## HUNGARIAN CUBES

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> AbSTRACT. We prove the consistency of the relation $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ when $\lambda<\mu=\operatorname{cf}(\mu)<\nu=\operatorname{cf}(\nu) \leq 2^{\mu}$.

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## 0. Introduction

The polarized cube relation $\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right) \rightarrow\left(\begin{array}{l}\varepsilon \\ \zeta \\ \eta\end{array}\right)$ says that for every coloring $d: \alpha \times \beta \times \gamma \rightarrow 2$ one can find $A \subseteq \alpha, B \subseteq \beta$ and $C \subseteq \gamma$ such that $\operatorname{otp}(A)=\varepsilon, \operatorname{otp}(B)=\zeta, \operatorname{otp}(C)=\eta$ and $d \upharpoonright(A \times B \times C)$ is constant. In his famous remark from [dF70], Pierre de Fermat says that Cubum autem in duos cubos, that is: a cube cannot split into two cubes. Though true for natural numbers, as proved by Wiles in [Wi195], it may not hold at infinite cardinals. Our goal is to prove positive combinatorial relations for cubes whose edges are infinite cardinals.

A necessary condition for a positive cube relation is the positive standard relation at all three pairs of cardinals mentioned in the cube. Thus if $\binom{\alpha}{\beta} \nrightarrow$ $\binom{\varepsilon}{\zeta}$ or $\binom{\beta}{\gamma} \nrightarrow\binom{\zeta}{\eta}$ or $\binom{\alpha}{\gamma} \nrightarrow\binom{\varepsilon}{\eta}$ then $\left(\begin{array}{l}\alpha \\ \beta \\ \gamma\end{array}\right) \nrightarrow\left(\begin{array}{l}\varepsilon \\ \zeta \\ \eta\end{array}\right)$. Moreover, positive relations at these pairs are necessary but insufficient and usually something very strong is needed for a positive cube relation in non-trivial cases.

The case $\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right) \rightarrow\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right)$ in which the required size of the monochromatic cube equals that of the domain of the coloring is called the strong cube relation. Since $\binom{\kappa}{\kappa} \leftrightarrow\binom{\kappa}{\kappa}$ we see that the strong relation $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ is possible only if $\lambda, \mu, \nu$ are all distinct. Our convention is that $\lambda \leq \mu \leq \nu$ so we may assume from now on that $\lambda<\mu<\nu$. If $\nu=\operatorname{cf}(\nu)>2^{\mu}$ then $\nu$ adds nothing to the validity of the cube relation and $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\binom{\nu}{\mu}$ iff $\binom{\mu}{\lambda} \rightarrow\binom{\mu}{\lambda}$ thus the cube problem reduces to a standard polarized relation.

The opposite situation in which $2^{\lambda}$ is relatively small does not reduce immediately to the case $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$. An easy case is $2^{\lambda}<\kappa<\mu<\nu$ where $\kappa$ is strongly compact and $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$ can be witnessed by sets which belong to some prescribed $\lambda$-complete ultrafilters over $\mu$ and $\nu$. In this case one has $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ but without this assumption the cube relation is more involved even if $2^{\lambda}$ is small. It seems that one has to distinguish between the case of $2^{\lambda}<\mu$ and the case of $2^{\lambda} \geq \mu$, and in both cases the cube relation $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ seems to be interesting.

The most challenging case seems to be $\nu \leq 2^{\lambda}$, see Question 4.1. In this paper we obtain the strong cube relation $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ where $\nu \leq 2^{\mu}$ but $2^{\lambda}$ is relatively small. We indicate, however, that the combinatorial theorem that we prove is phrased in a general way and might be applied to cases in which $\nu \leq 2^{\lambda}$ provided that the pertinent assumptions are forceable.

We consider in this paper only strong relations, namely relations of the form $\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right) \rightarrow\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right)$. The literature concerning cube polarized relations is sparse. An old negative relation under GCH appeared in [Sie51]. A famous problem from [EH71] is whether $\left(\begin{array}{l}\aleph_{1} \\ \aleph_{1} \\ \aleph_{1}\end{array}\right) \rightarrow\left(\begin{array}{l}\aleph_{0} \\ \aleph_{0} \\ \aleph_{0}\end{array}\right)$ and more generally whether
$\left(\begin{array}{c}\kappa^{+} \\ \kappa^{+} \\ \kappa^{+}\end{array}\right) \rightarrow\left(\begin{array}{c}\kappa \\ \kappa \\ \kappa\end{array}\right)$ for some $\kappa$, see [Wil77, p. 110]. However, cube relations in which the size of the monochromatic product is smaller than the size of the domain will not be discussed in the current paper.

Our notation is mostly standard, and follows [EHMR84]. We use the Jerusalem forcing notation, so we force upwards. We suggest [AM10] and [She94] for background in pcf theory and [Wil77] for basic background in partition calculus.

## 1. Preliminaries

In this section we garner some information needed for the forcing constructions of Theorem 3.3 and Proposition 4.2. The material in this section is either standard or appears explicitly in the papers to be cited, so this section can be skipped over by the cognoscenti.

Let $\kappa$ be a strongly compact cardinal. The defining property which stands behind the name of this cardinal is captured in the fact that the logic $\mathcal{L}_{\kappa \kappa}$ satisfies the $\kappa$-compactness theorem, but these cardinals can be characterized combinatorially using ultrafilters. We shall use the fact that $\kappa$ is strongly compact iff every $\kappa$-complete filter over every set $S$ extends to a $\kappa$-complete ultrafilter.

Strong compactness is a global property, as it says something about every cardinal above the strongly compact cardinal. We indicate that in our context only a local bounded version of strong compactness is needed. That is, we do not need the extension property of filters at every set $S$, but rather at a specific point. Thus the consistency strength of our results can be somewhat reduced.

A filter $\mathscr{F}$ over $\kappa$ is $\theta$-indecomposable iff for every partition $\left(A_{\alpha}: \alpha \in \theta\right)$ of $\kappa$ one can find an index set $I \in[\theta]^{<\theta}$ so that $\bigcup_{\alpha \in I} A_{\alpha} \in \mathscr{F}$. This property becomes powerful if one also requires some degree of completeness. Consider a measurable cardinal $\kappa$ and a normal ultrafilter $\mathscr{U}$ over $\kappa$. One can verify that $\mathscr{U}$ is $\theta$-indecomposable for every $\theta \in \kappa$. However, if $\theta$ is infinite then one can force the existence of $\theta$-indecomposable ultrafilters over non-measurable cardinals.

It has been proved in [She83] that one can force $\omega$-indecomposable ultrafilters over strongly inaccessible but not weakly compact cardinals. From large cardinal assumptions one can force various types of indecomposable ultrafilters over successor cardinals, see [BDM86]. These, and similar facts, were employed by Raghavan and Shelah in [RS20] for proving the consistency of $\mathfrak{u}_{\kappa}<2^{\kappa}$ where $\kappa$ is accessible. Let us recall the basic concept of $(\lambda, \kappa, \mu)$-filtration, labeled as [RS20, Definition 5]. We omitted the filter $\mathscr{D}$ from the definition, but this makes no essential difference.

