

$N = (2, 2)$ superfields and geometry revisited

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Abstract

We take a fresh look at the relation between generalised Kähler geometry and $N = (2, 2)$ supersymmetric sigma models in two dimensions formulated in terms of $(2, 2)$ superfields. Dual formulations in terms of different kinds of superfield are combined to give a formulation with a doubled target space and both the original superfield and the dual superfield. For Kähler geometry, we show that this doubled geometry is Donaldson's deformation of the holomorphic cotangent bundle of the original Kähler manifold. This doubled formulation gives an elegant geometric reformulation of the equations of motion. We interpret the equations of motion as the intersection of two Lagrangian submanifolds (or of a Lagrangian submanifold with an isotropic one) in the infinite dimensional symplectic supermanifold which is the analogue of phase space. We then consider further extensions of this formalism, including one in which the geometry is quadrupled, and discuss their geometry.

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1 Introduction

Superfields are smarter than we are!

Martin Roček told us this many times

Two dimensional $N = (2, 2)$ supersymmetric non-linear sigma models play a prominent role in string theory and in mathematical physics, with a beautiful relation between supersymmetry and the geometry of the target space. Zumino [1] was the first to realise, in 1979, that $N = (2, 2)$ supersymmetry constrains the target space of a non-linear sigma-model to be Kähler. Chiral superfields played a prominent role and the $N = (2, 2)$ Lagrangian density was identified with the Kähler potential. In [2], this was generalised to the case of sigma-models with Wess-Zumino (WZ) term and it was found that the general $N = (2, 2)$ models were described in terms of a bihermitian geometry. This geometry is now called generalised Kähler geometry. In [2], a $N = (2, 2)$ superfield formulation was found for a special class of generalised Kähler geometries (those with commuting complex structures that were simultaneously integrable) and it involved both chiral and twisted-chiral superfields. A further kind of superfield – the semichiral superfield – was introduced in [3] and in [4] the full $N = (2, 2)$ superfield description of generalised Kähler geometry was found (subject to a certain regularity assumption) and this superfield description was given a geometric realisation in terms of local coordinates (which are associated with the various superfields). For this general model, the $N = (2, 2)$ Lagrangian density is given in terms of single function, the generalised Kähler potential, which encodes all local geometry.

In parallel, there were interesting developments in mathematics, starting with the work of Hitchin [5] who introduced the notion of generalised complex structure. Gualtieri, in his PhD thesis [6], introduced the notion of generalised Kähler geometry and showed that this is the same as the bihermitian geometry of [2] (see also [7]). Later, a global interpretation of the generalised Kähler potential was given in [8] for the case of a generalised Kähler structure of symplectic type.

In this paper, we return to the superfield description of the $N = (2, 2)$ sigma model and find new geometrical structure. Strikingly, it is the superfield structure that suggests the new geometry.

We start with the Kähler geometry of the $N = (2, 2)$ sigma-model without WZ term. As well as the usual formulation in terms of chiral superfields, there is a dual formulation in terms of complex linear superfields.¹ In one formulation, the coordinates appear as the lowest component of chiral superfields, in the other as the lowest component of complex linear

¹As we will discuss later, there is, strictly speaking, only a full global dual formulation for a special class of target spaces.

superfields. We reformulate the general Kähler sigma-model in a ‘doubled target space’ with both chiral superfields and complex linear superfields. We then impose a relation between the two types of superfield such that the superspace constraint on either one implies the field equation of the other. This gives an elegant set of equations that captures the classical structure of the model. Moreover, for those target spaces for which there are two dual formulations, this makes the duality manifest. We show that the geometry of the doubled space is that of Donaldson’s deformation of the holomorphic cotangent bundle of the original Kähler manifold, while the relation between the two types of superfield constructs the original space as a Lagrangian submanifold of the deformed cotangent bundle.

Next, we lift this structure to the space of unconstrained maps from $(2, 2)$ superspace to the deformed cotangent bundle, which might be thought of as a kind of phase space. We show that this has a holomorphic symplectic structure. We show that the superspace constraints correspond to restricting to a holomorphic isotropic submanifold, so that the sigma model is specified by the intersection of an isotropic submanifold and a Lagrangian submanifold. We also give an extended formulation in which the submanifold specified by the superfield constraints is also Lagrangian.

We then extend this formulation to the case of $N = (2, 2)$ sigma-models with Wess-Zumino term and their generalised Kähler geometry. We find an enlarged formulation in which, as well as the chiral, twisted-chiral and semichiral superfields, we include the dual superfields. The model is specified by a constraint that selects a submanifold of this enlarged space and implies the field equations. We discuss the symplectic geometry of this and show that the model can be formulated as the intersection of two submanifolds of the space of supermaps to this enlarged phase space. (These submanifolds are either both Lagrangian, or one is Lagrangian and one is isotropic, depending on the choice of extended space.) We discuss several different enlarged spaces, each of which gives an extended formulation of the original sigma models.

In our discussion, we combine the superfield description from [4] with ideas coming from dualities [9] and global considerations from [10]. The superfield formalism imposes a very rigid structure on field transformations both at local and global levels and as a result constrains the kind of transition functions that are permitted. The gerbe structure found in [10] underpins the geometry, and we find that in some cases it leads to interesting spaces that might be considered as generalisations of fibre bundles. It will be interesting to compare all this with the global discussion in [8]. We think that the ideas we set out here can be further developed and generalised beyond the present paper.

The paper is organised as follows. In section 2 we review Kähler geometry and the formulation of the corresponding sigma model in terms of chiral superfields and the dual

formulation in terms of complex linear superfields. We then present a doubled formulation in terms of both kinds of superfields and show that the doubled target space is Donaldson's deformation of the holomorphic cotangent bundle. In section 3 we find a similar doubled formulation of the general $N = (2, 2)$ sigma model with WZ term; in this section, we present only a local analysis. In section 4 we discuss the global issues for this construction. Sections 5 and 6 extend the phase space further in constructions dictated by the superfield formalism. In Section 5 we consider a construction in which we quadruple some fields and double others, while in Section 6 we offer a democratic formulation in which all fields are quadrupled. Finally, in section 7 we summarise our results and outline open problems. In two appendices we collect some basic properties of the $N = (2, 2)$ superfield formalism and review some features of generalised Kähler geometry and its gerbe structure.

2 Kähler Geometry and the $N = 2$ Supersymmetric Sigma Model

In this section we consider the well known case of Kähler geometry and the corresponding $N = (2, 2)$ supersymmetric sigma models. We first review the local structure of the sigma model and its dual formulation. We then provide a system of equations combining the equations of motion and the superspace constraints in a unified way. This system has a manifest duality that interchanges field equations with constraints. The equations govern superfields taking values in a 'doubled space', with the original space arising as a Lagrangian submanifold. We show that this doubled space is Donaldson's deformed cotangent bundle and this facilitates the discussion of global issues. Finally, we propose a new geometric interpretation of the system in an infinite-dimensional setting.

2.1 The Kähler Sigma Model and its Dual Formulation

The $N = (2, 2)$ sigma model with Kähler target M is a theory of maps

$$\mathcal{N} = \{\phi : \mathbb{R}^{2|4} \longrightarrow M\} \quad (2.1)$$

from $N = (2, 2)$ superspace $\mathbb{R}^{2|4}$ with coordinates (x^m, θ^α) (where $\alpha = +, -$ are spinor indices and θ^α are complex) to the manifold M with local complex coordinates $(\phi^a, \bar{\phi}^{\bar{a}})$. The maps are locally specified by coordinate maps $\phi^a(x^m, \theta^\alpha)$ which are constrained to be chiral

$$\bar{D}_\pm \phi^a = 0, \quad D_\pm \bar{\phi}^{\bar{a}} = 0 \quad (2.2)$$

and the action is

$$S = \int d^2x d^4\theta K(\phi^a, \bar{\phi}^{\bar{a}}), \quad (2.3)$$

where the real function $K(\phi^a, \bar{\phi}^{\bar{a}})$ is the Kähler potential. See Appendix A for details of our superspace conventions and notation.

