

An unconventional deformation of the nonrelativistic spin-1/2 Fermi gas

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We explore a generalization of nonrelativistic fermionic statistics that interpolates between bosons and fermions, in which up to K particles may occupy a single-particle state. We show that it can be mapped exactly to K flavors of fermions with imaginary polarization. In particular, for $K=2$, we use such a mapping to derive the virial coefficients and relate them to those of conventional spin-1/2 fermions in an exact fashion. We also use the mapping to derive next-to-leading-order perturbative results for the pressure equation of state. Our results indicate that the $K=2$ particles are more strongly coupled than conventional spin-1/2 fermions, as measured by the interaction effects on the virial expansion and on the pressure equation of state. In the regime set by the unitary limit, the proposed $K=2$ deformation represents a universal many-body system whose properties remain largely unknown. In particular the system can be expected to become superfluid at a critical temperature T_c higher than that of the unitary limit. We suggest it may be possible to realize this system experimentally by engineering a polarized coupling to an electrostatic potential. Finally, we show that the $K=2$ system does not display a sign problem for determinantal Monte Carlo calculations, which indicates that T_c can at least in principle be calculated with conventional methods.

I. INTRODUCTION

Over the last two decades there has been considerable interest in the exploration of universality in nonrelativistic quantum many-body systems (besides the well-known cases close to continuous phase transitions). By far the most studied case, both in theory and experiments (see e.g. [1–3]), is that of spin-1/2 fermions in the unitary limit (a system of nonrelativistic spin-1/2 particles with a zero-range interaction tuned to the threshold of two-body bound-state formation, i.e. infinite scattering length). In such a situation, the property of universality stems from the lack of physical scales (and corresponding scale invariance) associated with the attractive interaction, as a system in this limit presents as many dimensionful parameters as a noninteracting gas (albeit also displaying strong pairing correlations and becoming superfluid at low enough temperature). In practice, fermions at unitarity are realized to an excellent approximation in ultracold atom experiments [3] and to a lesser extent in the dilute neutron matter layer of neutron stars [2].

Given the interest in these types of universal systems, it becomes a relevant question whether there are other related systems that are also universal. Examples of such cases were proposed by Nishida and Son in Refs. [4], where they showed that there is a one-dimensional realization of the unitary limit with four flavors and a fine-tuned four-body interaction. They also showed, in a previous publication [5], that fermions at unitarity obey a nonrelativistic conformal algebra for which anyons in two spatial dimensions provide a representation.

Motivated in part by the above developments, we explore here a definition of yet another type of particle statistics that presents nontrivial behavior at unitarity and in some sense interpolates between fermions and bosons. Generally speaking, systems of quantum particles with unconventional statistics have been studied

for many years (see e.g. [6]). The particular case we consider here is perhaps most directly related to the so-called Gentile statistics [7], which generalizes fermions and bosons by allowing single-particle states to hold at most K particles, where $K \rightarrow 1$ recovers the fermionic case and $K \rightarrow \infty$ the bosonic one (see also [8]).

As detailed below, our approach consists in starting with a fermionic system in a path-integral representation and define the generalized K statistics by modifying the integrand in a well-defined fashion. We show that such a prescription corresponds to K flavors of fermions with complex chemical potentials. We then explore the thermodynamics of the system for the case $K=2$ at unitarity using a perturbative approach as well as the virial expansion.

II. FORMALISM

A. Noninteracting systems

For completeness, we briefly review the noninteracting thermodynamics of fermions and bosons in the grand-canonical ensemble. The grand canonical partition function is

$$\mathcal{Z}_0 = \text{tr} \left[e^{-\beta(\hat{T} - \mu \hat{N})} \right], \quad (1)$$

where β is the inverse temperature, μ the chemical potential,

$$\hat{N} = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}, \quad (2)$$

is the particle number operator, and

$$\hat{T} = \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} \epsilon(\mathbf{p}), \quad (3)$$

is the kinetic energy operator. Here, $\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}}$ are, respectively, the creation and annihilation operators for particles of momentum \mathbf{p} , which satisfy the appropriate commutation or anticommutation relations. We will set

$$\epsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} \quad (4)$$

at the end of the calculation, but as we show below, it is useful to keep $\epsilon(\mathbf{p})$ as an arbitrary function.

