

Non-parametric estimation for the stochastic wave equation

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Abstract

The spatially dependent wave speed of a stochastic wave equation driven by space-time white noise is estimated using the local observation scheme. Given a fixed time horizon, we prove asymptotic normality for an augmented maximum likelihood estimator as the resolution level of the observations tends to zero. We show that the expectation and variance of the observed Fisher information are intrinsically related to the kinetic energy within an associated deterministic wave equation and prove an asymptotic equipartition of energy principle using the notion of asymptotic Riemann-Lebesgue operators.

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1 Introduction

1.1 Motivation

Let $A_\vartheta z := \operatorname{div}(\vartheta \nabla z)$, $\vartheta : \Lambda \rightarrow (0, \infty)$, be a weighted Laplacian in divergence form satisfying Dirichlet boundary conditions on an open and bounded domain $\Lambda \subset \mathbb{R}^d$ with C^2 -boundary. We consider the stochastic wave equation

$$\partial_{tt}^2 u(t) = A_\vartheta u(t) + \dot{W}(t), \quad t \in [0, T], \quad (1.1)$$

driven by Gaussian space-time white noise $(\dot{W}(t), t \in [0, T])$. This work is devoted to the non-parametric estimation of the spatially varying wave speed $\vartheta : \Lambda \rightarrow (0, \infty)$ based on local observations. The stochastic wave equation is of interest both from a theoretical and applied point of view, see e.g. [14, 12, 11, 17, 21] and the references therein.

This work is related to Liu and Lototsky [40] as well as Delgado-Vences and Pavon-Español [16], who inferred the parametric wave speed, i.e. assuming ϑ to be constant, of a one-dimensional stochastic wave equation based on spectral observations up to a finite time horizon $T > 0$. The spectral approach

is, however, restricted to operators A_ϑ whose leading order eigenfunctions are independent of ϑ and is not suitable for estimating a spatially varying wave speed. The constant wave speed of a stochastic wave equation was also inferred using the following two different large-time observation schemes. Using the ergodicity of the solution, Janák [31, 32] estimated the wave speed and damping of a strongly damped stochastic wave equation based on an observation window but required the asymptotic regime $T \rightarrow \infty$. Furthermore, Markussen [44] proposed an approximate likelihood approach and analysed asymptotic properties of the associated maximum likelihood estimator based on discrete observations at $t = \Delta, 2\Delta, \dots, n\Delta$, $\Delta > 0$, as $n \rightarrow \infty$. Other works on statistics for the stochastic wave equation, e.g. Shevchenko [55] and Shevchenko et al. [56], concern estimators of the unknown Hurst coefficient of a fractional stochastic wave equation using Malliavin calculus.

The local observation scheme was introduced in the parabolic case of the stochastic heat equation by Altmeyer and Reiß [4] and was used to construct rate optimal estimators of a spatially varying diffusivity. A first extension to semilinear parabolic SPDEs was achieved by Altmeyer, Cialenco, and Pasemann [7]. Based on multiple local measurements, Altmeyer, Tiepner, and Wahl [6] obtained optimal rates for estimating lower-order coefficients. Multiple local measurements were then used in Reiß, Strauch, and Trottner [51] to detect a change point within the diffusivity and by Strauch and Tiepner [58] to estimate the non-parametric velocity of a stochastic heat equation. Based on real data, the local observation scheme identified parameters in the Meinhardt model for cell repolarisation, see Altmeyer, Bretschneider, Janák, and Reiß [5]. Janák and Reiß [33] show that local measurements can also be used to estimate the diffusivity in the case of multiplicative noise. Aihara [2, 3] took a first step towards the estimation of a spatially varying parameter of an elastic operator in a damped stochastic hyperbolic system under noisy partial observations by considering a Kalman filtering problem and using the methods of sieves. Consistency, however, is only achieved for global observations in the large time regime $T \rightarrow \infty$.

To our knowledge, the statistical inference for a spatially varying wave speed ϑ has not yet been explored for a fixed time horizon $T < \infty$. In the parabolic case, to control the asymptotic behaviour of the observed Fisher information, Altmeyer and Reiß [4] use the fact that the heat semigroup naturally decays in time. It is a priori unclear if the local measurement approach is also applicable to the undamped stochastic wave equation, which is inherently energy-preserving. Intriguingly, we will show that the local observation scheme is a suitable tool for studying stochastic hyperbolic equations by relating the observed Fisher information to the energetic behaviour of an associated deterministic system.

More specifically, given a fixed finite time horizon $T > 0$, we will estimate the spatially dependent wave speed ϑ at $x_0 \in \Lambda$ using the augmented maximum likelihood approach (MLE) based on local observations $\langle u(t), K_{\delta, x_0} \rangle$, where the solution process u is tested against a kernel localised around x_0 . The augmented MLE is consistent with rate δ whilst satisfying a central limit theorem. The asymptotic variance of the limiting normal distribution depends on the

fixed time horizon T through T^{-2} , which is specific to the hyperbolic case and different from the parabolic dependence T^{-1} . This discrepancy highlights the difference between the energetically dissipative heat equation and the energetically stable wave equation. A similar influence of the dissipative behaviour of the underlying equation is also observed in the MLE's rate of convergence for the ordinary Ornstein-Uhlenbeck process or the harmonic oscillator. Indeed, the rate is \sqrt{T} in the ergodic and T in the energetically stable case; see Kutoyants [38, Proposition 3.46] and Lin and Lototsky [39].

In the hyperbolic case of the stochastic wave equation, the covariance structure of the local measurements involves rescaled operator cosine and sine families, which generalise the concept of semigroups to the hyperbolic case. Thus, we can analyse the asymptotic behaviour of the augmented MLE, which is linked through the observed Fisher information to the energetic behaviour within a scaled deterministic wave equation, by showing a scaled version of the equipartition of energy principle. The equipartition of energy was studied for abstract hyperbolic Cauchy problems by several works, including [26, 25, 22, 23, 24, 48, 57], and relies on the concept of Riemann-Lebesgue operators. In particular, we show that the scaled weighted Laplace operator $A_{\vartheta(\delta)}$ behaves asymptotically as $\delta \rightarrow 0$ like a Riemann-Lebesgue operator. In fact, our analysis of the observed Fisher information is based on the weak operator topology and not as in the parabolic case on norm bounds of the heat semigroup. This is accomplished using functional calculus and convergence results for the resolution of the identity associated with the scaled weighted Laplace operator on the growing spatial domain $\delta^{-1}(\Lambda - x_0)$.

The probabilistic backbone of the asymptotic results is a scaling limit for the observation processes as $\delta \rightarrow 0$. In the parametric case, this scaling limit can even be achieved in finite time due to the finite propagation speed of the wave equation.

1.2 An overview of the main results

We briefly discuss the main results of this work. The stochastic partial differential equation and all objects related to its solution are rigorously introduced in Section 2.

In the local observation scheme, for a given resolution $\delta > 0$, we assume to observe continuous time processes of the form

$$u_\delta(t) = \langle u(t), K_{\delta, x_0} \rangle_{L^2(\Lambda)}, \quad u_\delta^\Delta(t) = \langle u(t), \Delta K_{\delta, x_0} \rangle_{L^2(\Lambda)}, \quad t \in [0, T], \quad (1.2)$$

where $K_{\delta, x_0}(x) = \delta^{-d/2} K(\delta^{-1}(x - x_0))$ is a kernel localised around $x_0 \in \Lambda$. Abbreviate by $v_\delta(t) := \dot{u}_\delta(t)$ for $t \in [0, T]$ local measurements of the velocity field, which are retrieved as the time derivative of the continuously observed differentiable process $u_\delta(t)$. The formal relationship between the velocity field v and the amplitude u is analysed in Section 2.4.

In analogy to Altmeyer and Reiß [4], we consider the augmented maximum

likelihood estimator

$$\hat{\vartheta}_\delta(x_0) := \frac{\int_0^T u_\delta^\Delta(t) dv_\delta(t)}{\int_0^T (u_\delta^\Delta(t))^2 dt}, \quad \delta > 0, \quad (1.3)$$

This estimator can be motivated using the Girsanov theorem or a least squares criterion, c.f. Altmeyer and Reiß [4, Section 4.1]. In Theorem 5.6 of Section 5, we prove, under smoothness assumptions on the kernel K , the initial conditions and the wave speed ϑ , that the augmented MLE satisfies for $\delta \rightarrow 0$:

$$\delta^{-1}(\hat{\vartheta}_\delta(x_0) - \vartheta(x_0)) \xrightarrow{d} \mathcal{N} \left(\frac{\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}, \frac{4\vartheta(x_0)\|K\|_{L^2(\mathbb{R}^d)}^2}{T^2\|\nabla K\|_{L^2(\mathbb{R}^d)}^2} \right), \quad (1.4)$$

with $\beta^{(0)}$ given by (A.22). In Corollary 5.9, we provide reasonable assumptions for the kernel K such that the asymptotic bias of the augmented MLE vanishes, allowing us to construct asymptotic confidence intervals. Crucial in proving the asymptotic normality is the asymptotic behaviour of the observed Fisher information $I_\delta = \int_0^T u_\delta^\Delta(t)^2 dt$ involved in the error decomposition (5.4):

$$\hat{\vartheta}_\delta(x_0) - \vartheta(x_0) = \|K\|_{L^2(\mathbb{R}^d)} I_\delta^{-1} M_\delta + I_\delta^{-1} R_\delta,$$

where M_δ is a martingale term with quadratic variation I_δ , and R_δ characterises the remaining bias. The quadratic variation of the martingale $M_\delta/(\mathbb{E}[I_\delta])^{1/2}$ is given by $I_\delta/\mathbb{E}[I_\delta]$. For the asymptotic normality (1.4), we require the convergence $I_\delta/\mathbb{E}[I_\delta] \xrightarrow{\mathbb{P}} 1$ as $\delta \rightarrow 0$, as the martingale central limit theorem then implies $M_\delta/(\mathbb{E}[I_\delta])^{1/2} \xrightarrow{d} \mathcal{N}(0, 1)$. Therefore, we will analyse the expectation and variance of the observed Fisher information I_δ and the bias R_δ , which is due to the variability of the wave speed $\vartheta(\cdot)$ in space.

In the case of zero initial conditions, the observed Fisher information satisfies

$$\mathbb{E}[\delta^2 I_\delta] = \int_0^T \int_0^t \|S_{\vartheta, \delta}(\delta^{-1}r) \Delta K\|_{L^2(\Lambda_\delta)}^2 dr dt,$$

where $(S_{\vartheta, \delta}(t), t \in \mathbb{R})$ is the rescaled operator sine function associated with the operator $A_{\vartheta, \delta} = A_{\vartheta(\delta, \cdot)}$, which converges asymptotically to $\vartheta(0)\Delta$ on $H^2(\mathbb{R}^d)$. Using functional calculus, we can represent the operator sine function through

$$S_{\vartheta, \delta}(\delta^{-1}r) = (-A_{\vartheta, \delta})^{-1/2} \sin(\delta^{-1}r(-A_{\vartheta, \delta})^{1/2}), \quad r \geq 0.$$

Precise definitions of the operator cosine and sine are provided in Section 2.3, and the associated scaling properties are introduced in Lemma 3.1 of Section 3. We further show fixed-time scaling limits for both the deterministic and stochastic wave equation, c.f. Proposition 3.3 and Proposition 3.4.

A key result in harmonic analysis is the Riemann-Lebesgue lemma that states that the Fourier transform of an L^1 -function vanishes at infinity, c.f. Kahane [35]. This result is important for studying the wave equation as it

provides a tool for studying the long-term behaviour of the energy within the hyperbolic system. Using the Riemann-Lebesgue lemma, it can be shown that the limiting operator $\vartheta(x_0)\Delta$ on $H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ is a so-called Riemann-Lebesgue operator, satisfying the convergence

$$\sin(\delta^{-1}t(-\Delta)^{1/2}) \xrightarrow{w} 0, \quad \delta \rightarrow 0,$$

in the weak operator topology. In particular, if the generator is a Riemann-Lebesgue operator, long term, the kinetic and potential energy contribute equally to the total energy within the system. This effect is called the equipartition of energy.

In Section 4, we formally introduce Riemann-Lebesgue operators and show that asymptotically $A_{\vartheta,\delta}$ satisfies

$$\sin(\delta^{-1}t(-A_{\vartheta,\delta})^{1/2}) \xrightarrow{w} 0, \quad \delta \rightarrow 0,$$

see Proposition 4.5 for the precise notion of convergence associated with varying domains. Thus, we prove that $A_{\vartheta,\delta}$ inherits the Riemann-Lebesgue property of the limiting operator $\vartheta(0)\Delta$ and we obtain the convergence

$$\|S_{\vartheta,\delta}(\delta^{-1}t)\Delta K\|_{L^2(\Lambda_\delta)}^2 \rightarrow \frac{1}{2\vartheta(x_0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \delta \rightarrow 0, \quad t \in [0, T].$$

The scaling of the local observations as $\delta \rightarrow 0$, therefore, corresponds to the long-term energetic behaviour within a deterministic wave equation with first-order initial condition ΔK . Example 4.8 of Section 4, illustrates how the Riemann-Lebesgue lemma emerges when analysing local measurements of a stochastic wave equation on the unbounded spatial domain \mathbb{R}^d given a constant wave speed ϑ . It provides the blueprint on how the asymptotic properties of the statistical quantities in Section 5 are derived based on the analytical results of Section 4.

The finite-sample properties of the augmented MLE, which are demonstrated in the simulations of Section 6, are in line with our theoretical findings. Surprisingly, we can even detect the hyperbolic dependence T^{-2} within the empirical asymptotic variance of the augmented MLE. The appendix is devoted to more technical proofs of the well-posedness of the SPDE in Appendix A.1, spectral asymptotics in Appendix A.2, the asymptotic energetic behaviour in Appendix A.3 and the asymptotic properties of the observed Fisher information I_δ and the bias R_δ in the error decomposition for the augmented MLE in Appendix A.4.

2 The model

2.1 Notation

Let $\Lambda \subset \mathbb{R}^d$ be an open bounded and convex set with C^2 -boundary and consider the standard L^2 -space $(L^2(\Lambda), \langle \cdot, \cdot \rangle_{L^2(\Lambda)})$ equipped with the usual inner product. Let $H^k(\Lambda)$ be the L^2 -Sobolev spaces of order $k \in \mathbb{N}$ and define $H_0^1(\Lambda)$ as the

closure of $C_c^\infty(\bar{\Lambda})$ in $H^1(\Lambda)$. The notation $H_0^{-1}(\Lambda) = H_0^1(\Lambda)^*$ denotes the dual space of $H_0^1(\Lambda)$. Similarly, we will abbreviate by $H_0^\alpha(\Lambda)$ and $\alpha \in \mathbb{R}$ the associated fractional Sobolev spaces introduced in Kovács et al. [37, Section 3]. The space of $L^2(\Lambda)$ will be identified with its dual space $L^2(\Lambda)^*$. We write $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ for the Euclidean inner product and $|\cdot|_{\mathbb{R}^d}$ for the norm. If U and H are two Hilbert spaces, we abbreviate the space of bounded linear operators by $\mathcal{L}(U, H)$ and the Hilbert-Schmidt norm by $\|\cdot\|_{\text{HS}(U, H)}$. Let Δ be the Laplace operator on $L^2(\mathbb{R}^d)$ defined on the domain $D(\Delta) = H^2(\mathbb{R}^d)$. For $f \in L^1(\mathbb{R}^d)$, we define the Fourier transform by $\mathcal{F}[f](\omega) = \int_{\mathbb{R}^d} f(x) e^{i\langle \omega, x \rangle_{\mathbb{R}^d}} dx$ for $\omega \in \mathbb{R}^d$. We further use the usual extension of the Fourier transform to the isomorphism $\mathcal{F} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$.

2.2 The SPDE model

Throughout this work, we consider a fixed time horizon $T < \infty$. Consider the general second-order operator $A_\vartheta : H_0^1(\Lambda) \rightarrow H_0^{-1}(\Lambda)$ defined through

$$\langle A_\vartheta z_1, z_2 \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)} := -\langle \vartheta \nabla z_1, \nabla z_2 \rangle_{L^2(\Lambda)}, \quad z_1, z_2 \in H_0^1(\Lambda). \quad (2.1)$$

For $z \in H_0^1(\Lambda) \cap H^2(\Lambda) \subset L^2(\Lambda)$ we recover the usual second order elliptic operator $A_\vartheta z := \text{div}(\vartheta \nabla z) = \sum_{i=1}^d \partial_i(\vartheta \partial_i z)$ with Dirichlet boundary conditions. The aim of this work is the non-parametric estimation of the wave speed of the stochastic wave equation

$$\begin{cases} \partial_{tt}^2 u(t) = A_\vartheta u(t) + \dot{W}(t), & t \in (0, T], \\ u(t)|_{\partial\Lambda} = 0, & t \in (0, T], \\ u(0) = u_0, \dot{u}(0) = v_0, \end{cases} \quad (2.2)$$

with the wave speed ϑ satisfying **(2.1, Regularity, ϑ, α)** throughout, a cylindrical Wiener process $(W(t), t \in [0, T])$ on $L^2(\Lambda)$ and initial conditions $(u_0, v_0) \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$.

Assumption (2.1, Regularity, ϑ, α). Suppose that $\vartheta \in C^{1+\alpha}(\bar{\Lambda})$ for $\alpha > 0$ describes the space-dependent wave speed, satisfying $\min_{x \in \bar{\Lambda}} \vartheta(x) > 0$. If $d = 1$, we further assume $\alpha > 1/2$.

Assumption (2.2, Initial, u_0, v_0). Suppose $u_0 \in H_0^1(\Lambda) \cap H^2(\Lambda)$ and $v_0 \in H_0^1(\Lambda)$.

Remark 2.3.

- (i) The regularity Assumption **(2.1, Regularity, ϑ, α)** is analogous to the assumption on the diffusivity in Altmeyer and Reiß [4] because we require similar results based on the hypercontractivity of the heat semigroup. In particular, since we have

$$\|e^{t\Delta} u\|_{L^2(\mathbb{R}^d)} \lesssim \min(\|u\|_{L^2(\mathbb{R}^d)}, t^{-d/4} \|u\|_{L^1(\mathbb{R}^d)}), \quad u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

the decay in time of the heat semigroup $(e^{t\Delta}, t \geq 0)$ becomes stronger as the spatial dimension increases. The resulting decay is minimal in dimension one. Hence, we require a slightly higher order of regularity for the wave speed ϑ for $d = 1$.

- (ii) By the Assumption **(2.1, Regularity, ϑ, α)** the wave speed ϑ is lower and upper bounded on the domain Λ . Thus, the standard inner product on $H_0^1(\Lambda)$ can be replaced by the inner product $\langle \vartheta \nabla \cdot, \nabla \cdot \rangle_{L^2(\Lambda)}$ weighted with the wave speed ϑ . By Brezis [10, Remark 24], the weighted Dirichlet problem is as regular as the usual Dirichlet problem provided that $\vartheta \in C^1(\bar{\Lambda})$. The operator (2.1), incorporating the spatially varying wave speed ϑ , is bijective, and the associated existence results for the stochastic wave equation (2.2) will essentially reduce to the parametric case in Appendix A.1.

2.3 Operator cosine and sine functions

Consider the deterministic abstract wave equation

$$\partial_{tt}^2 u(t) = A_\vartheta u(t), \quad u(0) = u_0, \quad \dot{u}(0) = v_0, \quad t \in [0, T].$$

Setting $v = \partial_t u$, we may rewrite (2.3) system of two equations

$$\partial_t u(t) = v(t), \quad \partial_t v(t) = A_\vartheta u(t), \quad u(0) = u_0, \quad v(0) = v_0, \quad t \in [0, T].$$

In particular, the system (2.3) gives rise to the first-order Cauchy problem

$$\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_\vartheta & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad (u(0), v(0))^\top = (u_0, v_0)^\top,$$

on a suitable product space between Hilbert spaces called phase space. In the rest of this section, we will find a representation of the strongly continuous group associated with (2.3) on a suitable phase-space through operator cosine and sine functions, which are the hyperbolic counterpart to the strongly continuous semigroups associated with parabolic Cauchy problems. For an introduction to the theory of operator sine and cosine function, we refer to Arendt et al. [8].

Using the operator (2.1), we may interpret $H_0^{-1}(\Lambda)$ itself as a Hilbert space by defining the inner product

$$\langle l, l' \rangle_{H_0^{-1}(\Lambda)} := \langle A_\vartheta^{-1} l, A_\vartheta^{-1} l' \rangle_{H_0^1(\Lambda)}, \quad l, l' \in H_0^{-1}(\Lambda). \quad (2.3)$$

By Remark 2.3, we obtain the Gelfand triple

$$(H_0^1(\Lambda), L^2(\Lambda), H_0^{-1}(\Lambda)).$$

Arendt, Batty, Hieber, and Neubrander [8, Proposition 7.1.5] show that with **(2.1, Regularity, ϑ, α)** the operator A_ϑ defined in (2.1) is a bounded, self-adjoint operator and that $L^2(\Lambda) \times H_0^{-1}(\Lambda)$ is the phase space associated with the cosine function generated by A_ϑ on $H_0^{-1}(\Lambda)$ with the inner product (2.3).

