R} over R . Also, fix the minimal free resolution $F \xrightarrow{\simeq} k$ over R .

Let \overline{s} denote the element of highest $\mathfrak{m}\overline{R}$ -adic order in the socle of \overline{R} . That is to say, the $\mathfrak{m}\overline{R}$ -adic order of \overline{s} , denoted n , is exactly one less than the Loewy length of \overline{R} . Moreover, a standard lifting property guarantees the existence of a lift between the complexes of R -modules below:

$$\begin{array}{ccc} F & \xrightarrow{\exists} & A \\ \simeq \downarrow & & \downarrow \simeq \\ k & \xrightarrow{1+\overline{s}} & \overline{R} \end{array}.$$

In what follows we construct a particular lift $\sigma: F \rightarrow A$ with $\sigma(F) \subseteq \mathfrak{m}^n A$ provided that $\text{reg}(R) = \ell\ell_R(R/(\mathbf{x})) - 1$.

Lemma 2.6. *Assume (R, \mathfrak{m}, k) is a local ring with $\text{reg}(R) = \ell_R(R/(\mathbf{x})) - 1$ for some superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$. Set $\overline{R} = R/(\mathbf{x})$ and $n = \text{reg}(R)$, and let F denote the minimal R -free resolution of k and $A = \text{Kos}^R(\mathbf{x})$ be the minimal R -free resolution \overline{R} .*

For any nonzero $\overline{s} \in \mathfrak{m}^n \overline{R}$, the R -module map $k \mapsto \overline{R}$ given by $1 \mapsto \overline{s}$ admits a lift $\sigma: F \rightarrow A$ satisfying:

- (1) $\sigma: F \rightarrow A$ is map of complexes with $\sigma(F) \subseteq \mathfrak{m}^n A$;
- (2) $\text{cone}(\sigma)$ is the minimal free resolution of $\overline{R}/(\overline{s})$ over R .

Proof. First, we inductively construct σ . In degree zero, let $\sigma_0: F_0 = R \rightarrow R = A_0$ be given by multiplication by s where the image of s in \overline{R} is \overline{s} and $s \in \mathfrak{m}^n$. Now assume we have constructed $\sigma_0, \dots, \sigma_i$ with

$$\sigma_j(F_j) \subseteq \mathfrak{m}^n A_j \quad \text{and} \quad \sigma_{j-1} \partial_j^F = \partial_{j-1}^A \sigma_j$$

for each $j = 0, \dots, i$. Observe that

$$\sigma_i \partial_{i+1}^F(F_{i+1}) \subseteq \mathfrak{m}^{n+1} A_i \cap \ker \partial_i^A = \mathfrak{m}^{n+1} A_i \cap \partial^A(A_{i+1})$$

where the first containment holds by the inductive hypothesis and the equality holds since A is exact. As a consequence, combined with Lemma 2.4, we obtain

$$\sigma_i \partial_{i+1}^F(F_{i+1}) \subseteq \partial^A(\mathfrak{m}^n A_{i+1});$$

this is where the assumption that $\text{reg}(R) = \ell_R(R/(\mathbf{x})) - 1$ is used. Hence by the lifting property of a free R -module, there exists an R -linear map $\sigma_{i+1}: F_{i+1} \rightarrow A_{i+1}$ such that

$$\sigma_i \partial_{i+1}^F = \partial_{i+1}^A \sigma_{i+1} \quad \text{and} \quad \sigma_{i+1}(F_{i+1}) \subseteq \mathfrak{m}^n A_{i+1}.$$

This completes the construction of the lift $\sigma: F \rightarrow A$, and verifies (1).

For (2), set $C = \text{cone}(\sigma)$ and observe that the cone exact sequence

$$0 \rightarrow A \rightarrow C \rightarrow \Sigma F \rightarrow 0$$

induces the exact sequence

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow H_2(C) \rightarrow 0 \rightarrow 0 \rightarrow H_1(C) \rightarrow H_0(F) \xrightarrow{H_0(\sigma)} H_0(A) \rightarrow H_0(C) \rightarrow 0.$$

In particular, $H_i(C) = 0$ for $i > 1$. Furthermore, note that $H_0(\sigma)$ is the inclusion $k \hookrightarrow \overline{R}$ sending 1 to \overline{s} , and so it follows from the exact sequence above that

$$H_i(C) = \begin{cases} \overline{R}/(\overline{s}) & i = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Also, as σ is a map between free resolutions of R -modules, C is a nonnegatively graded complex of free R -modules and hence it is a free resolution of $\overline{R}/(\overline{s})$ over R . Finally, by definition of the differential of C , it is in fact minimal since both F and A are minimal R -complexes and $\sigma(F) \subseteq \mathfrak{m}^n A$; cf. (1). \square

Lemma 2.7. *Let (R, \mathfrak{m}, k) be a Cohen-Macaulay ring with a superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that $\text{reg}(R) = \ell_R(R/(\mathbf{x})) - 1$. Set $\overline{R} = R/(\mathbf{x})$ and let \overline{s} be a socle element of highest $\mathfrak{m}\overline{R}$ -adic order.*

For an R -module M with $\ell_R(M) < \ell_R(\overline{R})$, we have

$$\text{Tor}^R(M, \overline{R}/(\overline{s})) \cong \text{Tor}^R(M, \overline{R}) \oplus \Sigma \text{Tor}^R(M, k).$$

In particular, if M is a nonzero, finite length R -module with $\ell_R(M) < \ell_R(\overline{R})$ then $\text{projdim}_R M = \infty$.