For noninteracting fermions (one species),

$$\mathcal{Z}_{0,F} = \det[1 + z\mathcal{U}_0], \quad (5)$$

where $[\mathcal{U}_0]_{\mathbf{p},\mathbf{p}'} = e^{-\beta\epsilon(\mathbf{p})}\delta_{\mathbf{p},\mathbf{p}'}$ is a diagonal matrix in momentum space and $z = e^{\beta\mu}$ is the fugacity. Thus,

$$\ln \mathcal{Z}_{0,F} = \sum_{\mathbf{p}} \ln \left[1 + ze^{-\beta\epsilon(\mathbf{p})} \right], \quad (6)$$

which in the thermodynamic limit of large volume, yields a well-known integral expression that is commonly written in terms of the so-called Fermi function [9].

Similarly, for noninteracting bosons (again only one species),

$$\mathcal{Z}_{0,B} = \det[(1 - z\mathcal{U}_0)^{-1}], \quad (7)$$

such that

$$\ln \mathcal{Z}_{0,B} = \sum_{\mathbf{p}} \ln \left[\left(1 - ze^{-\beta\epsilon(\mathbf{p})} \right)^{-1} \right], \quad (8)$$

which is also usually written in integral form in the thermodynamic limit.

Based on the above, it seems natural to consider defining a quantum statistics of identical particles that interpolates between the above two cases and is such that at most K particles can occupy a given single-particle state. Such a statistics has been pursued by many authors in the past, perhaps the best known case being that of Gentile [7]. For a maximum of K particles per single-particle state, one obtains

$$\mathcal{Z}_{0,K} = \det \left[\sum_{n=0}^K z^n \mathcal{U}_0^n \right], \quad (9)$$

where the (single-flavor) fermionic case is recovered for $K = 1$ and the bosonic case for $K \rightarrow \infty$. As we show below, the $K = 2$ is different from spin-1/2 fermions. The polynomial in $x = z\mathcal{U}_0$ inside the determinant can naturally be written as

$$1 + x + x^2 + \dots + x^K = \frac{1 - x^{K+1}}{1 - x}, \quad (10)$$

which is an easy way to see that the roots of our polynomial are K of the $(K + 1)$ -th roots of unity, namely $\alpha_n = e^{i2\pi n/(K+1)}$, with $n = 1, 2, \dots, K$, such that

$$\ln \mathcal{Z}_{0,K} = \sum_{n=1}^K \sum_{\mathbf{p}} \ln \left(ze^{-\beta\epsilon(\mathbf{p})} - \alpha_n \right), \quad (11)$$

and therefore (dropping an additive constant which, in particular, vanishes if K is even),

$$\ln \mathcal{Z}_{0,K} = \sum_{n=1}^K \sum_{\mathbf{p}} \ln \left(1 + w_n e^{-\beta\epsilon(\mathbf{p})} \right), \quad (12)$$

where $w_n = -\alpha_n z$ which shows that the noninteracting K statistics corresponds to K fermionic species with complex fugacities.

As an example, consider the $K = 2$ case, where

$$\mathcal{Z}_{0,2} = \prod_{\mathbf{p}} \left| 1 + e^{-i\pi/3} z e^{-\beta\epsilon(\mathbf{p})} \right|^2, \quad (13)$$

such that total particle number is

$$N^{(0)} = z \frac{\partial \ln \mathcal{Z}_{0,2}}{\partial z} = \sum_{\mathbf{p}} 2\text{Re} \left[\frac{e^{-i\pi/3} z e^{-\beta\epsilon(\mathbf{p})}}{1 + e^{-i\pi/3} z e^{-\beta\epsilon(\mathbf{p})}} \right], \quad (14)$$

and such that the noninteracting occupation probabilities of the single-particle momentum states are

$$n_p^{(0)}(z) = 2\text{Re} \left[\frac{e^{-i\pi/3} z e^{-\beta\epsilon(\mathbf{p})}}{1 + e^{-i\pi/3} z e^{-\beta\epsilon(\mathbf{p})}} \right]. \quad (15)$$

In Fig. 1 we show the momentum distribution $n_p^{(0)}(z)$ for $K = 2$, alongside the distributions for $K = 1$ and $K = 3$ and the spin-1/2 noninteracting Fermi gas. Notably, there is a clear difference between the $K = 2$ case and the spin-1/2 Fermi gas. The former allows for higher occupations for $x < 0$ at the cost of lower occupations at $x > 0$; both distributions yield the same answer at $x = 0$. As we show below when discussing the virial expansion, there is a well-defined sense in which the $K = 2$ case is "more bosonic" than the spin-1/2 Fermi gas, even though the former is effectively just a complex- z deformation of the latter.