Definition 1.1. Filtrations.
A forcing notion $\mathbb{P}$ has $\left(\lambda, \kappa, \mu^{\prime}\right)$-filtration iff there exists a sequence of suborderings $\left(\mathbb{P}_{\alpha}: \alpha \in \mu^{\prime}\right)$ with the following properties:
(a) $\aleph_{0} \leq \lambda=\lambda^{<\lambda}<\operatorname{cf}\left(\mu^{\prime}\right)<\kappa<\mu^{\prime}$.
(b) $\mu^{\prime}$ is a strong limit cardinal.
(c) $\mathbb{P}$ is $\lambda^{+}$-cc.
(d) $\bigcup_{\alpha \in \mu^{\prime}} \mathbb{P}_{\alpha}=\mathbb{P}$.
(e) $\forall \alpha \in \mu^{\prime}, \mathbb{P}_{\alpha} \subseteq_{\mathrm{c}} \mathbb{P}$ and $\alpha<\beta<\mu^{\prime} \Rightarrow \mathbb{P}_{\alpha} \subseteq \mathbb{P}_{\beta}$.
$(f) \forall \alpha \in \mu^{\prime},\left|\mathbb{P}_{\alpha}\right|<\mu^{\prime}$.

Suppose that $\kappa$ is a regular cardinal. Raghavan and Shelah proved that if $\mathbb{P}$ has $\left(\lambda, \kappa, \mu^{\prime}\right)$-filtration and $\mathscr{D}$ is a $\operatorname{cf}\left(\mu^{\prime}\right)$-indecomposable uniform filter over $\kappa$ then every uniform ultrafilter $\mathscr{U}$ that extends $\mathscr{D}$ in the generic extension by $\mathbb{P}$ is generated by a set of size at most $\mu^{\prime}$, see [RS20, Theorem 7].

This statement is useful if one wishes to force $\mathfrak{u}_{\kappa} \leq \mu^{\prime}<2^{\kappa}$. In fact, simple instances of Cohen forcing which blow up $2^{\kappa}$ satisfy the requirements of Definition 1.1. One important feature for our construction is that one can force this with adding relatively large Cohen sets. In particular, if $\kappa$ is a Laver-indestructible supercompact cardinal and $\mu=\operatorname{cf}(\mu)>\kappa^{+}$then one can force $\mathfrak{u}_{\mu}<2^{\mu}$ by adding many Cohen sets of size greater than $\kappa$ and thus preserving the supercompactness of $\kappa$. Let us give a formal shape to the above discussion.

Proposition 1.2. Assume that:
(a) $\kappa$ is supercompact and Laver-indestructible.
(b) $\kappa^{+}<\theta=\operatorname{cf}(\theta)<\mu=\operatorname{cf}(\mu)$.
(c) $\mu^{\prime}$ is a strong limit singular cardinal.
(d) $\theta=\operatorname{cf}\left(\mu^{\prime}\right)$ and $\mu<\mu^{\prime}$.
(e) $\nu=\operatorname{cf}(\nu)>\mu^{\prime}$.

Then one can force $\mathfrak{u}_{\mu}<2^{\mu}=\nu$ by adding $\nu$-many Cohen subsets of $\theta$ thus preserving the supercompactness of $\kappa$ in the generic extension, provided that there exists a $\theta$-indecomposable filter over $\mu$ in the ground model.

We turn now to a short discussion concerning Mathias forcing from [Mat77] and the generalized Mathias forcing from [GS12b, Definition 3.1]. Let $\mathscr{U}$ be a non-principal ultrafilter over $\omega$. Mathias forcing relativized to $\mathscr{U}$ is denoted by $\mathbb{M}_{\mathscr{U}}$. A condition $p \in \mathbb{M}_{\mathscr{U}}$ is a pair $\left(s^{p}, A^{p}\right)$ where $s^{p} \in[\omega]^{<\omega}, A^{p} \in \mathscr{U}$ and $\max \left(s^{p}\right)<\min \left(A^{p}\right)$. If $p, q \in \mathbb{M}_{\mathscr{U}}$ then $p \leq q$ iff $s^{p} \subseteq s^{q}, A^{p} \supseteq A^{q}$ and $s^{q}-s^{p} \subseteq A^{p}$. If $G \subseteq \mathbb{M}_{\mathscr{U}}$ is $V$-generic then $\bigcup\left\{s^{p}: p \in G\right\}$ is a subset of $\omega$ called a Mathias real.

The advantage of relativizing Mathias forcing to an ultrafilter is that it becomes ccc and hence can be iterated with a finite support without collapsing cardinals. If one iterates $\mathbb{M}_{\mathscr{U}}$ over a model of CH then the splitting number $\mathfrak{s}$ becomes the (cofinality of the) length of the iteration. Hence in $V[G]$ if $\omega<\operatorname{cf}(\mu) \leq \mu<\mathfrak{s}$ then $\binom{\mu}{\omega} \rightarrow\binom{\mu}{\omega}$. But this forcing notion has a similar effect on cardinals above $\mathfrak{s}$. If one begins the iteration with blowing up $2^{\omega}$ to $\nu$ and then iterates $\mathbb{M}_{\mathscr{U}}$ say $\lambda$-many times where $\omega<\lambda=\operatorname{cf}(\lambda)<\nu$ then $\binom{\mu}{\omega} \rightarrow\binom{\mu}{\omega}$ holds in $V[G]$ for every $\lambda<\operatorname{cf}(\mu) \leq \mu \leq \nu$. The reason is that the $\lambda$-sequence of Mathias reals forms a $\subseteq^{*}$-decreasing sequence of subsets of $\omega$ and thus generates a non-principal ultrafilter over $\omega$ whose character is smaller than $\mu$. In particular, if $2^{\omega}=\nu$ and $\operatorname{cf}(\nu)>\lambda$ then $\binom{\nu}{\omega} \rightarrow\binom{\nu}{\omega}$ holds in $V[G]$. This will be used in Claim 3.1.

Mathias forcing generalizes to uncountable cardinals in lieu of $\omega$. Suppose that $\lambda$ is measurable and $\mathscr{U}$ is a $\lambda$-complete ultrafilter over $\lambda$. A single step of the generalized Mathias forcing consists of conditions of the form $\left(s^{p}, A^{p}\right)$
where $s^{p} \in[\lambda]^{<\lambda}, A^{p} \in \mathscr{U}$ and $\sup \left(s^{p}\right)<\min \left(A^{p}\right)$. The $\lambda$-completeness of $\mathscr{U}$ is needed for $\mathbb{M}_{\mathscr{U}}$ to be $\lambda$-closed, hence measurability is indispensable.

