A SIMPLE EXAMPLE OF PATHOLOGICAL FOLIATIONS IN SKEW-PRODUCT DIFFEOMORPHISMS

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Abstract. Inspired by examples of Katok and Milnor [2], we construct a simple example of skew-product volume preserving diffeomorphism where the center foliation is pathological in the sense that, there is a full measure set whose intersection with any center leaf contains at most one point.

1. Introduction

An interesting and intriguing phenomenon in dynamical systems is the pathological foliations. Roughly speaking, a foliation is pathological if there is a full volume set that meets every leaf of the foliation on a set of leaf-volume zero. In fact, there are examples with a full measure set that intersects each leaf only in a finite number of points. In [2], J. Milnor constructed an example, inspired by A. Katok, of such a non-absolutely continuous foliation on the unit square. In [6], M. Shub and A. Wilkinson found the same phenomenon on $\mathbb{T}^3$ with a different approach. They termed this phenomenon aptly *"Fubini's nightmare"*. Surprisingly, this phenomenon is persistant and robust under perturbations. We refer readers to a survey [4] written by F. R. Hertz, M. R. Hertz and R. Ures for more details. Saghin and Xia [5], using a different mechanism, showed more examples of systems with persistent non-absolute continuous center and weak unstable foliations, where these foliations are not necessarily compact. Pesin [3] also showed some examples of a non-absolutely continuous foliation in his book.

In this paper, we constructs a simple example of skew-product diffeomorphism where the center foliation is pathological. And our main theorem is as follows:

Theorem 1.1. *There exists a full measure set E on $[0,1]\times\mathbb{T}^2$, together with a family of disjoint curves Γ_β which fill out $[0,1]\times\mathbb{T}^2$, so that each curve Γ_β intersects the set E at most a single point.*

The idea of the proof of this theorem is inspired by Milnor's proof of Katok's paradoxical example, see [2]. We construct a path of Anosov area-preserving diffeomorphism f_p, $p\in[0,1]$ on $\mathbb{T}^2$ beginning with Arnold's cat map. And we construct Markov partitions for each f_p such that each Markov partition has two rectangles and the Lebesgue measure of one rectangle is

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strictly increasing with respect to p. Then using a (semi)conjugacy from a shift space, we can construct a full measure set E and a family of disjoint curves Γ_β satisfying the conditions in the theorem.

Different from all other examples in [6], [5], [3], our construction is elementary, involving no Lyapunov exponents.

In section 2, we discuss about some definitions and properties of Anosov diffeomorphisms and Markov partitions and especially introduce Arnold's cat map. In section 3, we prove our main theorem. The main idea is to perturb the Arnold's cat map in such a way that the areas of Markov petitions of the perturbed map are different.

2. Preliminaries

Our proof of the main theorem uses some basic properties of a hyperbolic toral automorphism which is an Anosov diffeomorphism. We will introduce some basic definitions in this section. The content of this section is based on Brin and Stuck's book [1].

Definition 2.1. Let M be a smooth Riemannian manifold and $f : M \to M$ be a diffeomorphism. A compact, f-invariant subset $\Lambda \subset M$ is call *hyperbolic* if there are $\lambda \in (0,1), C > 0$, and families of subspaces $E^s(x) \subset T_xM$ and $E^u(x) \subset T_xM$, $x \in \Lambda$, such that for every $x \in \Lambda$,

(1) $T_xM = E^s(x) \oplus E^u(x)$,
(2) $\|df_x^n v^s\| \le C\lambda^n \|v^s\|$ for every $v^s \in E^s(x)$ and $n \ge 0$,
(3) $\|df_x^{-n} v^u\| \le C\lambda^n \|v^u\|$ for every $v^u \in E^u(x)$ and $n \ge 0$,
(4) $df_x E^s(x) = E^s(f(x))$ and $df_x E^u(x) = E^u(f(x))$.

In particular, if $\Lambda = M$, then f is called an *Anosov diffeomorphism*

Hyperbolic toral automorphisms are examples of Anosov diffeomorphisms. Explicitly,

Definition 2.2. Let $M = \mathbb{T}^n$. Consider an $n \times n$ matrix A with determinant one and with integer entries. The matrix A induces a *toral automorphism*: $f_A : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{T}^n$ defined by $f_A x = Ax \bmod \mathbb{Z}^n$.