B. Interacting systems

While the above completely determines the thermodynamics of the non-interacting case without further need to invoke the algebraic properties of \hat{a} and \hat{a}^\dagger , the interacting case requires more care, as interactions will involve products of four or more of these operators.

In this work we will follow a different, non-algebraic route by defining the interacting system using the field-integral formulation of the many-body problem. [We do, however, carry out our derivations using the spin-1/2 fermionic case as a starting point.] Using a Hubbard-Stratonovich transformation [10, 11] (see also [12, 13]) to decouple the interaction in the fermionic case, one obtains an expression for \mathcal{Z} that involves a field integral over auxiliary-field configurations $\sigma(\mathbf{r}, t)$ in which the integrand takes the form of a product of two determinants:

$$\begin{aligned} \det(1 + z\mathcal{U}[\sigma]) \det(1 + z\mathcal{U}[\sigma]) \\ = \det(1 + 2z\mathcal{U}[\sigma] + z^2\mathcal{U}^2[\sigma]), \end{aligned} \quad (16)$$

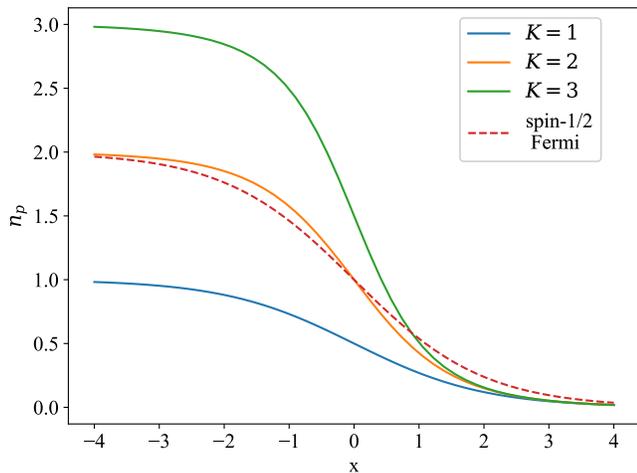


FIG. 1. Momentum distribution of the noninteracting $K = 1, 2, 3$ gas. Here, the x axis represents $x = \beta(p^2/(2m) - \mu)$. For comparison, we also show the noninteracting spin-1/2 Fermi gas result.

which represent noninteracting systems immersed in the external field $\sigma(\mathbf{r}, t)$. We then deform such a system to our $K = 2$ case by replacing the fermionic determinants as follows:

$$\begin{aligned} & \det(1 + z\mathcal{U}[\sigma]) \det(1 + z\mathcal{U}[\sigma]) \\ & \rightarrow \det(\alpha + z\mathcal{U}[\sigma]) \det(\alpha^* + z\mathcal{U}[\sigma]), \end{aligned} \quad (17)$$

where $\alpha = -e^{i2\pi/3} = e^{-i\pi/3}$ and where we have factored out the fugacity z explicitly.

The $\mathcal{U}[\sigma]$ matrices contain the product of N_τ exponentials of kinetic and potential energy operators along the imaginary-time direction:

$$\mathcal{U}[\sigma] = e^{-\tau T} e^{-\tau V[\sigma]} \dots e^{-\tau T} e^{-\tau V[\sigma]}, \quad (18)$$

where the inverse temperature is $\beta = N_\tau \tau$; T is the single-particle representation of the kinetic energy operator; and $V[\sigma]$ is the auxiliary potential resulting from the Hubbard-Stratonovich transformation.