In order to iterate $\mathbb{M}_{\mathscr{U}}$ one has to ensure that $\lambda$ remains measurable at each step of the iteration. If one begins with a Laver indestructible supercompact cardinal $\lambda$ then this can be done since $\mathbb{M}_{\mathscr{U}}$ will be $\lambda$-directedclosed. If one forces $2^{\lambda}=\nu=\operatorname{cf}(\nu)$ and then iterates $\mathbb{M}_{\mathscr{U}} \tau$-many times where $\tau=\operatorname{cf}(\tau)<\nu$ then one obtains $\binom{\nu}{\lambda} \rightarrow\binom{\nu}{\lambda}$ in the generic extension (and concomitantly $\binom{\mu}{\lambda} \rightarrow\binom{\mu}{\lambda}$ if $\left.\lambda<\mu=\operatorname{cf}(\mu)<\tau\right)$. We indicate that for iterating this type of Mathias forcing one has to choose (a name of) an ultrafilter at each step. Let us summarize the above discussion.
Corollary 1.3. Let $\lambda$ be a Laver-indestructible supercompact cardinal, and suppose that $\lambda<\tau=\operatorname{cf}(\tau)<\nu=\operatorname{cf}(\nu)$. Let $\mathbb{Q}$ be a $(<\lambda)$-support iteration of $\mathbb{M}_{\mathscr{O}}$ where $\mathscr{U}$ is a non-principal $\lambda$-complete ultrafilter over $\lambda$, the first step of $\mathbb{Q}$ forces $2^{\lambda}=\nu$ and the length of the iteration is $\tau$. Then $\binom{\mu}{\lambda} \rightarrow\binom{\mu}{\lambda}$ for every $\lambda<\mu=\operatorname{cf}(\mu)<\tau$ and every $\tau<\mu=\operatorname{cf}(\mu) \leq \nu$.

We conclude this section with a short description of a forcing notion which will be used within the proof of Theorem 3.3. This forcing notion comes from [GS12a] and it provides a good control of true cofinalities of certain sequences.
Definition 1.4. Let $\lambda$ be a supercompact cardinal. Let $\bar{\theta}=\left\langle\theta_{\alpha}: \alpha<\lambda\right\rangle$ be an increasing sequence of regular cardinals so that $2^{|\alpha|+\aleph_{0}}<\theta_{\alpha}<\lambda$ for every $\alpha<\lambda$.
(※) $p \in \mathbb{Q}_{\bar{\theta}}$ iff:
(a) $p=(\eta, f)=\left(\eta^{p}, f^{p}\right)$,
(b) $\lg (\eta)<\lambda$,
(c) $\eta \in \prod\left\{\theta_{\zeta}: \zeta<\ell g(\eta)\right\}$,
(d) $f \in \prod\left\{\theta_{\zeta}: \zeta<\lambda\right\}$,
(e) $\eta \triangleleft f$ (i.e., $\eta(\zeta)=f(\zeta)$ for every $\zeta<\ell g(\eta))$.
(コ) $p \leq_{\mathbb{Q}_{\bar{\theta}}} q$ iff $\left(p, q \in \mathbb{Q}_{\bar{\theta}}\right.$ and $)$
(a) $\eta^{p} \unlhd \eta^{q}$,
(b) $f^{p}(\varepsilon) \leq f^{q}(\varepsilon)$, for every $\varepsilon<\lambda$.

As proved in [GS12a], one can iterate $\mathbb{Q}_{\bar{\theta}}$ while increasing $2^{\lambda}$. Moreover, one can choose different sequences of the form $\bar{\theta}$ along the iteration. If $\lambda$ is singularized at the end then one obtains different sequences with different prescribed true cofinalities.

## 2. Terraced cubes and echeloned cubes

For an infinite cardinal $\lambda$ let us call the cube relation $\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right)$ the terraced cube relation at $\lambda$. Under AD the terraced relation holds at many infinite cardinals as proved in [Gar20]. In ZFC it is unknown whether $\binom{\lambda^{++}}{\lambda^{+}} \rightarrow\binom{\lambda^{++}}{\lambda^{+}}$is consistent for some $\lambda$, so we do not have a model in which we can even start checking the possibility of the positive terraced cube relation at $\lambda$.

It has been proved in [EHR65] that $2^{\lambda}=\lambda^{+} \operatorname{implies}\binom{\lambda^{+}}{\lambda} \nrightarrow\binom{\lambda^{+}}{\lambda}$, so a necessary assumption for the terraced relation is $2^{\lambda}>\lambda^{+}$and $2^{\lambda^{+}}>\lambda^{++}$. In this section we show that in some sense these assumptions are far from being sufficient. We shall prove that if $2^{\lambda}=\lambda^{++}$(so the necessary $2^{\lambda}>\lambda^{+}$ is satisfied) then $\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right) \nrightarrow\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right)$ no matter how large is $2^{\lambda^{+}}$or any other property of $\lambda^{+}$.

Theorem 2.1. Let $\lambda$ be an infinite cardinal.
If $2^{\lambda} \leq \lambda^{++}$then $\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right) \nrightarrow\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right)$.

Proof.
If $2^{\lambda}=\lambda^{+}$then $\binom{\lambda^{+}}{\lambda} \nrightarrow\binom{\lambda^{+}}{\lambda}$ and the failure of the terraced relation follows. Assume, therefore, that $2^{\lambda}=\lambda^{++}$and let $\left\{C_{\zeta}: \zeta \in \lambda^{++}\right\}$be an enumeration of $[\lambda]^{\lambda}$. For every $\alpha \in \lambda^{++}$let $\mathcal{F}_{\alpha}=\left\{C_{\zeta}: \zeta \in \alpha\right\}$, so $\left|\mathcal{F}_{\alpha}\right| \leq \lambda^{+}$. Let $\left\{C_{\zeta}^{\alpha}: \zeta \in \lambda^{+}\right\}$be an enumeration of $\mathcal{F}_{\alpha}$ of order type $\lambda^{+}$, using repetitions if needed. We may also assume, without loss of generality, that $\bigcup \mathcal{F}_{\alpha}=\lambda$.

By induction on $\alpha \in \lambda^{++}$we define $d_{\alpha}:\{\alpha\} \times \lambda^{+} \times \lambda \rightarrow 2$. Arriving at the ordinal $\alpha$ we define for every $\beta \in \lambda^{+}$the family $\mathcal{G}_{\beta}^{\alpha}=\left\{C_{\zeta}^{\alpha}: \zeta \in \beta\right\}$. Since $|\beta| \leq \lambda$ we can reenumerate the elements of $\mathcal{G}_{\beta}^{\alpha}$ by $\left\{C_{\beta \eta}^{\alpha}: \eta \in \lambda\right\}$, possibly with repetitions.

Now for every $\beta \in \lambda^{+}$we choose by induction on $\eta \in \lambda$ two distinct elements $i_{\beta \eta}^{\alpha}, j_{\beta \eta}^{\alpha} \in C_{\beta \eta}^{\alpha}$ so that $i_{\beta \eta}^{\alpha}, j_{\beta \eta}^{\alpha} \notin\left\{i_{\beta \sigma}^{\alpha}, j_{\beta \sigma}^{\alpha}: \sigma \in \eta\right\}$. The choice is possible since $\left|C_{\beta \eta}^{\alpha}\right|=\lambda$ and $\eta \in \lambda$. We define $d_{\alpha}\left(\alpha, \beta, i_{\beta \eta}^{\alpha}\right)=0$ and $d_{\alpha}\left(\alpha, \beta, j_{\beta \eta}^{\alpha}\right)=1$. At the end of the process, if $d_{\alpha}(\alpha, \gamma, \delta)$ is not defined yet for some $(\gamma, \delta) \in \lambda^{+} \times \lambda$ then we let $d_{\alpha}(\alpha, \gamma, \delta)=0$.