Proof. We are exactly in the setting of Lemma 2.6. By Lemma 2.6(2), $\text{cone}(\sigma)$ is the minimal free resolution of $\overline{R}/(\overline{s})$ over R . Hence we have the first isomorphism below:

$$\begin{aligned} \text{Tor}^R(M, \overline{R}/(\overline{s})) &\cong \text{H}(M \otimes_R \text{cone}(\sigma)) \\ &\cong \text{H}(\text{cone}(M \otimes_R \sigma)) \\ &\cong \text{H}(\text{cone}(M \otimes_R F \xrightarrow{0} M \otimes_R A)) \\ &= \text{H}(M \otimes_R A \oplus \Sigma(M \otimes_R F)) \\ &\cong \text{H}(M \otimes_R A) \oplus \Sigma \text{H}(M \otimes_R F) \\ &\cong \text{Tor}^R(M, \overline{R}) \oplus \Sigma \text{Tor}^R(M, k); \end{aligned}$$

the remaining isomorphisms are all immediate except for the third one. This isomorphism is where we use the hypothesis $\ell\ell_R(M) \leq n = \ell\ell_R(\overline{R}) - 1$ and Lemma 2.6(1) to establish the isomorphism of the underlying complexes.

Now by way of contradiction, assume that M is a nonzero finite length R -module of finite projective dimension. By the Auslander-Buchsbaum formula (and Nakayama's lemma), we have that $\text{Tor}_d^R(M, k) \neq 0$ and $\text{Tor}_i^R(M, -) = 0$ whenever $i > d$, where $d = \dim(R)$. However, observe that the already established isomorphism yields

$$\text{Tor}_{d+1}^R(M, \overline{R}/(\overline{s})) \cong \text{Tor}_{d+1}^R(M, \overline{R}) \oplus \text{Tor}_d^R(M, k),$$

giving us a contradiction. Hence, M must have infinite projective dimension. \square

Proof of Theorem 2.1. As discussed in Notation 2.3, we can assume R is a Cohen-Macaulay local ring with infinite residue field, and by our hypothesis, there exists a superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that $\text{reg}(R) = \ell\ell_R(R/(\mathbf{x})) - 1$. Hence we are in the context of Lemma 2.7, and so we obtain

$$\ell\ell_R(M) \geq g\ell\ell_R(R),$$

whenever M has nonzero finite length and finite projective dimension.

Next assume that M is a nonzero Cohen-Macaulay module having finite projective dimension over R . For any system of parameters \mathbf{x} on M , the R -module $M/(\mathbf{x})M$ is a nonzero finite length module with finite projective dimension. Now one need only apply Theorem 2.1 to $M/(\mathbf{x})M$ to deduce

$$\ell\ell_R(M/(\mathbf{x})M) \geq \text{reg}(R) + 1,$$

and when R has infinite residue field $\ell\ell_R(M/(\mathbf{x})M) \geq g\ell\ell_R(R)$. Since these inequalities hold for each system of parameters \mathbf{x} on M , we obtain the desired inequalities. \square

Remark 2.8. For a local ring (R, \mathfrak{m}) , write (R', \mathfrak{m}') for the inflation $R[t]_{\mathfrak{m}[t]}$ of R . We say R has *minimal regularity* if

$$\text{reg}(R') = \ell\ell_{R'}(R'/(\mathbf{x})) - 1$$

for some superficial system of parameters \mathbf{x} in $\mathfrak{m}' \setminus (\mathfrak{m}')^2$. When the ring has an infinite residue field, having minimal regularity is an intrinsic property. That is to say, a local ring (R, \mathfrak{m}) with infinite residue field has minimal regularity if and only if $\text{reg}(R) = \ell\ell_R(R/(\mathbf{x})) - 1$ for some superficial system of parameters \mathbf{x} in $\mathfrak{m} \setminus \mathfrak{m}^2$. Indeed, the backwards direction is clear, while the forward direction is actually a consequence of Theorem 2.1. Tracking through the proof of Theorem 2.1, one sees

that the theorem holds for rings having minimal regularity. It is easy to write down examples of such rings, however we do not know examples of *Cohen-Macaulay* local rings having minimal regularity that are not strict Cohen-Macaulay.