We therefore define the interacting $K = 2$ partition function as

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\sigma \det(z\mathcal{U}[\sigma] + \alpha) \det(z\mathcal{U}[\sigma] + \alpha^*) \\ &= \int \mathcal{D}\sigma \det(1 + z\mathcal{U}[\sigma] + z^2\mathcal{U}^2[\sigma]), \end{aligned} \quad (19)$$

where we see that the expected $K = 2$ noninteracting limit is recovered when the interaction is turned off. Rearranging, noting that $|\alpha|^2 = 1$, one obtains

$$\mathcal{Z} = \int \mathcal{D}\sigma \det(1 + w\mathcal{U}[\sigma]) \det(1 + w^*\mathcal{U}[\sigma]), \quad (20)$$

which shows once again that the $K = 2$ case is identical to that of spin-1/2 fermions with complex fugacities $w = \alpha z$ and w^* for each spin projection, respectively. In spite of the appearance of a complex fugacity,

the presence of its complex conjugate indicates that this system does not display a sign problem for conventional auxiliary-field Monte Carlo calculations, as long as the interaction is purely attractive. Notably, α is temperature-independent, which leads to a temperature-varying complex effective chemical potential, namely

$$\beta\mu_{\text{eff}} = \ln w = \beta\mu \pm i\frac{\pi}{3}, \quad (21)$$

for spin-up and spin-down, respectively. Thus, our $K = 2$ system is equivalent to a spin-1/2 Fermi gas with imaginary polarization.

It is well-known [14] that a chemical potential is equivalent to an imaginary A_0 gauge field. Thus, coupling to a constant A_0 gauge field is equivalent to an imaginary chemical potential, which is our interest here. Therefore, the above imaginary polarization can potentially be realized by engineering a coupling of spin-1/2 fermions in which the spins couple to an electrostatic field with an electrostatic potential difference of $2\pi/3$. Ideally, this coupling would be independent from the usual Feshbach resonance coupling that controls the inter-spin interaction.

In the next sections we will focus on this $K = 2$ case, tuning the interaction to the unitary limit. While it is generally accepted that polarized unitary fermions undergo a phase transition at some polarization, from a superfluid phase to a normal phase (possibly going through exotic superfluid phases), their behavior at imaginary polarization remains little explored (see however the seminal studies of Refs. [15, 16]).

In investigating the present deformation of the Fermi gas, we have kept the internal number of degrees of freedom constant, i.e. we compare $K = 2$ to the spin-1/2 Fermi gas. Another route could be to promote the individual spin degrees of freedom of the Fermi gas to $K = 2$, but this would result in a system with four internal degrees of freedom, which would be generally very difficult to compare with its spin-1/2 counterpart and possibly unstable depending on the form of the interaction (see below).

For $K > 2$, attractive interactions make the unitary-limit system unstable toward Thomas collapse (due to the formation of infinitely deep three-body bound states) [17] (see also [18]). However, one may still envision an appropriate modification of the interaction at short range along with a decoupling of the interaction to proceed with the mapping to fermions by factorization of the determinant as

$$\det \left[\sum_{n=0}^K z^n \mathcal{U}^n[\sigma] \right] = \prod_{n=1}^K \det [1 + w_n \mathcal{U}[\sigma]], \quad (22)$$

where $w_n = \alpha_n z$, and α_n , $n = 1, 2, \dots, K - 1$ is one of the (non-unity) $(K + 1)$ -th complex roots of unity. In other words, our system maps exactly onto a system of K fermions with complex chemical potentials w_n . For K even, the relevant roots of unity come in complex con-

jugate pairs, such that pairing the determinants accordingly one finds that there is no sign problem. For K odd, on the other hand, there will always be a determinant factor of the form

$$\det(1 - z\mathcal{U}[\sigma]), \quad (23)$$

which can be made real (in some cases, depending on the form of the interaction), but cannot be guaranteed in general to be positive definite (except under specific symmetry conditions; see e.g. Ref. [19] for a review); such a factor corresponds to the root $\alpha = -1$, which is present for all K odd.

The next few sections present our results for $K = 2$ for the virial coefficients up to fourth order, and a next-to-leading-order perturbative calculation of the pressure.