Define $d: \lambda^{++} \times \lambda^{+} \times \lambda \rightarrow 2$ by $d=\bigcup\left\{d_{\alpha}: \alpha \in \lambda^{++}\right\}$. Suppose that $A \in\left[\lambda^{++}\right]^{\lambda^{++}}, B \in\left[\lambda^{+}\right]^{\lambda^{+}}$and $C \in[\lambda]^{\lambda}$. Choose $\alpha \in A$ such that $C \in \mathcal{F}_{\alpha}$, so $C=C_{\zeta}^{\alpha}$ for some $\zeta \in \lambda^{+}$. Choose $\beta \in B$ such that $C_{\zeta}^{\alpha} \in \mathcal{G}_{\beta}^{\alpha}$, so $C=$ $C_{\zeta}^{\alpha}=C_{\beta \eta}^{\alpha}$ for some $\eta \in \lambda$. By the construction of $d$ there are $i_{\beta \eta}^{\alpha}, j_{\beta \eta}^{\alpha} \in C$ so that $d_{\alpha}\left(\alpha, \beta, i_{\beta \eta}^{\alpha}\right)=0$ and $d_{\alpha}\left(\alpha, \beta, j_{\beta \eta}^{\alpha}\right)=1$, hence $d^{\prime \prime}(A \times B \times C)=\{0,1\}$ as required.

Notice that $\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda^{++}\end{array}\right) \nrightarrow\left(\begin{array}{c}\lambda^{++} \\ \lambda^{+} \\ \lambda\end{array}\right)$ follows from $2^{\lambda} \leq \lambda^{++}$by a similar argument. One has to replace $[\lambda]^{\lambda}$ in the above proof by $\left[\lambda^{++}\right]^{\lambda}$ whose size is $\lambda^{++}$under the assumption $2^{\lambda} \leq \lambda^{++}$. Another generalization of the above theorem gives a negative $n$-cube relation whenever $2^{\lambda}<\lambda^{+n}$. We phrase the following:

Question 2.2. Is the positive strong terraced relation consistent with ZFC for some infinite cardinal $\lambda$ ?

We make the comment that if one can prove the negative terraced relation at $\lambda$ under the assumption that $2^{\lambda}=\lambda^{+3}$ then a negative answer to the above problem will be proved. Indeed, if $2^{\lambda}>\lambda^{+3}$ and the positive terraced relation holds at $\lambda$ then it will be preserved upon collapsing $2^{\lambda}$ to $\lambda^{+3}$.

We mentioned the fact that the negative relation $\binom{\lambda^{+}}{\lambda} \nrightarrow\binom{\lambda^{+}}{\lambda}$ follows from the assumption $2^{\lambda}=\lambda^{+}$, as proved by Erdős, Hajnal and Rado in [EHR65]. Naturally, they asked whether one can replace $\lambda^{+}$by $2^{\lambda}$ and prove, in ZFC, that $\binom{2^{\lambda}}{\lambda} \nrightarrow\binom{2^{\lambda}}{\lambda}$. It turned out that the answer is negative, and the positive relation $\binom{2^{\lambda}}{\lambda} \rightarrow\binom{2^{\lambda}}{\lambda}$ is consistent in many cases. An example of this positive relation will be seen later in the paper.

In the light of the negative terraced relation of Theorem 2.1 one may wonder about cube relations where the small parameter is $\lambda$ and the large parameter $\nu$ is above $\lambda^{++}$. In the next section we prove a combinatorial theorem which suggests a way to produce a positive relation. This will be followed by a forcing construction which gives one type of such a relation, where $2^{\lambda}$ is relatively small.

We move now to a generalized form of terraced relations. Suppose that $\lambda<\mu<\nu \leq 2^{\mu}$. A cube relation of the form $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\alpha \\ \beta \\ \gamma\end{array}\right)$ will be called an echeloned cube relation. As usual, if $\alpha=\nu, \beta=\mu$ and $\gamma=\lambda$ then the relation will be called strong. Strong terraced cube relations are the special case in which $\mu=\lambda^{+}$and $\nu=\lambda^{++}$. As in the previous section we focus on strong relations of the form $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$, namely the size of the monochromatic cube is the size of the domain of the coloring.

We shall prove that under large cardinal assumptions one obtains the consistency of a positive strong echeloned cube relation over $\lambda$. In the light of the result from the previous section, a reasonable assumption to begin with is $\nu>\lambda^{++}$. This will be discussed at the end of the proof. The small component $\lambda$ in our theorem will be a strong limit singular cardinal. In this section we shall prove that under some combinatorial assumptions one has a positive echeloned relation. In the next section we shall see that these assumptions are forceable.

In the theorem below, the cofinality of $\lambda$ is $\omega$. Countable cofinality is not essential for the main result, but it facilitates the argument since one can
assume GCH below $\lambda$ and yet increase $2^{\lambda}$. If one wishes to prove a similar result at singular cardinals with uncountable cofinality then GCH should be assumed at some appropriate sparse sequence of cardinals below $\lambda$. This issue will be dicussed briefly at the end of this section. The proof of the main theorem of this section is a modification of [GS16, Theorem 5.1]. However, we need a bit more in order to get the cube relation.

Theorem 2.3. Assume that:
(a) $\lambda>\operatorname{cf}(\lambda)=\omega$ and GCH holds below $\lambda$.
(b) $\lambda<\mu=\operatorname{cf}(\mu)<\nu=\operatorname{cf}(\nu) \leq 2^{\mu}$.
(c) $\left(\lambda_{n}: n \in \omega\right)$ is an increasing sequence of regular cardinals so that $\lambda=\bigcup_{n \in \omega} \lambda_{n}$.
(d) $\Upsilon_{\ell}=\operatorname{tcf}\left(\prod_{n \in \omega} \lambda_{n}^{+\ell}, J_{\omega}^{\mathrm{bd}}\right)$ for $\ell \in\{0,1,2\}$.
(e) $\mu \neq \Upsilon_{\ell}$ and $\nu>\Upsilon_{\ell}$ for $\ell \in\{0,1,2\}$.
(f) $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\aleph_{1}}$.