Given Arendt, Batty, Hieber, and Neubrander [8, Theorem 3.14.11], let $(C_\vartheta(t), t \in \mathbb{R})$ and $(S_\vartheta(t), t \in \mathbb{R})$, with $S(t) := \int_0^t C(s)ds$ for $t \in [0, T]$, be the operator cosine and sine-functions associated with the operator A_ϑ . We abbreviate by \mathcal{A}_ϑ the associated generator on the phase space $L^2(\Lambda) \times H_0^{-1}(\Lambda)$, given by

$$\begin{aligned} D(\mathcal{A}_\vartheta) &= D(A_\vartheta) \times L^2(\Lambda) = H_0^1(\Lambda) \times L^2(\Lambda), \\ \mathcal{A}_\vartheta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 0 & I \\ A_\vartheta & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ A_\vartheta z_1 \end{pmatrix}. \end{aligned} \quad (2.4)$$

The operator sine and cosine functions take the following values

$$\begin{aligned} S_\vartheta(\cdot)l &\in C(\mathbb{R}, L^2(\Lambda)), \quad l \in H_0^{-1}(\Lambda), \\ S_\vartheta(\cdot)z &\in C(\mathbb{R}, H_0^1(\Lambda)), \quad z \in L^2(\Lambda), \\ C_\vartheta(\cdot)z &\in C^1(\mathbb{R}, H_0^{-1}(\Lambda)) \cap C(\mathbb{R}, L^2(\Lambda)), \quad z \in L^2(\Lambda). \end{aligned}$$

In particular, by Arendt et al. [8, Theorem 3.14.11], the operator \mathcal{A}_ϑ defined through (2.4) generates a C_0 -group \mathcal{J}_ϑ on the phase-space $L^2(\Lambda) \times H_0^{-1}(\Lambda)$ given by

$$\mathcal{J}_\vartheta(t) = \begin{pmatrix} C_\vartheta(t) & S_\vartheta(t) \\ C'_\vartheta(t) & C_\vartheta(t) \end{pmatrix} = \begin{pmatrix} C_\vartheta(t) & S_\vartheta(t) \\ A_\vartheta S_\vartheta(t) & C_\vartheta(t) \end{pmatrix}, \quad t \in \mathbb{R}. \quad (2.5)$$

Remark 2.4. The operator cosine function is precisely the cosine applied to the square root of the negative of the generator by the functional calculus. In contrast, the operator sine is defined through $S(t) = \int_0^t C(s)ds$ for $t \in [0, T]$ and does not correspond to the operator sine as induced by the functional calculus. The operator sine $S(t)$ is only used because it emerges naturally in d'Alembert's formula and is therefore involved directly in the solution of a second-order abstract Cauchy problem. For an overview of the different types of notations for operator cosine and sine functions, we refer to Pandolfi [45].

2.4 Well-posedness for the SPDE

Throughout this section, we assume zero-initial conditions for simplicity:

$$u_0(x) = v_0(x) = 0, \quad x \in \Lambda. \quad (2.6)$$

All the results in this section immediately extend to initial conditions satisfying **(2.2, Initial, u_0, v_0)**. We begin by considering the particular case $d = 1$. The following result provides the existence and uniqueness of a mild function-valued solution to the stochastic wave equation on a bounded spatial domain.

Theorem 2.5 (Existence of the one-dimensional stochastic wave equation). *Assume zero initial conditions (2.6) in (2.2). Let $\Lambda \subset \mathbb{R}$ be an open and bounded one-dimensional spatial domain. Suppose that $(W(t), t \in [0, T])$ is a cylindrical Wiener process on $L^2(\Lambda)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, \mathbb{P})$. Then, (2.2) has a unique mild solution given by the variations of constants formula*

$$X(t) = \int_0^t \mathcal{J}_\vartheta(t-s) B dW(s), \quad t \geq 0, \quad (2.7)$$

where $(\mathcal{J}_\vartheta(t), t \in [0, T])$ is given by (2.5) and $B : L^2(\Lambda) \rightarrow L^2(\Lambda) \times H_0^{-1}(\Lambda)$ with $Bu = (0, u) \in L^2(\Lambda) \times (L^2(\Lambda))' \subset L^2(\Lambda) \times H_0^{-1}(\Lambda)$.

Proof of Theorem 2.5. This is an immediate corollary of Kovács, Larsson, and Saedpanah [37, Theorem 3.1 and Remark 3.2]. \square

By Da Prato and Zabczyk [13, Theorem 5.4], the mild solution (2.7) is also a weak solution and satisfies

$$\begin{aligned} & \langle X(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} \\ &= \int_0^t \langle X(s), \mathcal{A}_\vartheta^* U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} ds + \langle BW(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}, \quad U \in D(\mathcal{A}_\vartheta^*). \end{aligned} \tag{2.8}$$

Let us denote $(u(t), v(t))^\top := X(t)$. We obtain the following important dynamical representation by determining the adjoint of \mathcal{A}_ϑ and carefully differentiating between the inner product and dual-pairing associated with $H_0^{-1}(\Lambda)$.

Proposition 2.6 (Dynamic representation of the weak solution). *Assume zero initial conditions (2.6) in (2.2). Let $\Lambda \subset \mathbb{R}$ be an open and bounded one-dimensional spatial domain. For every function $z \in H_0^1(\Lambda) \cap H^2(\Lambda)$, we have*

$$\begin{aligned} \langle u(t), z \rangle_{L^2(\Lambda)} &= \int_0^t \langle v(s), z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)} ds, \\ \langle v(t), z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)} &= \int_0^t \langle u(s), A_\vartheta z \rangle_{L^2(\Lambda)} ds + \langle W(t), z \rangle_{L^2(\Lambda)}, \end{aligned} \tag{2.9}$$

for all $t \in [0, T]$ on the same set of probability one.

Proof of Proposition 2.6. The result is proved on page 31. \square

We now turn to $d > 1$. The arguments employed for Theorem 2.5 and Proposition 2.6 cannot be applied because

$$\int_0^T \|\mathcal{J}_\vartheta(t)B\|_{\text{HS}(L^2(\Lambda), L^2(\Lambda) \times H_0^{-1}(\Lambda))}^2 dt = \infty,$$

violates condition (5.3) of Da Prato and Zabczyk [13, Theorem 5.2], required for an informative version of Itô's isometry. In this case, the stochastic integral (2.7) is only well-defined as a distribution, and there does not exist any standard function-valued solution.

This issue is also prevalent in the case of the stochastic heat equation and was resolved in Altmeyer and Reiß [4, Proposition 2.1] by passing to a Gaussian process that preserves the covariance structure induced by the associated strongly continuous semigroup. The following result extends this argument to the setting of the stochastic wave equation.

Proposition 2.7 (Properties of the Gaussian process solution). *There is a centred Gaussian process $(\mathcal{V}(t, U), t \in [0, T], U \in L^2(\Lambda) \times H_0^{-1}(\Lambda))$ given by (A.4) with the covariance function*

$$\text{Cov}(\mathcal{V}(t, U), \mathcal{V}(t', U')) = \int_0^{t \wedge t'} \langle B^* \mathcal{J}_\vartheta^*(t-s)U, B^* \mathcal{J}_\vartheta^*(t'-s)U' \rangle_{L^2(\Lambda)} ds, \quad (2.10)$$

for $U, \tilde{U} \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$ and $t, t' \in [0, T]$. The process $(\mathcal{V}(t, U), U \in L^2(\Lambda) \times H_0^{-1}(\Lambda))$ is given by the sum of two Gaussian processes, $(u_\mathcal{V}(t, z), t \in [0, T], z \in L^2(\Lambda))$ and $(v_\mathcal{V}(t, z), t \in [0, T], z \in L^2(\Lambda))$, satisfying the dynamic

$$u_\mathcal{V}(t, z) = \int_0^t v_\mathcal{V}(s, z) ds \quad (2.11)$$

$$v_\mathcal{V}(t, z) = \int_0^t u_\mathcal{V}(s, A_\vartheta z) ds + \langle W(t), z \rangle_{L^2(\Lambda)}, \quad z \in H_0^1(\Lambda) \cap H^2(\Lambda). \quad (2.12)$$

Proof of Proposition 2.7. The result is proved on page 31. \square

Justified by this result, we will write $\langle u(t), z \rangle_{L^2(\Lambda)}$ and $\langle v(t), z \rangle$ throughout instead of $(u_\mathcal{V}(t, z))$ and $(v_\mathcal{V}(t, z))$ for $z \in L^2(\Lambda)$ and any $t \in [0, T]$ and consider the Gaussian process from (2.7) as the solution to the stochastic wave equation (2.2).

Remark 2.8 (Solution concepts).

- (i) Notice that the inclusion mappings $\iota_{r,s} : H_0^r(\Lambda) \rightarrow H_0^s(\Lambda)$ are Hilbert-Schmidt provided that $r - s > d/2$ for any $r > s$ and $r, s \in \mathbb{R}$, where $H_0^{-s}(\Lambda)$ are fractional Sobolev spaces of the negative order. In particular, there is a Hilbert-Schmidt embedding $\tilde{\iota} : L^2(\Lambda) \times H_0^{-1}(\Lambda) \rightarrow H_0^{-s}(\Lambda) \times H_0^{-(s+1)}(\Lambda)$, and we have

$$\int_0^T \|\tilde{\iota} \mathcal{J}(t) B\|_{\text{HS}(L^2(\Lambda), H_0^{-s}(\Lambda) \times H_0^{-(s+1)}(\Lambda))}^2 dt < \infty.$$

Consequently, (c.f. Remark 5.7 in Hairer [28]) the process $(X(t), t \in [0, T])$ defined by (2.7) takes values in $H_0^{-s}(\Lambda) \times H_0^{-(s+1)}(\Lambda)$. Note that the process \mathcal{V} from Proposition 2.7 is well-defined independently of the embedding space for $(X(t), t \in [0, T])$ and extends the linear form $U \mapsto \langle X(t), U \rangle$ from $C_c^\infty(\bar{\Lambda}) \times C_c^\infty(\bar{\Lambda})$ to $L^2(\Lambda) \times H_0^{-1}(\Lambda)$.

- (ii) For the unbounded spatial domain $\Lambda = \mathbb{R}$, function-valued solutions exist in a weighted L_ρ^2 -space where ρ is an integrable weight function, see Karczewska and Zabczyk [36]. In this case, Proposition 2.6 is not immediate as the change in the underlying norm also changes partial integration properties crucial to the behaviour of the Laplace operator. Instead, a random-field approach for the stochastic wave equation similar to the approach described in Walsh [59] can also lead to dynamic representations analogous to (2.9), see Delerue [15].

- (iii) For the case of space-time white noise, there is neither a function-valued solution (2.2) nor a random field solution to the stochastic wave equation for $d > 1$, see for instance Foondun, Khoshnevisan, and Nualart [20]. As we are only interested in the covariance structure of the process and a representation of the form (2.9), it is sufficient for our purpose to understand the behaviour of the distribution valued solution evaluated through more regular test functions.

Corollary 2.9 (Covariance structure of the contributing processes). *For $z, z' \in L^2(\Lambda)$ and $t, s \in [0, T]$, we have*

$$\begin{aligned} \text{Cov}(\langle u(t), z \rangle_{L^2(\Lambda)}, \langle u(s), z' \rangle_{L^2(\Lambda)}) &= \int_0^{t \wedge s} \langle S_\vartheta(t-r)z, S_\vartheta(s-r)z' \rangle_{L^2(\Lambda)} dr, \\ \text{Cov}(\langle v(t), z \rangle, \langle v(s), z' \rangle) &= \int_0^{t \wedge s} \langle C_\vartheta(t-r)z, C_\vartheta(s-r)z' \rangle_{L^2(\Lambda)} dr. \end{aligned}$$

Proof of Corollary 2.9. The result follows from immediately from Da Prato and Zabczyk [13, Proposition 4.28], Lemma A.3 and the definition of the processes $u_\mathcal{V}(t, z)$ and $v_\mathcal{V}(t, z)$. \square

3 Scaling limits

This section is devoted to deriving a scaling limit for the stochastic wave equation. As introduced in Altmeyer and Reiß [4], we fix $\delta > 0$ and define for any $z \in L^2(\mathbb{R}^d)$ the rescaling

$$\begin{aligned} \Lambda_\delta &:= \delta^{-1}\Lambda = \{\delta^{-1}x : x \in \Lambda\}, & \Lambda_0 &:= \mathbb{R}^d, \\ z_\delta(x) &:= \delta^{-d/2}z(\delta^{-1}x), & x &\in \mathbb{R}^d. \end{aligned} \tag{3.1}$$

For convenience only, we consider a localisation around zero and estimate the unknown wave speed ϑ at $\vartheta(0)$. If we wish to estimate ϑ at some different $x_0 \in \Lambda$ the rescaling (3.1) has to be shifted by x_0 as introduced in Altmeyer and Reiß [4]. The normalisation of the rescaling is arbitrary and satisfies $\|z_\delta\|_{L^2(\Lambda)} = \|z\|_{L^2(\mathbb{R}^d)}$ for convenience.

The rescaled generator $A_{\vartheta, \delta} := A_{\vartheta(\delta \cdot)}$ with $D(A_{\vartheta, \delta}) = H_0^1(\Lambda_\delta) \cap H^2(\Lambda_\delta)$ induces operator sine and cosine functions $(C_{\vartheta, \delta}(t), t \in [0, T])$ and $(S_{\vartheta, \delta}(t), t \in [0, T])$. Analogous to Altmeyer and Reiß [4, Lemma 3.1], the following result characterises the rescaling behaviour of operator cosine and sine functions acting on localised functions.

Lemma 3.1 (Rescaling of operator cosine and sine functions). *For $\delta > 0$:*

- (i) *If $z \in H_0^1(\Lambda_\delta) \cap H^2(\Lambda_\delta)$, then $A_\vartheta z_\delta = \delta^{-2}(A_{\vartheta, \delta} z)_\delta$.*
- (ii) *If $z \in L^2(\Lambda_\delta)$, then $S_\vartheta(t)z_\delta = \delta(S_{\vartheta, \delta}(\delta^{-1}t)z)_\delta$ and $C_\vartheta(t)z_\delta = (C_{\vartheta, \delta}(\delta^{-1}t)z)_\delta$.*

Proof of Lemma 3.1. It suffices to prove the result for any $z_1, z_2 \in C_c^\infty(\bar{\Lambda}_\delta)$. In particular, (i) is a special case of (i) in Altmeyer and Reiß [4, Lemma 3.1]. For (ii), we define

$$w(t) := \delta(S_{\vartheta,\delta}(\delta^{-1}t)z_1)_\delta + (C_{\vartheta,\delta}(\delta^{-1}t)z_2)_\delta, \quad t \in [0, T].$$

The first and second derivatives of w are given by

$$\begin{aligned} \dot{w}(t) &= (C_{\vartheta,\delta}(\delta^{-1}t)z_1)_\delta + \delta^{-1}(A_{\vartheta,\delta}S_{\vartheta,\delta}(\delta^{-1}t)z_2)_\delta \\ \ddot{w}(t) &= \delta^{-1}(A_{\vartheta,\delta}S_{\vartheta,\delta}(\delta^{-1}t)z_1)_\delta + \delta^{-2}(A_{\vartheta,\delta}C_{\vartheta,\delta}(\delta^{-1}t)z_2)_\delta \\ &= A_\vartheta\delta(S_{\vartheta,\delta}(\delta^{-1}t)z_1)_\delta + A_\vartheta(C_{\vartheta,\delta}(\delta^{-1}t)z_2)_\delta = A_\vartheta w(t), \quad t \in [0, T]. \end{aligned}$$

Given Arendt et al. [8, Corollary 3.14.8], we conclude from $w(0) = (z_1)_\delta$, $\dot{w}(0) = (z_2)_\delta$ and $w''(t) = A_\vartheta w(t)$ the identity $w(t) = S_\vartheta(t)(z_1)_\delta + C_\vartheta(t)(z_2)_\delta$ as the function w is the unique solution to the second-order abstract Cauchy problem

$$\partial_{tt}^2 g(t) = A_\vartheta g(t), \quad g(0) = (z_1)_\delta, \quad \dot{g}(0) = (z_2)_\delta, \quad t \in [0, T].$$

The result follows by setting z_1 or z_2 to zero, respectively. \square

The following example illustrates the extension of the rescaling Lemma 3.1 to an unbounded spatial domain in the one-dimensional case.

Example 3.2 (Parametric rescaling on the unbounded domain). Lemma 3.1 extends naturally to the setting of an unbounded domain. In that case, we extend our notation for $\delta = 0$ and suppose that $(C_{\vartheta,0}(t), t \in [0, T])$ and $(S_{\vartheta,0}(t), t \in [0, T])$ are the operator cosine and sine functions associated with the operator $\vartheta(0)\Delta$ defined on $H^2(\mathbb{R}^d)$. In order to simplify our notation, we will write $(S_{\vartheta(0)}(t), t \in [0, T])$ and $(C_{\vartheta(0)}(t), t \in [0, T])$ instead of $(C_{\vartheta,0}(t), t \in [0, T])$ and $(S_{\vartheta,0}(t), t \in [0, T])$, respectively.

Assume for simplicity that $d = 1$ and consider the left translation group $(T_{\vartheta(0)}(t)z)(x) = z(x + \vartheta(0)t)$ for $t \in [0, T]$, $z \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Using Arendt et al. [8, Example 3.14.15], we can represent the operator cosine function through $C_{\vartheta(0)}(t)z := (T_{\vartheta(0)}(t)z + T_{\vartheta(0)}(-t)z)/2$ for $z \in L^2(\mathbb{R})$ and $t \in [0, T]$. The operator sine function associated with $(C_{\vartheta(0)}(t), t \in [0, T])$ is then given by

$$\begin{aligned} [S_{\vartheta(0)}(t)z](x) &:= \int_0^t [C_{\vartheta(0)}(s)z](x) ds \\ &= \frac{1}{2} \int_0^t z(x - \vartheta(0)s) + z(x + \vartheta(0)s) ds, \quad t \in [0, T], \quad x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

We recover the rescaling Lemma 3.1 (ii) in the case of the unbounded one-dimensional spatial domain by applying the operator cosine to a localised function

$$\begin{aligned} [C_{\vartheta(0)}(t)z_\delta](x) &= \frac{1}{2}(z_\delta(x + \vartheta(0)t) + z_\delta(x - \vartheta(0)t)) \\ &= \frac{\delta^{-1/2}}{2}(z(\delta^{-1}x + \delta^{-1}\vartheta(0)t) + z(\delta^{-1}x - \delta^{-1}\vartheta(0)t)) \\ &= [C_{\vartheta(0)}(\delta^{-1}t)z]_\delta(x), \quad t \in [0, T], \quad x \in \mathbb{R}. \end{aligned}$$

Similarly, using (3.2), the rescaling of the operator sine function is obtained using the transformation theorem:

$$\begin{aligned}
[S_{\vartheta(0)}(t)z_{\delta}](x) &= \delta^{-1/2} \int_0^t z(\delta^{-1}x + \delta^{-1}\vartheta(0)s) + z(\delta^{-1}x - \delta^{-1}\vartheta(0)s)ds \\
&= \delta^{1/2} \int_0^{\delta^{-1}t} z(\delta^{-1}x + \vartheta(0)r) + z(\delta^{-1}x - \vartheta(0)r)dr \\
&= \delta\delta^{-1/2}[S_{\vartheta(0)}(\delta^{-1}t)z](\delta^{-1}x) = \delta[S_{\vartheta(0)}(\delta^{-1}t)z]_{\delta}(x),
\end{aligned}$$

for $t \in [0, T]$ and $x \in \mathbb{R}$. As expected, Lemma 3.1 still holds in the parametric case on the unbounded spatial domain \mathbb{R} , only that the rescaled families of operators do not depend themselves on the scaling parameter $\delta > 0$ through the unknown function.

Let us denote by

$$P_{\delta}z = \mathbb{1}_{\Lambda_{\delta}}z, \quad P_{\delta} : L^2(\mathbb{R}^d) \rightarrow L^2(\Lambda_{\delta}), \quad (3.3)$$

the orthogonal projection from $L^2(\mathbb{R}^d)$ onto $L^2(\Lambda_{\delta})$. For more details on the conventions associated with this projection, see also Remark A.5.

Using the asymptotic behaviour of the partition of unity associated with $A_{\vartheta, \delta}$ analysed in Appendix A.2, we obtain the following scaling behaviour for the operator cosine and sine functions ($C_{\vartheta, \delta}(t), t \in [0, T]$) and ($S_{\vartheta, \delta}(t), t \in [0, T]$).