The sigma-model has a remarkable dual formulation in terms of complex linear superfields, given in the superspace textbook [9] for the four-dimensional sigma model (see the discussion around eqs (4.5.10a,b)) and for the two-dimensional sigma model in [11]. First, note that the action (2.3) can be rewritten in a first-order form as

$$S_{\text{dual}} = \int d^2x d^4\theta \left[K(\Phi^b, \bar{\Phi}^{\bar{b}}) - \Sigma_a \Phi^a - \bar{\Sigma}_{\bar{a}} \bar{\Phi}^{\bar{a}} \right], \quad (2.4)$$

where Φ^b are unconstrained complex superfields and Σ_a are complex linear superfields satisfying the constraint

$$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0 \quad (2.5)$$

($\bar{\Sigma}_{\bar{a}}$ are the complex conjugate superfields). The field equation for Σ_a imposes the chirality constraint $\bar{\mathbb{D}}_{\pm} \Phi^a = 0$ so that integrating out the Σ_a 's recovers the original action (2.3) (on identifying the constrained Φ^a with the chiral superfields ϕ^a). On the other hand, the field equation for Φ^a is

$$\Sigma_a = \frac{\partial K}{\partial \Phi^a} \quad (2.6)$$

giving Σ_a as a function of $\Phi^a, \bar{\Phi}^{\bar{a}}$. If this can be inverted to give Φ^a as a function of $\Sigma_a, \bar{\Sigma}_{\bar{a}}$, then integrating out $\Phi^a, \bar{\Phi}^{\bar{a}}$ sets Φ^a to be the function $\Phi^a(\Sigma_a, \bar{\Sigma}_{\bar{a}})$ giving the dual formulation

$$S = \int d^2x d^4\theta \tilde{K}(\Sigma_a, \bar{\Sigma}_{\bar{a}}), \quad (2.7)$$

where \tilde{K} is the Legendre transform of K :

$$\tilde{K} = K(\Phi^b, \bar{\Phi}^{\bar{b}}) - \Sigma_a \Phi^a - \bar{\Sigma}_{\bar{a}} \bar{\Phi}^{\bar{a}}, \quad (2.8)$$

and (2.6) is inverted and used to write the RHS as a function of $\Sigma_a, \bar{\Sigma}_{\bar{a}}$. For this to be a good duality requires that the function $\Sigma_a(\Phi^a, \bar{\Phi}^{\bar{a}})$ is invertible and that the Legendre function is well-defined (not multivalued) and so in particular needs K to be a convex function. The general case in which it is not convex is interesting and will be discussed elsewhere.

Now the dual action (2.7) can itself be written in a first-order form as

$$\tilde{S}_{\text{dual}} = \int d^2x d^4\theta \left[\tilde{K}(\Phi_b, \bar{\Phi}_{\bar{b}}) - \phi^a \Phi_a - \bar{\phi}^{\bar{a}} \bar{\Phi}_{\bar{a}} \right], \quad (2.9)$$

where the Φ_b are complex unconstrained superfields and the ϕ^a are chiral superfields. The ϕ equation of motion imposes the constraint that Φ_a is a complex linear superfield, $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Phi_a =$

0, so that integrating out ϕ recovers the action (2.7) (on identifying the constrained Φ_a with the complex linear superfield Σ_a). On the other hand, the Φ field equation gives

$$\phi^a = \frac{\partial \tilde{K}}{\partial \Sigma_a}. \quad (2.10)$$

Integrating out the fields Φ_a requires solving this to determine Σ as a function of $\phi^a, \bar{\phi}^{\bar{a}}$ and substituting in the action (2.9) to arrive back at the original action (2.3) (for this to be well-defined, \tilde{K} should be convex).

2.2 Lagrangian System for the Kähler Sigma Model

The equations of motion following from the action (2.4) together with the superfield constraints can be recast as the following system of equations

$$\boxed{\Sigma_a = \frac{\partial K}{\partial \phi^a}, \quad \bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0, \quad \bar{\mathbb{D}}_{\pm} \phi^a = 0,} \quad (2.11)$$

where $K(\phi, \bar{\phi})$ is the Kähler potential. Indeed, the first two equations combine to give

$$\bar{\mathbb{D}}^2 \frac{\partial K}{\partial \phi^a} = 0, \quad (2.12)$$

which are precisely the equations of motion following from (2.3). Alternatively, the equations of motion derived from the dual action (2.9) can be recast as the following system

$$\boxed{\phi^a = \frac{\partial \tilde{K}}{\partial \Sigma_a}, \quad \bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0, \quad \bar{\mathbb{D}}_{\pm} \phi^a = 0,} \quad (2.13)$$

where $\tilde{K}(\Sigma, \bar{\Sigma})$ is the dual potential. Then the first and last equation combine to give the field equation for Σ .

For both systems (2.11) and (2.13) we have maps from superspace to a space with complex coordinates (ϕ^a, Σ_a) satisfying the constraints $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0$, $\bar{\mathbb{D}}_{\pm} \phi^a = 0$ and the model is defined by restricting to a subspace defined by the d complex equations (for a manifold of complex dimension d) $\Sigma_a = \frac{\partial K}{\partial \phi^a}$ or $\phi^a = \frac{\partial \tilde{K}}{\partial \Sigma_a}$.

The two systems of equations (2.11) and (2.13) are equivalent provided that the Legendre transform between K and \tilde{K} is well-defined. This duality between the two systems interchanges constraints with field equations. In the first system (2.11), we can regard ϕ as the fundamental field with $\bar{\mathbb{D}}_{\pm} \phi^a = 0$ viewed as a constraint and the other two equations combining to give the field equation for ϕ that follows from the action (2.3). For the dual system (2.13) we can regard Σ as the fundamental field with $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0$ viewed as a constraint

and the other two equations combining to give the field equation for Σ that follows from the action (2.7).

For example, in the simple case of a flat target space, the Kähler potential is quadratic, $K = \frac{1}{2} \sum_a \phi^a \bar{\phi}^{\bar{a}}$. Then $\Sigma_a = \frac{\partial K}{\partial \phi^a}$ gives $\Sigma_a = \bar{\phi}^{\bar{a}}$ so that the complex linear constraint on Σ , which is $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0$, gives the field equation for a free chiral superfield, $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \bar{\phi}^{\bar{a}} = 0$. The dual potential is $\tilde{K} = \frac{1}{2} \sum_a \Sigma_a \bar{\Sigma}_{\bar{a}}$ so that we again obtain $\Sigma_a = \bar{\phi}^{\bar{a}}$ and now the chiral constraint on ϕ implies the complex linear constraint on Σ , i.e. $\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0$.

This duality becomes rather subtle if K is not convex as then the Legendre transform \tilde{K} becomes multivalued and may even be singular. We will not discuss this duality further here and will return to this subject elsewhere. Instead, we will focus here on the system (2.11) which is defined (locally) for any Kähler manifold and not assume the existence of a dual potential \tilde{K} . We now turn to the global formulation of this system for any Kähler manifold.

2.3 Global Structure

We now look at the global structure of the system (2.11). First we discuss the symmetries of the equations (2.11). The chirality conditions allow the field redefinitions $\phi^a \rightarrow \phi'^a(\phi)$ corresponding to a holomorphic change of complex coordinates on M . Then $\Sigma_a = \frac{\partial K}{\partial \phi^a}$ requires that the linear complex superfield transforms as

$$\Sigma_a \rightarrow \Sigma'_a = \frac{\partial \phi^b}{\partial \phi'^a} \Sigma_b . \quad (2.14)$$

Note that this is compatible with the superfield structure (see Appendix): if Σ_a satisfies the complex linear constraint $\bar{\mathbb{D}}^2 \Sigma_a = 0$, then so does Σ'_a . Then Σ_a transforms like a holomorphic 1-form and (ϕ, Σ) transform under changes of coordinates in the same way as the holomorphic coordinates of the cotangent bundle. However, although it is tempting to suppose that (ϕ, Σ) correspond to complex coordinates on the cotangent bundle T^*M , there is a subtlety related to the way the coordinates Σ are glued.

The superspace action is unchanged if the Kähler potential is transformed by

$$K \rightarrow K' = K + f(\phi) + \bar{f}(\bar{\phi}) \quad (2.15)$$

for any holomorphic function $f(\phi)$. This is sometimes referred to as a Kähler gauge transformation. For this to be consistent with $\Sigma_a = \frac{\partial K}{\partial \phi^a}$, it is necessary that Σ_a transforms as

$$\Sigma_a \rightarrow \Sigma'_a = \Sigma_a + \frac{\partial}{\partial \phi^a} f(\phi) \quad (2.16)$$

so that Σ can be viewed as a connection one-form for these transformations. These transformations are important for the global structure, as the glueing between patches is through a Kähler gauge transformation composed with a holomorphic diffeomorphism.

For an open cover of the Kähler manifold with (contractible) open sets U_α the Kähler structure leads to a locally-defined Kähler potential K_α on each patch U_α . On the intersection of two patches $U_\alpha \cap U_\beta$ they are glued by a Kähler gauge transformation

$$K_\alpha(\phi, \bar{\phi}) - K_\beta(\phi, \bar{\phi}) = f_{\alpha\beta}(\phi) + \bar{f}_{\alpha\beta}(\bar{\phi}) \quad (2.17)$$

for some holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. This leads to a well-defined superspace action as the right hand side vanishes identically under the superspace integral. To preserve the condition $\Sigma_a = \frac{\partial K}{\partial \phi^a}$ we postulate the following glueing of Σ 's on the intersections

$$(\Sigma_a)_\beta = (\Sigma_a)_\alpha + \frac{\partial}{\partial \phi^a} f_{\alpha\beta}(\phi) \quad \text{on } U_\alpha \cap U_\beta, \quad (2.18)$$

which is again compatible with superfield constraints. The full glueing conditions combine these with a holomorphic diffeomorphism from the coordinates in U_α to those in U_β . In (2.17),(2.18) we suppress the diffeomorphism part of the transition function; this can be thought of as expressing all quantities in the relation in terms of the same coordinate system.²