3. GENERALIZED LOEWY LENGTH ALONG FLAT EXTENSIONS

A tremendous amount of research (see, for example, [23, 30, 31, 32] and the references therein) in commutative algebra has been motivated by the longstanding conjecture of Lech [29]: For a flat local extension $R \rightarrow S$, one has the following inequality on Hilbert-Samuel multiplicities:

$$e(R) \leq e(S).$$

Ma has established the conjecture for equicharacteristic rings of dimension at most three [31], and for all standard graded algebras localized at their homogeneous maximal ideal [32]. In light of Remark 1.2 and Lech's conjecture, we are led to (perhaps optimistically) conjecture the following.

Conjecture 3.1. *If $R \rightarrow S$ is a flat local extension between Cohen-Macaulay rings with infinite residue fields, then $gll_R(R) \leq gll_S(S)$.*

By combining [13, Theorem 2.1] and [19, Corollary 5.2], the conjecture is already known to hold when $R \rightarrow S$ is a flat local extension of Gorenstein, strict Cohen-Macaulay rings having infinite residue fields and a regular fiber. We give a substantial improvement below.

Theorem 3.2. *Suppose $R \rightarrow S$ is a local extension of Cohen-Macaulay rings with infinite residue fields, and assume $\text{flatdim}_R(S) < \infty$. If R is strict Cohen-Macaulay, then $gll_R(R) \leq gll_S(S)$. In particular, Conjecture 3.1 holds in this setting.*

Proof. By [19, Lemma 3.3] generalized Loewy length is invariant upon completion, so we can assume both R and S are complete. Now by [6, Theorem 1.1], there exists a Cohen factorization of φ :

$$R \xrightarrow{\iota} R' \xrightarrow{\varphi'} S$$

where ι weakly regular (that is to say, ι is flat with regular fiber) and φ' is surjective.

As ι is weakly regular, the map $\iota^{\mathfrak{g}}: R^{\mathfrak{g}} \rightarrow (R')^{\mathfrak{g}}$ is tangentially flat, in the sense that it is a flat extension of standard graded algebras with a symmetric algebra fiber; see [20, Theorem 1.2]. In particular, R' is strict Cohen-Macaulay. As a consequence, the second equality below holds:

$$(1) \quad gll_R(R) = \text{reg}(R) + 1 = \text{reg}(R') + 1 = gll_{R'}(R');$$

the outside equalities are from Corollary 2.2. Again, using that ι is weakly regular and $\text{flatdim}_R(S) < \infty$ we have that $\text{projdim}_{R'}(S) < \infty$; see [6, Lemma 3.2]. Also, since S is Cohen-Macaulay it follows that it is a Cohen-Macaulay R' -module. Therefore, we have

$$gll_S(S) \geq gll_{R'}(R') = gll_R(R),$$

where inequality uses Theorem 2.1, and the equality comes from (1). □

With Lemma 1.7, we obtain an immediate corollary in the following, where we refer the reader to the nice introduction on reductions and reduction numbers in [24, Chapter 8]. Furthermore, the next corollary verifies a conjecture of Hanes [18, Conjecture 3.1] when R is strict Cohen-Macaulay: *If $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a local*

extension of Cohen-Macaulay rings with infinite residue fields, then $r(\mathfrak{m}) \leq r(\mathfrak{n})$. Hanes had previously established the conjecture in the standard graded setting.

Corollary 3.3. *Suppose $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is a local extension of Cohen-Macaulay rings with infinite residue fields, and assume $\text{flatdim}_R(S) < \infty$. If R is strict Cohen-Macaulay, then $r(\mathfrak{m}) \leq r(\mathfrak{n})$ and $\text{reg}(R) \leq \text{reg}(S)$.*

Remark 3.4. The *Length Conjecture* of Iyengar–Ma–Walker [26, Conjecture 1] implies Lech’s conjecture in full generality for Cohen-Macaulay rings; see also [30, Chapter V]. The former posits that over a local ring R , any nonzero module of finite projective dimension has $\ell_R(M) \geq e(R)$. This is known to hold when R is a strict complete intersection (that is, the associated graded is a complete intersection) or when R is a localization of a standard graded algebra, at its homogeneous maximal ideal, over a perfect field of positive characteristic [26]. The Loewy Length Conjecture (Conjecture A) is the analog of the Length Conjecture; cf. Remark 1.2. Hence, it seems appropriate to ask the following.

Question 3.5. If the Loewy Length Conjecture holds for all Cohen-Macaulay rings having infinite residue field, does this imply Conjecture 3.1 holds?

The only thing to determine to answer the question in the positive is whether the generalized Loewy length remains the same along a weakly regular map between Cohen-Macaulay rings having infinite residue field; without infinite residue fields this is false because of the example of Hashimoto–Shida [19].

4. GENERAL BOUNDS

In this section, we show that there are lower bounds for the Loewy lengths of nonzero modules of finite projective dimension without any assumption on the associated graded ring. We give a significant strengthening of known results in Theorem 4.3, and then present an even stronger bound for local complete intersection rings in Theorem 4.6.