Proposition 3.3 (Deterministic scaling limits for the operator sine and cosine). *Grant (2.1, Regularity, ϑ, α). Then, for any $z \in L^2(\mathbb{R}^d)$, we have*

$$S_{\vartheta, \delta}(t)P_{\delta}z \rightarrow S_{\vartheta(0)}(t)z, \quad C_{\vartheta, \delta}(t)P_{\delta}z \rightarrow C_{\vartheta(0)}(t)z, \quad \delta \rightarrow 0, \quad t \in [0, T], \quad (3.4)$$

where ($C_{\vartheta(0)}(t), t \in [0, T]$) and ($S_{\vartheta(0)}(t), t \in [0, T]$) are the operator cosine and sine functions associated with the operator $\vartheta(0)\Delta$, see also page 12.

Proof of Proposition 3.3. The result is proved on page 35. \square

Similar to the scaling limit for the stochastic heat equation, see Altmeyer and Reiß [4, Theorem 3.6], we also obtain a scaling limit for the stochastic wave equation (2.2). Suppose we consider the scaled localised process up until a finite time horizon $T > 0$. Then, in the parametric case, as the wave equation has a finite speed of propagation. There exists some $\delta_T > 0$ such that for $\delta \in (0, \delta_T)$, the localised process associated with the bounded domain cannot be differentiated from the process associated with the unbounded spatial domain upon testing against a function compactly supported in Λ . These insights are summarised in the following result if we assume zero initial conditions.

Proposition 3.4 (Scaling limit for the stochastic wave equation). *Assume that $u_0 = 0$ and $v_0 = 0$. Consider the process $Z_{\delta}(t, z) := \delta^{-3/2}\langle u(\delta t), (P_{\delta}z)_{\delta} \rangle_{L^2(\Lambda)}$ for $z \in L^2(\mathbb{R}^d)$ and $t \geq 0$. For $t \in [0, T]$ and $z \in L^2(\mathbb{R}^d)$, let $\underline{Z}(t, z) = \langle \underline{u}(t), z \rangle_{L^2(\mathbb{R}^d)}$*

be the scaled localisation of the limiting process $(\underline{u}(t), t \in [0, T])$ solving the stochastic wave equation

$$\partial_{tt}^2 \underline{u}(t) = \vartheta(0) \Delta \underline{u}(t) + \dot{W}(t), \quad \underline{u}(0) = \dot{\underline{u}}(0) = 0, \quad t \geq 0,$$

as a Gaussian process in the sense of Proposition 2.7 with the space-time white-noise $(\dot{W}(t), t \in [0, T])$ on $L^2(\mathbb{R}^d)$.

- (i) Then, the finite-dimensional distributions of the process $(Z_\delta(t, z), t \geq 0, z \in L^2(\mathbb{R}^d))$ converge to those of $(\underline{Z}(t, z), t \geq 0, z \in L^2(\mathbb{R}^d))$.
- (ii) Let $T > 0$ be some fixed time horizon and assume that ϑ is constant. Let $z \in L^2(\mathbb{R}^d)$ be compactly supported in Λ . Then, there exists some $\delta_T = \delta_T(\vartheta_*, z) > 0$ such that the finite-dimensional distributions of the process $(Z_\delta(t, z), t \in [0, T])$ are identical to those of $(\underline{Z}(t, z), t \in [0, T])$, for any $0 < \delta < \delta_T$.

Proof of Proposition 3.4.

Step 1. Clearly, by Corollary 2.9 and Lemma 3.1 the process Z_δ is a centred Gaussian process with the covariance function

$$\begin{aligned} & \mathbb{E}[Z_\delta(t, z_1) Z_\delta(s, z_2)] \\ &= \delta^{-3} \int_0^{\delta(t \wedge s)} \langle S_\vartheta(\delta t - r)(P_\delta z_1)_\delta, S_\vartheta(\delta s - r)(P_\delta z_2)_\delta \rangle_{L^2(\Lambda)} dr \\ &= \delta^{-3} \int_0^{\delta(t \wedge s)} \langle \delta S_{\vartheta, \delta}(\delta^{-1}(\delta t - r)) P_\delta z_1, \delta S_{\vartheta, \delta}(\delta^{-1}(\delta s - r)) P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} dr \\ &= \int_0^{t \wedge s} \langle S_{\vartheta, \delta}(t - r) P_\delta z_1, S_{\vartheta, \delta}(s - r) P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} dr, \quad t, s \geq 0, \quad z_1, z_2 \in L^2(\mathbb{R}^d). \end{aligned} \tag{3.5}$$

Similarly, the covariance function of the process \underline{Z} is given by

$$\begin{aligned} & \mathbb{E}[\underline{Z}(t, z_1) \underline{Z}(s, z_2)] \\ &= \int_0^{t \wedge s} \langle S_{\vartheta(0)}(t - r) z_1, S_{\vartheta(0)}(s - r) z_2 \rangle_{L^2(\mathbb{R}^d)} dr, \quad t, s \geq 0, \quad z, z' \in L^2(\mathbb{R}^d). \end{aligned} \tag{3.6}$$

By (3.4) in Proposition 3.3, we obtain the convergence $S_{\vartheta, \delta}(\tau) P_\delta z_1 \rightarrow S_{\vartheta(0)}(\tau) z_1$ in $L^2(\mathbb{R}^d)$ for any $\tau \geq 0$ as $\delta \rightarrow 0$. The same is true for z_2 . In particular, by the representation $S_{\vartheta, \delta}(\tau) z = \int_0^\tau C_{\vartheta, \delta}(s) z ds$ and the boundedness of the cosine, we have with the functional calculus the upper bound

$$\sup_{0 < \delta \leq 1} \sup_{0 \leq \tau \leq T} \|S_{\vartheta, \delta}(\tau) z\|_{L^2(\Lambda_\delta)} \leq T \|z\|_{L^2(\mathbb{R}^d)} < \infty.$$

The result follows as (3.5) converges to (3.6) by the dominated convergence theorem.

Step 2. As we have assumed $\vartheta(x) = \vartheta_*$ for some constant $\vartheta_* > 0$ and all $x \in \Lambda$, it makes sense write $S_{\vartheta, \delta} = S_{\vartheta_*, \delta}$. As the wave equation has a finite

propagation speed, see Evans [19], and z is compactly supported, there exists some $\delta_T = \delta_T(\vartheta_*, z) > 0$ such that for all $0 < \delta < \delta_T$ we have

$$S_{\vartheta_*, \delta}(\tau)z = S_{\vartheta_*}(\tau)z, \quad \tau \in [0, T]. \quad (3.7)$$

Indeed, if $\delta_T > 0$ is chosen so small that the outer edge of the light cone induced by $S_{\vartheta_*, \delta}(\tau)z$ does not reach the boundary of the domain Λ_δ up until the finite time horizon T , $S_{\vartheta_*, \delta}(\tau)z$ cannot be differentiated from $S_{\vartheta_*}(\tau)z$ for any $\tau \in [0, T]$. The identity (3.7) then follows by the uniqueness of solutions to the wave equation; see Arendt, Batty, Hieber, and Neubrander [8, Corollary 3.14.8]. As a consequence, combining the representations (3.5) and (3.6) with the identity (3.7), we have

$$\mathbb{E}[Z_\delta(t, z)Z_\delta(s, z)] = \mathbb{E}[\underline{Z}(t, z)\underline{Z}(s, z)], \quad t, s \in [0, T], \quad 0 < \delta < \delta_T. \quad (3.8)$$

The result follows from (3.8) as two centred Gaussian processes with the same covariance function have the same law. \square

Remark 3.5 (Finite propagation speed).

- (i) Note that in the proof of Proposition 3.4 (ii) the identity (3.7) does not hold in the non-parametric situation. Even if the outer edges of the associated light cones do not reach the boundary of the domain Λ_δ , the associated PDEs are only identical in the limit as $\delta \rightarrow 0$ in general as $\vartheta(\delta \cdot)$ approximates $\vartheta(0)$ but is never quite identical to it provided that ϑ is not locally constant at zero.
- (ii) In the parametric case $\vartheta \equiv \vartheta_0 > 0$, Proposition 3.4 (ii) implies that the Hellinger-distance between the laws of processes $(Z_\delta(t, z), t \in [0, T])$ and $(\underline{Z}(t, z), t \in [0, T])$ is zero for $\delta \in (0, \delta_T)$. Note that $\delta_T > 0$ depends on the time horizon. Thus, the result is no longer accessible if, after rescaling, the time horizon depends on the resolution level δ itself. Thus, it is important to also understand the energetic behaviour of the rescaled trigonometric operator families as time increases, which will be the topic of the next section.

4 Asymptotic energetic behaviour

This section is devoted to understanding the energetic behaviour of the stochastic wave equation under rescaling. The remaining proofs and more technical results are postponed to Appendix A.3.

Consider a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ carrying a self-adjoint, negative, linear unbounded operator $(\mathcal{A}, D(\mathcal{A}))$. Goldstein [22] studied the energetic behaviour of abstract wave equations of the form

$$w''(t) = \mathcal{A}w(t), \quad w(0) = w_0 \in D(\mathcal{A}), \quad \dot{w}(0) = w_1 \in D((- \mathcal{A})^{1/2}). \quad (4.1)$$

The abstract Cauchy problem (4.1) is well-posed, and the total energy

$$\mathcal{E}_{\mathcal{A}} \equiv P(t) + K(t) := \|(-\mathcal{A})^{1/2}w(t)\|_{\mathcal{H}}^2 + \|\dot{w}(t)\|_{\mathcal{H}}^2, \quad t \in \mathbb{R}. \quad (4.2)$$

is preserved and does not depend on time $t \in \mathbb{R}$. Goldstein [22] discovered that the asymptotic equipartition of energy

$$\lim_{|t| \rightarrow \infty} P(t) = \lim_{|t| \rightarrow \infty} K(t) = \frac{\mathcal{E}_{\mathcal{A}}}{2} \quad (4.3)$$

holds if \mathcal{A} satisfies the Riemann-Lebesgue property defined through

$$\langle e^{it(-\mathcal{A})^{1/2}} z_1, z_2 \rangle_{\mathcal{H}} \rightarrow 0, \quad |t| \rightarrow \infty, \quad z_1, z_2 \in \mathcal{H}, \quad (4.4)$$

where the convergence is exactly the convergence $e^{it(-\mathcal{A})^{1/2}} \xrightarrow{w} 0$ in the weak operator topology as $|t| \rightarrow \infty$. Thus, asymptotically, the kinetic and potential energy contribute equally to the entire energy within the system. Operators satisfying condition (4.4) are called Riemann-Lebesgue operators. Goldstein [23] noticed that assumption (4.4) is a condition on the resolution of the identity $(E(\lambda), \lambda \in \mathbb{R})$ of \mathcal{A} , which lies strictly between continuity and absolute continuity of $E(\cdot)$. Thus, an operator like the Laplace operator on the unbounded spatial domain, which has a fully absolutely continuous spectrum as defined in Schmüdgen [54, Chapter 9.1], is a Riemann-Lebesgue operator.

The concept of Riemann-Lebesgue operators is not specific to the abstract wave equation and also extends to a large class of energy-preserving and hyperbolic equations like the abstract Schrödinger equation, see for instance Goldstein and Sandefur [26, 27], Sandefur and Payne [53], Picard and Seidler [48], Picard [49]. By Goldstein [24], the equipartition of energy (4.2) also holds in the sense of Cèsaro limits and for auto-correlations. Marcello D'Abbicco, Girardi, Ruiz Goldstein, A. Goldstein, and Romanelli [43] and Biler [9] employed similar techniques in the analysis of the behaviour of energies within abstract wave equations with certain types of damping.

Remark 4.1. Depending on the context, sometimes the Riemann-Lebesgue operator property (4.4) is defined directly using the operator $e^{it\mathcal{A}}$ and not using the root $(-\mathcal{A})^{1/2}$, see for instance Goldstein and Sandefur [26]. We will use the above convention because we consider the stochastic wave equation, and the operator root turns up naturally.

Consider the Laplace operator Δ on $L^2(\mathbb{R}^d)$ with the domain $D(\Delta) = H^2(\mathbb{R}^d)$. Using the Riemann-Lebesgue lemma, we show that the Laplace operator Δ is a Riemann-Lebesgue operator.

Lemma 4.2 (The Laplace operator is a Riemann-Lebesgue operator). *We have $e^{it(-\Delta)^{1/2}} \xrightarrow{w} 0$ as $|t| \rightarrow \infty$, i.e.,*

$$\langle e^{it(-\Delta)^{1/2}} z_1, z_2 \rangle_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad |t| \rightarrow \infty, \quad z_1, z_2 \in L^2(\mathbb{R}^d). \quad (4.5)$$

Proof of Lemma 4.2. Since $\sigma(-\Delta) = [0, \infty)$ we observe with $g_t(x) = e^{it\sqrt{x}}$, that $e^{it(-\Delta)^{1/2}}z = \mathcal{F}^{-1}(M_{g_t \circ p}\mathcal{F}(z))$, where $p(x) = |x|^2$ and

$$M_{g_t \circ p}[\tilde{z}](\omega) := (g_t \circ p)(\omega)\tilde{z}(\omega), \quad \omega \in \mathbb{R}^d, \quad \tilde{z} \in L^2(\mathbb{R}^d),$$

is the standard multiplication operator; see Schmüdgen [54, Proposition 8.2]. By polarisation, it is sufficient to prove the result for $z = z_1 = z_2 \in L^2(\mathbb{R}^d)$. Using Plancherel's identity, we obtain

$$\begin{aligned} \langle e^{it(-\Delta)^{1/2}}z, z \rangle_{L^2(\mathbb{R}^d)} &= (2\pi)^d \langle M_{g_t \circ p}\mathcal{F}(z), \mathcal{F}(z) \rangle_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} e^{it|\omega|} |\mathcal{F}(z)|^2(\omega) d\omega \rightarrow 0, \quad |t| \rightarrow \infty, \end{aligned} \quad (4.6)$$

where the last convergence follows from the generalised Riemann-Lebesgue Lemma in Kahane [35] as $|\mathcal{F}(z)|^2(\cdot) \in L^1(\mathbb{R}^d)$. \square

Remark 4.3 (Spectrum of Riemann-Lebesgue operators).

- (i) We have already mentioned that for a self-adjoint operator to be a Riemann Lebesgue operator, it is sufficient that it has a purely absolutely continuous spectrum. Any self-adjoint differential operator on the unbounded domain \mathbb{R}^d that can be diagonalised using the Fourier transform has a purely absolutely continuous spectrum; see Schmüdgen [54, Corollary 9.4]. However, any Riemann-Lebesgue operator must have at least a fully continuous spectrum. The Laplace operator on a bounded spatial domain Λ naturally has eigenvalues. At these eigenvalues, the resolution of the identity has jumps and is discontinuous. Therefore, the spectrum is not fully continuous, and the weak convergence cannot hold for all elements of $L^2(\Lambda)$. Thus, the Laplace operator on a bounded spatial domain is not a Riemann-Lebesgue operator in the sense of (4.4). See also the proof of (iii) in Goldstein [22].
- (ii) Even if a generator \mathcal{A} , for example, the Laplace operator on a bounded domain, is not a Riemann-Lebesgue operator, the equipartition of energy may still hold in the Cesaro sense

$$\lim_{|T| \rightarrow \infty} \frac{1}{T} \int_0^T P(t) dt = \lim_{|T| \rightarrow \infty} \frac{1}{T} \int_0^T K(t) dt = \frac{\mathcal{E}_{\mathcal{A}}}{2},$$

see for instance Goldstein [23]. This is intuitive as the potential and kinetic energy may oscillate and may never converge, while these oscillations do not interfere with a Cesaro limit.

Since the Laplace operator on $L^2(\mathbb{R}^d)$ with domain $D(\Delta) = H^2(\mathbb{R}^d)$ is a Riemann-Lebesgue operator, Goldstein [22] shows for $\mathcal{A} = \Delta$ in (4.1) and (4.3) that we have the asymptotic equipartition of energy

$$\lim_{|t| \rightarrow \infty} \|(-\Delta)^{1/2}w(t)\|_{L^2(\mathbb{R}^d)}^2 = \lim_{|t| \rightarrow \infty} \|w'(t)\|_{L^2(\mathbb{R}^d)}^2 = \frac{\mathcal{E}_{\Delta}}{2}. \quad (4.7)$$

Remark 4.4 (Different notions of energy). In (4.2), the total energy (in \mathcal{H}) within the system of interest is preserved and can be decomposed into the total potential energy $\|(-\mathcal{A})^{1/2}w_0\|_{\mathcal{H}}^2$ and the total kinetic energy $\|w_1\|_{\mathcal{H}}^2$. If zero belongs to the resolvent set of the operator \mathcal{A} , one can also define the energy in the dual space $D((-\mathcal{A})^{1/2})^*$:

$$\tilde{\mathcal{E}}_{\mathcal{A}} = \|w(t)\|_{\mathcal{H}}^2 + \|(-\mathcal{A})^{-1/2}\dot{w}(t)\|_{\mathcal{H}}^2 \equiv \tilde{P}(t) + \tilde{K}(t), \quad (4.8)$$

which also does not depend on $t \in \mathbb{R}$ and leads to the same type of asymptotic equipartition of energy.

By Euler's formula, the function $z \mapsto e^{iz}$ can be represented using the cosine and the sine functions, leading in turn to representations of the sine and cosine in terms of the exponential function. The functional calculus then reveals a natural relation between operator cosine and sine functions and the Riemann-Lebesgue property (4.4), given by

$$\cos(t(-\mathcal{A})^{1/2}) = \frac{e^{it(-\mathcal{A})^{1/2}} + e^{-it(-\mathcal{A})^{1/2}}}{2}, \quad t \geq 0, \quad (4.9)$$

and

$$(-\mathcal{A})^{-1/2} \sin(t(-\mathcal{A})^{1/2}) = \frac{(-\mathcal{A})^{-1/2}(e^{it(-\mathcal{A})^{1/2}} - e^{-it(-\mathcal{A})^{1/2}})}{2i}, \quad t \geq 0.$$

This realisation is fundamental to the proof of the equipartition result presented in Goldstein [22] and is also essential in this work.

The rescaled versions of the operator cosine and sine functions characterised by Lemma 3.1 incorporate the scaling δ^{-1} in time and are associated with the rescaled generator $A_{\vartheta, \delta}$. Therefore, it seems natural to expect an energetic behaviour similar to (4.4) and (4.7) for the trigonometric families acting on rescaled functions. To this end, we show the following asymptotic version of the Riemann-Lebesgue property.

Proposition 4.5 (Approximate Riemann-Lebesgue property). *For any $z_1, z_2 \in L^2(\mathbb{R}^d)$, we have*

$$\langle e^{i\delta^{-1}t(-A_{\vartheta, \delta})^{1/2}} P_{\delta} z_1, P_{\delta} z_2 \rangle_{L^2(\Lambda_{\delta})} \rightarrow 0, \quad \delta \rightarrow 0, \quad t \in \mathbb{R},$$

where the orthogonal projection P_{δ} is defined through (3.3).

Proof of Proposition 4.5. The result is proved on page 34. \square

Remark 4.6 (Strongly continuous groups associated with varying domains). Analysing strongly continuous families of operators associated with varying domains leads to many technical challenges, which were solved by Altmeyer and Reiß [4] using the Feynman-Kac theorem. Given a fixed Hilbert space, the Trotter-Kato approximation theorem, for instance Engel and Nagel [18, Theorem 4.8], characterises the relation between the convergence of semigroups, generators and resolvents. In Ito and Kappel [30] and Ito and Kappel [29, Chapter

4], a Trotter-Kato approximation theorem is proved for varying Banach spaces. These results can also be applied to the convergence of semigroups associated with varying domains. Indeed, Ito and Kappel [30] show that the convergence of semigroups associated with a varying spatial domain is equivalent to an adapted version of the strong resolvent convergence. Ito and Kappel [29, Proposition 3.1] and Weidmann [60, Theorem 1] provide concrete conditions under which this resolvent convergence holds.

Now that we have shown an asymptotic version of the Riemann-Lebesgue property, we are ready to analyse the asymptotic behaviour of the operator cosine and sine under rescaling. In our situation, the analysis is more involved as the spatial domain Λ_δ also grows in parallel to the temporal increase as $\delta \rightarrow 0$, and the asymptotic Riemann-Lebesgue property Proposition 4.5 is required to prove an asymptotic characterisation of the energetic behaviour within the wave equation. The following result is an instructive example of how to extend the results by Goldstein [22] to varying domains.

Proposition 4.7 (Asymptotic energetic behaviour for the scaled cosine). *Consider $z_1, z_2 \in L^2(\mathbb{R}^d)$. Then, we have*

$$\langle C_{\vartheta, \delta}(\delta^{-1}t)P_\delta z_1, C_{\vartheta, \delta}(\delta^{-1}t)P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \rightarrow \frac{1}{2} \langle z_1, z_2 \rangle_{L^2(\mathbb{R}^d)}, \quad \delta \rightarrow 0, \quad t \in \mathbb{R}, \quad (4.10)$$

and

$$\langle C_{\vartheta, \delta}(\delta^{-1}t)P_\delta z, C_{\vartheta, \delta}(\delta^{-1}s)P_\delta z \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad \delta \rightarrow 0, \quad t \neq s \in \mathbb{R} \setminus \{0\},$$

where the orthogonal projection P_δ is defined through (3.3).