III. RESULTS: THE VIRIAL EXPANSION

The virial expansion (see e.g. [9]) is an expansion of \mathcal{Z} in powers of the fugacity z , such that

$$\mathcal{Z} = \sum_{n=0}^{\infty} Q_n z^n, \quad (24)$$

where Q_n is the n -particle canonical partition function, and

$$\ln \mathcal{Z} = Q_1 \sum_{n=1}^{\infty} b_n z^n, \quad (25)$$

where b_n are the virial coefficients, typically written in terms of Q_m with $m \leq n$; for example, $b_1 = 1$, while

$$b_2 = \frac{Q_2}{Q_1} - \frac{Q_1}{2!}, \quad (26)$$

and

$$b_3 = \frac{Q_3}{Q_1} + b_2 Q_1 - \frac{Q_1^2}{3!}, \quad (27)$$

and so forth, where $Q_1 = V/\lambda_T^3$ in 3D. The above expressions for b_2 and b_3 are independent of the quantum statistics.

For reference, we note that the b_n for noninteracting fermions and bosons in homogeneous space are given, respectively, by

$$b_{0,F,n} = (-1)^{n+1} \frac{1}{n^{5/2}}, \quad (28)$$

and

$$b_{0,B,n} = \frac{1}{n^{5/2}}. \quad (29)$$

It is not difficult to calculate the virial coefficients of the noninteracting case for arbitrary K and notice that they are identical to those of the bosonic case, except

that for every n that is multiple of $K + 1$ one obtains an extra sign and an overall factor of K , i.e.

$$b_{0,K,m(K+1)} = -K \frac{1}{n^{5/2}}, \quad (30)$$

where $m = 1, 2, 3, \dots$, which also captures the expected results both at $K = 1$ and $K \rightarrow \infty$.

It is straightforward to derive relations between the virial coefficients of the $K = 2$ gas and the spin-1/2 Fermi gas by noting that the virial expansion for the complex-polarized Fermi gas takes the form

$$\ln(\mathcal{Z}/\mathcal{Z}_0) = 2Q_{1,0}^F \sum_{n=2}^{\infty} \sum_{m+j=n} \Delta b_{m,j}^F z_{\uparrow}^m z_{\downarrow}^j, \quad (31)$$

where $Q_{1,0}^F = V/\lambda_T^3 = Q_1$, $z_{\uparrow} = ze^{i\alpha}$, and $z_{\downarrow} = ze^{-i\alpha}$, and $\Delta b_{m,j}^F$ are the virial coefficients of the polarized spin-1/2 Fermi gas. Thus, identifying the powers of z against the $K = 2$ expression

$$\ln(\mathcal{Z}/\mathcal{Z}_0) = Q_{1,0}^F \sum_{n=2}^{\infty} \Delta b_n^{K=2} z^n, \quad (32)$$

we find

$$\Delta b_2^{K=2} = 2\Delta b_{11}^F = 2\Delta b_2^F, \quad (33)$$

$$\Delta b_3^{K=2} = 2\Delta b_{21}^F = \Delta b_3^F, \quad (34)$$

$$\Delta b_4^{K=2} = -2\Delta b_{31}^F + 2\Delta b_{22}^F \neq \Delta b_4^F, \quad (35)$$

where for completeness we note that $\Delta b_4^F = 2\Delta b_{31}^F + \Delta b_{22}^F$.

Using known results at unitarity from Refs. [20–22], we find

$$\Delta b_2^{K=2} = \sqrt{2}, \quad (36)$$

$$\Delta b_3^{K=2} = -0.3551\dots, \quad (37)$$

$$\Delta b_4^{K=2} = -0.435\dots, \quad (38)$$

where we only quote enough digits for the purposes of this work. It is evident from the above, however, that for $K = 2$ at unitarity, the fourth-order coefficient is larger in magnitude than the third-order coefficient. This is in contrast to the conventional spin-1/2 unitary Fermi gas, where that type of behavior shows up at one higher order (i.e. between fourth and fifth).

For future reference, we note that the second-order virial expansion of the pressure at unitarity reads

$$\beta\Delta PV = \ln(\mathcal{Z}/\mathcal{Z}_0) = Q_{1,0}^F \sqrt{2} z^2, \quad (39)$$

such that, using the noninteracting result $\beta P_0 V = Q_1(z + b_2^0 z^2 + \dots)$, we obtain

$$\frac{P}{P_0} = 1 + \frac{\Delta P}{P_0} = 1 + \frac{\sqrt{2} z^2}{z + b_2^0 z^2 + \dots} = 1 + \Delta b_2^{K=2} z \quad (40)$$