4.1. For a Gorenstein local ring R , if M is a nonzero finite length module having finite projective dimension, then

$$\ell\ell_R(M) \geq \text{ord}(R);$$

here $\text{ord}(R)$, the order of R , is minimal \mathfrak{m}_Q -adic order of the kernel of a minimal Cohen presentation $(Q, \mathfrak{m}_Q) \rightarrow \widehat{R}$. This was first established in the case that R is also assumed to be strict Cohen-Macaulay with infinite residue field in [5, Theorem 1.1], and for general Gorenstein rings in [33, Theorem 1.1].

4.2. For an element f in a local ring (R, \mathfrak{m}) we write $\text{ord}(f)$ for its \mathfrak{m} -adic order. Define the *max-order of R* , denoted $\max \text{ord}(R)$, to be the supremum over all nonnegative integers n such that R admits a minimal Cohen presentation $\widehat{R} \cong Q/I$ where I has a minimal generator f of order n . We always have inequalities

$$\text{ord}(R) \leq \max \text{ord}(R) \leq \text{reg}(R) + 1,$$

where the first is by definition. The second inequality holds since $\max \text{ord}(f)$ is the degree of a minimal homogeneous generator for the kernel of a surjective map from a standard graded polynomial ring to R^{e} .

Next is one of the main result of the section; this removes the Gorenstein hypothesis and improves the bound in 4.1 in a rather drastic way. The proof is similar to that of [28, Proposition 1.2] which identifies homology classes in Tor-modules depending on the Loewy length of a module.

Theorem 4.3. *Assume R is a local ring, and fix a minimal Cohen presentation $\widehat{R} \cong Q/(f_1, \dots, f_t)$ and a nonzero finite length R -module M .*

- (1) *If $\ell\ell_R(M) < \text{ord}(R)$, then $\text{cx}_R(M) \geq t$.*
- (2) *If M has finite projective dimension, then $\ell\ell_R(M) \geq \max \text{ord}(R)$.*

Above, recall the complexity of an R -module M is

$$\text{cx}_R(M) = \inf\{d \in \mathbb{N} \mid \beta_n^R(M) \leq an^{d-1} \text{ for some } a > 0 \text{ and all } n\};$$

this is a measure of the polynomial growth of the Betti numbers of M over R . In particular $\text{cx}_R(M) = 0$ if and only if $\text{projdim}_R(M) < \infty$. See [3, Chapter 5] for more on this and other asymptotic homological invariants defined over local rings; see also [7].

Proof of Theorem 4.3. We make use of a well-known dg algebra structure on the minimal free resolution of the residue field k over R ; see [3, Chapter 6], or [17], for more details and any unexplained notation or terminology.

The essential point is that the minimal free resolution of k over R has the form $R\langle X \rangle$ where as a graded R -algebra it is the free divided power algebra with $X = X_1, X_2, X_3, \dots$, and X_i consists of variables of degree i . The differential is determined by its values on the variables and extended via the Leibniz rule and respecting divided powers. We are particularly interested in the differential on the divided power subalgebra on X_2 , which can be described explicitly as follows.

There is no harm in assuming R is complete and so using the minimal Cohen presentation $R = Q/(f_1, \dots, f_t)$, fix a minimal set of generators x_1, \dots, x_d of \mathfrak{m}_Q , and write

$$f_i = \sum \tilde{a}_{ij} x_j \quad \text{with } \tilde{a}_{ij} \in \mathfrak{m}_Q.$$

Then $X_1 = \{e_1, \dots, e_d\}$, $X_2 = \{y_1, \dots, y_t\}$ and we have

$$\partial(y_i) = \sum a_{ij} e_j \quad \text{for each } i = 1, \dots, t$$

where a_{ij} is the image of \tilde{a}_{ij} in \mathfrak{m} , the maximal ideal of R ; this calculation is classical and due to Tate [37, Theorem 4], but it is also explained in the references above. As a consequence, the differential on the divided power monomial $y_1^{(c_1)} \dots y_t^{(c_t)}$ is

$$(2) \quad \partial(y_1^{(c_1)} \dots y_t^{(c_t)}) = \sum_{i=1}^t \sum_{j=1}^d a_{ij} e_j y_1^{(c_1)} \dots y_i^{(c_i-1)} \dots y_t^{(c_t)}.$$