Moreover, if all eigenvalues of A are away from the unit circle, then f_A is a *hyperbolic toral automorphism*.

The best know example of a hyperbolic automorphism is Arnold's cat map that is $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

And the eigenvalues of A is $\lambda = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda^{-1} = \frac{3-\sqrt{5}}{2}$. The corresponding eigenvectors are $\nu_\lambda = (1, \frac{\sqrt{5}-1}{2})$ and $\nu_{\lambda^{-1}} = (1, \frac{-\sqrt{5}-1}{2})$.

To figure out the features of the Arnold's cat map, we need to introduce more concepts.

Definition 2.3. Let $f : M \to M$ be an Anosov diffeomorphism. For every $x \in M$, the *(global) stable* and *unstable manifolds* of x are defined by

$$W^s(x) := \{y \in M \colon \operatorname{dist}(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x) := \{y \in M \colon \operatorname{dist}(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

And for any $\rho > 0$, we define

$$W^s(x, \rho) := \{y \in M : \operatorname{dist}(f^n(x), f^n(y)) < \rho, \forall n \in \mathbb{N}_0\},$$
$$W^u(x, \rho) := \{y \in M : \operatorname{dist}(f^{-n}(x), f^{-n}(y)) < \rho, \forall n \in \mathbb{N}_0\}.$$

With these definitions, we can check that if x_0 is a fixed point of f, we have

$$W^s(x_0) = \bigcup_{n\in\mathbb{N}} (f)^{-n} \left(W^s(x_0, \rho)\right)$$
$$W^u(x_0) = \bigcup_{n\in\mathbb{N}} (f)^{n} \left(W^u(x_0, \rho)\right).$$

Another concept that we would like to introduce is called a Markov partition. It allows us to study the dynamics of f using symbolic dynamics.

Definition 2.4. Let $f : M \to M$ be an Anosov diffeomorphism. A collection of subset of M, $\mathcal{R} = \{R_1, \dots, R_n\}$ is called a *Markov partition* for (M, f) if

(1) $\operatorname{cl}(\operatorname{int} R_i) = R_i$ for each R_i;
(2) $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ for $i \neq j$;
(3) $M = \bigcup_i R_i$;
(4) If $x \in \operatorname{int} R_i, f(x) \in R_j$, then

$$f(W^s(x, R_i)) \subseteq W^s(f(x), R_j), \quad f^{-1}(W^u(f(x), R_j)) \subseteq W^u(x, R_i),$$

where $W^s(x, R) := W^s(x, \rho) \cap R$ and $W^u(x, R) := W^u(x, \rho) \cap R$, and $\rho > 0$ satisfies $W^s(x, \rho) \cap W^u(y, \rho)$ is a single point in R for all points x, y in R.

As an example, we can construct a Markov partition for Arnold's cat map f_A by draw segments of stable and unstable manifolds of the fixed point $(0, 0)$ until they cross sufficiently many times and separate $\mathbb{T}^2$ into two disjoint rectangles R_0 and R_1: R_0 consists of two parts B_1 and B_2; R_1 consists of three parts A_1, A_2 and A_3. See figure 1.

If we consider the fundamental domain in Figure 1 as $[0, 1] \times [0, 1]$, we would like to mention that the preimage of the segment p_0p_3 is the segment $\{(x, y) \in p_0p_3 \big| x \in [1 - \frac{2}{5+\sqrt{5}}, 1]\}$. The preimage of the segment p_0p_2 is contained in the segment $\{(x, y) \in p_0p_2 \big| x \in [0, 1 - \frac{2}{5+\sqrt{5}}]\}$. And the image of the segment p_0p_1 is contained in p_0p_1 where the x-coordinate of p_1 is less than $1 - \frac{2}{5+\sqrt{5}}$.