Proof of Proposition 4.7. We abbreviate by $R_\delta(t) = e^{it(-A_{\vartheta, \delta})^{1/2}}$ the unitary group on the complex Hilbert space $L^2(\Lambda_\delta)$ for $t \in \mathbb{R}$. As in (4.9), the operator cosine can be represented using $R_\delta(t)$ through

$$C_{\vartheta, \delta}(\delta^{-1}t)P_\delta z = \frac{1}{2}(R_\delta(\delta^{-1}t) + R_\delta(-\delta^{-1}t))P_\delta z, \quad t \in \mathbb{R}.$$

Let $t, s \in \mathbb{R} \setminus \{0\}$ be arbitrary. Then, we observe

$$\begin{aligned} & \langle C_{\vartheta, \delta}(\delta^{-1}t)P_\delta z_1, C_{\vartheta, \delta}(\delta^{-1}s)P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &= \frac{1}{4} \langle (R_\delta(\delta^{-1}t) + R_\delta(-\delta^{-1}t))P_\delta z_1, (R_\delta(\delta^{-1}s) + R_\delta(-\delta^{-1}s))P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &= \frac{1}{4} \langle (R_\delta(\delta^{-1}s) + R_\delta(-\delta^{-1}s))^* \circ (R_\delta(\delta^{-1}t) + R_\delta(-\delta^{-1}t))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &= \frac{1}{4} \langle (R_\delta(-\delta^{-1}s) + R_\delta(\delta^{-1}s)) \circ (R_\delta(\delta^{-1}t) + R_\delta(-\delta^{-1}t))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &= \frac{1}{4} \left(\langle R_\delta(\delta^{-1}(t-s))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} + \langle R_\delta(\delta^{-1}(t+s))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \right. \\ & \quad \left. + \langle R_\delta(-\delta^{-1}(t+s))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} + \langle R_\delta(\delta^{-1}(s-t))P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \right). \end{aligned}$$

Notice that for $t \neq s$, the group $R_\delta(\cdot)$ is evaluated at non-zero times. Hence, by Proposition 4.5, each summand converges to zero as $\delta \rightarrow 0$. If $t = s$, we have

$$\begin{aligned} \langle C_{\vartheta,\delta}(\delta^{-1}t)P_\delta z_1, C_{\vartheta,\delta}(\delta^{-1}s)P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} &= \frac{1}{2} \langle P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &\quad + \langle R_\delta(-2t\delta^{-1})P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} + \langle R_\delta(2t\delta^{-1})P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \\ &\rightarrow \frac{1}{2} \langle z_1, z_2 \rangle_{L^2(\mathbb{R}^d)}, \quad \delta \rightarrow 0, \end{aligned}$$

Thus, we have shown both of the desired convergences. \square

Similar results to Proposition 4.7 for the operator cosine are proved for the operator sine in Appendix A.3.

The covariance structure of the stochastic wave equation is characterised using the operator cosine and sine functions, see Corollary 2.9. The following example illustrates how the Riemann-Lebesgue lemma and the asymptotic behaviour of energy within the associated deterministic equations emerge naturally when analysing the covariance structure of the stochastic wave equation tested against localised functions.

Example 4.8 (Wave equation on the unbounded domain). For some $\vartheta_0 > 0$ and a fixed time horizon $T > 0$, consider the one-dimensional stochastic wave equation

$$\partial_{tt}^2 \underline{u}(t, x) = \vartheta_0 \partial_{xx}^2 \underline{u}(t, x) + \mathscr{W}(t, x), \quad t \in (0, T], \quad x \in \mathbb{R}, \quad (4.11)$$

with zero initial conditions, driven by space-time white noise \mathscr{W} on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. We interpret the driving white noise as a centred Gaussian random field $(\mathscr{W}(A), A \in \mathcal{B}_b([0, T] \times \mathbb{R}))$ with the covariance structure $\mathbb{E}[\mathscr{W}(A)\mathscr{W}(B)] = \lambda_{[0, T] \times \mathbb{R}}(A \cap B)$, where $\mathcal{B}_b([0, T] \times \mathbb{R})$ denotes the bounded Borel subsets and $\lambda_{[0, T] \times \mathbb{R}}$ the Lebesgue measure of $[0, T] \times \mathbb{R}$. In view of Walsh [59] a continuous mild solution to (4.11) exists as an $(\mathcal{F}_t)_{t \in [0, T]}$ -predictable random field given by

$$\underline{u}(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \mathscr{W}(ds, dy), \quad t \in [0, T], \quad x \in \mathbb{R},$$

where

$$G_t(x) := \frac{1}{2\sqrt{\vartheta_0}} \mathbb{1}(|x| \leq \sqrt{\vartheta_0}t), \quad x \in \mathbb{R}, \quad t \in [0, T], \quad (4.12)$$

is the fundamental solution (Green's function) to the associated deterministic problem. The Fourier transform of the Green's function (4.12) is given by

$$\mathcal{F}(G_t)(\omega) = \frac{\sin(t\sqrt{\vartheta_0}|\omega|)}{\sqrt{\vartheta_0}|\omega|}, \quad t \in [0, T], \quad \omega \in \mathbb{R} \setminus \{0\}.$$

For some kernel $K \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
& \mathbb{E}[\langle \underline{u}(t), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})} \langle \underline{u}(s), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})}] \\
&= \delta^{-4} \int_0^{t \wedge s} \int_{\mathbb{R}} \langle G_{t-r}(\cdot - y), (K'')_\delta \rangle_{L^2(\mathbb{R})} \langle G_{s-r}(\cdot - y), (K'')_\delta \rangle_{L^2(\mathbb{R})} dy dr \\
&= \frac{\delta^{-4}}{2\pi} \int_0^{t \wedge s} \int_{\mathbb{R}} \frac{\sin((t-r)\sqrt{\vartheta_0}|\omega|) \sin((s-r)\sqrt{\vartheta_0}|\omega|)}{\vartheta_0|\omega|^2} |\mathcal{F}((K'')_\delta)|^2(\omega) d\omega dr \\
&= \frac{\delta^{-2}}{2\pi\vartheta_0} \int_0^{t \wedge s} \int_{\mathbb{R}} \sin(\delta^{-1}(t-r)\sqrt{\vartheta_0}|\omega|) \sin(\delta^{-1}(s-r)\sqrt{\vartheta_0}|\omega|) |\mathcal{F}(K')|^2(\omega) d\omega dr
\end{aligned} \tag{4.13}$$

where we have used the stochastic Fubini theorem and Plancherel's identity. Note that rescaling in this way via the Fourier transform and the transformation theorem is analogous to Example 3.2. Note that $|\mathcal{F}(K')|^2 \in L^1(\mathbb{R}^d)$. We observe

$$\begin{aligned}
& \int_0^{t \wedge s} \sin(\delta^{-1}(t-r)\sqrt{\vartheta_0}|\omega|) \sin(\delta^{-1}(s-r)\sqrt{\vartheta_0}|\omega|) dr \\
&= \frac{(t \wedge s)}{2} \cos(\delta^{-1}(t-s)\sqrt{\vartheta_0}|\omega|) \\
&+ \delta \frac{\sin(\delta^{-1}|t-s|\sqrt{\vartheta_0}|\omega|) - \sin(\delta^{-1}(t+s)\sqrt{\vartheta_0}|\omega|)}{4\sqrt{\vartheta_0}|\omega|}.
\end{aligned} \tag{4.14}$$

Thus, if $t \neq s$ the generalised Riemann-Lebesgue lemma in Kahane [35] applied to the inner integral in (4.13), shows

$$\delta^2 \mathbb{E}[\langle \underline{u}(t), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})} \langle \underline{u}(s), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})}] \rightarrow 0.$$

Otherwise, if $t = s$ Fubini's theorem yields with (4.14):

$$\delta^2 \mathbb{E}[\langle \underline{u}(t), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})} \langle \underline{u}(s), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})}] \rightarrow \frac{t}{2\vartheta_0} \|K'\|_{L^2(\mathbb{R})}^2.$$

Consequently, the observed Fisher information satisfies

$$\delta^2 \mathbb{E} \left(\int_0^T \langle \underline{u}(t), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})}^2 dt \right) \rightarrow \frac{T^2 \|K'\|_{L^2(\mathbb{R})}^2}{4\vartheta_0}, \quad \delta \rightarrow 0, \tag{4.15}$$

and

$$\text{Var} \left(\delta^2 \int_0^T \langle \underline{u}(t), \delta^{-2}(K'')_\delta \rangle_{L^2(\mathbb{R})}^2 dt \right) \rightarrow 0, \quad \delta \rightarrow 0. \tag{4.16}$$

The convergences (4.15) and (4.16) are already sufficient for obtaining a central limit theorem for an augmented MLE in the parametric case on an unbounded domain.

In the next section, we formalise the estimation procedure and extend Example 4.8 to a bounded spatial domain and the spatially varying wave-speed ϑ .

5 Estimating the wave speed

Based on Altmeyer and Reiß [4], we will study an augmented maximum likelihood estimator $\hat{\vartheta}_\delta$, which was adapted to the setting of the stochastic wave equation. In contrast to the stochastic heat equation, the augmented MLE will involve both local measurements of the amplitude and the velocity.

We begin by stating the following regularity assumption.

Assumption (5.1, Kernel, z). Suppose that $z \in H^2(\mathbb{R}^d)$ has compact support in Λ_δ for some $\delta > 0$ and satisfies $\int_{\mathbb{R}^d} z(x)dx = 0$ for $d = 1$.

Fix some kernel K satisfying (5.1, Kernel, K). By inserting the localised kernel K_δ into the dynamic representation of the weak solution from Proposition 2.7, we obtain the real-valued processes $(u_\delta(t), t \in [0, T])$, $(u_\delta^\Delta(t), t \in [0, T])$ and $(v_\delta(t), t \in [0, T])$, called local measurements:

$$\begin{aligned} u_\delta(t) &:= \langle u(t), K_\delta \rangle_{L^2(\Lambda)}, & v_\delta(t) &:= \langle v(t), K_\delta \rangle, \\ u_\delta^\Delta(t) &:= \langle u(t), \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)}, & t &\in [0, T]. \end{aligned} \quad (5.1)$$

We observe $u_\delta(t)$ continuously in time $t \in [0, T]$ for a known kernel K chosen by the statistician. By Proposition 2.7, the local measurements (5.1) satisfy the dynamic

$$u_\delta(t) = \int_0^t v_\delta(s)ds, \quad v_\delta(t) = \int_0^t \langle u(s), A_\vartheta K_\delta \rangle_{L^2(\Lambda)} ds + \|K\|_{L^2(\mathbb{R}^d)} \overline{W}(t), \quad (5.2)$$

where $(\overline{W}(t), t \in [0, T])$ defined through $\overline{W}(t) = \langle W(t), K_\delta \rangle_{L^2(\Lambda)} / \|K_\delta\|_{L^2(\Lambda)}$ is a scalar Brownian motion and $t \in [0, T]$.

Remark 5.2. Note that the local observations of the velocity, i.e. $(v_\delta(t), t \in [0, T])$ do not need to be observed as $(v_\delta(t), t \in [0, T])$ can be recovered as a limit of the corresponding difference quotient of $(u_\delta(t), t \in [0, T])$ because $u_\delta(t) = \int_0^t v_\delta(s)ds$. Furthermore, we can approximate the measurements $(u_\delta^\Delta(t), t \in [0, T])$ based on $(u_\delta(t), t \in [0, T])$ observed in a neighbourhood of zero, as discussed in the introduction of the local observations in Altmeyer and Reiß [4].

In the deterministic parametric situation $A_\vartheta = \vartheta \Delta$ of (5.2) without any noise, the parameter could be recovered through the local measurements satisfying $\dot{v}_\delta(t) = \vartheta u_\delta^\Delta(t)$ for $t \in [0, T]$. Thus, in the situation with noise, a least squares ansatz suggests the heuristic minimisation problem

$$\hat{\vartheta}_\delta = \operatorname{argmin}_\vartheta \int_0^T (\dot{v}_\delta(t) - \vartheta u_\delta^\Delta(t))^2 dt,$$

leading via the corresponding normal equations to the estimator

$$\hat{\vartheta}_\delta := \frac{\int_0^T u_\delta^\Delta(t) dv_\delta(t)}{\int_0^T (u_\delta^\Delta(t))^2 dt}, \quad \delta > 0. \quad (5.3)$$

The estimator can also be motivated by the Girsanov-type arguments as outlined in Altmeyer and Reiß [4, Section 4.1] and will therefore also be called augmented MLE. Using the dynamic representation (5.2), the error decomposition for the augmented MLE is given by

$$\hat{\vartheta}_\delta - \vartheta(0) = \|K\|_{L^2(\mathbb{R}^d)} I_\delta^{-1} M_\delta + I_\delta^{-1} R_\delta, \quad (5.4)$$

where we introduce the following notations.

- *Observed Fisher information:* We define the observed Fisher information as

$$I_\delta := \int_0^T u_\delta^\Delta(t)^2 dt, \quad \delta > 0. \quad (5.5)$$

- *Martingale part:* The martingale part of $\hat{\vartheta}_\delta$ is given by

$$M_\delta := \int_0^T u_\delta^\Delta(t) d\bar{W}(t), \quad \delta > 0.$$

- *Remaining bias:* The remaining bias is given by

$$R_\delta := \int_0^T u_\delta^\Delta(t) \langle u(t), (A_\vartheta - \vartheta(0)\Delta)K_\delta \rangle_{L^2(\Lambda)} dt. \quad (5.6)$$

Next, we show that the observed Fisher information and the remaining bias converge to deterministic constants. All propositions are proved in Appendix A.4.

Proposition 5.3 (Asymptotics for the observed Fisher information). *Grant Assumptions (5.1, Kernel, K) and (2.2, Initial, u_0, v_0). The expectation and covariance of the observed Fisher information satisfy*

$$\mathbb{E}[\delta^2 I_\delta] \rightarrow \frac{T^2}{4\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \text{Var}(\delta^2 I_\delta) \rightarrow 0, \quad \delta \rightarrow 0.$$

Proof of Proposition 5.3. The result is proved on page 43. \square

Remark 5.4. Consider the second-order abstract Cauchy problem

$$w(t) = A_{\vartheta,\delta} w(t), \quad w(0) = 0, \quad \dot{w}(0) = \Delta K, \quad t \in \mathbb{R}.$$

As the first-order initial condition is zero, the associated total energy in $H_0^{-1}(\Lambda)$ from Remark 4.4 corresponds to the total kinetic energy in $H_0^{-1}(\Lambda)$, satisfying

$$\tilde{\mathcal{E}}_{A_{\vartheta,\delta}} = \|(-A_{\vartheta,\delta})^{-1/2} \Delta K\|_{L^2(\Lambda_\delta)}^2 \rightarrow \frac{1}{\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \delta \rightarrow 0,$$

where the convergence follows from Lemma A.9 as K satisfies (5.1, Kernel, K). Thus, the limiting expectation of the observed Fisher information scaled by δ^2 is proportional to the limiting total kinetic energy in $H^{-1}(\mathbb{R}^d)$ within the Cauchy problem (5.4).

Proposition 5.5 (Asymptotics for the remaining bias). *Grant Assumptions (5.1, Kernel, K) and (2.2, Initial, u_0, v_0). The remaining bias satisfies*

$$\delta^{-1}(I_\delta)^{-1}R_\delta \xrightarrow{\mathbb{P}} \frac{\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}, \quad \delta \rightarrow 0, \quad (5.7)$$

where $\beta^{(0)}$ is defined through (A.22).

Proof of Proposition 5.5. The result is proved on page 45. \square

Note that in the parametric case $\vartheta \equiv \vartheta_0 > 0$, the bias term is just zero, and the asymptotic bias derived through (5.7) is also zero.

Theorem 5.6 (Asymptotic normality of the augmented MLE). *Grant Assumptions (5.1, Kernel, K) and (2.2, Initial, u_0, v_0). Then, the augmented MLE (5.3) satisfies*

$$\delta^{-1}(\hat{\vartheta}_\delta - \vartheta(0)) \xrightarrow{d} \mathcal{N} \left(\frac{\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}, \frac{4\vartheta(0)\|K\|_{L^2(\mathbb{R}^d)}^2}{T^2\|\nabla K\|_{L^2(\mathbb{R}^d)}^2} \right), \quad \delta \rightarrow 0,$$

where $\beta^{(0)}$ is defined through (A.22).

Remark 5.7 (Rate of convergence). The rate of convergence of order δ is the same as in the case of the stochastic heat equation obtained by Altmeyer and Reiß [4, Proposition 5.2]. In the spectral observation scheme based on the first N -Fourier modes of the solution process, the rate of convergence is $N^{-3/2}$ for the maximum likelihood estimator analysed by Liu and Lototsky [40, Theorem 3.1]. Heuristically, the convergence rate δ of the augmented MLE $\hat{\vartheta}_\delta$ can be regarded as a single-mode version of the maximum likelihood estimator in the spectral case.

Proof of Theorem 5.6. In view of (5.4), we begin by introducing the error decomposition

$$\begin{aligned} \delta^{-1}(\hat{\vartheta}_\delta - \vartheta(0)) &= \delta^{-1}\|K\|_{L^2(\mathbb{R}^d)}(I_\delta)^{-1}M_\delta + \delta^{-1}I_\delta^{-1}R_\delta \\ &= \left(\frac{M_\delta}{\sqrt{\mathbb{E}[I_\delta]}} \right) \left(\frac{I_\delta}{\mathbb{E}[I_\delta]} \right)^{-1} (\delta^2\mathbb{E}[I_\delta])^{-1/2} \|K\|_{L^2(\mathbb{R}^d)} + \delta^{-1}I_\delta^{-1}R_\delta. \end{aligned}$$

The quadratic variation of the martingale $Y^{(\delta)} := M_\delta/(\mathbb{E}[I_\delta])^{1/2}$ is given by $\langle Y^{(\delta)} \rangle = I_\delta/\mathbb{E}[I_\delta]$. The standard continuous martingale central limit theorem, see for instance Pasemann [46, Theorem A.1], shows that $Y^{(\delta)} \xrightarrow{d} \mathcal{N}(0, 1)$ if $\langle Y^{(\delta)} \rangle \xrightarrow{\mathbb{P}} 1$ as $\delta \rightarrow 0$. In view of Proposition 5.3 and Chebyshev's inequality we obtain $I_\delta/\mathbb{E}[I_\delta] \xrightarrow{\mathbb{P}} 1$ and thus $Y^{(\delta)} \xrightarrow{d} \mathcal{N}(0, 1)$ as $\delta \rightarrow 0$. The result follows with Proposition 5.5 and using Slutsky's theorem. \square

Remark 5.8 (Dependence of the asymptotic variance on the time horizon). In the case of the stochastic heat equation, Altmeyer and Reiß [4, Proposition 5.2] shows that increasing the time horizon T will lead to a decrease of the asymptotic variance of the limiting normal distribution in the associated central limit theorem of order T^{-1} . This effect is even more prevalent for the stochastic wave equation, as the asymptotic variance scales with T^{-2} , which is also visible in the spectral approach. Indeed, the asymptotic variances from Lototsky [42, Theorem 1.1] and Liu and Lototsky [40, Theorem 3.1] depend on the time horizon through the factors $2T^{-1}$ and $4T^{-2}$, respectively.

The rate of convergence of the MLE in the case of the ordinary Ornstein-Uhlenbeck process or the harmonic oscillator is \sqrt{T} in the ergodic and T in the energetically stable case; see Kutoyants [38, Proposition 3.46] and Lin and Lototsky [39]. The rate of estimation associated with the Fourier modes of the corresponding SPDE is then inherited by the augmented MLE through the dependence of asymptotic variance on the time horizon.

Up to a constant, the asymptotic bias is determined by

$$\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)} = -\langle \Delta K, \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}, \quad (5.8)$$

where $\beta^{(0)}$ is defined through (A.22). Suppose that $K \in H^4(\mathbb{R}^d)$. Then, in view of Altmeyer and Reiß [4, Lemma A.3], (5.8) can be rewritten as

$$-\langle \Delta K, \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)} = \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d}, |\nabla \Delta K(x)|_{\mathbb{R}^d}^2 \rangle_{L^2(\mathbb{R}^d)}.$$

If $\nabla \Delta K$ is symmetric, i.e. $|\nabla \Delta K(-x)|_{\mathbb{R}^d} = |\nabla \Delta K(x)|_{\mathbb{R}^d}$ for $x \in \mathbb{R}^d$, then the asymptotic bias vanishes. Note that the asymptotic bias is different in the case of the heat equation and involves the term $-\frac{1}{2} \langle K, \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}$ and not (5.8). As described by Altmeyer and Reiß [4, Lemma A.3], this leads to the requirement that ∇K is symmetric in contrast to our assumption that $\nabla \Delta K$ is symmetric.