Now assume $\ell\ell_R(M) < \text{ord}(R) = n$, then in the Cohen presentation above each $f_i \in \mathfrak{m}_Q^n$ and so we can assume $a_{ij} \in \mathfrak{m}^{n-1}$. In particular, from (2) it follows that

$$\partial(y_1^{(c_1)} \dots y_t^{(c_t)}) \otimes m = \sum_{i=1}^t \sum_{j=1}^d a_{ij} e_j y_1^{(c_1)} \dots y_i^{(c_i-1)} \dots y_t^{(c_t)} \otimes m = 0$$

as each $a_{ij} \in \mathfrak{m}^{n-1} \subseteq \text{ann}_R(M)$. Hence, for any $m \in M \setminus \mathfrak{m}M$, we have a cycle

$$y_1^{(c_1)} \dots y_t^{(c_t)} \otimes m$$

that cannot be a boundary as $\partial(R\langle X \rangle \otimes_R M) \subseteq \mathfrak{m}R\langle X \rangle \otimes_R M$. Therefore, we have obtained the inequality below:

$$\beta_i^R(M) = \text{rank}_k \text{Tor}_i^R(k, M) = \text{rank}_k H_i(R\langle X \rangle \otimes_R M) \geq \text{rank}_k(\Gamma_i \otimes_k M/\mathfrak{m}M)$$

where $\Gamma = k\langle X_2 \rangle$, the free divided power algebra on the degree two variables X_2 . The Hilbert series for Γ (over k) is $(1-z^2)^{-t}$ and so the desired result on complexity has been established.

Next, assume $\ell\ell_R(M) < \max \text{ord}(R) = n$, then in the Cohen presentation above at least one f_i belongs to \mathfrak{m}_Q^n and so for that i , we can assume $a_{ij} \in \mathfrak{m}^{n-1}$ for each j . In this case, from (2) it follows that

$$\partial(y_i^{(c)} \otimes m) = \sum_{j=1}^t a_{ij} e_j y_i^{(c)} \otimes m = 0$$

as each $a_{ij} \in \mathfrak{m}^{n-1} \subseteq \text{ann}_R(M)$. Again, the minimality of $R\langle X \rangle \otimes_R M$ gives us cycles that are not boundaries in each even degree:

$$\{y_i^{(c)} \otimes m \mid c \geq 0 \text{ and } m \in M \setminus \mathfrak{m}M\}.$$

Finally, recalling the homology of $R\langle X \rangle \otimes_R M$ is $\text{Tor}^R(k, M)$ we have shown that $\text{projdim}_R(M) = \infty$, as claimed. \square

Remark 4.4. Theorem 4.3 shows that when R is a local complete intersection ring and M is a nonzero R -module with $\ell\ell_R(M) < \text{ord}(R)$, then M is extremal in the sense of Avramov [2]. Roughly speaking, this means the Betti sequence of M grows at the same rate as the Betti sequence of the residue field (the latter has maximal growth, in a precise sense, among all finitely generated R -modules). In light of this and Theorem 4.3, we ask the following.

Question 4.5. For a local ring R , if a nonzero R -module M has $\ell\ell_R(M) < \text{ord}(R)$, is M extremal?

We end the article with a lower bound for Loewy lengths over complete intersection rings. We give two proofs: the first proof is closer to the proof of Theorem 2.1, while the second proof, suggested to us by Mark Walker, uses a construction also used to establish the Length Conjecture for strict complete intersection rings in [30, Corollary V.29].

Theorem 4.6. *Assume R is a complete intersection ring of codimension c . If M is a nonzero Cohen-Macaulay module of finite projective dimension, then*

$$g\ell\ell_R(M) \geq \sum_{i=1}^c \text{ord}(f_i) - c + 1$$

where $\widehat{R} \cong Q/(f_1, \dots, f_c)$ is any minimal Cohen presentation of R .

Remark 4.7. The bound in Theorem 4.6 is usually lower than $g\ell\ell_R(R)$ of a local complete intersection ring that is not strictly Cohen-Macaulay. For example, in [12], DeStefani showed that the following one dimensional complete intersection k -algebra, with k a field,

$$R = \frac{k[[x, y, z]]}{(x^2 - y^5, xy^2 + yz^3 - z^5)}$$

has $g\ell_R(R) = 6$ and a calculation, using `Macaulay2`, shows $\text{reg}(R) = 6$. In particular, R cannot be strict Cohen-Macaulay; one can also see this by calculating $R^{\mathfrak{g}}$. In fact, by Theorem 2.1, it follows that R does not have minimal regularity (in the sense defined in Remark 2.8). Here, the bound from Theorem 4.6 is

$$\text{ord}(x^2 - y^5) + \text{ord}(xy^2 + yz^3 - z^5) - 1 = 4.$$

Remark 4.8. In Theorem 4.6, each $\text{ord}(f_i)$ is at least two and so Theorem 4.6 implies $\ell\ell_R(M) \geq c+1$. Hence the bound from Theorem 4.6 strengthens the already known bounds for *modules* over complete intersection rings in [5]; the bounds from the latter are known to hold for the sum of the Loewy lengths of the homology modules of perfect complexes, and is tight as the Koszul complex K on the maximal ideal always has $\sum \ell\ell_R H_i(K) = c + 1$.

First proof of Theorem 4.6. First, we need a bit of notation; see [3] for background on dg modules in commutative algebra.