Carrying out the expansion now up to z^3 order, we obtain

$$\begin{aligned} \frac{P}{P_0} &= 1 + \Delta b_2^{K=2} z + \left[-\frac{\Delta b_2^{K=2}}{4\sqrt{2}} + \Delta b_3^{K=2} \right] z^2 \\ &+ \left[-\frac{\Delta b_3^{K=2}}{4\sqrt{2}} + \Delta b_4^{K=2} + \Delta b_2^{K=2} \left(\frac{1}{32} + \frac{2}{9\sqrt{3}} \right) \right] z^3. \end{aligned}$$

IV. RESULTS: PERTURBATIVE APPROACH

To complement our VE results of the previous section, we present here a second-order perturbation theory calculation of the thermodynamics of our $K = 2$ system for arbitrary w . To this end, our starting point is

$$\mathcal{Z} = \int \mathcal{D}\sigma \det(1 + w\mathcal{U}[\sigma]) \det(1 + w^*\mathcal{U}[\sigma]). \quad (41)$$

To expand \mathcal{Z} perturbatively, we use a discretization of the imaginary-time direction, as in Ref. [23], and expand $\mathcal{U}[\sigma]$ in powers of the bare coupling constant C , such that

$$\mathcal{U}[\sigma] = \mathcal{U}_0 + C\mathcal{U}_1[\sigma] + C^2\mathcal{U}_2[\sigma] + \dots, \quad (42)$$

where, we define C by

$$e^{-\tau V[\sigma]} = 1 + CM[\sigma], \quad (43)$$

and $M[\sigma]$ contains all the non-trivial dependence on σ and its form will depend on the specific choice of Hubbard-Stratonovich transformation. Then

$$\begin{aligned} \det(1 + w\mathcal{U}[\sigma]) &= \det(1 + w\mathcal{U}_0) \\ &\times \det [1 + CX_1[\sigma] + C^2X_2[\sigma]], \end{aligned} \quad (44)$$

$$(45)$$

where

$$X_k[\sigma] = \frac{w\mathcal{U}_k[\sigma]}{1 + w\mathcal{U}_0}. \quad (46)$$

Therefore, at order C^2 ,

$$\begin{aligned} \det(1 + w\mathcal{U}[\sigma]) &= 1 + C\text{tr}X_1[\sigma] + C^2\text{tr}X_2[\sigma] \\ &+ \frac{C^2}{2} [\text{tr}^2X_1[\sigma] - \text{tr}X_1^2[\sigma]]. \end{aligned} \quad (47)$$

Integrating over σ to get \mathcal{Z} , we obtain

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\sigma \det(1 + w\mathcal{U}[\sigma]) \det(1 + w^*\mathcal{U}[\sigma]) \\ &= \mathcal{Z}_0 [1 + C\Delta_1(w, w^*) + C^2\Delta_2(w, w^*)], \end{aligned} \quad (48)$$

where the noninteracting partition function is

$$\mathcal{Z}_0 = \det(1 + w\mathcal{U}_0) \det(1 + w^*\mathcal{U}_0), \quad (49)$$

the first-order contribution in C is

$$\Delta_1(w, w^*) = \int \mathcal{D}\sigma (\text{tr}X_1[\sigma] + c.c.) \quad (50)$$

and the second-order term is

$$\begin{aligned} \Delta_2(w, w^*) &= \int \mathcal{D}\sigma [\text{tr}X_1[\sigma]\text{tr}X_1^*[\sigma] + \text{tr}X_2[\sigma] \\ &+ \frac{1}{2} (\text{tr}^2X_1[\sigma] - \text{tr}X_1^2[\sigma] + c.c.)]. \end{aligned} \quad (51)$$

Since we assume the interaction to be only a two-body interaction, only terms with even powers of w will contribute to the final result. In the above equations, it can be shown that, indeed, $\Delta_1 = 0$. Similarly, the X_2 term in Δ_2 also vanishes. The remaining contribution in Δ_2 is what is conventionally called the first-order perturbation theory result, which boils down to

$$\mathcal{Z}/\mathcal{Z}_0 = 1 + C^2Vn(w)n(w^*), \quad (52)$$

where

$$n(w) = \frac{1}{V} \sum_p n_p(w), \quad (53)$$

and

$$n_p(w) = \frac{we^{-\beta\epsilon(p)}}{1 + we^{-\beta\epsilon(p)}}. \quad (54)$$

In the large-volume limit,

$$n(w) \rightarrow \frac{V}{(2\pi)^3} \int dp n_p(w), \quad (55)$$

and setting $\epsilon(p) = p^2/(2m)$ the integral can be evaluated and is proportional to $Li_{\frac{3}{2}}(-w)$.