The structure here is a deformation of the holomorphic cotangent bundle and we follow Donaldson's construction of this [12] (see also [8]). With local complex coordinates $(z_\alpha^a, p_{a\alpha})$ on T^*U_α , we define a 1-form $p_\alpha = p_{a\alpha} dz_\alpha^a$ on each patch U_α with the holomorphic affine glueing relation for the fibres over $U_\alpha \cap U_\beta$

$$p_\beta = p_\alpha + \partial f_{\alpha\beta}, \quad (2.19)$$

which satisfy the cocycle condition. Expressing all terms in the same coordinate system, this can be written as

$$(p_a)_\beta = (p_a)_\alpha + \partial_a f_{\alpha\beta}(z). \quad (2.20)$$

The total space is constructed from patches of the form T^*U_α glued together using these transition functions. That is, we can define the holomorphic symplectic affine bundle $(\mathcal{Z}, \omega^{(2,0)})$ as

$$\mathcal{Z} = \left(\coprod_{\alpha} T^*U_\alpha \right) / \sim \quad (2.21)$$

with the equivalence relation $p_\alpha \sim p_\beta$ defined by (2.20). The bundle \mathcal{Z} inherits a canonical holomorphic symplectic structure from T^*M

$$\omega^{(2,0)} = dp_a \wedge dz^a, \quad (2.22)$$

which is well-defined under the identification. In other words, the transformations between patches (2.20) are symplectomorphisms and so the symplectic structure $\omega^{(2,0)}$ is globally

²We will similarly suppress the diffeomorphism part of glueing conditions throughout this paper.

defined on \mathcal{Z} . The choice of Kähler potential is encoded in the choice of a global (non-holomorphic) section $\mathcal{L} : M \rightarrow \mathcal{Z}$ that is given by

$$p_a dz^a = \partial K . \quad (2.23)$$

This equation specifies a globally defined submanifold \mathcal{L} of \mathcal{Z} that is Lagrangian with respect to $\text{Re}(\omega^{(2,0)})$ and symplectic with respect to $\text{Im}(\omega^{(2,0)})$. For a more detailed discussion we refer the reader to [12] and [8].³

2.4 Interpretation

The system (2.11) is then one of supermaps $\mathbb{R}^{2|4} \rightarrow \mathcal{L}$ with $p(x, \theta, \bar{\theta})$ satisfying the complex linear constraints and $z(x, \theta, \bar{\theta})$ satisfying the chiral constraints. Let us reformulate this observation in geometrical terms.

We introduce the infinite dimensional space of unconstrained supermaps from $\mathbb{R}^{2|4}$ to the holomorphic symplectic affine bundle

$$\mathcal{M} = \{\Phi : \mathbb{R}^{2|4} \rightarrow \mathcal{Z}\} \quad (2.24)$$

with local holomorphic Darboux coordinates Φ^a, Φ_a on \mathcal{Z} with Φ^a corresponding to complex coordinates on M (referred to as z^a in the last subsection) while Φ_a are complex coordinates along the fibre (referred to as p_a in the last subsection). On the space of unconstrained supermaps given locally by $\Phi^a(x, \theta, \bar{\theta}), \Phi_a(x, \theta, \bar{\theta})$, we denote the exterior derivative by δ so that the basic 1-forms are $\delta\Phi^a, \delta\Phi_a$ and use \wedge for the wedge product. Then this infinite dimensional space is equipped with the holomorphic symplectic structure

$$\Omega = \int d^2x d^4\theta \delta\Phi_a \wedge \delta\Phi^a , \quad (2.25)$$

which arises from the holomorphic symplectic structure $\omega^{(2,0)}$ (2.22) on \mathcal{Z} .

The main idea here is that the system of equations (2.11) can be interpreted as the intersection of two infinite dimensional Lagrangian submanifolds of \mathcal{M} . A Lagrangian submanifold is isotropic (i.e. the symplectic structure restricted to the submanifold vanishes) and satisfies a maximality condition. In the finite dimensional setting, a Lagrangian submanifold of a manifold of dimension $2d$ is an isotropic submanifold with dimension d . In the infinite dimensional setting, a Lagrangian subspace of a symplectic vector space is an isotropic subspace whose symplectic complement that is also isotropic. The extension of this to infinite

³Here and in the rest of the paper we follow conventions which allow us to minimise the appearance of factors of i in our formulae. In some other works different conventions are used in which the roles of $\text{Re}(\omega^{(2,0)})$ and $\text{Im}(\omega^{(2,0)})$ are interchanged. We use the same letter for the section and the Lagrangian submanifold since they are two ways of viewing the same submanifold.

dimensional manifolds is subtle, but intuitively the maximality of a Lagrangian submanifold means that if one tries to enlarge the space then the property of being isotropic is lost.

The first equation (and its complex conjugate) in (2.11) corresponds to

$$\Phi_a = \frac{\partial K}{\partial \bar{\Phi}^a}, \quad (2.26)$$

which defines a real Lagrangian submanifold of \mathcal{M} with respect to $\text{Re}(\Omega)$ (here K is a real function of $(\Phi^b, \bar{\Phi}^{\bar{b}})$). This infinite dimensional Lagrangian submanifold is induced by a finite dimensional Lagrangian submanifold \mathcal{L} of \mathcal{Z} . K can be promoted to become the generating function of an infinite dimensional space, as we show in next subsection.

On the other hand, the superfield constraints in (2.11) specify a holomorphic isotropic submanifold with respect to Ω . To see that this submanifold is isotropic we evaluate the holomorphic symplectic form on the subspace in which the superfields are constrained to obtain

$$\Omega|_{\bar{\mathbb{D}}_{\pm}\Phi^a=0, \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\Phi_a=0} = \int d^2x d^4\theta \delta\Phi_a \wedge \delta\bar{\Phi}^a = 0, \quad (2.27)$$

where we have used the representation (A.16) for these constrained superfields. The integral vanishes identically (possibly up to boundary terms, but we assume boundary conditions that eliminate these). For example, (A.16) implies that $\delta\bar{\Phi}^a$, which is chiral on the constrained submanifold, can be written as $\delta\bar{\Phi}^a = \bar{\mathbb{D}}^2 W^a$ for some W^a , so that on integrating by parts the integrand in (2.27) becomes $W^a \wedge \bar{\mathbb{D}}^2 \delta\Phi_a$ which vanishes on the submanifold as $\delta\Phi_a$ satisfies the complex linear constraints there. Thus we see that these superfield constraints correspond to restricting to a holomorphic isotropic submanifold of \mathcal{M} . In the discussion below we show that the corresponding submanifold in an extended space is actually holomorphic Lagrangian.

2.5 Solving the linear constraints and extended space

The complex linear constraint

$$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0 \quad (2.28)$$

is solved by

$$\Sigma_a = \bar{\mathbb{D}}_{\alpha} \Psi_a^{\alpha} \quad (2.29)$$

for some unconstrained spinor superfields $\Psi_a^{\alpha} = (\Psi_a^+, \Psi_a^-)$. Then the first-order action (2.4) can be rewritten as

$$S_{\text{dual}} = \int d^2x d^4\theta \left[K(\Phi^b, \bar{\Phi}^{\bar{b}}) + \Psi_a^{\alpha} \bar{\mathbb{D}}_{\alpha} \Phi^a + \bar{\Psi}_{\bar{a}}^{\alpha} \mathbb{D}_{\alpha} \bar{\Phi}^{\bar{a}} \right]. \quad (2.30)$$

The superfields Ψ_a^{α} are lagrange multipliers imposing the chirality constraint $\bar{\mathbb{D}}_{\pm} \Phi^a = 0$. This can be written as

$$S_{\text{dual}} = \mathcal{K} + 2\text{Re}(\mathcal{H}), \quad (2.31)$$

where

$$\mathcal{K} = \int d^2x d^4\theta K(\Phi^a, \bar{\Phi}^{\bar{a}}) \quad (2.32)$$

and

$$\mathcal{H} = \int d^2x d^4\theta \Psi_a^\alpha \bar{D}_\alpha \Phi^a = \int d^2x d^4\theta (-i\Psi_{-a} \bar{D}_+ \Phi^a + i\Psi_{+a} \bar{D}_- \Phi^a) . \quad (2.33)$$

Note that the fields Ψ_a^α transform as (1,0) forms under holomorphic coordinate transformations for M so that this action is covariant.

We now return to the geometrical setup. We can enlarge the Kähler target space M with the coordinates $(z^a, \bar{z}^{\bar{a}})$ by introducing odd additional coordinates $(\psi_{+a}, \bar{\psi}_{+\bar{a}})$ and $(\psi_{-a}, \bar{\psi}_{-\bar{a}})$ for two copies of the fibre of the odd cotangent bundle, so that we have $(\Pi T^* \oplus \Pi T^*)M$. (Here we follow the standard notation in which Π indicates the parity-reversed fibre.) Next we introduce the cotangent bundle to this supermanifold

$$T^*\left((\Pi T^* \oplus \Pi T^*)M\right) \quad (2.34)$$

and denote new fibre coordinates by $(p_a, \psi_+^a, \psi_-^a)$ (together with their complex conjugates). This supermanifold is equipped with the canonical holomorphic symplectic structure

$$\omega'^{(2,0)} = dp_a \wedge dz^a - id\psi_+^a \wedge d\psi_{-a} + id\psi_-^a \wedge d\psi_{+a} . \quad (2.35)$$

Here the odd coordinates ψ_\pm^a transform as fibre coordinates for the odd tangent bundle. The transformations of p_a are a bit more complicated, the p_a transform as section of T^*M plus an additional term which is quadratic in ψ . This supermanifold has been studied explicitly and the detailed formulae for the transformation of all coordinates, together with other properties, can be found in [13] (e.g. see Remark 3.3.2 in [13] for explicit formulae in the real case). Next we deform this cotangent bundle in the same way as described in subsection 2.3 using the identification (2.20) for the even coordinate p_a . Thus we obtain the deformed cotangent bundle which is now the supermanifold \mathcal{Z}' and is equipped with the holomorphic symplectic structure given again by formula (2.35) (in Darboux coordinates).