Corollary 5.9 (Confidence interval). *Assume that the asymptotic bias is zero in the setting of Theorem 5.6. For some $\bar{\alpha} \in (0, 1)$ the confidence interval around $\vartheta(0)$, given by*

$$I_{1-\bar{\alpha}} := \left[\hat{\vartheta}_\delta - \delta \sqrt{\hat{\vartheta}_\delta} \frac{2\|K\|_{L^2(\mathbb{R}^d)}}{T\|\nabla K\|_{L^2(\mathbb{R}^d)}} q_{1-\bar{\alpha}/2}, \hat{\vartheta}_\delta + \delta \sqrt{\hat{\vartheta}_\delta} \frac{2\|K\|_{L^2(\mathbb{R}^d)}}{T\|\nabla K\|_{L^2(\mathbb{R}^d)}} q_{1-\bar{\alpha}/2} \right],$$

with the standard normal $(1 - \bar{\alpha}/2)$ -quantile $q_{1-\bar{\alpha}/2}$, has asymptotic coverage $1 - \bar{\alpha}$ for $\delta \rightarrow 0$.

Proof of Corollary 5.9. By Theorem 5.6 we not only obtain the asymptotic normality but also $\hat{\vartheta}_\delta \xrightarrow{\mathbb{P}} \vartheta(0)$. Thus, we may apply Slutsky's lemma and obtain

$$\delta^{-1} \left(\hat{\vartheta}_\delta \frac{4\|K\|_{L^2(\mathbb{R}^d)}^2}{T^2\|\nabla K\|_{L^2(\mathbb{R}^d)}^2} \right)^{-1/2} (\hat{\vartheta}_\delta - \vartheta(0)) \xrightarrow{d} \mathcal{N}(0, 1), \quad \delta \rightarrow 0,$$

as we have assumed that the asymptotic bias is zero. Consequently, we obtain

$$\mathbb{P}(\vartheta(0) \in I_{1-\bar{\alpha}}) \rightarrow 1 - \bar{\alpha}, \quad \delta \rightarrow 0. \quad \square$$

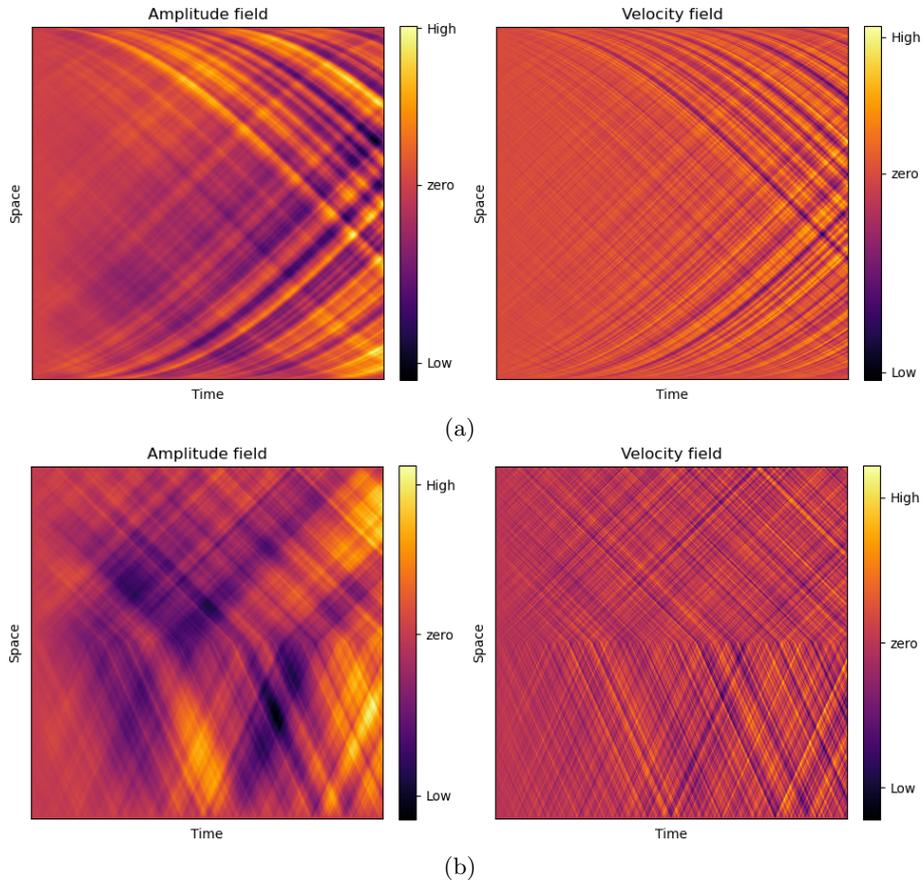


Figure 1: Simulation of the amplitude and velocity field of (6.1) with spatially dependent wave speed (6.2) in (a) and (6.3) in (b)

6 Numerical illustration

This section is dedicated to the illustration of the main results. We consider the stochastic wave equations on the 1-dimensional bounded domain $\Lambda = (0, 1)$ up to the time horizon $T = 1$:

$$\partial_{tt}^2 u(t, x) = \partial_x(\vartheta(x)\partial_x u(t, x)) + \dot{W}(t, x), \quad t \in [0, 1], \quad x \in (0, 1). \quad (6.1)$$

Unless stated otherwise, we assume zero first and second-order initial conditions. Based on the results of Lord et al. [41, Section 10.5] and the work of Quersardanyons and Sanz-Solé [50], we employ a semi-implicit Euler scheme with a finite difference approximation of the second spatial derivative on the uniform grid

$$\{(t_k, y_j) : t_k = k/N, y_j = j/M, \quad k = 0, \dots, N, j = 0, \dots, M\},$$

where the spatial and temporal resolutions are $M = 10^3$ and $N = M^2$, respectively.

Smooth wave speed: In Figure 1a, we simulate the stochastic wave equation (6.1) with the spatially dependent wave speed

$$\vartheta_a(x) = 4x(1-x) + 0.01, \quad x \in (0, 1). \quad (6.2)$$

In this case, the wave speed of the stochastic wave equation is increasing towards the centre of the interval. The amplitude field is much smoother than the velocity field as the white noise enters the stochastic wave equation through the velocity field. As both the first and second-order initial conditions are zero, energy is added to the system through the noise, and the observed pattern results from accumulating fluctuations.

Piecewise constant wave speed: In Figure 1b, we simulate the stochastic wave equation (6.1) with parameters

$$\vartheta_b(x) = \frac{1}{2} \mathbb{1}_{(0,1/2]}(x) + \mathbb{1}_{(1/2,1)}(x), \quad x \in (0, 1). \quad (6.3)$$

This situation arises if a wave travels from one medium (air) into another (water) and back again. The wave is partially transmitted as it passes from one medium into the other medium. Indeed, as the wave speed in both media is different, we observe a change of angle at the interface reminiscent of Snell's law. This case is not covered by the theory as ϑ_b is not differentiable in $1/2$. Still, it can be considered an intriguing limiting case reminiscent of the situation of change point detection discussed in Reiß, Strauch, and Trottner [51].

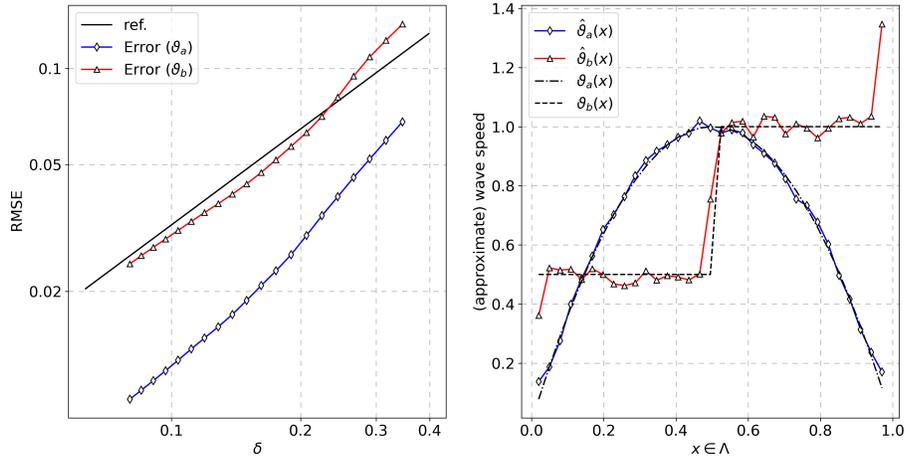
As in Altmeyer and Reiß [4], we consider the kernels $K = \varphi'''$ based on the smooth bump function:

$$\varphi(x) = \exp\left(\frac{-12}{(1-x^2)}\right), \quad x \in (-1, 1). \quad (6.4)$$

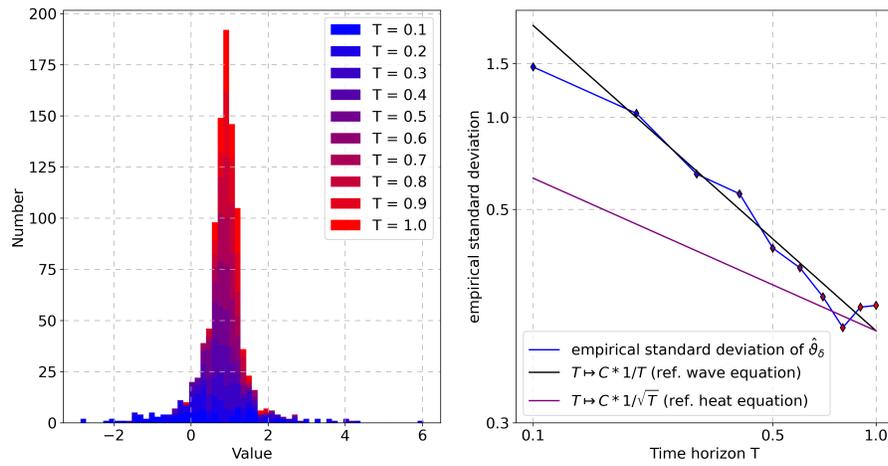
For $\delta \in (0.05, 0.5)$ and $x_0 \in (0, 1)$, we can empirically approximate the local measurements u_{δ, x_0}^Δ and v_{δ, x_0} based on the kernel K and compute the associated augmented MLE accordingly.

Figure 2a provides a \log_{10} - \log_{10} plot of the root mean squared estimation error as $\delta \rightarrow 0$ based on 1000 Monte-Carlo runs. The convergence rate of δ is achieved for both ϑ_a and ϑ_b . On the other hand, the (right) side of Figure 2a displays the true wave speed compared to its approximation. When approaching the discontinuity in the wave speed ϑ_b , the accuracy of the estimation shrinks significantly.

Figure 2b visualises in the case of (6.2) the asymptotic normality result of Theorem 5.6 and the dependence of the asymptotic variance on the time horizon. The right-hand side of Figure 2b displays the empirical standard deviation of the asymptotic distribution approximated based on 500 Monte-Carlo runs for each displayed time horizon. In contrast, the left-hand side shows how the asymptotic normal distribution concentrates around the true value as the associated time horizon increases. As expected by Theorem 5.6 and Remark 5.8, the asymptotic



(a)



(b)

Figure 2: (a) Monte-Carlo simulation of the augmented MLE. (left) \log_{10} - \log_{10} plot of root mean squared estimation error at $x_0 = 0.6$. (b)(left) histogram plot of the estimator for different values of T . (right) Monte-Carlo simulation of the asymptotic variance associated with augmented MLE.

variance in the case of the augmented MLE depends differs from the parabolic case and depends on the time horizon through $1/T^2$ and not through $1/T$ as in the case of the stochastic heat equation.

We conclude this section with the following table, which illustrates the asymptotic coverage of the confidence interval $I_{1-\bar{\alpha}}$ derived in Corollary 5.9 for different values of α and δ based on 500 Monte-Carlo simulations.

Empirical probability of $\vartheta(0) \in I_{1-\bar{\alpha}}$			
/	$\delta \geq 0.2$	$\delta = 0.1$	$\delta = 0.09$
$\bar{\alpha} = 0.1$	≈ 1	0.93	0.89
$\bar{\alpha} = 0.05$	≈ 1	0.98	0.97

As δ decreases, the observations are in line with the converge $\mathbb{P}(\vartheta(0) \in I_{1-\bar{\alpha}}) \rightarrow 1 - \bar{\alpha}$ as $\delta \rightarrow 0$. For larger values of δ , the confidence intervals have a greater size, resulting in larger empirical coverage probabilities.

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A Remaining proofs

A.1 Well-posedness proofs

Based on page 415 ff. in Arendt et al. [8], the following result summarises some crucial properties of the isometric isomorphism A_ϑ defined through (2.1).

Lemma A.1 (Some Gelfand properties).

- (i) *The triple $(H_0^1(\Lambda), L^2(\Lambda), H_0^{-1}(\Lambda))$ is a Gelfand triple, i.e. the inclusions $H_0^1(\Lambda) \hookrightarrow L^2(\Lambda)$ and $L^2(\Lambda) \hookrightarrow H_0^{-1}(\Lambda)$ are bounded linear operators.*
- (ii) *For $z_2 \in L^2(\Lambda)$ and $z_1 \in H_0^1(\Lambda)$, we have*

$$\langle A_\vartheta z_1, z_2 \rangle_{H_0^{-1}(\Lambda)} = -\langle z_2, z_1 \rangle_{L^2(\Lambda)}.$$

- (iii) *For $z \in L^2(\Lambda)$ and $l \in H_0^{-1}(\Lambda)$ we have*

$$\langle l, z \rangle_{H_0^{-1}(\Lambda)} = -\langle A_\vartheta^{-1} l, z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)} = -\langle z, A_\vartheta^{-1} l \rangle_{L^2(\Lambda)}. \quad (\text{A.1})$$

- (iv) *For $l \in H_0^{-1}(\Lambda)$ and $z \in H_0^1(\Lambda)$, we have*

$$\langle l, -A_\vartheta z \rangle_{H_0^{-1}(\Lambda)} = \langle l, z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)}.$$

Proof of Lemma A.1. The result is proved in Arendt, Batty, Hieber, and Neubrander [8, 415 ff. and Proposition 7.1.5]. \square

Lemma A.2 (Properties of the wave generator). *The generator \mathcal{A}_ϑ is skew-adjoint, i.e. $\mathcal{A}_\vartheta^* = -\mathcal{A}_\vartheta$ and $D(\mathcal{A}_\vartheta^*) = D(\mathcal{A}_\vartheta)$.*

Proof of Lemma A.2. Let $U = (z_1, z_2)^\top \in H_0^1(\Lambda) \times L^2(\Lambda)$ and $\tilde{U} = (\tilde{z}_1, \tilde{z}_2)^\top \in H_0^1(\Lambda) \times L^2(\Lambda)$. By applying Lemma A.1 (ii), we immediately obtain

$$\begin{aligned} \langle \mathcal{A}_\vartheta U, \tilde{U} \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} &= \langle z_2, \tilde{z}_1 \rangle_{L^2(\Lambda)} + \langle A_\vartheta z_1, \tilde{z}_2 \rangle_{H_0^{-1}(\Lambda)} \\ &= -\langle z_2, A_\vartheta \tilde{z}_1 \rangle_{H_0^{-1}(\Lambda)} - \langle z_1, \tilde{z}_2 \rangle_{L^2(\Lambda)} \\ &= \langle U, -\mathcal{A}_\vartheta \tilde{U} \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}. \end{aligned}$$

Thus, $\mathcal{A}_\vartheta^* = -\mathcal{A}_\vartheta$ and $D(\mathcal{A}_\vartheta^*) = H_0^1(\Lambda) \times L^2(\Lambda)$. \square

Lemma A.3 (Unitary group). *The group $(\mathcal{J}_\vartheta(t), t \in \mathbb{R})$ is unitary and satisfies*

$$\mathcal{J}_\vartheta(t)^* = \mathcal{J}_\vartheta(-t) = \begin{pmatrix} C_\vartheta(t) & -S_\vartheta(t) \\ -C_\vartheta'(t) & C_\vartheta(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Proof of Lemma A.3. We have already shown in Lemma A.2 that the generator \mathcal{A}_ϑ is skew-adjoint. Thus, we conclude that $(\mathcal{J}_\vartheta(t), t \in \mathbb{R})$ is unitary by Engel and Nagel [18, Theorem 3.24]. As an immediate consequence, we observe

$$\mathcal{J}_\vartheta(t)^* = \mathcal{J}_\vartheta(t)^{-1} = \mathcal{J}_\vartheta(-t), \quad t \in \mathbb{R}.$$

By definition, we have $C_\vartheta(t) = C_\vartheta(-t)$ for all $t \in \mathbb{R}$, see Arendt, Batty, Hieber, and Neubrander [8, p.209]. With this and the transformation theorem, we also observe $S_\vartheta(-t) = -S_\vartheta(t)$ and $C'_\vartheta(-t) = A_\vartheta S_\vartheta(-t) = -A_\vartheta S_\vartheta(t) = -C'_\vartheta(t)$ for $t \in \mathbb{R}$. \square

Lemma A.4 (Adjoint of the component inclusion). *The adjoint of the operator $B : L^2(\Lambda) \rightarrow L^2(\Lambda) \times H_0^{-1}(\Lambda)$, defined through $Bu := (0, u)^\top$, is given by $B^*(z, l)^\top := -A_\vartheta^{-1}l$ for $(z, l)^\top \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$.*

Proof of Lemma A.4. Given Lemma A.1, the operator $B : L^2(\Lambda) \rightarrow L^2(\Lambda) \times H_0^{-1}(\Lambda)$ defined through $z \mapsto (0, z)^\top$ satisfies

$$\langle Bz, U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} = \langle z, l \rangle_{H_0^{-1}(\Lambda)} = \langle z, -A_\vartheta^{-1}l \rangle_{L^2(\Lambda)} = \langle z, B^*U \rangle_{L^2(\Lambda)} \quad (\text{A.2})$$

for $U = (\tilde{z}, l)^\top \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$ and $z \in L^2(\Lambda)$, where $B^* : L^2(\Lambda) \times H_0^{-1}(\Lambda) \rightarrow L^2(\Lambda)$ defined through $B^*(U_1, U_2)^\top := -A_\vartheta^{-1}U_2$ is the adjoint of B . \square

Proof of Proposition 2.6. By the definition of the weak solution, we have

$$\begin{aligned} & \langle X(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} \\ &= \int_0^t \langle X(s), \mathcal{A}_\vartheta^* U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} ds + \langle BW(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}, \end{aligned}$$

for $U \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$ and $t \in [0, T]$. For functions $z \in H_0^1(\Lambda) \cap H^2(\Lambda)$, we find $A_\vartheta z \in L^2(\Lambda) \subset H_0^{-1}(\Lambda)$. In view of Lemma A.1 and by setting $U = (z, -A_\vartheta z) \in L^2(\Lambda) \times (L^2(\Lambda))'$, we have

$$\begin{aligned} \langle X(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} &= \langle u(t), z \rangle_{L^2(\Lambda)} + \langle v(t), -A_\vartheta z \rangle_{H_0^{-1}(\Lambda)} \\ &= \langle u(t), z \rangle_{L^2(\Lambda)} + \langle v(t), z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)}, \\ \langle X(s), \mathcal{A}_\vartheta^* U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} &= \langle X(s), -\mathcal{A}_\vartheta(z, -A_\vartheta z)^\top \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} \\ &= -\langle u(t), -A_\vartheta z \rangle_{L^2(\Lambda)} - \langle v(t), A_\vartheta z \rangle_{H_0^{-1}(\Lambda)} \\ &= \langle u(t), A_\vartheta z \rangle_{L^2(\Lambda)} + \langle v(t), z \rangle_{H_0^{-1}(\Lambda), H_0^1(\Lambda)}, \\ \langle BW(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} &= \langle W(t), -A_\vartheta z \rangle_{H_0^{-1}(\Lambda)} \\ &= \langle W(t), z \rangle_{L^2(\Lambda)}. \end{aligned} \quad (\text{A.3})$$

The result is obtained through linear separation using $U = (z, 0)^\top$ and $U = (0, -A_\vartheta z)^\top$, respectively. \square

Proof of Proposition 2.7.