Thus, the change in pressure is given at this order by

$$\beta\Delta PV = \ln(\mathcal{Z}/\mathcal{Z}_0) = C^2Vn(w)n(w^*), \quad (56)$$

such that our final result, setting $\epsilon(p) = p^2/(2m)$, is

$$\frac{P}{P_0} = 1 + \mathcal{C} \left| Li_{\frac{3}{2}}(-w) \right|^2 [2\text{Re}Li_{\frac{5}{2}}(-w)]^{-1}, \quad (57)$$

where \mathcal{C} is a dimensionless coupling to be renormalized as explained below. Note that we have used the noninteracting result

$$\beta P_0 V = \ln \mathcal{Z}_0 = 2\text{Re} \sum_p \ln [1 + we^{-\beta\epsilon(p)}], \quad (58)$$

where the last sum, in the large-volume limit, is proportional to $Li_{\frac{5}{2}}(-w)$.

To renormalize \mathcal{C} , we use the VE result of the previous section by choosing a renormalization point z_0 . For our $K = 2$ case, with $\alpha = e^{-i\pi/3}$, the second-order VE for the pressure reads

$$\frac{P}{P_0} = 1 + \sqrt{2}z, \quad (59)$$

and therefore

$$\mathcal{C} = \sqrt{2}z_0 \left| Li_{\frac{3}{2}}(-e^{-i\pi/3}z_0) \right|^{-2} [2\text{Re}Li_{\frac{5}{2}}(-e^{-i\pi/3}z_0)], \quad (60)$$

which fixes \mathcal{C} in Eq. (57).

In Fig. 2 we show our results for the pressure P in units of P_0 , comparing the second, third, and fourth-order virial expansions with our perturbative result for P/P_0 as a function of $\beta\mu$, at unitarity for $K = 2$.

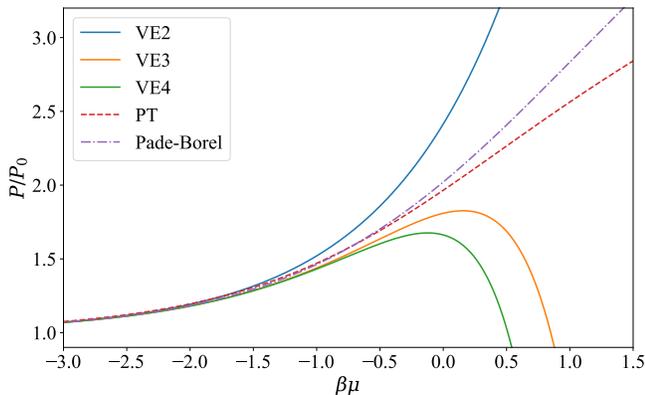


FIG. 2. Pressure P of the $K = 2$ gas at unitarity, in units of its noninteracting counterpart P_0 as a result of four different calculations: virial expansion at second, third, and fourth orders, and first-order perturbative result. We also show the result of a Pade-Borel resummation of the fourth-order virial expansion.

Using the above expressions, we may also access the momentum distribution. For that purpose, we restore a generic dispersion relation $\epsilon(p)$ instead of $p^2/(2m)$, which allows us to reinterpret the expressions for \mathcal{Z} as generating functionals for expectation values of the occupation probability n_p . For instance, in the noninteracting limit,

$$n_p^{(0)} = -\frac{1}{\beta} \frac{\delta \ln \mathcal{Z}_0}{\delta \epsilon(p)} = 2\text{Re}[n_p(w)]. \quad (61)$$

The interaction effects on the above are given by

$$\begin{aligned} \Delta n_p &= -\frac{1}{\beta} \frac{\delta \ln(\mathcal{Z}/\mathcal{Z}_0)}{\delta \epsilon(p)} \\ &\propto \text{Re} \left[n(w) \frac{(w^*)^{-1} e^{\beta \epsilon(p)}}{(1 + (w^*)^{-1} e^{\beta \epsilon(p)})^2} \right]. \end{aligned} \quad (62)$$

In Fig. 3, we show the above correction Δn_p relative to its conventional spin-1/2 counterpart $\Delta n_p^{\text{Fermi}}$, for three different fugacities, as a function of $x = \beta(p^2/(2m) - \mu)$.