Thus, if we perturb Arnold's cat map into f_p in the region $\{(x, y) \big| x \in (a, b)\}$, where $1 - \frac{2}{5+\sqrt{5}} < a < b < 1$. Then $(0, 0)$ is still a fixed point of f_p and if we draw stable manifold and unstable manifold of $(0, 0)$, we can get

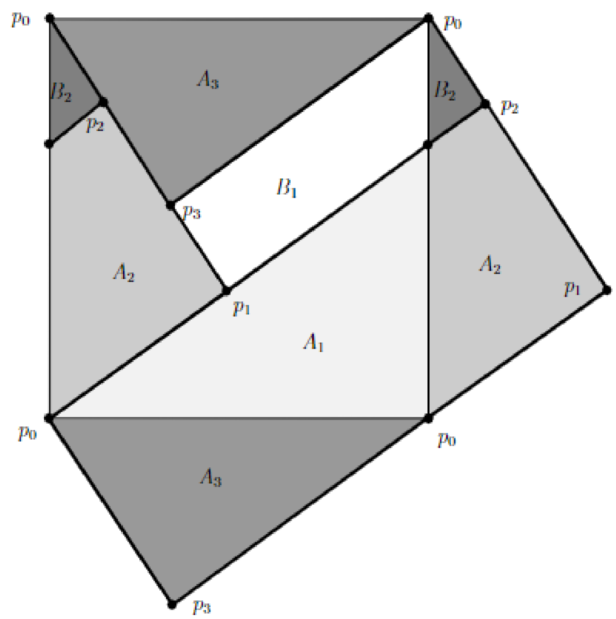

FIGURE 1. Markov partition of Arnold's cat map

p_0p_2 and p_0p_1 will not change. And p_0p_3 will be perturbed for the perturbed map f_p. This inspires us the proof of our main theorem.

3. Proof of Main theorem

Step 1: Our first step is to construct a continuous path of area-preserving Anosov diffeomorphism f_p, $p \in [0,1]$ on $\mathbb{T}^2$ beginning with the Arnold's cat map.

Let f_A be the Arnold's cat map on $\mathbb{T}^2$ that is

$$f_A(x,y) = (2x+y, x+y) \bmod 1.$$

Take real number $1 - \frac{2}{5+\sqrt{5}} < a < 1$ and $\delta > 0$ small enough such that $1 - \frac{2}{5+\sqrt{5}} < a - \delta < a + \delta < 1$. Then define a bump function $C(x)$ on $[0,1]$ such that

(1) $C(x) \equiv 0$ on $[0, a-\delta] \cup [a+\delta, 1]$;
(2) $0 < C(x) \leq 1$ on $(a-\delta, a+\delta)$;

Now, for any $p \in [0,1]$, we define a function $\phi_p(x)$ on $[0,1]$ such that $\phi_p(x) = p\epsilon_0 C(x)$. Then we define a diffeomorphism f_p on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that

$$f_p(x,y) = (2x+y-\phi_p(x), x+y-\phi_p(x)) \bmod 1.$$

Note that $f_0 = f_A$. And for a fixed sufficiently small $\epsilon_0 > 0$, we can see that for any $p \in [0,1]$, f_p is an Anosov diffeomorphism since a small perturbations of an Anosov diffeomorphism is also Anosov. Also, for any $p \in [0,1]$, it follows from the Jacobian of f_p equal to 1 that f_p is an area-preserving diffeomorphism. Moreover, $f_p(x,y) = f_A(x.y)$ for $x(\text{ mod } 1) \in [0, a-\delta] \cup [a+\delta, 1]$. In conclusion, f_p, $p \in [0,1]$ is a path of area-preserving Anosov diffeomorphism.

Step 2: Our second step is to construct Markov partitions $\mathcal{R}_p$ with two rectangles for f_p for each $p \in [0,1]$.