Step 1 (The Gaussian process). We define the Gaussian process $\mathcal{V}(t, U)$ parameterised by $t \in [0, T]$ and $U \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$ through

$$\mathcal{V}(t, U) := \int_0^t \langle \mathcal{J}_\vartheta^*(t-s)U, BdW(s) \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}, \quad (\text{A.4})$$

for $U \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$ and $t \in [0, T]$. By definition, \mathcal{V} is a well-defined centred Gaussian process, and Itô's isometry (Da Prato and Zabczyk [13, Proposition 4.28]) shows (2.10). The proof of Hairer [28, Proposition 6.7] reveals that the process (A.4) emerges when passing from an almost surely integrable mild solution to a weak solution. In particular, the process satisfies the dynamic behaviour

$$\mathcal{V}(t, U) = \int_0^t \mathcal{V}(s, \mathcal{A}_\vartheta^* U) ds + \langle BW(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}, \quad U \in D(\mathcal{A}_\vartheta^*). \quad (\text{A.5})$$

Step 2 (Decomposition into component processes). Using the adjoint of B and the unitary group $(\mathcal{J}(t), t \in [0, T])$ computed in Lemma A.4 and Lemma A.3, we may rewrite the process (A.4) as

$$\begin{aligned} \mathcal{V}(t, U) &= \int_0^t \langle \mathcal{J}_\vartheta^*(t-s)U, B \cdot \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)} dW(s) \\ &= \int_0^t \langle B^* \mathcal{J}_\vartheta^*(t-s)U, \cdot \rangle_{L^2(\Lambda)} dW(s) \\ &= \int_0^t \langle -A_\vartheta^{-1}(-C'_\vartheta(t-s)z + C_\vartheta(t-s)l), dW(s) \rangle_{L^2(\Lambda)} \\ &= \int_0^t \langle S_\vartheta(t-s)z, dW(s) \rangle_{L^2(\Lambda)} + \int_0^t \langle -A_\vartheta^{-1}C_\vartheta(t-s)l, dW(s) \rangle_{L^2(\Lambda)}, \end{aligned} \quad (\text{A.6})$$

for any $U = (z, l)^\top \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$. Thus, for $z \in L^2(\Lambda)$, we set

$$u_\mathcal{V}(t, z) := \int_0^t \langle S_\vartheta(t-s)z, dW(s) \rangle_{L^2(\Lambda)}, \quad v_\mathcal{V}(t, z) = \int_0^t \langle C_\vartheta(t-s)z, dW(s) \rangle_{L^2(\Lambda)},$$

and obtain $\mathcal{V}(t, U) := u_\mathcal{V}(t, z) + v_\mathcal{V}(t, -A_\vartheta^{-1}l)$ from (A.6), as the operator cosine commutes with its generator.

Step 3 (Dynamic behaviour). By Lemma A.2, the generator of the unitary group $(\mathcal{J}_\vartheta(t), t \in \mathbb{R})$ is skew-adjoint. Thus, representing the weak dynamic (A.5) using the processes $u_\mathcal{V}$ and $v_\mathcal{V}$, we obtain for $U = (z_1, z_2)^\top \in D(\mathcal{A}_\vartheta) = D(A_\vartheta) \times L^2(\Lambda)$:

$$\begin{aligned} &u_\mathcal{V}(t, z_1) + u_\mathcal{V}(t, -A_\vartheta^{-1}z_2) \\ &= \int_0^t u_\mathcal{V}(s, -z_2) + v_\mathcal{V}(s, z_1) ds + \langle BW(t), U \rangle_{L^2(\Lambda) \times H_0^{-1}(\Lambda)}. \end{aligned}$$

Because of (A.3), setting $U = (z_1, z_2)^\top = (0, -A_\vartheta z)^\top$ and $U = (z_1, z_2)^\top = (z, 0)^\top$ for $z \in H_0^1(\Lambda) \cap H^2(\Lambda)$ yields (2.11) and (2.12), respectively. \square

A.2 Proof of spectral asymptotics

For an overview or introduction to the theory of spectral measures, we refer to Schmüdgen [54] and Rudin [52]. By the spectral theorem for unbounded operators, see for instance Schmüdgen [54, Theorem 5.1], there exists a unique spectral

measure $\tilde{E}_{\mathcal{A}}(\cdot)$ associated with an unbounded self-adjoint operator $(\mathcal{A}, D(\mathcal{A}))$ such that

$$\mathcal{A} = \int_{\mathbb{R}} \lambda dE_{\mathcal{A}}(\lambda),$$

with the resolution of the identity $E_{\mathcal{A}}(\lambda) = \tilde{E}_{\mathcal{A}}((-\infty, \lambda])$. By the functional calculus for unbounded operators, we may then define the operator

$$f(\mathcal{A}) = \int_{\mathbb{R}} f(\lambda) dE_{\mathcal{A}}(\lambda), \quad D(f(\mathcal{A})) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 d\langle E_{\mathcal{A}}(\lambda)x, x \rangle < \infty \right\}.$$

For an overview of the properties of the functional calculus for unbounded operators, see Schmüdgen [54, Theorem 5.9] and Rudin [52, Theorem 13.24].

Abbreviate by $(E_{\delta}(\lambda), \lambda \geq 0)$ and $(E(\lambda), \lambda \geq 0)$ the resolution of identity associated with the operator $-A_{\vartheta, \delta}$ and $-\vartheta(0)\Delta$, respectively. Recall further the orthogonal projection $P_{\delta} : L^2(\mathbb{R}^d) \rightarrow L^2(\Lambda_{\delta})$ defined in (3.3).

Remark A.5 (Projections, subspaces and rescaling).

- (i) We can identify the space $L^2(\Lambda)$ with the subspace of $L^2(\mathbb{R}^d)$ consisting of those functions which vanish a.e. on the complement of Λ . Note that this is also possible for Sobolev spaces. Indeed, we can identify $H_0^1(\Lambda)$ with a subspace of $H^1(\mathbb{R}^d)$ by extending functions by zero. For $z \in H_0^1(\Lambda)$, we define

$$\tilde{z}(x) := \begin{cases} z(x) & \text{if } x \in \Lambda, \\ 0 & \text{else.} \end{cases}$$

Then, in view of Adams and Fournier [1, Chapter 5], we have $\tilde{z} \in H^1(\mathbb{R}^d)$ and $\nabla \tilde{z} = \widetilde{\nabla z}$. As a consequence, the \sim can be omitted throughout.

- (ii) Notice that by the convexity of Λ , we have $\Lambda_{\delta'} \subset \Lambda_{\delta}$ for all $\delta \in (0, \delta')$. In particular, if some function $z \in L^2(\mathbb{R}^d)$ has compact support in $\Lambda_{\delta'}$, its support is thus also contained in Λ_{δ} for $0 < \delta < \delta'$. In fact, we have $z \in L^2(\Lambda_{\delta})$ and $z_{\delta} \in L^2(\Lambda)$ for $0 < \delta < \delta'$. Crucially, the fact that $P_{\delta} z = z$ for $0 < \delta < \delta'$ will allow us to simplify notation considerably as we are starting out with some function in $L^2(\mathbb{R}^d)$, which is already contained in $L^2(\Lambda_{\delta})$ for δ sufficiently small.

- (iii) For any $z \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \|z - P_{\delta} z\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |z(x) - \mathbb{1}_{\Lambda_{\delta}}(x)z(x)|^2 dx \\ &= \int_{\mathbb{R}^d \setminus \Lambda_{\delta}} |z(x)|^2 dx \rightarrow 0, \quad \delta \rightarrow 0. \end{aligned} \tag{A.7}$$

Lemma A.6 (A strong convergence for the resolution of identities). *We have for any $z \in L^2(\mathbb{R}^d)$:*

$$\|E_{\delta}(\lambda)P_{\delta} z - E(\lambda)z\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0, \quad \delta \rightarrow 0, \quad \lambda \geq 0. \tag{A.8}$$

Proof of Lemma A.6. By Weidmann [60, Theorem 2], the convergence of the resolutions of identity in (A.8) follows from the strong resolvent convergence

$$\|(-A_{\vartheta,\delta} - \lambda)^{-1}P_{\delta}z - (-\vartheta(0)\Delta - \lambda)^{-1}z\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (\text{A.9})$$

A sufficient condition for (A.9) is given by Weidmann [60, Theorem 1]. Indeed, consider some $\varphi \in D_0 = C_c^\infty(\mathbb{R}^d)$ in the core of the Laplace operator. Then, there exists some $\delta_0(z)$ such that for any $0 < \delta < \delta_0(z)$, we have $\varphi \in H_0^1(\Lambda_\delta) \cap H^2(\Lambda_\delta)$ and $-A_{\vartheta,\delta}\varphi \rightarrow -\vartheta(0)\Delta\varphi$ in $L^2(\mathbb{R}^d)$. In particular, the assumptions of Weidmann [60, Theorem 1 and Theorem 2] are satisfied and (A.8) follows. \square

Corollary A.7. *For $z \in L^2(\mathbb{R}^d)$, we have*

$$\int_0^\infty g(\lambda)d\langle E_\delta(\lambda)P_\delta z - E(\lambda)z, z \rangle_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \delta \rightarrow 0, \quad (\text{A.10})$$

for any bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{C}$.

Proof of Corollary A.7. Consider the finite measures

$$E_{z,\delta}(B) := \langle E_\delta(B)P_\delta z, P_\delta z \rangle_{L^2(\Lambda_\delta)}, \quad B \in \mathcal{B}(\mathbb{R}), \quad \delta \geq 0, \quad (\text{A.11})$$

with the distribution functions $E_{z,\delta}(\lambda) = \langle E_\delta(\lambda)P_\delta z, P_\delta z \rangle_{L^2(\Lambda_\delta)}$ for $\delta \geq 0$. Using (A.8) and the fact that Δ has fully absolutely continuous spectrum, we observe for any $\lambda \geq 0$ that

$$\begin{aligned} |E_{z,\delta}(\lambda) - E_{z,0}(\lambda)| &= |\langle E_\delta(\lambda)P_\delta z - E(\lambda)z, P_\delta z \rangle_{L^2(\mathbb{R}^d)}| \\ &\leq \|E_\delta(\lambda)P_\delta z - E(\lambda)z\|_{L^2(\mathbb{R}^d)} \|z\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \delta \rightarrow 0, \end{aligned} \quad (\text{A.12})$$

where the convergence follows immediately from Lemma A.6. The convergence of the distribution function yields a weak convergence of the associated measures (A.11). For any bounded and continuous function $g : \mathbb{R} \rightarrow \mathbb{C}$ the convergence (A.12) implies (A.10). \square

Proof of Proposition 4.5. We wish to show that for any $z_1, z_2 \in L^2(\mathbb{R}^d)$, we have

$$\langle e^{i\delta^{-1}t(-A_{\vartheta,\delta})^{1/2}}P_\delta z_1, P_\delta z_2 \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad \delta \rightarrow 0, \quad t \in \mathbb{R},$$

where the orthogonal projection P_δ is defined through (3.3). By polarisation, we may assume that $z = z_1 = z_2$.

The assumptions of Lemma A.6 are satisfied and Corollary A.7 is applicable. With $|e^{i\delta^{-1}t\sqrt{\lambda}}| \leq 1$ and the fact that the Laplace operator is a Riemann-Lebesgue operator, c.f. (4.5) in Lemma 4.2, we observe

$$\begin{aligned} \langle e^{i\delta^{-1}t(-A_{\vartheta,\delta})^{1/2}}P_\delta z, P_\delta z \rangle_{L^2(\Lambda_\delta)} &= \int_0^\infty e^{i\delta^{-1}t\sqrt{\lambda}}d\langle E_\delta(\lambda)P_\delta z, P_\delta z \rangle_{L^2(\Lambda_\delta)} \\ &= \int_0^\infty e^{i\delta^{-1}t\sqrt{\lambda}}d\langle E_\delta(\lambda)P_\delta z, P_\delta z \rangle_{L^2(\mathbb{R}^d)} \\ &= \int_0^\infty e^{i\delta^{-1}t\sqrt{\lambda}}d\langle E_\delta(\lambda)P_\delta z - E(\lambda)z, P_\delta z \rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \langle e^{i\delta^{-1}t(-\Delta)^{1/2}}P_\delta z, P_\delta z \rangle_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \delta \rightarrow 0, \end{aligned}$$

where we have used (A.7) in Remark A.5 and (A.10) from Corollary A.7 in the last step. \square

Proof of Proposition 3.3. As $(C_{\vartheta,\delta}(t), t \in [0, T])$ and $(S_{\vartheta,\delta}(t), t \in [0, T])$ are the operator cosine and sine functions associated with the operator $A_{\vartheta,\delta}$ for $\delta \geq 0$, we can represent them using the resolutions of identities associated with $-A_{\vartheta,\delta}$ and $-\vartheta(0)\Delta$, respectively:

$$\begin{aligned} & \|S_{\vartheta,\delta}(\tau)P_\delta z - S_{\vartheta(0)}(\tau)z\|_{L^2(\mathbb{R}^d)}^2 \\ &= \int_0^\infty \left| \frac{\sin(\tau\sqrt{\lambda})}{\sqrt{\lambda}} \right|^2 d\langle (E_\delta(\lambda)P_\delta - E(\lambda))z, z \rangle_{L^2(\mathbb{R}^d)}, \\ & \|C_{\vartheta,\delta}(\tau)P_\delta z - C_{\vartheta(0)}(\tau)z\|_{L^2(\mathbb{R}^d)}^2 \\ &= \int_0^\infty |\cos(\tau\sqrt{\lambda})|^2 d\langle (E_\delta(\lambda)P_\delta - E(\lambda))z, z \rangle_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{A.13}$$

Thus, as both the cosine and the mapping $\lambda \rightarrow \sin(\tau\sqrt{\lambda})/\sqrt{\lambda}$ are bounded, we can apply Corollary A.7 and both expressions in (A.13) converge to zero. \square

A.3 Proof of asymptotic energy results

Unless stated otherwise, all limits are for $\delta \rightarrow 0$. For $z \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we define the norm

$$\|z\|_{L^1 \cap L^2(\mathbb{R}^d)} := \|z\|_{L^1(\mathbb{R}^d)} + \|z\|_{L^2(\mathbb{R}^d)},$$

and for z with partial derivatives up to second order in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ set

$$\|z\|_{W_{1,2}^2(\mathbb{R}^d)} := \|z + |\nabla z| + \Delta z\|_{L^1 \cap L^2(\mathbb{R}^d)}.$$

In order to use Altmeyer and Reiß [4, Lemma A.6], we require the following assumption.

Assumption (A.8, Approx, $w^{(\delta)}$, z). Suppose z satisfies (5.1, Kernel, z). Let $w^{(\delta)} \in L^2(\mathbb{R}^d)$ have compact support in $\Lambda_{\delta'}$ for some $\delta' > 0$ and

$$\|w^{(\delta)} - \Delta z\|_{L^1 \cap L^2(\mathbb{R}^d)} \leq C\delta^\alpha \|z\|_{W_{1,2}^2(\mathbb{R}^d)}$$

for all $0 < \delta \leq \delta'$.

We emphasize that we use (A.8, Approx, w^δ , z) as a declaration and assertion of a $w^{(\delta)}$ and z such that (A.8, Approx, w^δ , z) is satisfied.

Lemma A.9 (Limits for fractions and inverse). *Fix z and $w^{(\delta)}$ satisfying (5.1, Kernel, z) and (A.8, Approx, w^δ , z). Assume for $\gamma > 0$ that $\gamma > 1 - d/4 - \alpha/2$. Then, as $\delta \rightarrow 0$ we have*

- (i) $\|(-A_{\vartheta,\delta})^{-1/2}w^{(\delta)} - (-\vartheta(0)\Delta)^{-1/2}\Delta z\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0$,
- (ii) $\|(-A_{\vartheta,\delta})^{-1}\Delta z - (-\vartheta(0)\Delta)^{-1}\Delta z\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0$,

$$(iii) \|(-A_{\vartheta,\delta})^{-1+\gamma}w^{(\delta)} - (-\vartheta(0)\Delta)^{-1+\gamma}z\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0,$$

$$(iv) \sup_{0<\delta\leq 1} \|(-A_{\vartheta,\delta})^{-1}\Delta z\|_{L^2(\mathbb{R}^d)} < \infty,$$

$$(v) \sup_{0<\delta\leq 1} \|(-A_{\vartheta,\delta})^{-1/2}w^{(\delta)}\|_{L^2(\mathbb{R}^d)} < \infty.$$

Proof of Lemma A.9.

Step 1 (Semigroup representations). Using the semigroup representation for fractional powers of operators in Pazy [47, Chapter 2.6], we observe

$$\begin{aligned} (-A_{\vartheta,\delta})^{-1/2}w^{(\delta)} &= \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2}T_\delta(t)w^{(\delta)}dt, \\ (-A_{\vartheta,\delta})^{-1}\Delta z &= \frac{1}{\Gamma(1)} \int_0^\infty T_\delta(t)\Delta zdt, \\ (-A_{\vartheta,\delta})^{-1+\gamma}w^{(\delta)} &= \frac{1}{\Gamma(1-\gamma)} \int_0^\infty t^{-\gamma}T_\delta(t)w^{(\delta)}dt, \end{aligned}$$

where $(T_\delta(t), t \geq 0)$ is the strongly continuous semigroup generated by $A_{\vartheta,\delta}$ on $L^2(\Lambda_\delta)$.

Step 2 (Pointwise convergence of the integrand). Using Altmeyer and Reiß [4, Proposition 3.5], we immediately obtain

$$\begin{aligned} \|T_\delta(t)w^{(\delta)} - T_0(t)\Delta z\|_{L^2(\mathbb{R}^d)} &\rightarrow 0, \quad \delta \rightarrow 0, \\ \|T_\delta(t)\Delta z - T_0(t)\Delta z\|_{L^2(\mathbb{R}^d)} &\rightarrow 0, \quad \delta \rightarrow 0. \end{aligned}$$

Step 3 (Upper bounds for dominated convergence). Given (5.1, Kernel, z) and Altmeyer and Reiß [4, Lemma A.6] and the absence of a first and zeroth order perturbation term, we observe

$$\|t^{-1/2}T_\delta(t)w^{(\delta)}\|_{L^2(\Lambda_\delta)} \lesssim (t^{-1/2} \wedge t^{-1/2-d/4-\alpha/2})\|z\|_{W_{1,2}^2(\mathbb{R}^d)}, \quad (\text{A.14})$$

$$\|T_\delta(t)\Delta z\|_{L^2(\Lambda_\delta)} \lesssim (1 \wedge t^{-1/2-d/4-\alpha/2})\|z\|_{W_{1,2}^2(\mathbb{R}^d)}, \quad (\text{A.15})$$

$$\|t^{-\gamma}T_\delta(t)w^{(\delta)}\|_{L^2(\Lambda_\delta)} \lesssim (1 \wedge t^{-\gamma-d/4-\alpha/2})\|z\|_{W_{1,2}^2(\mathbb{R}^d)}. \quad (\text{A.16})$$

Step 4 (Upper bounds and dominated convergence). We will now argue that the latter upper bounds (A.14), (A.15) and (A.16) are integrable on $(0, \infty)$.

Case 1 ($d = 1$). In the one-dimensional case, we have

$$\begin{aligned} -1/2 - d/4 - \alpha/2 &= -3/4 - \alpha/2 < -1, \\ -\gamma - \alpha/2 - 1/4 &< -1, \end{aligned} \quad (\text{A.17})$$

because $\alpha > \frac{1}{2}$ by (2.1, Regularity, ϑ, α) and $\gamma > \frac{3}{4} - \alpha/2$.

Case 2 ($d > 1$). In this case $\alpha > 0$ is sufficient since $-1/2 - d/4 \leq -1$ implies

$$\begin{aligned} -1/2 - d/4 - \alpha/2 &< -1, \\ -\gamma - \alpha/2 - \frac{d}{4} &< -1. \end{aligned} \quad (\text{A.18})$$

Note that for $d \geq 4$, the assumption on γ is always satisfied.

By (A.17) and (A.18) the upper bounds (A.14), (A.15) and (A.16) are integrable on $(0, \infty)$. The upper bounds are independent of the variable δ , which implies (iv) and (v). The convergences (i), (ii) and (iii) follow by the dominated convergence theorem using the upper bounds (A.14), (A.15) and (A.16) as dominants. \square

Proposition A.10 (Asymptotic behaviour of energy for the operator sine). *Grant the Assumptions (5.1, Kernel, z_j) and (A.8, Approx, $w_j^{(\delta)}$, z_j) for $j = 1, 2$.*

(i) *Asymptotic equipartition of energy: For any $r \in \mathbb{R}$, we have*

$$\langle S_{\vartheta, \delta}(\delta^{-1}r)w_1^{(\delta)}, S_{\vartheta, \delta}(\delta^{-1}r)w_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow \frac{1}{2\vartheta(0)} \langle \nabla z_1, \nabla z_2 \rangle_{L^2(\mathbb{R}^d)}, \quad \delta \rightarrow 0.$$

(ii) *Slow-fast orthogonality: For any $r_1, r_2 \in \mathbb{R}$ with $r_1 \neq r_2$, we have*

$$\langle S_{\vartheta, \delta}(\delta^{-1}r_1)w_1^{(\delta)}, S_{\vartheta, \delta}(\delta^{-1}r_2)w_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad \delta \rightarrow 0.$$

Proof of Proposition A.10.