We now extend the discussion of the previous subsection. Instead of using the target space \mathcal{Z} we work with the supermanifold \mathcal{Z}' . We introduce the infinite dimensional space of supermaps from $\mathbb{R}^{2|4}$ to \mathcal{Z}'

$$\mathcal{M}' = \{\mathbb{R}^{2|4} \longrightarrow \mathcal{Z}'\} \quad (2.36)$$

with superfields $(\Phi^a, \Phi_a, \Psi_\pm^a, \Psi_{\pm a})$ (plus their complex conjugates). Here the superfields Φ^a are associated with coordinates z^a , the superfields Φ_a are associated with p_a , the superfields Ψ_\pm^a are associated with ψ_\pm^a and finally the superfields $\Psi_{\pm a}$ are associated with $\psi_{\pm a}$. The transformation rules for the superfields follow from those of the coordinates of the supermanifold \mathcal{Z}' .

The space \mathcal{M}' is equipped with the holomorphic symplectic form

$$\Omega_{\text{large}} = \int d^2x d^4\theta \left(\delta\Phi_a \wedge \delta\Phi^a - i\delta\Psi_+^a \wedge \delta\Psi_{-a} + i\delta\Psi_-^a \wedge \delta\Psi_{+a} \right). \quad (2.37)$$

(Note that this formalism with an enlarged phase space has some analogies to the BV/BRST formalism.) We now use generating functions to define Lagrangian submanifolds. We define a holomorphic Lagrangian submanifold of \mathcal{Z}' through the holomorphic generating function of $(\Phi^a, \Psi_{+a}, \Psi_{-a})$ given by \mathcal{H} defined in (2.33) which gives

$$\Phi_a = \frac{\delta\mathcal{H}}{\delta\Phi^a} = i\bar{\mathbb{D}}_+ \Psi_{-a} - i\bar{\mathbb{D}}_- \Psi_{+a}, \quad \Psi_+^a = \frac{\delta\mathcal{H}}{\delta\Psi_{-a}} = \bar{\mathbb{D}}_+ \Phi^a, \quad \Psi_-^a = \frac{\delta\mathcal{H}}{\delta\Psi_{+a}} = \bar{\mathbb{D}}_- \Phi^a. \quad (2.38)$$

A real Lagrangian submanifold with respect to $\text{Re}(\Omega_{\text{large}})$ can be defined using the real generating function of $(\Phi^a, \Psi_{+a}, \Psi_{-a})$ given by \mathcal{K} in (2.32) so that we have

$$\Phi_a = \frac{\delta\mathcal{K}}{\delta\Phi^a} = \frac{\partial\mathcal{K}}{\partial\Phi^a}, \quad \Psi_+^a = \frac{\delta\mathcal{K}}{\delta\Psi_{-a}} = 0, \quad \Psi_-^a = \frac{\delta\mathcal{K}}{\delta\Psi_{+a}} = 0. \quad (2.39)$$

Remarkably, combining (2.38) and (2.39) and using (2.29) we arrive at the system of equations (2.11). We interpret the equations of motion as the intersection of a real Lagrangian submanifold with a holomorphic Lagrangian submanifold. By enlarging the space from \mathcal{M} to \mathcal{M}' we are able to encode both of these submanifolds in terms of generating functions.

3 Generalised Kähler geometry and sigma models

Next we generalise the discussion of the last section to the case of the general $N = (2, 2)$ non-linear sigma model with target space a generalised Kähler manifold. These are manifolds equipped with two complex structures, I_+ and I_- , and a metric that is hermitian with respect to both of them. They are also equipped with a closed 3-form H ; see [7] for details of the geometry. In this section we will focus on the local structure and we postpone the discussion of global geometrical issues until the next section.

3.1 The Generalised Kähler Sigma Model and its Dual Formulation

The general $N = (2, 2)$ non-linear sigma model can be written in terms of four types of $N = (2, 2)$ superfields [4]. These are the chiral and anti-chiral fields

$$\bar{\mathbb{D}}_{\pm}\phi^a = 0, \quad \mathbb{D}_{\pm}\bar{\phi}^{\bar{a}} = 0, \quad (3.1)$$

the twisted chiral and twisted anti-chiral fields

$$\bar{\mathbb{D}}_+\chi^{a'} = 0, \quad \mathbb{D}_-\chi^{a'} = 0, \quad \mathbb{D}_+\bar{\chi}^{\bar{a}'} = 0, \quad \bar{\mathbb{D}}_-\bar{\chi}^{\bar{a}'} = 0, \quad (3.2)$$

the left semichiral fields

$$\bar{\mathbb{D}}_+ X_L^n = 0, \quad \mathbb{D}_+ \bar{X}_L^{\bar{n}'} = 0, \quad (3.3)$$

and finally the right semichiral fields

$$\bar{\mathbb{D}}_- X_R^{n'} = 0, \quad \mathbb{D}_- \bar{X}_R^{\bar{n}'} = 0. \quad (3.4)$$

Each superfield has a bosonic component that is interpreted as one of the coordinates of the target space, so the split into four kinds of superfields corresponds to a split of the coordinates into four different kinds. This is always possible locally, and will be possible globally if we assume that all relevant Poisson structures are regular [4]; we shall assume that this is the case for the rest of the paper. Then the coordinates are $\varphi^A = (\phi^a, \chi^{a'}, X_L^n, X_R^{n'})$ plus their complex conjugates and the indices split as $A = (a, a', n, n')$, $\bar{A} = (\bar{a}, \bar{a}', \bar{n}, \bar{n}')$. There are equal numbers of left semi-chirals X_L and right semi-chirals X_R . Then the general $N = (2, 2)$ sigma model action is

$$S = \int d^2x d^4\theta K(\phi, \bar{\phi}, \chi, \bar{\chi}, X_L, \bar{X}_L, X_R, \bar{X}_R), \quad (3.5)$$

where K is real function of the superfields; see [4] for further explanation.

In analogy with the Kähler case (2.4) we can rewrite the action in first-order form. We replace the constrained superfields $\varphi^A = (\phi^a, \chi^{a'}, X_L^n, X_R^{n'})$ with unconstrained superfields $\Phi^A = (\Phi^a, \Phi^{a'}, \Phi_L^n, \Phi_R^{n'})$ and introduce constrained lagrange multiplier fields $\varphi_A = (\Sigma_a, \Lambda_{a'}, Y_{Ln}, \bar{Y}_{R\bar{n}'})$ (plus their complex conjugates) whose field equations impose the appropriate constraints on the fields Φ^A . The action is

$$S_{\text{dual}} = \int d^2x d^4\theta \left(K(\Phi, \bar{\Phi}) - \varphi_A \Phi^A - \bar{\varphi}_{\bar{A}} \bar{\Phi}^{\bar{A}} \right), \quad (3.6)$$

which can be expanded to give

$$S_{\text{dual}} = \int d^2x d^4\theta \left(K(\Phi, \bar{\Phi}) - \Sigma_a \Phi^a - \bar{\Sigma}_{\bar{a}} \bar{\Phi}^{\bar{a}} - \Lambda_{a'} \Phi^{a'} - \bar{\Lambda}_{\bar{a}'} \bar{\Phi}^{\bar{a}'} \right. \\ \left. - Y_{Ln} \Phi^n - \bar{Y}_{L\bar{n}} \bar{\Phi}^{\bar{n}} - Y_{Rn'} \Phi^{n'} - \bar{Y}_{R\bar{n}'} \bar{\Phi}^{\bar{n}'} \right). \quad (3.7)$$

As seen in the last section, the Σ_a are complex linear superfields. The constraints on the other fields are as follows: the $\Lambda_{a'}$ are twisted complex linear superfields, the Y_{Ln} are left semichiral superfields and the $Y_{Rn'}$ are right semichiral superfields (see the appendix for the definitions). The field equations for Σ , Λ , Y_L and Y_R impose the appropriate chirality constraints on the Φ 's: they impose that Φ^a is chiral, $\Phi^{a'}$ is twisted chiral, Φ_L^n is left semi-chiral and $\Phi_R^{n'}$ is right semi-chiral. Then integrating out the fields φ_A we recover the original action (3.5) (after renaming the fields $\Phi^A = (\Phi^a, \Phi^{a'}, \Phi_L^n, \Phi_R^{n'})$ as $\varphi_A = (\Sigma_a, \Lambda_{a'}, Y_{Ln} \Phi^n, \bar{Y}_{R\bar{n}'})$).