We can assume R is complete and hence $R \cong Q/(f_1, \dots, f_c)$ with f_1, \dots, f_c a regular sequence in the regular ring (Q, \mathfrak{m}_Q) . Let $\mathfrak{m}_Q = (t_1, \dots, t_c, x_1, \dots, x_d)$ where the image of x_1, \dots, x_d is a maximal regular sequence in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$, and write

$$f_i = \sum_{j=i}^c a_{ij}t_j + \sum_{j=1}^d b_{ij}x_j$$

with each a_{ij} and b_{ij} belonging to $\mathfrak{m}_Q^{\text{ord}(f_i)-1}$. Set

$$\begin{aligned} E &= \text{Kos}^Q(\mathbf{f}) = Q\langle e_1, \dots, e_c \mid \partial e_i = f_i \rangle \\ \tilde{A} &= E \otimes_Q \text{Kos}^Q(\mathbf{x}) = Q\langle e_1, \dots, e_c, e'_1, \dots, e'_d \mid \partial e_i = f_i, \partial e'_i = x_i \rangle \\ K &= \text{Kos}^Q(\mathbf{t}, \mathbf{x}) = Q\langle e''_1, \dots, e''_c, e'_1, \dots, e'_d \mid \partial e''_i = t_i, \partial e'_i = x_i \rangle, \end{aligned}$$

which are dg E -modules; the E -actions on E, \tilde{A} are the obvious ones and the E action on K is given by the map on dg Q -algebras $E \rightarrow K$ determined by

$$e_i \mapsto \sum_{j=i}^c a_{ij}e''_j + \sum_{j=1}^d b_{ij}e'_j.$$

Finally, consider the map $\alpha: \tilde{A} \rightarrow K$ of dg E -modules extending the map above and sending each e'_j in \tilde{A} to e'_j in K .

Note that $\alpha_i = \bigwedge^i \alpha_1: \tilde{A}_i \rightarrow K_i$ and now using that each a_{ij}, b_{ij} is in $\mathfrak{m}_Q^{\text{ord}(f_i)-1}$, a direct calculation shows

$$(3) \quad \alpha(E_c \otimes_Q \text{Kos}^Q(\mathbf{x})) \subseteq \mathfrak{m}_Q^n K \quad \text{where} \quad n = \sum_{i=1}^c \text{ord}(f_i) - c.$$

Also, it is well known (see, for instance, [15, Exercise 21.23]) that the cone of

$$\Sigma^{c+d}\alpha^\vee: \Sigma^{c+d}K^\vee \rightarrow \Sigma^{c+d}\tilde{A}^\vee,$$

is a Q -free resolution of $R/(\mathbf{x}, \det(a_{ij}))$ where $(-)^{\vee} = \text{Hom}_Q(-, Q)$. Using the self-duality of Koszul complexes, $\Sigma^{c+d}\alpha^\vee$ can be regarded as a dg E -module map from K to \tilde{A} ; write $\tilde{\sigma}: K \rightarrow \tilde{A}$ for this map using these identifications of each Koszul complex

with its dual. By (3), composing $\tilde{\sigma}$ with the projection of $\tilde{A} \rightarrow E_0 \otimes_Q \text{Kos}^Q(\mathbf{x})$ factors as the map of graded Q -modules:

$$(4) \quad \begin{array}{ccc} K & \xrightarrow{\tilde{\sigma}} & \tilde{A} \\ \downarrow & & \downarrow \\ \mathfrak{m}_Q^n E_0 \otimes_Q \text{Kos}^Q(\mathbf{x}) & \hookrightarrow & E_0 \otimes_Q \text{Kos}^Q(\mathbf{x}); \end{array}$$

here we used that the identification of $\Sigma^{c+d}\tilde{A}$ with \tilde{A} , restricts to an isomorphism

$$\Sigma^{c+d}(E_c \otimes_Q \text{Kos}^Q(\mathbf{x}))^\vee \cong \Sigma^c E_c^\vee \otimes_Q \Sigma^d (\text{Kos}^Q(\mathbf{x}))^\vee \cong E_0 \otimes_Q \text{Kos}^Q(\mathbf{x}).$$

Now letting $\Gamma := R\langle y_1, \dots, y_c \rangle$ where each y_i is a degree two divided power variable, $1 \otimes \tilde{\sigma}: \Gamma \otimes_Q^\tau K \rightarrow \Gamma \otimes_Q^\tau \tilde{A}$ is a map of R -complexes; here, for a dg E -module N , the R -complex $\Gamma \otimes_Q^\tau N$ is the construction of Eisenbud [14] and Shamash [36] (see also [4, Section 2]). Explicitly, $\Gamma \otimes_Q^\tau N$ is the free graded R -module $\Gamma \otimes_Q N$ with differential

$$y_1^{(h_1)} \dots y_c^{(h_c)} \otimes n \mapsto y_1^{(h_1)} \dots y_c^{(h_c)} \otimes \partial^N(n) + \sum_{i=1}^c y_1^{(h_1)} \dots y_i^{(h_i-1)} \dots y_c^{(h_c)} \otimes e_i n.$$