Then $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$.
Proof.
Suppose that $d: \nu \times \mu \times \lambda \rightarrow 2$ is given. We fix three scales as follows. Let $\bar{f}=\left(f_{\alpha}: \alpha \in \Upsilon_{2}\right)$ be a scale in $\prod_{n \in \omega} \lambda_{n}^{++}$, let $\bar{g}=\left(g_{\varepsilon}: \varepsilon \in \Upsilon_{1}\right)$ be a scale in $\prod_{n \in \omega} \lambda_{n}^{+}$and let $\bar{h}=\left(h_{\delta}: \delta \in \Upsilon_{0}\right)$ be a scale in $\prod_{n \in \omega} \lambda_{n}$. For every $\alpha \in \nu, \beta \in \mu, n \in \omega$ and $i \in\{0,1\}$, define:

$$
A_{\alpha \beta n}^{i}=\left\{\gamma \in \lambda_{n}^{+}: d(\alpha, \beta, \gamma)=i\right\}
$$

For every $n \in \omega$ let $\left\{S_{j}^{n}: j \in \lambda_{n}^{++}\right\}$be an enumeration of $\mathcal{P}\left(\lambda_{n}^{+}\right)$. For every $\alpha \in \nu, \beta \in \mu, i \in\{0,1\}$ we define a function $f_{\alpha \beta}^{i} \in \prod_{n \in \omega} \lambda_{n}^{++}$by $f_{\alpha \beta}^{i}(n)=\min \left\{j \in \lambda_{n}^{++}: A_{\alpha \beta n}^{i}=S_{j}^{n}\right\}$. The color $i$ can be removed by defining $f_{\alpha \beta}(n)=\max \left\{f_{\alpha \beta}^{i}(n): i<2\right\}$.

Thus we have defined $\mu$ functions for each $\alpha \in \nu$ of the form $f_{\alpha \beta}$. Applying assumption (e) we can choose $f_{\alpha} \in \bar{f}$ so that $\beta \in \mu \Rightarrow f_{\alpha \beta}<J_{\omega}^{\text {bd }} f_{\alpha}$, and we do this for every $\alpha \in \nu$. Since $\Upsilon_{2}<\nu=\operatorname{cf}(\nu)$, there exists $A_{0} \in[\nu]^{\nu}$ and a fixed $f \in \bar{f}$ so that $\alpha \in A_{0} \Rightarrow f_{\alpha}=f$. By increasing each $f(n)$ if needed we may assume that $f(n) \geq \lambda_{n}^{+}$for every $n \in \omega$.

For every $\alpha \in \nu, \beta \in \mu$ we choose $n_{\alpha \beta} \in \omega$ so that $n \geq n_{\alpha \beta} \Rightarrow f_{\alpha \beta}(n)<$ $f(n)$. The mapping $(\alpha, \beta) \mapsto n_{\alpha \beta}$ admits a monochromatic product of size $\nu \times \mu$ by virtue of $(f)$, so choose $A_{1} \in\left[A_{0}\right]^{\nu}, B_{1} \in[\mu]^{\mu}$ and $n_{1} \in \omega$ such that $(\alpha, \beta) \in A_{1} \times B_{1} \Rightarrow n_{\alpha \beta}=n_{1}$.

Observe that $\mid\left\{S_{j}^{n}: j \in f(n) \mid=\lambda_{n}^{+}\right.$for every $n \in \omega$, so we can reenumerate these sets by $\left\{T_{j}^{n}: j \in \lambda_{n}^{+}\right\}$. For every $\alpha \in A_{1}, \beta \in B_{1}$ and $i \in\{0,1\}$ define $g_{\alpha \beta}^{i} \in \prod_{n \in \omega} \lambda_{n}^{+}$by $g_{\alpha \beta}^{i}(n)=\min \left\{j \in \lambda_{n}^{+}: A_{\alpha \beta n}^{i}=T_{j}^{n}\right\}$ and then let $g_{\alpha \beta}(n)=\max \left\{g_{\alpha \beta}^{i}(n): i<2\right\}$ for every $n \in \omega$. By the same reasoning as before we choose $A_{2} \in\left[A_{1}\right]^{\nu}, B_{2} \in\left[B_{1}\right]^{\mu}, n_{2} \in \omega$ and $g \in \bar{g}$ such that $\alpha \in A_{2}, \beta \in B_{2} \Rightarrow g_{\alpha \beta}<_{J_{\omega}^{\mathrm{bd}}} g$. Moreover, $g(n) \geq \lambda_{n}$ for every $n \in \omega, n_{1} \leq n_{2}$
and $g_{\alpha \beta}(n)<g(n)$ whenever $\alpha \in A_{2}, \beta \in B_{2}$ and $n \geq n_{2}$. These statements follow from the properties of $\bar{g}$ and assumption $(f)$.

We need another round of the same process, so we focus now on the set $\left\{T_{j}^{n}: j \in g(n)\right\}$ for every $n \in \omega$ whose size is $\lambda_{n}$. By reenumerating these sets as $\left\{W_{j}^{n}: j \in \lambda_{n}\right\}$ we define for each $\alpha \in A_{2}, \beta \in B_{2}$ and $i \in\{0,1\}$ the function $h_{\alpha \beta}^{i} \in \prod_{n \in \omega} \lambda_{n}$ in a similar fashion. Namely, $h_{\alpha \beta}^{i}(n)=\min \{j \in$ $\left.\lambda_{n}: A_{\alpha \beta n}^{i}=W_{j}^{n}\right\}$ and then $h_{\alpha \beta}(n)=\max \left\{h_{\alpha \beta}^{i}(n): i<2\right\}$ for every $n \in \omega$. At last, we shrink ourselves to $A_{3} \in\left[A_{2}\right]^{\nu}, B_{3} \in\left[B_{2}\right]^{\mu}$ and we choose $n_{2} \leq n_{3} \in \omega$ and $h \in \prod_{n \in \omega} \lambda_{n}, h \in \bar{h}$ so that $\alpha \in A_{3}, \beta \in B_{3} \Rightarrow h_{\alpha \beta}<_{J \omega \mathrm{bd}} h$ and $h_{\alpha \beta}(n)<h(n)$ whenever $\alpha \in A_{3}, \beta \in B_{3}$ and $n \geq n_{3}$.

For every $n \in \omega$ we define an equivalence relation $e_{n}$ on the ordinals of $\lambda_{n}^{+}$as follows:

$$
\gamma_{0} e_{n} \gamma_{1} \quad \text { iff } \quad \forall j \in h(n), \gamma_{0} \in W_{j}^{n} \Leftrightarrow \gamma_{1} \in W_{j}^{n} .
$$

Since we are assuming GCH below $\lambda$ we see that the number of equivalence classes of $e_{n}$ is less than $\lambda_{n}^{+}$. Hence one can choose for every $n \in \omega$ an equivalence class $E_{n}$ of $e_{n}$ and a color $i_{\alpha \beta}^{n} \in\{0,1\}$ for every $\alpha \in A_{3}, \beta \in B_{3}$ such that $\left|E_{n}\right|=\lambda_{n}^{+}$and $\gamma \in E_{n} \Rightarrow d(\alpha, \beta, \gamma)=i_{\alpha \beta}^{n}$.

For every $\alpha \in A_{3}, \beta \in B_{3}$ we choose an infinite set $u_{\alpha \beta} \subseteq \omega$ and a color $i_{\alpha \beta} \in\{0,1\}$ such that $n \in u_{\alpha \beta} \Rightarrow i_{\alpha \beta}^{n}=i_{\alpha \beta}$. Applying assumption $(f)$ we can find $A \in\left[A_{3}\right]^{\nu}, B \in\left[B_{3}\right]^{\mu}, u \in[\omega]^{\omega}$ and a fixed color $i \in\{0,1\}$ such that $\alpha \in A, \beta \in B, n \in u \Rightarrow i_{\alpha \beta}=i$. We indicate that here we use the fact that we have $\aleph_{1}$-many colors in assumption $(f)$. Let $C=\bigcup\left\{E_{n}: n_{3} \leq n \in u\right\}$, so $|C|=\mu$ as $u$ is unbounded in $\omega$. It follows that $d^{\prime \prime}(A \times B \times C)=\{i\}$, so the proof is accomplished.