Step 1 (Representation using Riemann-Lebesgue operators). We abbreviate by $R_\delta(t) = e^{it(-A_{\vartheta, \delta})^{1/2}}$ the unitary group on the complex Hilbert space $L^2(\Lambda_\delta)$ for $t \in \mathbb{R}$. The operator sine can be represented using $R_\delta(t)$ through

$$S_{\vartheta, \delta}(\delta^{-1}\tau)w_j^{(\delta)} = \frac{(-A_{\vartheta, \delta})^{-1/2}}{2i} (R_\delta(\delta^{-1}\tau) - R_\delta(-\delta^{-1}\tau))w_j^{(\delta)}, \quad \tau \in \mathbb{R}, \quad j \in \{1, 2\}.$$

Setting $\xi_j^{(\delta)} = (-A_{\vartheta, \delta})^{-1/2}w_j^{(\delta)}$, we observe

$$\begin{aligned} & 4 \langle S_{\vartheta, \delta}(\delta^{-1}r_1)\xi_1^{(\delta)}, S_{\vartheta, \delta}(\delta^{-1}r_2)\xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \langle (R_\delta(\delta^{-1}r_1) - R_\delta(-\delta^{-1}r_1))\xi_1^{(\delta)}, (R_\delta(\delta^{-1}r_2) - R_\delta(-\delta^{-1}r_2))\xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \langle (R_\delta(\delta^{-1}r_2) - R_\delta(-\delta^{-1}r_2))^* \circ (R_\delta(\delta^{-1}r_1) - R_\delta(-\delta^{-1}r_1))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \langle (R_\delta(-\delta^{-1}r_2) - R_\delta(\delta^{-1}r_2)) \circ (R_\delta(\delta^{-1}r_1) - R_\delta(-\delta^{-1}r_1))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \langle R_\delta(\delta^{-1}(r_1 - r_2))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} + \langle R_\delta(\delta^{-1}(r_2 - r_1))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &\quad - \langle R_\delta(\delta^{-1}(r_1 + r_2))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} - \langle R_\delta(-\delta^{-1}(r_1 + r_2))\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)}. \end{aligned} \tag{A.19}$$

Step 2 (Convergences). We have (5.1, Kernel, z_j), (A.8, Approx, $w_j^{(\delta)}$, z_j) and (2.1, Regularity, ϑ , α). Thus, all assumptions of Lemma A.9(i) are satisfied, and we obtain

$$\|\xi_j^{(\delta)} - (-\vartheta(0))^{-1/2}(-\Delta)^{1/2}z_j\|_{L^2(\mathbb{R}^d)}^2 \rightarrow 0, \quad \delta \rightarrow 0, \quad j = 1, 2. \tag{A.20}$$

Given Remark A.5 and using the convergence in norm (A.20), we can reduce the desired convergence of the expression

$$\langle R_\delta(\delta^{-1}\tau)\xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad \delta \rightarrow 0,$$

for any real number $\tau \in \mathbb{R} \setminus \{0\}$ to the convergence

$$\langle R_\delta(\delta^{-1}\tau)(-\Delta)^{1/2}z_{j_1}, (-\Delta)^{1/2}z_{j_2} \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad \delta \rightarrow 0, \quad j_1, j_2 \in \{1, 2\}. \quad (\text{A.21})$$

Clearly, (A.21) follows from the asymptotic Riemann-Lebesgue principle (Proposition 4.5). For (ii) every summand in (A.19) vanishes since $r_1 \neq r_2$. Similarly, for $r = r_1 = r_2$, we obtain the remaining term

$$\begin{aligned} \frac{1}{2} \langle \xi_1^{(\delta)}, \xi_2^{(\delta)} \rangle_{L^2(\Lambda_\delta)} &\rightarrow \frac{1}{2\vartheta(0)} \langle (-\Delta)^{1/2}z_1, (-\Delta)^{1/2}z_2 \rangle_{L^2(\mathbb{R}^d)} \\ &= \frac{1}{2\vartheta(0)} \langle \nabla z_1, \nabla z_2 \rangle_{L^2(\mathbb{R}^d)}, \quad \delta \rightarrow 0. \quad \square \end{aligned}$$

For the analysis of the bias, we define the functions:

$$\begin{aligned} \beta^{(\delta)}(x) &:= \delta^{-1}(A_{\vartheta, \delta} - \vartheta(0)\Delta)K(x), \quad \delta > 0, \\ \beta^{(0)}(x) &:= \Delta(\langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d} K)(x) - \langle \nabla \vartheta(0), \nabla K(x) \rangle_{\mathbb{R}^d}, \quad x \in \mathbb{R}^d. \end{aligned} \quad (\text{A.22})$$

Using (A.22), we determine the asymptotic behaviour of expressions emerging in the analysis of the observed Fisher information (5.5) and the bias (5.6).

Proposition A.11 (Asymptotics for the emerging energetic expressions). *Grant Assumption (5.1, Kernel, K). Let $\beta^{(0)}$ and $\beta^{(\delta)}$ be defined through (A.22). As $\delta \rightarrow 0$ we obtain the following convergences.*

(i) For $t \in \mathbb{R} \setminus \{0\}$, we have the asymptotic equipartitions

$$\|S_{\vartheta, \delta}(\delta^{-1}t)\Delta K\|_{L^2(\Lambda_\delta)}^2 \rightarrow \frac{1}{2\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad (\text{A.23})$$

$$\langle S_{\vartheta, \delta}(\delta^{-1}t)\Delta K, S_{\vartheta, \delta}(\delta^{-1}t)\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow \frac{1}{2\vartheta(0)} \langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}. \quad (\text{A.24})$$

(ii) For $s, t \in \mathbb{R}$ with $s \neq t$ fixed, we have the slow-fast orthogonality

$$\langle S_{\vartheta, \delta}(\delta^{-1}t)\Delta K, S_{\vartheta, \delta}(\delta^{-1}s)\Delta K \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad (\text{A.25})$$

$$\langle S_{\vartheta, \delta}(\delta^{-1}t)\Delta K, S_{\vartheta, \delta}(\delta^{-1}s)\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow 0, \quad (\text{A.26})$$

$$\langle S_{\vartheta, \delta}(\delta^{-1}t)\beta^{(\delta)}, S_{\vartheta, \delta}(\delta^{-1}s)\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \rightarrow 0. \quad (\text{A.27})$$

Proof of Proposition A.11. The result is a corollary of Proposition A.10 by setting $v_{j_1}^{(\delta)} = \beta^{(\delta)}$ or $v_{j_2}^{(\delta)} = \Delta K$ for $j_1, j_2 \in \{1, 2\}$. In particular, the convergences (A.23) and (A.24), and (A.25), (A.26) and (A.27) follow from the asymptotic equipartition (i) and the asymptotic orthogonality (ii) in Proposition A.10, respectively. The conditions of Proposition A.10 are trivially satisfied for ΔK in view of (5.1, Kernel, K). On the other, we refer to Altmeyer and Reiß [4, Lemma A.5] for the existence of a function z such that (A.8, Approx, β^δ, z) is satisfied. \square

Lemma A.12 (Uniform upper bounds operator sine). *Suppose that K satisfies (5.1, Kernel, K). Then, we have*

$$\|S_{\vartheta,\delta}(\delta^{-1}r)\Delta K\|_{L^2(\Lambda_\delta)}^2 \leq \sup_{0<\delta\leq 1} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_\delta)}^2 < \infty, \quad (\text{A.28})$$

$$\|S_{\vartheta,\delta}(\delta^{-1}r)\beta^{(\delta)}\|_{L^2(\Lambda_\delta)}^2 \leq \sup_{0<\delta\leq 1} \|(-A_{\vartheta,\delta})^{-1/2}\beta^{(\delta)}\|_{L^2(\Lambda_\delta)}^2 < \infty, \quad (\text{A.29})$$

where $\beta^{(\delta)}$ is defined through (A.22).

Proof of Lemma A.12. By (5.1, Kernel, K) and (2.1, Regularity, ϑ , α), both suprema are finite by (iv) and (v) in Lemma A.9. The result now follows immediately using the functional calculus from the boundedness of the sine through

$$\begin{aligned} \|S_{\vartheta,\delta}(\delta^{-1}r)z\|_{L^2(\Lambda_\delta)}^2 &= \|\sin(\delta^{-1}r(-A_{\vartheta,\delta})^{1/2})(-A_{\vartheta,\delta})^{-1/2}z\|_{L^2(\Lambda_\delta)}^2 \\ &\leq \|(-A_{\vartheta,\delta})^{-1/2}z\|_{L^2(\Lambda_\delta)}^2 \end{aligned}$$

for any $z \in L^2(\Lambda_\delta)$. □

A.4 Asymptotics for the augmented MLE

We decompose the mild solution to the stochastic wave equation with non-zero initial conditions into a deterministic and a stochastic part given by

$$u(t) = C_\vartheta(t)u_0 + S_\vartheta(t)v_0 + \tilde{u}(t), \quad t \in [0, T], \quad (\text{A.30})$$

$$v(t) = C'_\vartheta(t)u_0 + C_\vartheta(t)v_0 + \tilde{v}(t), \quad t \in [0, T], \quad (\text{A.31})$$

where $(\tilde{u}(t), \tilde{v}(t))$ is a solution to the stochastic wave equation with zero-initial conditions. Based on the solution with zero-initial conditions, we introduce the notation

$$\begin{aligned} \tilde{u}_\delta^\Delta(t) &:= \langle \tilde{u}(t), \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)}, \\ \tilde{I}_\delta &:= \int_0^T \tilde{u}_\delta^\Delta(t)^2 dt, \\ \tilde{R}_\delta &:= \int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), (A_\vartheta - \vartheta(0)\Delta)K_\delta \rangle_{L^2(\Lambda)} dt. \end{aligned}$$

Notice that the local measurements depend on the initial conditions through

$$u_\delta^\Delta(t) = \tilde{u}_\delta^\Delta(t) + \mathcal{L}_\delta^C(t) + \mathcal{L}_\delta^S(t), \quad t \in [0, T], \quad (\text{A.32})$$

where

$$\begin{aligned} \mathcal{L}_\delta^C(t) &:= \mathcal{L}_\delta^C(u_0, \Delta K)(t) := \langle C(t)u_0, \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)} \\ \mathcal{L}_\delta^S(t) &:= \mathcal{L}_\delta^S(v_0, \Delta K)(t) := \langle S(t)v_0, \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)}. \end{aligned}$$

Lemma A.13 (Dependence of the observed Fisher information on the initial conditions). *For any fixed $\delta > 0$ and deterministic $(u_0, v_0) \in L^2(\Lambda) \times H_0^{-1}(\Lambda)$, we have:*

(i) *The observed Fisher information I_δ depends on the initial conditions through*

$$\begin{aligned} I_\delta &= \tilde{I}_\delta + 2(\langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C \rangle_{L^2([0,T])} + \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^S \rangle_{L^2([0,T])}) \\ &\quad + \|\mathcal{L}_\delta^C\|_{L^2([0,T])}^2 + \|\mathcal{L}_\delta^S\|_{L^2([0,T])}^2 + 2\langle \mathcal{L}_\delta^C, \mathcal{L}_\delta^S \rangle_{L^2([0,T])}. \end{aligned} \quad (\text{A.33})$$

(ii) *The expectation of the observed Fisher information I_δ satisfies*

$$\mathbb{E}[I_\delta] = \mathbb{E}[\tilde{I}_\delta] + \|\mathcal{L}_\delta^C\|_{L^2([0,T])}^2 + \|\mathcal{L}_\delta^S\|_{L^2([0,T])}^2 + 2\langle \mathcal{L}_\delta^C, \mathcal{L}_\delta^S \rangle_{L^2([0,T])}. \quad (\text{A.34})$$

Proof of Lemma A.13. The result follows immediately using the Binomial formula's and noticing

$$\mathbb{E}[\langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C \rangle_{L^2([0,T])}] = \mathbb{E}[\langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^S \rangle_{L^2([0,T])}] = 0. \quad \square$$

Lemma A.14 (Bounds for the deterministic parts). *Grant (2.2, Initial, u_0, v_0). Then, we have*

$$\|\mathcal{L}_\delta^C + \mathcal{L}_\delta^S\|_{L^2([0,T])}^2 \in \mathcal{O}(1), \quad \delta \rightarrow 0. \quad (\text{A.35})$$

Proof of Lemma A.14.

Step 1 (Rewriting \mathcal{L}_δ^C and \mathcal{L}_δ^S). Using properties of the rescaling (Lemma 3.1), we observe

$$\begin{aligned} &\langle C_\vartheta(t)u_0, \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)} \\ &= \delta^{-2} \langle u_0, (C_{\vartheta,\delta}(\delta^{-1}t)\Delta K)_\delta \rangle_{L^2(\Lambda)} \\ &= \delta^{-2} \langle (u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)\Delta K \rangle_{L^2(\Lambda_\delta)} \\ &= \delta^{-2} \langle (-A_{\vartheta,\delta})(u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1}\Delta K \rangle_{L^2(\Lambda_\delta)} \\ &= \langle (-A_\vartheta u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1}\Delta K \rangle_{L^2(\Lambda_\delta)} \end{aligned}$$

and

$$\begin{aligned} &\langle S_\vartheta(t)v_0, \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)} \\ &= \delta^{-2} \langle v_0, \delta(S_{\vartheta,\delta}(\delta^{-1}t)\Delta K)_\delta \rangle_{L^2(\Lambda)} \\ &= \delta^{-1} \langle (-A_{\vartheta,\delta})^{1/2}(v_0)_{\delta^{-1}}, S_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1/2}\Delta K \rangle_{L^2(\Lambda_\delta)} \\ &= \langle ((-A_\vartheta)^{1/2}v_0)_{\delta^{-1}}, S_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1/2}\Delta K \rangle_{L^2(\Lambda_\delta)}. \end{aligned}$$

Step 2 (Upper bound using Cauchy-Schwarz). The Cauchy-Schwarz inequality

yields

$$\begin{aligned}
& |\langle C_{\vartheta}(t)u_0, \delta^{-2}(\Delta K)_{\delta} \rangle_{L^2(\Lambda)}| \\
& \leq \|(-A_{\vartheta}u_0)_{\delta^{-1}}\|_{L^2(\Lambda_{\delta})} \|C_{\vartheta,\delta}(\delta^{-1}t)\|_{\mathcal{L}(L^2(\Lambda_{\delta}), L^2(\Lambda_{\delta}))} \|(-A_{\vartheta,\delta})^{-1}\Delta K\|_{L^2(\Lambda_{\delta})} \\
& \leq \|A_{\vartheta}u_0\|_{L^2(\Lambda)} \|(-A_{\vartheta,\delta})^{-1}\Delta K\|_{L^2(\Lambda_{\delta})} \tag{A.36}
\end{aligned}$$

$$\begin{aligned}
& |\langle S_{\vartheta}(t)v_0, \delta^{-2}(\Delta K)_{\delta} \rangle_{L^2(\Lambda)}| \\
& \leq \|((-A_{\vartheta})^{1/2}v_0)_{\delta^{-1}}\|_{L^2(\Lambda_{\delta})} \|S_{\vartheta,\delta}(\delta^{-1}t)\|_{\mathcal{L}(L^2(\Lambda_{\delta}), L^2(\Lambda_{\delta}))} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_{\delta})} \\
& \leq \|(-A_{\vartheta})^{1/2}v_0\|_{L^2(\Lambda)} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_{\delta})} T. \tag{A.37}
\end{aligned}$$

Step 3 (Upper bounds remain finite). Clearly both $A_{\vartheta}u_0$ and $(-A_{\vartheta})^{1/2}v_0$ are elements of the Hilbert space $L^2(\Lambda)$. Thus, their $\|\cdot\|_{L^2(\Lambda)}$ norm remains finite. Furthermore, by Lemma A.9 both $(-A_{\vartheta,\delta})^{-1}\Delta K$ and $(-A_{\vartheta,\delta})^{-1/2}\Delta K$ converge in $L^2(\mathbb{R})$, implying

$$\sup_{0 < \delta < \delta'} \|(-A_{\vartheta,\delta})^{-1}\Delta K\|_{L^2(\Lambda_{\delta})} < \infty, \quad \sup_{0 < \delta < \delta'} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_{\delta})} < \infty.$$

The result follows from (A.36) and (A.37) as the time integral of the squared upper bound remains finite on a finite time horizon. \square

Lemma A.15 (Expectation and variance of observed Fisher information \tilde{I}_{δ}). *Grant (5.1, Kernel, K). The expectation of the observed Fisher information satisfies*

$$\delta^2 \mathbb{E}[\tilde{I}_{\delta}] = \int_0^T \int_0^t \|S_{\vartheta,\delta}(\delta^{-1}r)\Delta K\|_{L^2(\Lambda_{\delta})}^2 dr dt, \quad \delta > 0, \tag{A.38}$$

and the variance of the observed Fisher information is given by

$$\begin{aligned}
& \text{Var}(\delta^2 \tilde{I}_{\delta}) \\
& = 2 \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\Delta K, S_{\vartheta,\delta}(\delta^{-1}(s-r))\Delta K \rangle_{L^2(\Lambda_{\delta})} dr \right)^2 ds dt.
\end{aligned}$$

Proof of Lemma A.15. Using Fubini's theorem and Lemma A.3, we observe by Da Prato and Zabczyk [13, Proposition 4.28]:

$$\begin{aligned}
\mathbb{E}[\delta^2 \tilde{I}_{\delta}] & = \delta^2 \int_0^T \text{Var}(\tilde{u}_{\delta}^{\Delta}(t)) dt = \delta^2 \int_0^T \int_0^t \|S_{\vartheta}(t-s)\Delta K_{\delta}\|_{L^2(\Lambda)}^2 ds dt \\
& = \delta^{-2} \int_0^T \int_0^t \|S_{\vartheta}(t-s)(\Delta K)_{\delta}\|_{L^2(\Lambda)}^2 ds dt. \tag{A.39}
\end{aligned}$$

Applying the rescaling of the operator sine function in Lemma 3.1 (ii) to (A.39) yields

$$\begin{aligned}
\mathbb{E}[\delta^2 \tilde{I}_{\delta}] & = \int_0^T \int_0^t \|(S_{\vartheta,\delta}(\delta^{-1}(t-s))\Delta K)_{\delta}\|_{L^2(\Lambda)}^2 ds dt \\
& = \int_0^T \int_0^t \|S_{\vartheta,\delta}(\delta^{-1}r)\Delta K\|_{L^2(\Lambda_{\delta})}^2 dr dt.
\end{aligned}$$

Similarly, using Wick's formula (Janson [34, Theorem 1.28]), we obtain for the covariance

$$\begin{aligned}
& \text{Var}(\delta^2 \tilde{I}_\delta) \\
&= 2\delta^4 \int_0^T \int_0^T \text{Cov}(\tilde{u}_\delta^\Delta(s), \tilde{u}_\delta^\Delta(t))^2 \text{d}s \text{d}t \\
&= 2\delta^4 \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_\vartheta(t-r) \Delta K_\delta, S_\vartheta(s-r) \Delta K_\delta \rangle_{L^2(\Lambda)} \text{d}r \right)^2 \text{d}s \text{d}t \\
&= 2\delta^{-4} \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_\vartheta(t-r) (\Delta K)_\delta, S_\vartheta(s-r) (\Delta K)_\delta \rangle_{L^2(\Lambda)} \text{d}r \right)^2 \text{d}s \text{d}t
\end{aligned} \tag{A.40}$$

Another application of the rescaling of the operator sine function in Lemma 3.1 (ii) to (A.40) amounts to

$$\begin{aligned}
& \text{Var}(\delta^2 \tilde{I}_\delta) \\
&= 2\delta^{-4} \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_{\vartheta, \delta}(t-r) (\Delta K)_\delta, S_{\vartheta, \delta}(s-r) (\Delta K)_\delta \rangle_{L^2(\Lambda)} \text{d}r \right)^2 \text{d}s \text{d}t \\
&= 2 \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_{\vartheta, \delta}(\delta^{-1}(t-r)) \Delta K, S_{\vartheta, \delta}(\delta^{-1}(s-r)) \Delta K \rangle_{L^2(\Lambda_\delta)} \text{d}r \right)^2 \text{d}s \text{d}t.
\end{aligned}$$

□

Lemma A.16 (Expectation and variance of the remaining bias \tilde{R}_δ). *Grant (5.1, Kernel, K). The expectation of the remaining bias satisfies*

$$\mathbb{E}[\delta \tilde{R}_\delta] = \int_0^T \int_0^t \langle S_{\vartheta, \delta}(\delta^{-1}s) \Delta K, S_{\vartheta, \delta}(\delta^{-1}s) \beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \text{d}s \text{d}t, \tag{A.41}$$

and its variance is given by

$$\begin{aligned}
\text{Var}(\delta \tilde{R}_\delta) &= \int_0^T \int_0^T (\text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta^{(\delta)} \rangle_{L^2(\Lambda)}))^2 \\
&\quad + \text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta^{(\delta)} \rangle_{L^2(\Lambda)}) \text{Cov}(\tilde{u}_\delta^\Delta(s), \langle \tilde{u}(t), \beta^{(\delta)} \rangle_{L^2(\Lambda)}) \text{d}t \text{d}s,
\end{aligned} \tag{A.42}$$

where

$$\begin{aligned}
& \text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta^{(\delta)} \rangle_{L^2(\Lambda)}) \\
&= \int_0^{t \wedge s} \langle S_{\vartheta, \delta}(\delta^{-1}(t-r)) \Delta K, S_{\vartheta, \delta}(\delta^{-1}(s-r)) \beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \text{d}r,
\end{aligned}$$

with $\beta^{(\delta)}$ is defined through (A.22).