Set $F = \Gamma \otimes_Q^\tau K$, which by [4, Theorem 2.4] (see also [37]), is the minimal free resolution of k over R . Also, there is an isomorphism of complexes

$$\Gamma \otimes_Q^\tau \tilde{A} \cong R\langle e_1, \dots, e_c, e'_1, \dots, e'_d, y_1, \dots, y_c \mid \partial e_i = 0, \partial e'_i = x_i, \partial y_i = e_i \rangle$$

and so we have quasi-isomorphisms

$$\Gamma \otimes_Q^\tau \tilde{A} \xrightarrow{\cong} R \otimes_E \tilde{A} \cong \text{Kos}^R(\mathbf{x}) \xrightarrow{\cong} R/(\mathbf{x}).$$

Set $A = R \otimes_E \tilde{A}$ and let σ be the composition of chain maps

$$F = \Gamma \otimes_Q^\tau K \xrightarrow{1 \otimes \tilde{\sigma}} \Gamma \otimes_Q^\tau \tilde{A} \xrightarrow{\cong} A.$$

Forgetting differentials, σ factors as the following map of graded Q -graded modules:

$$\begin{array}{ccc} F & \xrightarrow{1 \otimes \tilde{\sigma}} & \Gamma \otimes_Q^\tau \tilde{A} \\ \downarrow & \searrow \sigma & \downarrow \\ R \otimes_Q K & \xrightarrow{1 \otimes \sigma} & R \otimes_Q \tilde{A} \twoheadrightarrow R \otimes_E \tilde{A} = A \end{array}$$

and so by (4) we have that $\sigma(F) \subseteq \mathfrak{m}_R^n A$. Also, observe that $H_0(\sigma): k \hookrightarrow R/(\mathbf{x})$, is given by $1 \mapsto \det(a_{ij})$, and so $\text{cone}(\sigma)$ is a free resolution of $N := R/(\mathbf{x}, \det(a_{ij}))$; cf. the proof of Lemma 2.6(1). Therefore, the same argument used to establish Lemma 2.7 shows that for each i we have isomorphisms

$$\text{Tor}_i^R(M, N) \cong \text{Tor}_i^R(M, R/(\mathbf{x})) \oplus \text{Tor}_{i-1}^R(M, k)$$

whenever M is a finite length module with $\ell_R(M) \leq n$; hence, if M is also assumed to be nonzero and of finite projective dimension we reach a contradiction through applications of the Auslander–Buchsbaum formula (again, as argued in Lemma 2.7). It remains to repeat the argument from Corollary 2.2 to establish the desired lower bound for perfect modules of positive dimension. \square

Before presenting the second proof of Theorem 4.6, we need the following construction from [21]; see also [30, Theorem V.27].

4.9. Let (Q, \mathfrak{m}) be a Cohen-Macaulay ring and $f \in \mathfrak{m}^d$ a regular element. Then for some integer $s \geq 1$, there exists a filtration of $R = Q/(f)$ -modules

$$0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_d = R^s$$

such that $U_{i-1} \subseteq \mathfrak{m}U_i$ and each U_i/U_{i-1} is a maximal Cohen-Macaulay (abbreviated, as usual, to MCM) R -module having finite projective dimension over Q . The condition on U_i/U_{i-1} is equivalent to $\text{projdim}_Q(U_i/U_{i-1}) = 1$.

Lemma 4.10. *Suppose (Q, \mathfrak{m}) is a Cohen-Macaulay ring and $R = Q/(f_1, \dots, f_c)$ where f_1, \dots, f_c is a Q -regular sequence. Set*

$$n = \sum_{i=1}^c \text{ord}(f_i) - c + 1.$$

Then for some integer $t \geq 1$, there exists a filtration of R -modules

$$0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = R^t$$

such that $U_{i-1} \subseteq \mathfrak{m}U_i$ and U_i/U_{i-1} is an MCM R -module for $1 \leq i \leq n$.

Proof. We induct on c . The base case follows by 4.9. Let $R' = Q/(f_1, \dots, f_{c-1})$ and

$$n' = \sum_{i=1}^{c-1} \text{ord}(f_i) - (c-1) + 1.$$

By the induction hypothesis, for some $t' \geq 1$, we have a filtration of R' -modules

$$0 = U'_0 \subseteq U'_1 \subseteq \dots \subseteq U'_{n'} \cong (R')^{t'}$$

where $U'_{i-1} \subseteq \mathfrak{m}U'_i$ and U'_i/U'_{i-1} is an MCM R' -module. Since R' is Cohen-Macaulay and $f = f_c$ regular on R' in $\mathfrak{m}^{\text{ord}(f)}R'$, we can apply 4.9 to obtain a filtration of $R = R'/(f)$ -modules

$$0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_{\text{ord}(f)} \cong R^t$$

for some $t \geq 1$ where each $V_{i-1} \subseteq \mathfrak{m}V_i$ and $\text{projdim}_{R'}(V_i/V_{i-1}) = 1$; in particular, $\text{projdim}_{R'} V_1 = 1$.