At sufficiently low- and high- x , $\Delta n_p^{\text{Fermi}}$ tends to zero. Thus, we note that Δn_p also decreases to zero at sufficiently low- and high- x . This indicates that the maximum occupation number (i.e. 2 for $K = 2$) is not modified by the interactions, at least away from $x = 0$. Around $x = 0$, interaction effects may violate the maximum occupation number for sufficiently strong interactions or sufficiently high z , but those effects are beyond our perturbative analysis.

At high fugacities, interaction effects are substantially enhanced for $K = 2$ relative to the Fermi gas, which is

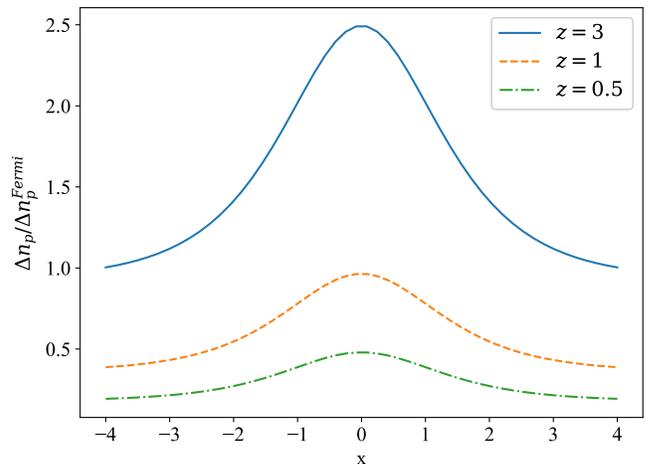


FIG. 3. Momentum distribution first-order perturbative correction for $K = 2$, relative to the conventional spin-1/2 Fermi gas. Here, the x axis represents $x = \beta(p^2/(2m) - \mu)$. For comparison, results are shown for various fugacities $z = 0.25, 1.0, \text{ and } 3.0$.

important because at sufficiently high z , these systems are expected to become superfluid. Based on our results for Δn_p , we anticipate that the critical temperature for $K = 2$ will be higher than that of the spin-1/2 Fermi gas at the same interaction strength, as the interparticle attraction is effectively stronger for the $K = 2$ gas.

V. SUMMARY AND CONCLUSIONS

In this work, we have explored a generalization of non-relativistic fermionic statistics that interpolates between bosons and fermions, for which up to K particles can occupy a single-particle state. We have shown that it can be mapped exactly to K flavors of fermions with a specific temperature-dependent imaginary polarization, i.e. the difference in density among the flavors is an imaginary quantity. In particular, for $K = 2$, we use the mapping to derive the virial coefficients and relate them to those of conventional spin-1/2 fermions in an exact fashion. We also use the mapping to derive next-to-leading-order perturbative results for the pressure equation of state. Our results indicate that the $K = 2$ particles are more strongly coupled than conventional spin-1/2 fermions, as measured by the interaction effects on the virial expansion and the pressure.

At unitarity, the proposed $K = 2$ system is a universal many-body system whose properties remain largely unknown. In particular the system can be expected to become superfluid at a critical temperature T_c higher than that of the conventional, unpolarized unitary limit [24–27]. Indeed, the mean-field study of Ref. [15] found that, for an imaginary polarization of $\pi/3$ [see Eq. (21)], the critical temperature is about 30% higher than in the unpolarized case. If that percent change applies

once all fluctuations are accounted for, one may expect $T_c/\epsilon_F \simeq 0.2$ for the $K = 2$ system (where ϵ_F is the Fermi energy). By engineering a polarized coupling to an electrostatic potential, it may be possible to realize this system in a controlled fashion via ultracold atoms, where the question of T_c can be explored experimentally. Finally, we showed that the $K = 2$ system does not display a sign problem for determinantal Monte Carlo calcula-

tions, which indicates that the precise value of T_c can at least in principle be determined with conventional methods.

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