For any $p \in [0,1]$, since $(0,0)$ is a fixed point of f_p, one way to construct a Markov partition for f_p is to draw segments of $W_p^u((0,0))$ and $W_p^s((0,0))$ until they cross sufficiently many times and separate $\mathbb{T}^2$ into two disjoint (curvilinear) rectangles. Explicitly, we view $\mathbb{T}^2 = [0,1] \times [0,1]/\sim$, where we identify $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$. Then we construct Markov partitions as follows (see Figure 2):

1. From $(0,0)$ draw the line $l_p(0,0) = W_p^u((0,0))$ in the unit square and stops when it hits the boundary of the unit square.
2. Form $(0,1)$ draw the line $l_p(0,1) = W_p^s((0,0))$ in the unit square and stops when it hits $l_p(0,0)$.
3. From $(1,1)$ draw the line $l_p(1,1) = W_p^u((0,0))$ in the unit square and stops when it hits $l_p(0,1)$.
4. Finally using symmetry, draw the extension l_p of the line $l_p(0,0)$ in the unit square and stops when it hits $l_p(0,1)$ to complete the rectangles.

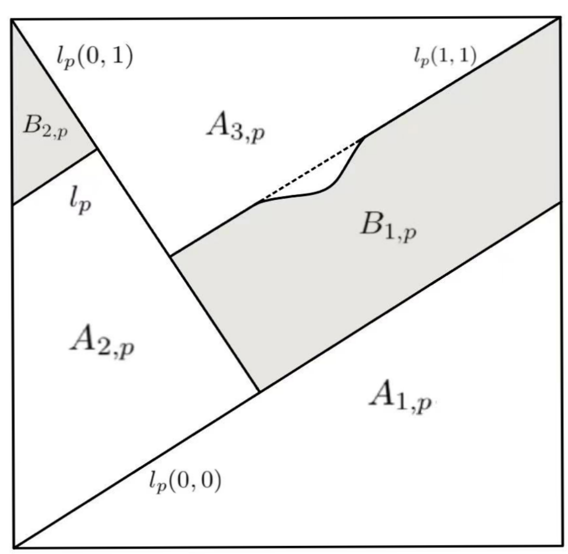

FIGURE 2. Markov partition of f_p

We denote the rectangle consisting of $B_{1,p}$ and $B_{2,p}$ by R_p^0, and denote the rectangle consisting of $A_{1,p}$, $A_{2,p}$ and $A_{3,p}$ by R_p^1. Then $\mathcal{R}_p = \{R_p^0, R_p^1\}$ is a Markov partition for f_p.

Note that when $p = 0$, the Markov partition $\mathcal{R}_0$ is exactly the Markov partition for the Arnold's cat map we introduced in last section. And since for any $p \in [0, 1]$,

$$W_p^s((0,0)) = \bigcup_{n\in\mathbb{N}} (f_p)^{-n}\left(W_p^s((0,0),\rho)\right)$$
$$W_p^u((0,0)) = \bigcup_{n\in\mathbb{N}} (f_p)^{n}\left(W_p^u((0,0),\rho)\right).$$

and for a sufficient small $\rho > 0$, $W_p^i((0,0),\rho) = W_0^i((0,0),\rho)$, $i = u, s$, we can find how the stable manifold and unstable manifold of $(0,0)$ change when p changes from 0 to 1.

Recall that $\forall p \in [0,1]$, $f_p(x,y) = f_0(x.y)$ for $x(\mod 1) \in [0, a-\delta] \cup [a+\delta, 1]$. By our construction of f_p, since $a - \delta > 1 - \frac{2}{5+\sqrt{5}}$, we can check that $l_p(0,0) = l_0(0,0)$, $l_p(0,1) = l_0(0,1)$ and $l_p = l_0$. And next let's figure out how $l_p(1,1)$ changes when p changes.

To simplify the calculation, let's consider the fundamental domain as $[-1,0]\times[-1,0]$. Then $l_p(1,1)$ in this fundamental domain is a line from $(0,0)$ and stops when it hits the stable manifold of $(0,0)$. Let $\lambda = \frac{3+\sqrt{5}}{2}$, then we can see that $l_0(1,1)$ in this fundamental domain is the segment $\{(x,y)\big|y = \frac{-1+\sqrt{5}}{2}x, x \in [-\lambda\frac{2}{5+\sqrt{5}}, 0]\}$. Thus, by our construction of f_p, for any $p \in [0,1]$, the segment

$$\begin{aligned} l_p(1,1) =& f_p(\{(x,y)\big|y = \frac{-1+\sqrt{5}}{2}x, x \in [-\frac{2}{5+\sqrt{5}}, 0]\}) \\ =& \begin{cases} l_0(1,1) & \text{for } x \in [-\lambda\frac{2}{5+\sqrt{5}}, \lambda(a-\delta-1)] \cup [\lambda(a+\delta-1), 0] \\ f_p(\{(x,y)\big|y = \frac{-1+\sqrt{5}}{2}x, x \in [a-\delta-1, a+\delta-1]\}), & \text{otherwise} \end{cases} \end{aligned}$$