The above theorem is based on several assumptions, and one may wonder whether these assumptions are forceable. In the next section we shall see that if there are enough supercompact cardinals in the ground model then the answer is positive.

We make the comment that one can modify the proof and incorporate singular cardinals with uncountable cofinality. The required changes are choosing an appropriate increasing sequence $\left(\lambda_{\varepsilon}: \varepsilon \in \operatorname{cf}(\lambda)\right)$, replacing GCH below $\lambda$ by GCH only at $\lambda_{\varepsilon}$ and $\lambda_{\varepsilon}^{+}$for every $\varepsilon \in \operatorname{cf}(\lambda)$, and increasing the number of colors in assumption $(f)$ from $\aleph_{1}$ to $2^{\operatorname{cf}(\lambda)}$.

## 3. Forcing our assumptions

As indicated in the introduction, in order to prove the consistency of $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ one has to make sure that all possible pairs satisfy the pertinent positive relation. The relations in which $\lambda$ is involved are relatively easy to force when $\lambda$ is a strong limit singular cardinal. This is done in [GS16, Theorem 5.1]. Actually, the proof of the main theorem in the previous section is based on the proof of that theorem, upon adding the assumption $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\aleph_{1}}$ and handling cubes. So basically one has to force the assumptions of [GS16, Theorem 5.1] and concomitantly the additional assumption (f).

The challenging relation which we need is the relation $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$, and what is more $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\aleph_{1}}$. If $\mu$ is a singular cardinal with small cofinality then $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$ will follow from $\binom{\nu}{\operatorname{cf}(\mu)} \rightarrow\binom{\nu}{\operatorname{cf}(\mu)}$ which can be easily arranged. However, this is not what the poet meant. Hence we are asking for a regular cardinal $\mu$ (and similarly, a regular cardinal $\nu$ ) with respect to our cube relation. In such cases, it is harder to force $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$.

It has been proved in [GS14] that if $\mathfrak{r}_{\mu}<\nu$ and $\operatorname{cf}(\nu)>\mathfrak{r}_{\mu}$ then $\binom{\nu}{\mu} \rightarrow$ $\binom{\nu}{\mu}$. However, we need more than two colors which makes life a bit more complicated. Rather than $\mathfrak{r}_{\mu}$ we shall work with $\mathfrak{u}_{\mu}$ (these characteristics are similar and actually it is unknown whether they can be separated where $\left.\mu>\aleph_{0}\right)$. Recall that a base of a uniform ultrafilter $\mathscr{U}$ over $\mu$ is a subset $\mathcal{B}$ of $\mathscr{U}$ such that for every $A \in \mathscr{U}$ one can find $B \in \mathcal{B}$ for which $B \subseteq A$. The character $\operatorname{Ch}(\mathscr{U})$ is the minimal cardinality of a base of $\mathscr{U}$. The ultrafilter number $\mathfrak{u}_{\mu}$ is the minimal value of $\operatorname{Ch}(\mathscr{U})$ for some uniform ultrafilter over $\mu$. If we restrict our attention to complete ultrafilters then we can get more colors from the assumption $\mathfrak{u}_{\mu}<2^{\mu}$ as mirrored by the following:
Claim 3.1. Suppose that $\mathscr{U}$ is a $\theta$-complete uniform ultrafilter over $\mu, \partial=$ $\operatorname{Ch}(\mathscr{U})<\nu=\operatorname{cf}(\nu)$ and $\chi<\theta$. Then $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\chi}$.
Proof.
Let $\mathcal{B}=\left\{B_{\delta}: \delta \in \partial\right\} \subseteq \mathscr{U}$ be a base for $\mathscr{U}$. Suppose that $c: \nu \times \mu \rightarrow \chi$ is a coloring. For every $\alpha \in \nu$ and every $i \in \chi$ let $A_{\alpha}^{i}=\{\beta \in \mu: c(\alpha, \beta)=i\}$. Since $\mathscr{U}$ is $\theta$-complete and $\chi<\theta$ one can find for every $\alpha \in \nu$ a color $i(\alpha)$ such that $A_{\alpha}^{i(\alpha)} \in \mathscr{U}$. Since $\mathcal{B}$ is a base of $\mathscr{U}$ one can find $\delta(\alpha) \in \partial$ such that $B_{\delta(\alpha)} \subseteq A_{\alpha}^{i(\alpha)}$.

Choose $A^{\prime} \in[\nu]^{\nu}$ and a fixed color $i \in \chi$ such that $\alpha \in A^{\prime} \Rightarrow i(\alpha)=i$. Since $\partial<\nu=\operatorname{cf}(\nu)$ one can find a fixed $\delta \in \partial$ and $A \in\left[A^{\prime}\right]^{\nu}$ such that $\alpha \in A \Rightarrow \delta(\alpha)=\delta$. Let $B=B_{\delta}$, and recall that $|B|=\mu$ since $B \in \mathscr{U}$. But now we are done since $c^{\prime \prime}(A \times B)=\{i\}$.
$\square_{3.1}$
The ability to force $\mathfrak{u}_{\mu}<2^{\mu}$ and moreover with ultrafilters which possess some degree of completeness is supplied by [RS20]. Suppose that $\kappa$ is a

Laver-indestructible supercompact cardinal, $\lambda^{\prime}<\kappa$ and $\mu>\kappa$. Assume further that $\mathscr{D}$ is a $\lambda^{\prime}$-complete $\theta$-indecomposable uniform filter over $\mu$, where $\theta=\operatorname{cf}\left(\mu^{\prime}\right)$ and $\mu^{\prime}$ is a strong limit singular cardinal above $\mu$. It is shown in [RS20] that there are forcing notions which make $2^{\mu}>\mu^{\prime}$ and every uniform ultrafilter $\mathscr{U}$ which extends $\mathscr{D}$ in the generic extension satisfies $\mathrm{Ch}(\mathscr{U}) \leq \mu^{\prime}<2^{\mu}$. This can be forced by adding Cohen subsets of the correct size, thus one can force $\mathfrak{u}_{\mu}<2^{\mu}$.

Now if $\kappa$ is Laver indestructible and the forcing $\mathbb{P}$ is $\kappa$-directed-closed then $\kappa$ remains supercompact in $V[G]$ where $G \subseteq \mathbb{P}$ is $V$-generic. Therefore, one can choose a uniform $\lambda^{\prime}$-complete ultrafilter which extends $\mathscr{D}$ and apply Claim 3.1. Our strategy will be to force $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$ for some $\lambda^{\prime}$ and then extend the universe once again in order to obtain the appropriate pcf structure. It is important at this stage to make sure that the relation $\binom{\nu}{\mu} \rightarrow$ $\binom{\nu}{\mu}_{\lambda^{\prime}}$ is preserved by the second step of our forcing. This will be ensured by the following:
Lemma 3.2. Assume that $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$ where $\lambda^{\prime}=\lambda^{\prime<\lambda^{\prime}}$. Let $\mathbb{P}$ be a $\lambda$-cc forcing notion where $\lambda<\lambda^{\prime}$ and assume further that $|\mathbb{P}|<\lambda<\lambda^{\prime}$. Let $G \subseteq \mathbb{P}$ be generic over $V$. Then the positive relation $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$, holds in $V[G]$.