Proof of Lemma A.16. With (A.22), we observe

$$\begin{aligned}
\mathbb{E}[\delta \tilde{R}_\delta] &= \mathbb{E} \left(\delta \int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), (A_\vartheta - \vartheta(0)\Delta)K_\delta \rangle_{L^2(\Lambda)} dt \right) \\
&= \mathbb{E} \left(\delta \int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), \delta^{-2}((A_\vartheta - \vartheta(0)\Delta)K)_\delta \rangle_{L^2(\Lambda)} dt \right) \\
&= \mathbb{E} \left(\int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), \delta^{-1}((A_\vartheta - \vartheta(0)\Delta)K)_\delta \rangle_{L^2(\Lambda)} dt \right) \\
&= \mathbb{E} \left(\int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \right).
\end{aligned} \tag{A.43}$$

As before, an application of the rescaling of the operator sine function in Lemma 3.1 (ii) to the representation (A.43) implies

$$\begin{aligned}
\mathbb{E}[\delta \tilde{R}_\delta] &= \int_0^T \int_0^t \langle S_\vartheta(t-s)\Delta K_\delta, S_\vartheta(t-s)\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} ds dt \\
&= \delta^{-2} \int_0^T \int_0^t \langle S_\vartheta(t-s)(\Delta K)_\delta, S_\vartheta(t-s)\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} ds dt \\
&= \delta^{-2} \int_0^T \int_0^t \langle \delta(S_{\vartheta,\delta}(\delta^{-1}(t-s))\Delta K)_\delta, \delta(S_{\vartheta,\delta}(\delta^{-1}(t-s))\beta_\delta^{(\delta)})_\delta \rangle_{L^2(\Lambda)} ds dt \\
&= \int_0^T \int_0^t \langle S_{\vartheta,\delta}(\delta^{-1}(t-s))\Delta K, S_{\vartheta,\delta}(\delta^{-1}(t-s))\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} ds dt.
\end{aligned}$$

Using Wick's formula (Janson [34, Theorem 1.28]), we further have

$$\begin{aligned}
\text{Var}(\delta \tilde{R}_\delta) &= \text{Var} \left(\int_0^T \tilde{u}_\delta^\Delta(t) \langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \right) \\
&= \int_0^T \int_0^T (\text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}))^2 \\
&\quad + \text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}) \text{Cov}(\tilde{u}_\delta^\Delta(s), \langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}) ds dt.
\end{aligned}$$

Using the same rescaling arguments for the operator sine function in Lemma 3.1 (ii), we obtain

$$\begin{aligned}
&\text{Cov}(\tilde{u}_\delta^\Delta(t), \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}) \\
&= \int_0^{t \wedge s} \langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\Delta K, S_{\vartheta,\delta}(\delta^{-1}(s-r))\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} dr. \quad \square
\end{aligned}$$

Proof of Proposition 5.3. Given (5.1, Kernel, K) and (2.2, Initial, u_0, v_0), we will show that the expectation and covariance of the observed Fisher infor-

mation satisfies

$$\mathbb{E}[\delta^2 I_\delta] \rightarrow \frac{T^2}{4\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \text{Var}(\delta^2 I_\delta) \rightarrow 0, \quad \delta \rightarrow 0.$$

We begin by considering the case of zero initial conditions.

Step 1 (Zero initial conditions). If we assume zero initial conditions, we need to show

$$\mathbb{E}[\delta^2 \tilde{I}_\delta] \rightarrow \frac{T^2}{4\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \text{Var}(\delta^2 \tilde{I}_\delta) \rightarrow 0, \quad \delta \rightarrow 0. \quad (\text{A.44})$$

The representations (A.38) and (A.15) in Lemma A.15 are given by

$$\delta^2 \mathbb{E}[\tilde{I}_\delta] = \int_0^T \int_0^t \|S_{\vartheta,\delta}(\delta^{-1}r)\Delta K\|_{L^2(\Lambda_\delta)}^2 dr dt, \quad \delta > 0,$$

and

$$\begin{aligned} & \text{Var}(\delta^2 \tilde{I}_\delta) \\ &= 2 \int_0^T \int_0^T \left(\int_0^{t \wedge s} \langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\Delta K, S_{\vartheta,\delta}(\delta^{-1}(s-r))\Delta K \rangle_{L^2(\Lambda_\delta)} dr \right)^2 ds dt. \end{aligned}$$

The limits (A.23) and (A.25) in Proposition A.11 (i) and (ii) show the pointwise convergence of the innermost integrands provided that $t \neq s$. Note that the diagonal $\{(t, s) : t = s \text{ for } t, s \in [0, T]\}$ is a set of Lebesgue measure zero, and we have almost sure convergence of the integrand. By (A.28) in Lemma A.12, we have

$$\sup_{0 < \delta \leq 1} \|S_{\vartheta,\delta}(\delta^{-1}r)\Delta K\|_{L^2(\Lambda_\delta)}^2 \leq \sup_{0 < \delta \leq 1} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_\delta)}^2 < \infty, \quad (\text{A.45})$$

and

$$\begin{aligned} & \sup_{0 < \delta \leq 1} |\langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\Delta K, S_{\vartheta,\delta}(\delta^{-1}(s-r))\Delta K \rangle_{L^2(\Lambda_\delta)}| \\ & \leq \sup_{0 < \delta \leq 1} \|S_{\vartheta,\delta}(\delta^{-1}(s-r))\Delta K\|_{L^2(\Lambda_\delta)} \|S_{\vartheta,\delta}(\delta^{-1}(t-r))\Delta K\|_{L^2(\Lambda_\delta)} \\ & \leq \left(\sup_{0 < \delta \leq 1} \|(-A_{\vartheta,\delta})^{-1/2}\Delta K\|_{L^2(\Lambda_\delta)}^2 \right)^2 < \infty. \end{aligned} \quad (\text{A.46})$$

As both (A.45) and (A.46) are finite and independent of all time variables, the convergences in (A.44) follow using the dominated convergence theorem.

Step 2 (Non-zero initial conditions). The expectation of the observed Fisher information given by (A.34) in Lemma A.13 satisfies

$$\delta^2 \mathbb{E}[I_\delta] = \delta^2 \mathbb{E}[\tilde{I}_\delta] + \delta^2 \|\mathcal{L}_\delta^C + \mathcal{L}_\delta^S\|_{L^2([0,T])}^2 \rightarrow \frac{T^2}{4\vartheta(0)} \|\nabla K\|_{L^2(\mathbb{R}^d)}^2, \quad \delta \rightarrow 0,$$

where the convergence follows immediately from the convergence of $\mathbb{E}[\delta^2 \tilde{I}_\delta]$ in (A.44) and with (A.35) in Lemma A.13. By (A.33) we obtain for the variance:

$$\begin{aligned} \text{Var}(\delta^2 I_\delta) &= \text{Var}(\delta^2 \tilde{I}_\delta + 2\delta^2 \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C + \mathcal{L}_\delta^S \rangle_{L^2([0,T])}) \\ &= \text{Var}(\delta^2 \tilde{I}_\delta) + 2\text{Cov}(\delta^2 \tilde{I}_\delta, 2\delta^2 \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C + \mathcal{L}_\delta^S \rangle_{L^2([0,T])}) \\ &\quad + \text{Var}(2\delta^2 \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C + \mathcal{L}_\delta^S \rangle_{L^2([0,T])}). \end{aligned}$$

By the first step, we already know that $\text{Var}(\delta^2 \tilde{I}_\delta)$ converges to zero. Thus, as the covariance term can be bounded in terms of the variances, it suffices to observe

$$\begin{aligned} \text{Var}(2\delta^2 \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C + \mathcal{L}_\delta^S \rangle_{L^2([0,T])}) &= \mathbb{E} \left((2\delta^2 \langle \tilde{u}_\delta^\Delta, \mathcal{L}_\delta^C + \mathcal{L}_\delta^S \rangle_{L^2([0,T])})^2 \right) \\ &\leq 4\mathbb{E}[\delta^2 \tilde{I}_\delta] \delta^2 \|\mathcal{L}_\delta^C + \mathcal{L}_\delta^S\|_{L^2([0,T])}^2 \rightarrow 0, \quad \delta \rightarrow 0, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality, (A.35) in Lemma A.14 and Step 1. \square

Proof of Proposition 5.5. Throughout this proof, recall the notation for $\beta^{(\delta)}$ and $\beta^{(0)}$ defined in (A.22):

$$\begin{aligned} \beta^{(\delta)}(x) &= \delta^{-1}(A_{\vartheta,\delta} - \vartheta(0)\Delta)K(x), \quad \delta > 0, \\ \beta^{(0)}(x) &= \Delta(\langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d} K)(x) - \langle \nabla \vartheta(0), \nabla K(x) \rangle_{\mathbb{R}^d}, \quad x \in \mathbb{R}^d. \end{aligned}$$

We wish to show under the Assumptions (5.1, Kernel, K) and (2.2, Initial, u_0, v_0) that the remaining bias satisfies

$$\delta^{-1}(I_\delta)^{-1}R_\delta \xrightarrow{\mathbb{P}} \frac{\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}, \quad \delta \rightarrow 0.$$

Step 1 (Zero initial conditions). By Proposition 5.3 we have $\tilde{I}_\delta/\mathbb{E}[\tilde{I}_\delta] \xrightarrow{\mathbb{P}} 1$. Applying the dominated convergence theorem as well as (A.24) and (A.26) in Proposition A.11 to (A.41) and (A.42) in Lemma A.15 respectively, yields the convergences

$$\mathbb{E}[\delta \tilde{R}_\delta] \rightarrow \frac{T^2}{4\vartheta(0)} \langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}, \quad \text{Var}(\delta \tilde{R}_\delta) \rightarrow 0, \quad \delta \rightarrow 0.$$

Note that we have used that both (A.45) and (A.46) remain valid if ΔK is replaced by $\beta^{(\delta)}$ given Lemma A.9 (v) and the dominated convergence theorem is applicable. Thus, with Chebyshev's inequality, we also have $\tilde{R}_\delta/\mathbb{E}[\tilde{R}_\delta] \xrightarrow{\mathbb{P}} 1$ as $\delta \rightarrow 0$. An application of Slutsky's lemma and the continuous mapping theorem shows

$$\delta^{-1}(\tilde{I}_\delta)^{-1}\tilde{R}_\delta = (\mathbb{E}[\tilde{I}_\delta]\tilde{I}_\delta^{-1})(\delta^2\mathbb{E}[\tilde{I}_\delta])^{-1}\mathbb{E}[\delta\tilde{R}_\delta]\tilde{R}_\delta(\mathbb{E}[\tilde{R}_\delta])^{-1} \xrightarrow{\mathbb{P}} \frac{\langle \nabla K, (\beta^{(0)})' \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}.$$

Next, we consider the situation with non-zero initial conditions.

Step 2 (Decomposition and connection to localisation). Observe that

$$\begin{aligned}\delta R_\delta &= \delta \int_0^T u_\delta^\Delta(t) \langle u(t), (A_\vartheta - \vartheta(0)\Delta)K_\delta \rangle_{L^2(\Lambda)} dt \\ &= \int_0^T u_\delta^\Delta(t) \langle u(t), \delta^{-1}((A_\vartheta - \vartheta(0)\Delta)K)_\delta \rangle_{L^2(\Lambda)} dt \\ &= \int_0^T u_\delta^\Delta(t) \langle u(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt.\end{aligned}$$

Using (A.30) and (A.32), we decompose $\delta R_\delta = \delta \tilde{R}_\delta + V_{1,\delta} + V_{2,\delta} + V_{3,\delta}$, where

$$\begin{aligned}V_{1,\delta} &:= \int_0^T (\mathcal{L}_\delta^C(t) + \mathcal{L}_\delta^S(t)) \langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \\ V_{2,\delta} &:= \int_0^T (\mathcal{L}_\delta^C(t) + \mathcal{L}_\delta^S(t)) \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \\ V_{3,\delta} &:= \int_0^T \tilde{u}_\delta^\Delta(t) \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt.\end{aligned}$$

We immediately obtain the decomposition

$$\delta^{-1}(I_\delta)^{-1}R_\delta = \delta^{-1}(I_\delta)^{-1}\tilde{R}_\delta + \delta^{-2}(I_\delta)^{-1}(V_{1,\delta} + V_{2,\delta} + V_{3,\delta}).$$

In particular, by Proposition 5.3, we have

$$\delta^{-1}(I_\delta)^{-1}\tilde{R}_\delta \xrightarrow{\mathbb{P}} \frac{\langle \nabla K, \nabla \beta^{(0)} \rangle_{L^2(\mathbb{R}^d)}}{\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}, \quad \delta \rightarrow 0.$$

Thus, as $\delta^{-2}(I_\delta)^{-1} \in \mathcal{O}_{\mathbb{P}}(1)$ in view of Proposition 5.3, we have to show

$$V_{1,\delta} + V_{2,\delta} + V_{3,\delta} \xrightarrow{\mathbb{P}} 0, \quad \delta \rightarrow 0.$$

Step 3 (Convergence of $V_{3,\delta}$). By the linearity of the integral, it suffices to show the convergence in probability for both individual terms in the sum

$$V_{3,\delta} = \int_0^T \tilde{u}_\delta^\Delta(t) \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt + \int_0^T \tilde{u}_\delta^\Delta(t) \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt.$$

By applying the Cauchy-Schwarz inequality to the time integral, we have

$$\begin{aligned}&\mathbb{E} \left[\left(\int_0^T \tilde{u}_\delta^\Delta(t) \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \right)^2 \right] \\ &\leq \mathbb{E}[\delta^2 \tilde{I}_\delta] \int_0^T \delta^{-2} \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}^2 dt\end{aligned} \tag{A.47}$$

and

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \tilde{u}_\delta^\Delta(t) \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \right)^2 \right] \\ & \leq \mathbb{E}[\delta^2 \tilde{I}_\delta] \int_0^T \delta^{-2} \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}^2 dt. \end{aligned}$$

By the functional calculus, both the operator cosine and sine are self-adjoint on $L^2(\Lambda)$, and the scaling property Lemma 3.1 of the operator sine and cosine yields

$$\begin{aligned} \delta^{-1} \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} &= \delta^{-1} \langle u_0, C_\vartheta(t)\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} \\ &= \delta^{-1} \langle u_0, (C_{\vartheta,\delta}(\delta^{-1}t)\beta^{(\delta)})_\delta \rangle_{L^2(\Lambda)} \\ &= \delta^{-1} \langle (u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}, \end{aligned}$$

and

$$\delta^{-1} \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} = \langle (v_0)_{\delta^{-1}}, S_{\vartheta,\delta}(\delta^{-1}t)\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}.$$

Choose $1/2 > \gamma > 3/4 - \alpha/2$ and observe by the fractional rescaling property, for instance in Altmeyer, Cialenco, and Pasemann [7, Lemma 16], that

$$\begin{aligned} & \delta^{-1} \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} \\ &= \delta^{-1} \langle (-A_{\vartheta,\delta})^{1-\gamma}(u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1+\gamma}\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \delta^{2(1-\gamma)-1} \langle ((-A_\vartheta)^{1-\gamma}u_0)_{\delta^{-1}}, C_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1+\gamma}\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}, \end{aligned}$$

and

$$\begin{aligned} & \delta^{-1} \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} \\ &= \langle (-A_{\vartheta,\delta})^{1/2}(v_0)_{\delta^{-1}}, S_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1/2}\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)} \\ &= \delta \langle ((-A_\vartheta)^{1/2}v_0)_{\delta^{-1}}, S_{\vartheta,\delta}(\delta^{-1}t)(-A_{\vartheta,\delta})^{-1/2}\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}. \end{aligned}$$

Another application of the Cauchy-Schwarz inequality amounts to

$$\begin{aligned} & |\delta^{-1} \langle C_\vartheta(t)u_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}| \\ & \leq \delta^{1-2\gamma} \|((-A_\vartheta)^{1-\gamma}u_0)_{\delta^{-1}}\|_{L^2(\Lambda_\delta)} \|C_\vartheta(\delta^{-1}t)\|_{\mathcal{L}(L^2(\Lambda_\delta))} \|(-A_{\vartheta,\delta})^{-1+\gamma}\beta^{(\delta)}\|_{L^2(\Lambda_\delta)}, \end{aligned} \tag{A.48}$$

and

$$\begin{aligned} & |\delta^{-1} \langle S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}| \\ & \leq \delta \|(-A_\vartheta)^{1/2}v_0\|_{L^2(\Lambda)} \|S_{\vartheta,\delta}(\delta^{-1}t)\|_{\mathcal{L}(L^2(\Lambda_\delta))} \|(-A_{\vartheta,\delta})^{-1/2}\beta^{(\delta)}\|_{L^2(\Lambda_\delta)}. \end{aligned} \tag{A.49}$$

All norms in (A.48) and (A.49) remain bounded as $\delta \rightarrow 0$ by Lemma A.9 and the rescaling property. As $\gamma < \frac{1}{2}$, we notice that $1 - 2\gamma > 0$. Thus, both terms (A.48) and (A.49) and subsequently in (A.47) converge to zero.

Step 4 (Convergence of $V_{2,\delta}$). Using Cauchy-Schwarz, we obtain for the deterministic part:

$$\begin{aligned}
& |V_{2,\delta}|^2 \\
&= \left| \int_0^T \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \delta^{-2}(\Delta K)_\delta \rangle_{L^2(\Lambda)} \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} dt \right|^2 \\
&\leq \int_0^T \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \delta^{-1}(\Delta K)_\delta \rangle_{L^2(\Lambda)}^2 dt \int_0^T \langle C_\vartheta(t)u_0 + S_\vartheta(t)v_0, \delta^{-1}\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}^2 dt.
\end{aligned} \tag{A.50}$$

By (A.35) in Lemma A.14 the first factor in (A.50) remains asymptotically bounded. Thus, the convergence follows exactly as for (A.47) from the upper bounds (A.49) and (A.48).

Step 5 (Convergence of $V_{1,\delta}$). Using Cauchy-Schwarz we obtain

$$\begin{aligned}
& \mathbb{E}[|V_{1,\delta}|^2] = \text{Var}(V_{1,\delta}) \\
&= \int_0^T \int_0^T \text{Cov} \left((\mathcal{L}_\delta^C(t) + \mathcal{L}_\delta^S(t)) \langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}, \right. \\
&\quad \left. (\mathcal{L}_\delta^C(s) + \mathcal{L}_\delta^S(s)) \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)} \right) ds dt \\
&\lesssim \left(\int_0^T \int_0^T |\text{Cov}(\langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}, \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)})| ds dt \right)^{1/2} \|\mathcal{L}_\delta^C + \mathcal{L}_\delta^S\|_T^2.
\end{aligned}$$

We may upper bound this covariance through

$$\begin{aligned}
& |\text{Cov}(\langle \tilde{u}(t), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}, \langle \tilde{u}(s), \beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)})| \\
&\leq \int_0^{t \wedge s} |\langle S_\vartheta(t-r)\beta_\delta^{(\delta)}, S_\vartheta(s-r)\beta_\delta^{(\delta)} \rangle_{L^2(\Lambda)}| dr \\
&= \delta^2 \int_0^{t \wedge s} |\langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\beta^{(\delta)}, S_{\vartheta,\delta}(\delta^{-1}(s-r))\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}| dr.
\end{aligned} \tag{A.51}$$

Using (A.29) in Lemma A.12 and the Cauchy-Schwarz inequality, we obtain the upper bound

$$\begin{aligned}
& \sup_{0 < \delta \leq 1} |\langle S_{\vartheta,\delta}(\delta^{-1}(t-r))\beta^{(\delta)}, S_{\vartheta,\delta}(\delta^{-1}(s-r))\beta^{(\delta)} \rangle_{L^2(\Lambda_\delta)}| \\
&\leq \left(\sup_{0 < \delta \leq 1} \|(-A_{\vartheta,\delta})^{-1/2}\beta^{(\delta)}\|_{L^2(\mathbb{R}^d)} \right)^2 < \infty.
\end{aligned} \tag{A.52}$$

Using the upper bound (A.52) the dominated convergence theorem and (A.27) in Proposition A.11, the integral (A.51) converges to zero. Consequently, the bound Lemma A.14 shows $\text{Var}(V_{1,\delta}) \rightarrow 0$ as $\delta \rightarrow 0$ since the other factor remains bounded as $\delta \rightarrow 0$ by (A.35) in Lemma A.14. \square

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