Alternatively we can integrate out the Φ 's to arrive at the dual action

$$S_{\text{dual}} = \int d^2x d^4\theta \tilde{K}(\Sigma, \bar{\Sigma}, \Lambda, \bar{\Lambda}, Y_L, \bar{Y}_L, Y_R, \bar{Y}_R) , \quad (3.8)$$

where $\tilde{K}(\Sigma, \bar{\Sigma}, \Lambda, \bar{\Lambda}, Y_L, \bar{Y}_L, Y_R, \bar{Y}_R)$ is a generalised Legendre transform of K .⁴ As in the last section, the duality needs the Legendre transform to be well-defined, which requires K to be convex.

3.2 Lagrangian System for the Generalised Kähler Sigma Model

The field equations that follow from (3.7) can be re-expressed as the following system of equations for constrained superfields $\varphi^A = (\phi^a, \chi^{a'}, X_L^n, X_R^{n'})$ and $\varphi_A = (\Sigma_a, \Lambda_{a'}, Y_{Ln}, \bar{Y}_{Rn'})$:

$$\varphi_A = \frac{\partial K}{\partial \varphi^A} , \quad (3.9)$$

where $K(\varphi^A, \bar{\varphi}^{\bar{A}})$ is the generalised Kähler potential. Explicitly, this gives the set of equations

$$\begin{aligned} \Sigma_a &= \frac{\partial K}{\partial \phi^a} , & \Lambda_{a'} &= \frac{\partial K}{\partial \chi^{a'}} , & Y_{Ln} &= \frac{\partial K}{\partial X_L^n} , & Y_{Rn'} &= \frac{\partial K}{\partial X_R^{n'}} \\ \bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a &= 0 , & \bar{\mathbb{D}}_+ \mathbb{D}_- \Lambda_{a'} &= 0 , & \bar{\mathbb{D}}_+ Y_{Ln} &= 0 , & \bar{\mathbb{D}}_- Y_{Rn'} &= 0 \\ \bar{\mathbb{D}}_{\pm} \phi^a &= 0 , & \bar{\mathbb{D}}_+ \chi^{a'} &= \mathbb{D}_- \chi^{a'} = 0 , & \bar{\mathbb{D}}_+ X_L^n &= 0 , & \bar{\mathbb{D}}_- X_R^{n'} &= 0 \end{aligned} \quad (3.10)$$

together with the complex conjugate equations. Regarding these as equations for the constrained superfields φ^A , the constraints on φ_A applied to $\frac{\partial K}{\partial \varphi^A}$ give the field equations for φ^A that follow from the action (3.5).

If a Legendre transform \tilde{K} exists, then the equations can also be cast in the dual form

$$\varphi^A = \frac{\partial \tilde{K}}{\partial \varphi_A} , \quad (3.11)$$

where $\tilde{K}(\varphi_A, \bar{\varphi}_{\bar{A}})$ is the Legendre transform of the generalised Kähler potential. Explicitly, this gives the set of equations

$$\begin{aligned} \phi^a &= \frac{\partial \tilde{K}}{\partial \Sigma_a} , & \chi^{a'} &= \frac{\partial \tilde{K}}{\partial \Lambda_{a'}} , & X_L^n &= \frac{\partial \tilde{K}}{\partial Y_{Ln}} , & X_R^{n'} &= \frac{\partial \tilde{K}}{\partial Y_{Rn'}} \\ \bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a &= 0 , & \bar{\mathbb{D}}_+ \mathbb{D}_- \Lambda_{a'} &= 0 , & \bar{\mathbb{D}}_+ Y_{Ln} &= 0 , & \bar{\mathbb{D}}_- Y_{Rn'} &= 0 \\ \bar{\mathbb{D}}_{\pm} \phi^a &= 0 , & \bar{\mathbb{D}}_+ \chi^{a'} &= \mathbb{D}_- \chi^{a'} = 0 , & \bar{\mathbb{D}}_+ X_L^n &= 0 , & \bar{\mathbb{D}}_- X_R^{n'} &= 0 \end{aligned} \quad (3.12)$$

together with the complex conjugate equations. We leave this dual system for future discussion and concentrate here on the equations (3.10).

⁴This duality and the action (3.7) have also been considered by U. Lindström, in unpublished work.

3.3 Local symplectic interpretation

We now take a first look at the symplectic geometry associated with the system (3.10). We will investigate the local geometry in this subsection and postpone discussion of the global structure to the following section. In particular, we restrict ourselves to a single coordinate patch U with coordinates φ^A, φ_A . This coordinate patch has a complex structure with respect to which $(\varphi^A, \varphi_A) = (\phi^a, \Sigma_a, \chi^{a'}, \Lambda_{a'}, X_L^n, Y_{Ln}, X_R^{n'}, Y_{Rn'})$ are holomorphic coordinates and their complex conjugates are anti-holomorphic coordinates.

Following our discussion of the Kähler case, we re-cast the theory in terms of unconstrained superfields, replacing the constrained superfields φ^A, φ_A with unrestricted fields Φ^A, Φ_A (together with their complex conjugates $\bar{\Phi}^{\bar{A}}, \bar{\Phi}_{\bar{A}}$). First, we regard Φ^A, Φ_A as coordinates on U . Then U has a holomorphic symplectic structure

$$\omega^{(2,0)} = d\Phi_A \wedge d\Phi^A . \quad (3.13)$$

Following our discussion of the Kähler case, we now consider unrestricted superfields $\Phi^A(x, \theta, \bar{\theta}), \Phi_A(x, \theta, \bar{\theta})$ which map superspace to the set U . Of course, we will be interested in the generalisation to superfields mapping to the whole space rather than to the subset U , but for now we work with this restricted case. Then the symplectic form $\omega^{(2,0)}$ lifts to

$$\Omega = \int d^2x d^4\theta \delta\Phi_A \wedge \delta\Phi^A . \quad (3.14)$$

As in the Kähler case, the first line in equation (3.10) corresponds to a submanifold of the space of supermaps to U

$$\Phi_A = \frac{\partial K}{\partial \Phi^A} \quad (3.15)$$

that is real Lagrangian with respect to $\text{Re}(\Omega)$. The superfield constraints in (3.10) define a holomorphic isotropic submanifold, as Ω vanishes when restricted to superfields satisfying the constraints:

$$\Omega \Big|_{\text{constraints}} = \int d^2x d^4\theta \left(\delta\Phi_a \wedge \delta\Phi^a + \delta\Phi_{a'} \wedge \delta\Phi^{a'} + \delta\Phi_n \wedge \delta\Phi^n + \delta\Phi_{n'} \wedge \delta\Phi^{n'} \right) = 0 , \quad (3.16)$$

where we have used the superfield representations (A.16). Thus we see that the corresponding submanifold is holomorphic isotropic with respect to the complex structure on U . Using the language of generating functions we now show that the corresponding manifold arises as a Lagrangian submanifold of an enlarged space.

As in the Kähler case, we enlarge our space by adding additional fermionic fields Ψ_{+A}, Ψ_{-A} and take the symplectic form to be

$$\Omega_{\text{large}} = \int d^2x d^4\theta \left(\delta\Phi_A \wedge \delta\Phi^A - i\delta\Psi_+^A \wedge \delta\Psi_{-A} + i\delta\Psi_-^A \wedge \delta\Psi_{+A} \right) . \quad (3.17)$$

Next we choose two generating functions which depend on $(\Phi^A, \Psi_{+A}, \Psi_{-A})$. We define a real generating function

$$\mathcal{K} = \int d^2x d^4\theta K(\Phi^A, \bar{\Phi}^{\bar{A}}) , \quad (3.18)$$

which gives rise to a real Lagrangian submanifold with respect to $\text{Re}(\Omega)$. We also define the holomorphic generating function

$$\mathcal{H} = \int d^2x d^4\theta \left(i\Psi_{-a}\bar{\mathbb{D}}_+\Phi^a - i\Psi_{+a}\bar{\mathbb{D}}_-\Phi^a + i\Psi_{-a'}\bar{\mathbb{D}}_+\Phi^{a'} - i\Psi_{+a'}\bar{\mathbb{D}}_-\Phi^{a'} + i\Psi_{-n}\bar{\mathbb{D}}_+\Phi^n - i\Psi_{+n'}\bar{\mathbb{D}}_-\Phi^{n'} \right) , \quad (3.19)$$

which defines a holomorphic Lagrangian submanifold. The intersection of these two Lagrangian submanifolds give rise to the equations (3.10).

However, this local discussion, attractive though it is, does not in general extend to the full space as the local complex structure on U used here does not in general extend to a complex structure on the whole space, because the transition functions mix coordinates that are holomorphic with respect to the local complex structure with ones that are anti-holomorphic.

4 Global issues

In this section, we seek a global structure that is consistent with the equations (3.10). Recall that in the Kähler case we considered the sigma model's symmetries – the holomorphic diffeomorphisms and the Kähler gauge transformations – and saw in section 2 how requiring these to be symmetries of the equations (2.11) fixed the transformation of Σ under these transformations. This then led to a global structure in which these symmetries were used in the glueing relations. In this section, we will study the symmetries of the equations (3.10) and attempt to define a global structure by using these symmetries as glueing relations in the overlaps of patches.