We claim that the following filtration of R -modules of length n has all the desired properties:

$$U'_0 \otimes V_1 \subseteq U'_1 \otimes V_1 \subseteq \dots \subseteq U'_{n'} \otimes V_1 \subseteq U'_{n'} \otimes V_2 \subseteq \dots \subseteq U'_{n'} \otimes V_{\text{ord}(f)}.$$

Indeed, $U'_0 \otimes V_1 = 0$ and $U'_{n'} \otimes V_{\text{ord}(f)} \cong R^{tt'}$, and for $i \geq 1$ every term in the filtration is of the form

$$U'_{n'} \otimes V_i \cong (R')^{t'} \otimes V_i \cong V_i^{t'};$$

as a consequence, there are the desired containments $U'_{n'} \otimes V_{i-1} \subseteq \mathfrak{m}U'_{n'} \otimes V_i$ and

$$\text{projdim}_Q(U'_{n'} \otimes V_i/U'_{n'} \otimes V_{i-1}) = \text{projdim}_Q(V_i^{t'}/V_{i-1}^{t'}) = c$$

where the last equality used $\text{projdim}_{R'}(V_i/V_{i-1}) = 1$; thus each of these subquotients are MCM R -modules. Therefore, it remains to verify the first n' -steps satisfy the desired properties as well.

To this end, for each i , as U'_i/U'_{i-1} is MCM over R' and $\text{projdim}_{R'} V_1 < \infty$ it follows that $\text{Tor}_{>0}^{R'}(U'_i/U'_{i-1}, V_1) = 0$ and hence the maps along the bottom are injective:

$$\begin{array}{ccc} & \mathfrak{m}U'_i \otimes_{R'} V_1 & \\ & \nearrow & \searrow \\ U'_{i-1} \otimes_{R'} V_1 & \xrightarrow{\quad} & U'_i \otimes_{R'} V_1 \end{array}$$

and so $U'_{i-1} \otimes V_1 \subseteq \mathfrak{m}U'_i \otimes V_1 = \mathfrak{m}(U'_i \otimes V_1)$. It only remains to observe

$$\begin{aligned} \text{depth}(U'_i/U'_{i-1}) &= \text{depth}(U'_i/U'_{i-1} \otimes_{R'}^{\mathfrak{L}} V_1) + \text{projdim}_{R'}(V_1) \\ &= \text{depth}(U'_i/U'_{i-1} \otimes_{R'} V_1) + 1 \\ &= \text{depth}(U'_i \otimes V_1/U'_{i-1} \otimes V_1) + 1 \end{aligned}$$

where the first equality uses the derived depth formula [25, Corollary 2.2], and second equality uses that $\text{projdim}_{R'}(V_1) = 1$ and $\text{Tor}_{>0}^{R'}(U'_i/U'_{i-1}, V_1) = 0$. We need only note that U'_i/U'_{i-1} is an MCM R' -module to deduce that $U'_i \otimes V_1/U'_{i-1} \otimes V_1$ is MCM over R . \square

Second proof of Theorem 4.6. We can assume $R \cong Q/(f_1, \dots, f_c)$ with f_1, \dots, f_c a regular sequence in the regular ring Q . By Lemma 4.10, there exists a filtration by R -submodules

$$0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = R^t, \quad \text{with } n = \sum_{i=1}^c \text{ord}(f_i) - c + 1,$$

such that each $U_{i-1} \subseteq \mathfrak{m}U_i$ and U_i/U_{i-1} is an MCM R -module, and $t \geq 1$. Arguing as in the proof of Lemma 4.10, since $\text{projdim}_R M < \infty$ and each subquotient of the filtration is an MCM R -module, we obtain inclusions $U_{i-1} \otimes M \subseteq \mathfrak{m}(U_i \otimes M)$ for each i . In particular,

$$0 \neq U_1 \otimes M \subseteq \mathfrak{m}U_2 \otimes M \subseteq \dots \subseteq \mathfrak{m}^{n-1}U_n \otimes M = \mathfrak{m}^{n-1}M^t,$$

and thus, $\ell\ell_R(M) \geq n$. Again, it remains to repeat the argument from Corollary 2.2 to establish the desired lower bound for perfect modules of positive dimension. \square

In fact, the second proof establishes a bound in the relative setting. That is to say, when a Cohen-Macaulay local ring R admits a deformation, there is the following uniform lower bound on the Loewy length of nonzero modules of finite projective dimension.

Theorem 4.11. *Assume a Cohen-Macaulay local ring R is a deformation:*

$$R \cong Q/(f_1, \dots, f_c) \quad \text{with } f_1, \dots, f_c \text{ a regular sequence.}$$

For any nonzero Cohen-Macaulay module M of finite projective dimension, we have

$$g\ell_R(M) \geq \sum_{i=1}^c \text{ord}(f_i) - c + 1. \quad \square$$

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