Proof.
Let $c: \nu \times \mu \rightarrow \lambda^{\prime}$ be a new coloring and let $\underset{\sim}{c}$ be a name of $c$. For every $\alpha \in \nu, \beta \in \mu$ let $\mathcal{A}_{\alpha \beta}$ be a maximal antichain of conditions which force a value to $\underset{\sim}{c}(\alpha, \beta)$. Define $f: \nu \times \mu \rightarrow[\mathbb{P}]^{<\lambda}$ by $f(\alpha, \beta)=\mathcal{A}_{\alpha \beta}$, so $f \in V$. Since $|\mathbb{P}|^{<\lambda}<\lambda^{\prime}$ there are $A_{0} \in[\nu]^{\nu}, B_{0} \in[\mu]^{\mu}$ and $\mathcal{A} \in[\mathbb{P}]^{<\lambda}$ such that $f^{\prime \prime}\left(A_{0} \times B_{0}\right)=\mathcal{A}$. Let $p$ be the unique condition in $G \cap \mathcal{A}$, and observe that $p \Vdash \underset{\sim}{c} \upharpoonright\left(A_{0} \times B_{0}\right)=f \upharpoonright\left(A_{0} \times B_{0}\right)$. Since $V \models\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$ there are $A \in\left[A_{0}\right]^{\nu}$ and $B \in\left[B_{0}\right]^{\mu}$ for which $f \upharpoonright(A \times B)$ is constant. If follows that $p \Vdash \underset{\sim}{c} \upharpoonright(A \times B)$ is constant, so $V[G] \models\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$, as required.

Equipped with the above lemma, we can phrase and prove the main result of this section:

Theorem 3.3. Assuming the existence of two supercompact cardinals and a measurable cardinal above them in the ground model, one can force the echeloned cube relation $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ when $\lambda<\mu=\operatorname{cf}(\mu)<\nu=\operatorname{cf}(\nu)=2^{\mu}$. Proof.
We commence with a pair of supercompact cardinals $\lambda<\kappa$, where both are Laver-indestructible, according to [Lav78]. We fix a regular cardinal $\lambda^{\prime}$ which satisfies $\lambda^{\prime}=\lambda^{\prime<\lambda^{\prime}}$, and $\lambda^{+}<\lambda^{\prime}<\kappa$. We fix a measurable cardinal $\mu>\kappa$, and a singular strong limit cardinal $\mu^{\prime}>\mu$ such that $\theta=\operatorname{cf}\left(\mu^{\prime}\right)$ and $\kappa<\theta<\mu$. Finally, let $\mathscr{D}$ be a $\lambda^{\prime}$-complete $\theta$-indecomposable filter over $\mu$. The existence of such a filter over $\mu$ follows from the fact that $\mu$ is measurable.

Let $\mathbb{P}$ be a forcing notion which adds Cohen sets and satisfies the requirements of [RS20] while making $2^{\mu}>\mu^{\prime}$ and $2^{\mu}$ is regular, say $2^{\mu}=\nu$. One can use here the usual Cohen forcing $\mathbb{C}_{\nu \theta}$. We add sufficiently large Cohen sets as described in Proposition 1.2, so that the forcing will be $\kappa$-directedclosed and hence the supercompactness of $\kappa$ will be preserved. Let $G \subseteq \mathbb{P}$ be $V$-generic, and let $\mathscr{U}$ be a $\lambda^{\prime}$-complete ultrafilter which extends $\mathscr{D}$ in $V[G]$. This is possible since $\lambda^{\prime}<\kappa$ and $\kappa$ is supercompact in $V[G]$.

By [RS20] we see that $\mathfrak{u}_{\mu}<2^{\mu}=\nu$ in the generic extension. Notice that the essential issue here is the existence of an ultrafilter $\mathscr{U}$ over $\mu$ such that $\mathrm{Ch}(\mathscr{U})<2^{\mu}=\nu$. Hence from Claim 3.1 we infer that $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$ holds in $V[G]$. We define, in $V[G]$, a forcing notion $\mathbb{Q}$ which increases $2^{\lambda}$ to $\nu$ and forces the rest of the assumptions of Theorem 2.3. For this end, we use the forcing notion of [GS12a]. For obtaining $2^{\lambda}=\nu$ choose an ordinal $\Upsilon \in\left(\nu, \nu^{+}\right)$whose cofinality is the desired value in the interval $\left(\lambda, 2^{\lambda}\right)$ and set $\Upsilon$ as the length of the iteration. In the generic extension one has $2^{\lambda}=\nu$ and one can make sure that the true cofinalities assume the values required in Theorem 2.3(e), see [GS12a].

Let $H \subseteq \mathbb{Q}$ be $V[G]$-generic. Notice that $\kappa$ is not supercompact anymore in $V[G][H]$, and $\lambda$ becomes a strong limit singular cardinal. However, the relation $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$, holds in $V[G][H]$ due to Lemma 3.2. Hence all the assumptions of Theorem 2.3 hold in $V[G][H]$ and the positive cube relation $\left(\begin{array}{l}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ follows.

An alternative way to perform the last step of the above forcing construction would be to extend $V[G]$ by the extender-based Prikry forcing as introduced in [GM94]. In this direction one has to begin with a strong limit singular cardinal $\lambda$ which is an $\omega$-limit of measurable cardinals in the ground model, and then add $\nu$-many Prikry sequences. The control on the true cofinalities required for the assumptions of Theorem 2.3 is a bit more involved.

## 4. Some open problems

In this section we collect some questions related to our results. The main problem is the following:
Question 4.1. Is it consistent that $\lambda<\mu=\operatorname{cf}(\mu)<\nu=\operatorname{cf}(\nu) \leq 2^{\lambda}$ and $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ holds?

Our echeloned strong relation is based on a singular cardinal $\lambda$. One may wonder whether such a cube relation is possible where $\lambda$ is regular. In the proposition below we show that one can force the three necessary pairs for such a cube relation to hold simultaneously. This can be considered as a first step toward the desired echeloned relation.
Proposition 4.2. It is consistent that $\lambda$ is supercompact, $\mu=\operatorname{cf}(\mu)>$ $\lambda, \nu=\operatorname{cf}(\nu)>\mu, 2^{\lambda}=\nu$ and the relations $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu},\binom{\mu}{\lambda} \rightarrow\binom{\mu}{\lambda},\binom{\nu}{\lambda} \rightarrow\binom{\nu}{\lambda}$ hold simultaneously.