There are two kinds of symmetry here. The first consists of those diffeomorphisms $\Phi^A \rightarrow \Phi'^A(\Phi^A, \bar{\Phi}^{\bar{A}})$ that are compatible with the choice of coordinates for symplectic foliations and with the various superfield constraints; these transformations are given in (A.17). Requiring these to extend to symmetries of (3.10) then determines the transformations of the fields φ_A , and these are then used in the glueing relations. The second kind of symmetry arises from the fact that the generalised Kähler potential is not uniquely defined, but is only defined up to transformations that generalise (2.15). On the intersection of two patches $U_\alpha \cap U_\beta$ the generalised Kähler potentials are related by the transformations [10]

$$K_\alpha - K_\beta = F_{\alpha\beta}^+(\phi, \chi, X_L) + \bar{F}^+(\bar{\phi}, \bar{\chi}, \bar{X}_L) + F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) + \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi, \bar{X}_R) , \quad (4.1)$$

since the combinations on the right hand side vanish identically after superintegration. For this to be a symmetry of the system of equations (3.10) we require that the fields φ_A are glued on the intersection of two patches $U_\alpha \cap U_\beta$ as follows

$$(\Sigma_a)_\beta = (\Sigma_a)_\alpha + \frac{\partial}{\partial \phi^a} \left(F_{\alpha\beta}^+(\phi, \chi, X_L) + F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) \right), \quad (4.2)$$

$$(\Lambda_{a'})_\beta = (\Lambda_{a'})_\alpha + \frac{\partial}{\partial \chi^{a'}} \left(F_{\alpha\beta}^+(\phi, \chi, X_L) + \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi, \bar{X}_R) \right), \quad (4.3)$$

$$(Y_{Ln})_\beta = (Y_{Ln})_\alpha + \frac{\partial}{\partial X_L^n} F_{\alpha\beta}^+(\phi, \chi, X_L), \quad (4.4)$$

$$(Y_{Rn'})_\beta = (Y_{Rn'})_\alpha + \frac{\partial}{\partial X_R^{n'}} F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R), \quad (4.5)$$

(These glueing relations are to be composed with the transition functions expressing the change of coordinates between the two patches.) As explained in Appendix B, the above shifts satisfy the cocycle conditions on the triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$.

Both kinds of symmetry should be compatible with the superfield redefinitions (A.17) allowed by the chirality constraints. It will be useful to state the most general superfield redefinitions that are compatible with the superfield constraints (we suppress all indices for the sake of clarity)

$$\begin{aligned} \phi &\longrightarrow f_1(\phi), \\ \Sigma &\longrightarrow f_2(\phi)\Sigma + f_3(\phi, \chi, X_L, Y_L) + f_4(\phi, \bar{\chi}, X_R, Y_R), \\ \chi &\longrightarrow f_5(\chi), \\ \Lambda &\longrightarrow f_6(\chi)\Lambda + f_7(\phi, \chi, X_L, Y_L) + f_8(\bar{\phi}, \chi, \bar{X}_R, \bar{Y}_R), \\ X_L &\longrightarrow f_9(\phi, \chi, X_L, Y_L), \\ X_R &\longrightarrow f_{10}(\phi, \bar{\chi}, X_R, Y_R), \\ Y_L &\longrightarrow f_9(\phi, \chi, X_L, Y_L), \\ Y_R &\longrightarrow f_{10}(\phi, \bar{\chi}, X_R, Y_R), \end{aligned} \quad (4.6)$$

where f_i are arbitrary functions of their arguments. We will use these field redefinitions as a guiding principle in our discussion of the global issues.

4.1 Sigma models with semi-chiral fields only

We consider first the special case in which the generalised Kähler geometry is described in terms of semi-chiral fields only. The system of equations (3.10) for the case with only semi-chiral fields reduces to

$$\begin{aligned}
Y_{Ln} &= \frac{\partial K}{\partial X_L^n} , & Y_{Rn'} &= \frac{\partial K}{\partial X_R^{n'}} \\
\bar{\mathbb{D}}_+ Y_{Ln} &= 0 , & \bar{\mathbb{D}}_- Y_{Rn'} &= 0 \\
\bar{\mathbb{D}}_+ X_L^n &= 0 , & \bar{\mathbb{D}}_- X_R^{n'} &= 0
\end{aligned} \tag{4.7}$$

We now look at the symmetries of this system. The constraints on the superfields allow the following transformations

$$X_L^n \rightarrow f^n(X_L, Y_L) , \quad Y_{Ln} \rightarrow g_n(X_L, Y_L) , \tag{4.8}$$

$$X_R^{n'} \rightarrow h^{n'}(X_R, Y_R) , \quad Y_{Rn'} \rightarrow k_{n'}(X_R, Y_R) , \tag{4.9}$$

for arbitrary functions f, g, h, k and the shifts (4.4)-(4.5) are just a special form of these transformations. This suggests that there is no linear structure associated with the Y directions so that, unlike the Kähler case, the conjugate coordinates are not fibre coordinates for some vector bundle.

The correct global interpretation of the system (4.7) follows from [4]. We now briefly review the relevant geometry. We introduce local coordinates $Z^A = (\zeta_L^n, \zeta_R^{n'})$ on the manifold M corresponding to the lowest components of the superfields $\varphi^A = (X_L^n, X_R^{n'})$ together with variables $P_A = (\pi_{Ln}, \pi_{Rn'})$ corresponding to the lowest components of the superfields $\varphi_A = (Y_{Ln}, Y_{Rn'})$. The space M has two complex structures I_\pm and it was shown in [4] that (ζ_L^n, π_{Ln}) can be taken as coordinates for M and that moreover they are holomorphic coordinates with respect to I_+ . The symplectic form $d\pi_{Ln} \wedge d\zeta_L^n$ is then holomorphic with respect to I_+ . Similarly, $(\zeta_R^{n'}, \pi_{Rn'})$ can be also taken as coordinates for M and these are holomorphic coordinates with respect to I_- . The symplectic structure $d\pi_{Rn'} \wedge d\zeta_R^{n'}$ is holomorphic with respect to I_- . The transformation from the coordinates (ζ_L, π_L) to the coordinates (ζ_R, π_R) is a canonical transformation that is generated by the generalised Kähler potential $K(\zeta_L, \zeta_R)$.

This extends to a global formulation. In each patch U for M , there are I_+ -holomorphic coordinates (ζ_L, π_L) and in each overlap $U \cap U'$ the coordinates (ζ_L, π_L) in U and the coordinates (ζ'_L, π'_L) in U' are related by I_+ -holomorphic reparameterisations, $\zeta'_L = \zeta'_L(\zeta_L, \pi_L)$, $\pi'_L = \pi'_L(\zeta_L, \pi_L)$. Similarly, I_- -holomorphic coordinates $(\zeta_R^{n'}, \pi_{Rn'})$ can also be introduced for each patch and the transition functions for the coordinates (ζ_R, π_R) are I_- -holomorphic reparameterisations $\zeta'_R = \zeta'_R(\zeta_R, \pi_R)$, $\pi'_R = \pi'_R(\zeta_R, \pi_R)$. These transition functions give glueing relations for the corresponding superfields that are consistent with (4.8), so that they preserve the constraints. Then in each patch, there is a generalised Kähler potential $K(\zeta_L, \zeta_R)$ generating the transformation between the I_+ -holomorphic coordinates (ζ_L^n, π_{Ln}) and the I_- -holomorphic coordinates $(\zeta_R^{n'}, \pi_{Rn'})$ and the potentials in overlapping patches have the glueing relations (4.1).

This can be elegantly reformulated in terms of a double space $\mathcal{Z} = M \times M$ with coordinates $(\zeta_L, \pi_L, \zeta_R, \pi_R)$. The first M is taken to have coordinates (ζ_L, π_L) which are holomorphic with respect to I_+ . It has the symplectic form

$$\omega_+^{(2,0)} = d\pi_{L_n} \wedge d\zeta_L^n \quad (4.10)$$

which is holomorphic with respect to I_+ . The second M is taken to have coordinates (ζ_R, π_R) that are holomorphic with respect to I_- and an I_- -holomorphic symplectic form

$$\omega_-^{(2,0)} = d\pi_{R_{n'}} \wedge d\zeta_R^{n'} . \quad (4.11)$$

The space $\mathcal{Z} = M \times M$ is itself equipped with the complex structure

$$I = I_+ \times I_- \quad (4.12)$$

and the symplectic structure

$$\omega^{(2,0)} = dP_A \wedge dZ^A = d\pi_{L_n} \wedge d\zeta_L^n + d\pi_{R_{n'}} \wedge d\zeta_R^{n'} , \quad (4.13)$$

which is holomorphic with respect to I . The splitting of the coordinates into P and Z is a choice of polarization and P, Z are Darboux coordinates. It will be useful to write this as

$$\mathcal{Z} = M_+ \times M_- , \quad (4.14)$$

where $M_+ = (M, I_+)$ and $M_- = (M, I_-)$.

Then the Lagrangian submanifold of \mathcal{Z} defined by the equation with a real function $K(\zeta_L^n, \bar{\zeta}_L^n, \zeta_R^{n'}, \bar{\zeta}_R^{n'})$

$$P_A = \frac{\partial K}{\partial Z^A} \quad (4.15)$$

gives a diagonally embedded submanifold M , with coordinates $(\zeta_L^n, \zeta_R^{n'})$, and this is the sigma model target space.