Moreover, if we write $l_p(1,1) = \{(X, Y(X))\big|X \in [-\lambda\frac{2}{5+\sqrt{5}}, 0]\}$, we can calculate the area of the domain bounded by $l_p(1,1)$ and $l_0(1,1)$ for each $p \in [0,1]$:

$$\begin{aligned} \text{Area}_p =& \int_{a-\delta-1}^{a+\delta-1} \frac{\sqrt{5}+1}{2}\frac{\sqrt{5}+3}{2}x dx - \int_{\lambda(a-\delta-1)}^{\lambda(a+\delta-1)} Y_p(X)dX \\ =& \int_{a-\delta-1}^{a+\delta-1} \frac{\sqrt{5}+1}{2}\frac{\sqrt{5}+3}{2}x dx - \int_{a-\delta-1}^{a+\delta-1} (\frac{\sqrt{5}+1}{2}x - \phi_p(x))(\frac{\sqrt{5}+3}{2} - \frac{d\phi_p}{dx}(x))dx \\ =& \int_{a-\delta-1}^{a+\delta-1} \phi_p(x)dx + \int_{a-\delta-1}^{a+\delta-1} \frac{\sqrt{5}+1}{2}(x\phi_p(x))'dx - \int_{a-\delta-1}^{a+\delta-1} \frac{1}{2}(\phi_p^2(x))'dx \\ =& \int_{a-\delta-1}^{a+\delta-1} \phi_p(x)dx. \end{aligned}$$

Thus, if we denote the Lebesgue measure on $\mathbb{T}^2$ by m, then for each $p \in [0,1]$, the Markov partition $\mathcal{R}_p$ defined above consists of two rectangles R_p^0 with $m(R_p^0) = \frac{5-\sqrt{5}}{10} - \text{Area}_p$ and R_p^1 with $m(R_p^1) = \frac{5+\sqrt{5}}{10} + \text{Area}_p$. Since Area_p is strictly increasing when p increases from 0 to 1, $m(R_p^1)$ is strictly increasing when p increases from 0 to 1.

In conclusion, for any $p \in [0,1]$, we construct a Markov partition $\mathcal{R}_p = \{R_p^0, R_p^1\}$ for f_p, and $m(R_p^1)$ is strictly increasing when p increases from 0 to 1.

Step 3: Our next step is to relate $(\mathbb{T}^2, f_p, m)$ to a shift space.

For any $p \in [0,1]$, by our construction of Markov partition $\mathcal{R}_p = \{R_p^0, R_p^1\}$, we can see that

$$f_p(\text{int}(R_p^i)) \cap \text{int}(R_p^j) \neq \emptyset, \forall i, j = 0, 1.$$

Thus, the transition matrix defined by $\mathcal{R}_p$ and f_p is

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

And this matrix B generates a shift of finite type:

$$\Sigma_B = \{(b_i)_{i \in \mathbb{Z}} : b_i \in \{0,1\}, B_{b_i b_{i+1}} = 1 \ \forall i \in \mathbb{Z}\},$$

equipped with a left-side shift σ. And it can be shown that $(\mathbb{T}^2, f_p, m)$ is (semi)conjugate to the shift space $(\Sigma_B, \sigma, \mu_p)$, where the measure of a cylinder $[b_k b_{k+1} \ldots b_n])$ is defined by

$$\mu_p([b_k b_{k+1} \ldots b_n]) = m(R_p^{b_k}) \frac{m(f_p(R_p^{b_k}) \cap R_p^{b_{k+1}})}{m(R_p^{b_k})} \cdots \frac{m(f_p(R_p^{b_{n-1}}) \cap R_p^{b_n})}{m(R_p^{b_{n-1}})}.$$