Proof.
Suppose that $\lambda, \kappa$ are supercompact cardinals and $\lambda<\kappa$. Assume further that both are Laver-indestructible. Fix a regular cardinal $\lambda^{\prime}<\kappa$ so that $\lambda^{\prime}>\lambda^{+}$. Choose a regular cardinal $\mu>\kappa$ which carries a $\lambda^{\prime}$-complete $\theta$ indecomposable ultrafilter, where $\theta=\operatorname{cf}\left(\mu^{\prime}\right)$ for some strong limit singular cardinal $\mu^{\prime}>\mu$ such that $\kappa<\theta<\mu$. Let $\mathscr{D}$ be such an ultrafilter and let $\mathbb{P}$ be a forcing notion which satisfies [RS20, Definition 5] while making $2^{\mu}=\nu>\mu^{\prime}$ with $\nu=\operatorname{cf}(\nu)$, and while keeping the supercompactness of $\lambda$ and $\kappa$.

Let $G \subseteq \mathbb{P}$ be $V$-generic and let $\mathscr{U}$ be a $\lambda^{\prime}$-complete ultrafilter which extends $\mathscr{D}$ in $V[G]$. From [RS20] we infer that $\operatorname{Ch}(\mathscr{U})<\nu=2^{\mu}$ and hence $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$. We force now with a forcing notion $\mathbb{Q}$ to make $2^{\lambda}=\nu$ and $\mu<\mathfrak{s}_{\lambda}<\nu$ using the generalized Mathias forcing, see [GS14]. The chain condition of $\mathbb{Q}$ is less than $\lambda^{\prime}$. Hence if $H \subseteq \mathbb{Q}$ is $V[G]$-generic then $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}_{\lambda^{\prime}}$ holds in $V[G][H]$. Likewise, $\binom{\mu}{\lambda} \rightarrow\binom{\mu}{\lambda}$ and $\binom{\nu}{\lambda} \rightarrow\binom{\nu}{\lambda}$ hold in $V[G][H]$ since $\mu<\mathfrak{s}_{\lambda}<\nu$, see Corollary 1.3.

Let us try to understand what is still missing if one wishes to obtain $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$. A natural attempt would be to fix one of the coordinates and to work with the positive relation of the remaining pair. Let $d: \nu \times \mu \times \lambda \rightarrow 2$ be a coloring, and suppose that we work with $\nu \times \mu$. So for every $\gamma \in \lambda$ we let $d_{\gamma}=d \upharpoonright(\nu \times \mu \times\{\gamma\})$ and by the relation $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$ we have $A_{\gamma} \in[\nu]^{\nu}, B_{\gamma} \in[\mu]^{\mu}$ such that $d_{\gamma} \upharpoonright\left(A_{\gamma} \times B_{\gamma}\right)$ is constant. The problem is that $A_{\gamma}$ and $B_{\gamma}$ might be different for each $\gamma \in \lambda$, and by forcing $2^{\lambda}=\nu$ we see that there is no hope to choose $A_{\gamma}$ or $B_{\gamma}$ from a sufficiently complete ultrafilter over $\nu$ and $\mu$ respectively.
Question 4.3. Is it consistent that $\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \lambda\end{array}\right)$ where $\lambda<\mu<\nu \leq 2^{\lambda}$ and $\lambda, \mu, \nu$ are regular cardinals?

In the specific case of $\lambda=\aleph_{0}$ we can get something a bit better. Begin by forcing $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$ where $\mu$ and $\nu$ are regular and sufficiently large. Now let $\mathbb{P}$ be a $c c c$ forcing notion which forces $\mathfrak{u}=\aleph_{1}<2^{\aleph_{0}}=\nu^{+}$. Let $G \subseteq \mathbb{P}$ be $V$-generic, and let $\mathscr{U}$ be a uniform ultrafilter over $\omega$ such that $\operatorname{Ch}(\mathscr{U})=\aleph_{1}$. The relation $\binom{\nu}{\mu} \rightarrow\binom{\nu}{\mu}$ is preserved in $V[G]$ due to the chain condition. The relations $\binom{\nu}{\omega} \rightarrow\binom{\nu}{\omega}$ and $\binom{\mu}{\omega} \rightarrow\binom{\mu}{\omega}$ hold in $V[G]$ since $\mathfrak{u}<\mu, \nu$ and $\mu, \nu$ are regular. Moreover, $\binom{\nu}{\omega} \rightarrow\binom{\nu}{\mathscr{U}}$ and $\binom{\mu}{\omega} \rightarrow\binom{\mu}{\mathscr{U}}$ hold.

Suppose that $d: \nu \times \mu \times \omega \rightarrow 2$, and for every $\alpha \in \nu$ let $d_{\alpha}=d \upharpoonright$ $(\{\alpha\} \times \mu \times \omega)$. It follows that $d_{\alpha}$ is constant over $B_{\alpha} \times C_{\alpha}$ where $B_{\alpha} \in[\mu]^{\mu}$ and $C_{\alpha} \in \mathscr{U}$. Since $\operatorname{Ch}(\mathscr{U})=\aleph_{1}$ one can find $A \in[\nu]^{\nu}$ and a fixed $C \in \mathscr{U}$ such that $\alpha \in A \Rightarrow C \subseteq C_{\alpha}$. The problem is with the $B_{\alpha}$ s which might be far from each other and it seems very hard to force them to satisfy the finite intersection property. However, this discussion points to the possibility that the countable case would be easier:
Question 4.4. Is it consistent that $\left(\begin{array}{c}\nu \\ \mu \\ \omega\end{array}\right) \rightarrow\left(\begin{array}{c}\nu \\ \mu \\ \omega\end{array}\right)$ where $\mu, \nu$ are regular and $\nu \leq 2^{\omega}$ ?

In the previous sections we dealt with cube relations of three parameters. One may consider $n$-dimensional cubes where $n>3$. The case of $n=4$ seems interesting since we can say something about it if we drop the axiom of choice. Suppose that $\aleph_{\ell}$ is measurable for every $\ell \in\{1,2,3\}$, this is known to be consistent with ZF. Using the method of [Gar20] we can prove that in such models we have the 4 -cube relation at the cardinals $\left(\aleph_{0}, \aleph_{1}, \aleph_{2}, \aleph_{3}\right)$. Namely, we can get the terraced 4-cube relation. In the framework of ZFC we should consider echeloned 4-cube relations and it seems like a good challenge since such a relation requires a positive 3 -cube relation at every pertinent triple.

Question 4.5. Is it consistent with ZFC that a positive 4-cube relation holds at $(\kappa, \lambda, \mu, \nu)$ where $\lambda, \mu, \nu$ are regular cardinals above $\kappa$ and $\nu \leq 2^{\kappa}$ ?

In the framework of ZF one may wish to consider terraced $n$-cube relations for $n>4$. An interesting endeavor would be a positive $\omega$-cube relation. As a starting point one may wonder how many consecutive measurable cardinals are possible. The existence of four consecutive measurable cardinals is an open problem in ZF, but maybe a negative $\omega$-statement can be proved.

Question 4.6. Is it consistent relative to ZF that there is an infinite cardinal $\kappa$ so that $\kappa^{+n}$ is measurable for every $n \in \omega$ ?

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