The space $\mathcal{Z} = M_+ \times M_-$ is then the setting for the equations (4.7). Then X_L, Y_L are supermaps to the first factor M_+ and X_R, Y_R are supermaps to the second factor M_- while X_L, X_R are supermaps to the Lagrangian submanifold M .

We now reformulate this in terms of *unconstrained* superfields Φ_A, Φ^A where $\Phi^A = (\zeta_L(x, \theta, \bar{\theta}), \zeta_R(x, \theta, \bar{\theta}))$ and $\Phi_A = (\pi_L(x, \theta, \bar{\theta}), \pi_R(x, \theta, \bar{\theta}))$. The space of supermaps

$$\mathcal{M} = \{\Phi : \mathbb{R}^{2|4} \longrightarrow M_+ \times M_-\} \quad (4.16)$$

is equipped with the holomorphic symplectic structure

$$\Omega = \int d^2x d^4\theta \delta\Phi_A \wedge \delta\Phi^A . \quad (4.17)$$

The equations of motion (4.7) then specify the intersection of two submanifolds of \mathcal{M} . The submanifold determined by

$$\Phi_A = \frac{\partial K}{\partial \Phi^A} \quad (4.18)$$

is a real Lagrangian submanifold with respect to $\text{Re}(\Omega)$, while the submanifold specified by the semi-chiral superfield constraints is a holomorphic isotropic submanifold with respect to Ω . As before, enlarging the space of supermaps by introducing auxiliary fermionic superfields gives a formulation in which the submanifold (of the enlarged space) specified by the semi-chiral superfield constraints is Lagrangian.

4.2 Sigma models with semi-chiral fields and chiral fields

For our next example, we consider the case in which there are chiral and semichiral superfields but there are no (χ, Λ) -fields in the equations (3.10). As we shall see, this combines features of the Kähler case (with only chiral fields) with features of the purely semi-chiral case from the previous subsection. The most general field redefinitions (A.17) compatible with the constraints can be written in this case as follows

$$\begin{aligned} \phi &\longrightarrow f_1(\phi) , \\ \Sigma &\longrightarrow f_2(\phi)\Sigma + f_3(\phi, X_L, Y_L) + f_4(\phi, X_R, Y_R) , \\ X_L &\longrightarrow f_9(\phi, X_L, Y_L) , \quad Y_L \longrightarrow \tilde{f}_9(\phi, X_L, Y_L) , \\ X_R &\longrightarrow f_{10}(\phi, X_R, Y_R) , \quad Y_R \longrightarrow \tilde{f}_{10}(\phi, X_R, Y_R) , \end{aligned} \quad (4.19)$$

(together with their complex conjugates) for some functions f_i . Importantly, there is no mixing between barred and unbarred fields in this case. Thus the holomorphic superfields $(\phi, \Sigma, X_L, Y_L, X_R, Y_R)$ are glued to each other and the transition functions do not depend on the conjugate fields and thus we have a complex structure on the space of superfields. Here the glueing combines two models, the deformed holomorphic cotangent bundle from the Kähler case for the (ϕ, Σ) coordinates and the product $M \times M$ for the semi-chiral coordinates.

We now explain the construction in more detail. We introduce local coordinates $Z^A = (z^a, \zeta_L^n, \zeta_R^{n'})$ on the manifold M corresponding to the lowest components of the superfields $\varphi^A = (\phi^a, X_L^n, X_R^{n'})$ together with variables $P_A = (p_a, \pi_{Ln}, \pi_{Rn'})$ corresponding to the lowest components of the superfields $\varphi_A = (\Sigma_a, Y_{Ln}, Y_{Rn'})$. The glueing relations (4.1) for the generalised Kähler potential $K(z, \bar{z}, \zeta_L, \zeta_R, \bar{\zeta}_L, \bar{\zeta}_R)$ in an overlap $U \cap U'$ become in this case

$$K' - K = F^+(z, \zeta_L) + F^-(z, \zeta_R) + \text{complex conjugate} \quad (4.20)$$

where K is the potential on U and K' is the potential on U' . With Z^A the coordinates of M , the P_A can be taken to be defined by

$$P_A = \frac{\partial K}{\partial Z^A} \quad (4.21)$$

Then (4.20) implies that the glueing conditions for P_A in an overlap $U \cap U'$ must be

$$p'_a = p_a + \frac{\partial}{\partial z^a} [F^+(z, \zeta_L) + F^-(z, \zeta_R)] , \quad (4.22)$$

$$\begin{aligned} \pi'_{Ln} &= \pi_{Ln} + \frac{\partial}{\partial \zeta_L^n} F^+(z, \zeta_L) , \\ \pi'_{Rn'} &= \pi_{Rn'} + \frac{\partial}{\partial \zeta_R^{n'}} F^-(z, \zeta_R) , \end{aligned} \quad (4.23)$$

where F^\pm are the functions appearing in (4.20) and these glueing relations satisfy the cocycle condition in triple overlaps (see Appendix B). Note that (4.23) are a special case of the transformations (4.26). Then the variables (Z^A, P_A) can be regarded as coordinates on a manifold, which we refer to as the ‘phase space’. We construct this phase space \mathcal{Z} explicitly below.

On M we can use the coordinates $Z^A = (z^a, \zeta_L^n, \zeta_R^{n'})$, or the I_+ -holomorphic coordinates (z, ζ_L, π_L) or the I_- -holomorphic coordinates (z, ζ_R, π_R) . The manifold M is foliated by subspaces of constant z . The coordinates on each leaf can be taken to be ζ_L, ζ_R or $\rho_L = (\zeta_L, \pi_L)$ or $\rho_R = (\zeta_R, \pi_R)$. Choosing the leaf coordinates to be $\rho_L = (\zeta_L, \pi_L)$, the transition functions in an overlap $U \cap U'$ are I_+ -holomorphic and of the form

$$z' = z'(z) , \quad \rho'_L = \rho'_L(z, \rho_L) , \quad (4.24)$$

which is consistent with (4.19). Alternatively, choosing the coordinates on each leaf to be $\rho_R = (\zeta_R, \pi_R)$, the transition functions in an overlap $U \cap U'$ are I_- -holomorphic of the form

$$z' = z'(z) , \quad \rho'_R = \rho'_R(z, \rho_R) . \quad (4.25)$$

We now construct a space \widehat{M} in which the leaves of this foliation are ‘doubled’. For each open set U of an atlas for M we can choose coordinates (z, ρ_L) or (z, ρ_R) adapted to the foliation. Then for each such U we introduce a space \widehat{U} with coordinates z, ρ_L, ρ_R . Then \widehat{M} is constructed by glueing together the patches \widehat{U} with transition functions

$$z' = z'(z) , \quad \rho'_L = \rho'_L(z, \rho_L) , \quad \rho'_R = \rho'_R(z, \rho_R) . \quad (4.26)$$

Note that the foliated structure of M is essential for this construction of \widehat{M} . This space \widehat{M} can be regarded as a quotient of $M_+ \times M_-$. With coordinates (z, ρ_L) for M_+ and (z', ρ_R) for M_- , taking the quotient by the relation $z \sim z'$ gives \widehat{M} .

The next step is to construct a bundle \mathcal{Z} over \widehat{M} with fibre coordinates p_a and glueing conditions (4.22). It is important that these satisfy the cocycle condition in triple overlaps.

Then \mathcal{Z} has coordinates Z^A, P_A and the transition functions are all holomorphic in Z^A, P_A and so this endows \mathcal{Z} with a complex structure. Moreover, \mathcal{Z} has a holomorphic symplectic structure

$$\omega^{(2,0)} = dP_A \wedge dZ^A = dp_a \wedge dz^a + d\pi_{L_n} \wedge d\zeta_L^n + d\pi_{R_{n'}} \wedge d\zeta_R^{n'} , \quad (4.27)$$

which is invariant under the glueing (4.22) supplemented by the diffeomorphisms (4.23). The equation

$$P_A = \frac{\partial K}{\partial Z^A} \quad (4.28)$$

specifies a real Lagrangian submanifold of \mathcal{Z} with respect to $\text{Re}(\omega^{(2,0)})$, which is the original manifold M .

We now consider the space of unrestricted supermaps $\Phi^A = Z^A(x, \theta, \bar{\theta})$, $\Phi_A = P_A(x, \theta, \bar{\theta})$

$$\mathcal{M} = \{ \Phi : \mathbb{R}^{2|4} \longrightarrow \mathcal{Z} \} , \quad (4.29)$$

which has the holomorphic symplectic structure

$$\Omega = \int d^2x d^4\theta \delta\Phi_A \wedge \delta\Phi^A . \quad (4.30)$$

The submanifold determined by

$$\Phi_A = \frac{\partial K}{\partial \Phi^A} \quad (4.31)$$

is a real Lagrangian submanifold with respect to $\text{Re}(\Omega)$ and is the space of unconstrained supermaps to M . The superfield constraints that $z^a(x, \theta, \bar{\theta}) = \phi^a$ is chiral, $p_a(x, \theta, \bar{\theta}) = \Sigma_a$ is complex linear, $\zeta_L^n(x, \theta, \bar{\theta}) = X_L^n$, $\pi_{L_n}(x, \theta, \bar{\theta}) = Y_{L_n}$ are left-semi-chiral and $\zeta_R^{n'}(x, \theta, \bar{\theta}) = X_R^{n'}$, $\pi^{R_{n'}}(x, \theta, \bar{\theta}) = Y_{R_{n'}}$ are right-semi-chiral specify a submanifold of \mathcal{M} that is a holomorphic isotropic submanifold with respect to Ω .