And this (semi)conjugacy is defined as follows: for any sequence

$$b = (\ldots, b_{-1}, b_0, b_1, \ldots) \in \Sigma_B,$$

the (semi)conjugate $\psi_p : \Sigma_B \to \mathbb{T}^2$ maps this sequence to a point $\psi_p(b) = \bigcap_{k=-\infty}^{\infty} f_p^k(R_p^{b_{-k}})$ in $\mathbb{T}^2$. Conversely, for any point $z \in \mathbb{T}^2$, one can also define a *symbol sequence* $(\ldots, b_{-1}, b_0, b_1, \ldots)$ associated with z and f_p such that $b_k = i$ if $f_p^k(z) \in R_p^i$, though it may not be unique. In terms of this coding, f_p corresponds to the left shift map σ.

To prove our main theorem, we define a measurable set E such that

$$E = \left\{ (p,z) \in [0,1] \times \mathbb{T}^2 \;\middle|\; \begin{array}{l} \lim\limits_{n\to\infty} \frac{b_0 + \cdots + b_n}{n} = m(R_p^1), \\ \text{where } (\ldots, b_{-1}, b_0, b_1, \ldots) \text{ is a symbol} \\ \text{sequence associated with } z \text{ and } f_p. \end{array} \right\}.$$

Now we say E is a well-defined and full measure set in $[0,1] \times \mathbb{T}^2$. Because for each fixed p, f_p is a smooth area-preserving Anosov diffeomorphism, and hence it is ergodic. By Birkhoff Ergodic theorem, for m-a.e.$z \in \mathbb{T}^2$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{R_p^1}(f_p^k(z)) = m(R_p^1),$$

where $\chi_{R_p^1}$ is the characteristic function of R_p^1. And if we denote the union of the boundaries of R_p^0 and R_p^1 by $\partial\mathcal{R}_p$, for $z \in \mathbb{T}^2 \setminus \bigcup_{i\in\mathbb{Z}} f_p^i(\partial\mathcal{R}_p)$, the symbol sequence associated with z and f_p is unique, and in terms of coding by f_p, the above limit is exactly

$$\lim_{n\to\infty} \frac{b_0 + \cdots + b_n}{n}.$$

Therefore, let C_p denote the torus $\{p\} \times \mathbb{T}^2 \subset [0,1] \times \mathbb{T}^2$. Then the intersection of E with each C_p has two-dimensional Lebesgue measure 1. So, it follows from Fubini's Theorem that E has full three-dimensional Lebesgue measure.

Next, we define a family of curves $\{\Gamma_\beta\}_{\beta\in\mathbb{T}^2}$ as follows. When we take ϵ_0 small enough in step 1, then for any $p \in [0,1]$, we can get a unique homeomorphism h_p on $\mathbb{T}^2$ such that $f_p = h_p \circ f_0 \circ h_p^{-1}$ since Anosov diffeomorphisms are C^1-structurally stable and f_p is a continuous path beginning with f_0. Then for any $\beta \in \mathbb{T}^2$, we define

$$\Gamma_\beta := \{(p, h_p(\beta)) \in [0,1] \times \mathbb{T}^2 \,\Big|\, p \in [0,1]\}.$$

Then $\{\Gamma_\beta\}_{\beta\in\mathbb{T}^2}$ is a family of disjoint curves which fill out $[0,1] \times \mathbb{T}^2$. And note that a symbol sequence of β under f_0 is $(\ldots, \beta_{-1}, \beta_0, \beta_1, \ldots)$ and the symbol sequence of $h_p(\beta)$ under f_p is the same sequence $(\ldots, \beta_{-1}, \beta_0, \beta_1, \ldots)$. Then each Γ_β can intersect the measurable set E in at most a single point $(p, h_p(\beta))$. Because $m(R_p^1)$ is strictly increasing when p increases from 0 to 1, and a given symbol sequence can have at most one limit

$$\lim_{n\to\infty} \frac{b_0 + \cdots + b_n}{n} = m(R_p^1)$$

which determines p.

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