As in the previous cases, enlarging the space of supermaps by introducing auxiliary fermionic superfields Ψ should give a formulation in which the submanifold (of the enlarged space) specified by the superfield constraints is Lagrangian. This construction is straightforward locally. Globally, a problem which may arise with the Ψ fields is that of defining them globally since we need to understand the global structures associated with our construction and this may require extra input from the geometry, in particular a better understanding of the global issues related to symplectic realisations.

There is another interesting case to consider consisting of the system (3.10) with no (ϕ, Σ) superfields. If we write the general field redefinitions (A.17) for this specific case then we

have

$$\begin{aligned}
\chi &\longrightarrow f_5(\chi) , \\
\Lambda &\longrightarrow f_6(\chi)\Lambda + f_7(\chi, X_L) + f_8(\chi, \bar{X}_R) , \\
X_L &\longrightarrow f_9(\chi, X_L, Y_L) , \quad Y_L \longrightarrow \tilde{f}_9(\chi, X_L, Y_L) , \\
\bar{X}_R &\longrightarrow f_{10}(\chi, \bar{X}_R, \bar{Y}_R) , \quad \bar{Y}_R \longrightarrow \tilde{f}_{10}(\chi, \bar{X}_R, \bar{Y}_R) .
\end{aligned}$$

In these transformations the superfields $(\chi, \Lambda, X_L, Y_L, \bar{X}_R, \bar{Y}_R)$ do not mix with superfields $(\bar{\chi}, \bar{\Lambda}, \bar{X}_L, \bar{Y}_L, X_R, Y_R)$. Thus we can claim that the set $(\chi, \Lambda, X_L, Y_L, \bar{X}_R, \bar{Y}_R)$ corresponds to holomorphic coordinates and we again have a well-defined holomorphic symplectic structure

$$\Omega = \int d^2x d^4\theta \left(\delta\Phi_{a'} \wedge \delta\Phi^{a'} + \delta\Phi_n \wedge \delta\Phi^n + \delta\bar{\Phi}_{\bar{n}'} \wedge \delta\bar{\Phi}^{\bar{n}'} \right) \quad (4.32)$$

and the discussions from the previous case go through similarly, after interchanging $n' \leftrightarrow \bar{n}'$. Thus the system (3.10) without (ϕ, Σ) can again be interpreted as the intersection of real Lagrangian and holomorphic isotropic (or Lagrangian) submanifolds but with a modified holomorphic symplectic structure.

These two cases, either without (χ, Λ) or without (ϕ, Σ) , have target spaces which are generalised Kähler manifolds of symplectic type and the corresponding geometry has been studied in [8]. For any generalised Kähler manifold of symplectic type a ‘doubled space’ was constructed using a holomorphic Morita equivalence in [8] that is a holomorphic symplectic manifold. Here we have also constructed a ‘doubled’ holomorphic symplectic manifold (\mathcal{Z}, ω) and we conjecture that the two constructions in fact agree. As far as we can see, the present discussion is consistent with the global considerations in [8].

4.3 Chiral and twisted chiral case

We now consider the case of generalised Kähler geometry for which the corresponding sigma model is formulated only in terms of chiral and twisted chiral fields. We again investigate the global structure of the system (3.10) without X and Y fields by examining the symmetries of these equations. The chirality conditions allow the field redefinitions $\phi^a \rightarrow \phi'^a(\phi)$, $\chi^{a'} \rightarrow \chi'^{a'}(\chi)$ and the linear complex superfields Σ_a and twisted linear complex superfields $\Lambda_{a'}$ transform under these coordinate transformations as

$$\Sigma_a \rightarrow \Sigma'_a = \frac{\partial\phi^b}{\partial\phi'^a} \Sigma_b , \quad \Lambda_{a'} \rightarrow \Lambda'_{a'} = \frac{\partial\chi^{b'}}{\partial\chi'^{a'}} \Lambda_{b'} . \quad (4.33)$$

Note that this is compatible with the superfield structure (see Appendix A). In addition to these symmetries we have the following glueing of Σ 's and Λ 's

$$(\Sigma_a)_\beta = (\Sigma_a)_\alpha + \frac{\partial}{\partial \phi^a} \left(F_{\alpha\beta}^+(\phi, \chi) + F_{\alpha\beta}^-(\phi, \bar{\chi}) \right), \quad (4.34)$$

$$(\Lambda_{a'})_\beta = (\Lambda_{a'})_\alpha + \frac{\partial}{\partial \chi^{a'}} \left(F_{\alpha\beta}^+(\phi, \chi) + \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi) \right). \quad (4.35)$$

The structure here is again a deformation \mathcal{Z} of the cotangent bundle of the generalised Kähler manifold M , generalising that considered in the previous section. However there is an important difference as we do not have a holomorphic structure on \mathcal{Z} since the above transformations mix barred and unbarred fields. For a patch U_α of M (with $2d$ the real dimension of M), we introduce d local complex coordinates $Z_\alpha^A = (z_\alpha^a, w_\alpha^{a'})$ (corresponding to the lowest components of the superfields $\varphi^A = (\phi^a, \chi^{a'})$). Then for the cotangent bundle $T^*(U_\alpha) = U_\alpha \times \mathbb{R}^{2d}$ we introduce d complex fibre coordinates $(P_A)_\alpha = ((p_a)_\alpha, (r_{a'})_\alpha)$ (corresponding to the lowest components of the superfields $\varphi_A = (\Sigma_a, \Lambda_{a'})$). We then construct the deformed cotangent bundle \mathcal{Z} by glueing together the patches $T^*(U_\alpha)$ with the following transition conditions in $T^*(U_\alpha \cap U_\beta)$:

$$(p_a)_\beta = (p_a)_\alpha + \frac{\partial}{\partial z^a} \left(F_{\alpha\beta}^+ + F_{\alpha\beta}^- \right), \quad (4.36)$$

$$(r_{a'})_\beta = (r_{a'})_\alpha + \frac{\partial}{\partial w^{a'}} \left(F_{\alpha\beta}^+ + \bar{F}_{\alpha\beta}^- \right), \quad (4.37)$$

where the functions F^\pm have the following dependence on the coordinates:

$$F_{\alpha\beta}^+ = F_{\alpha\beta}^+(z^a, w^{a'}), \quad F_{\alpha\beta}^- = F_{\alpha\beta}^-(z^a, \bar{w}^{a'}). \quad (4.38)$$

The properties of the glueing in (4.1) were studied in [10] where it was shown that they are ultimately related to gerbes. One consequence of [10] is that the above glueing relations (4.36)-(4.37) satisfy the cocycle conditions in triple overlaps and thus they are consistent glueing rules for a bundle (see Appendix B).

In each patch $T^*(U_\alpha)$ there is, as the notation suggests, a complex structure that acts as $+i$ on (dZ^A, dP_A) , together with a holomorphic symplectic structure

$$\omega^{(2,0)} = dP_A \wedge dZ^A. \quad (4.39)$$

However, the transition functions (4.36)-(4.37) mix the holomorphic coordinates (Z^A, P_A) with the antiholomorphic ones $(\bar{Z}^{\bar{A}}, \bar{P}_{\bar{A}})$ so the local complex structure and holomorphic symplectic structure do not extend to holomorphic structures on \mathcal{Z} . Thus there is no *manifest* complex structure on \mathcal{Z} in general, although of course the generalised Kähler manifold has two complex structures I_\pm . The bundle is then a real bundle with fibres \mathbb{R}^{2d} and can't be

viewed as a holomorphic bundle using the complex structure on each patch. However, there is a well-defined real symplectic form

$$2 \operatorname{Re}(\omega^{(2,0)}) = dP_A \wedge dZ^A + d\bar{P}_{\bar{A}} \wedge d\bar{Z}^{\bar{A}}. \quad (4.40)$$

Explicit calculation shows that this symplectic form is invariant under the transformations in the transition functions (4.36)-(4.37), so that they are symplectomorphisms. Then $\operatorname{Re}(\omega)$ is globally well-defined on the deformed cotangent bundle \mathcal{Z} . In this way we obtain an affine symplectic bundle \mathcal{Z} . The transition functions here are closely related to gerbes, see [10]. Moreover, it is not clear how the space \mathcal{Z} encodes generalised Kähler geometry of M (this is known for the previous examples).

Next we define the infinite dimensional space of supermaps from $\mathbb{R}^{2|4}$ to the symplectic affine bundle constructed above

$$\mathcal{M} = \{ \Phi : \mathbb{R}^{2|4} \longrightarrow \mathcal{Z} \}, \quad (4.41)$$

which is equipped with the real symplectic form

$$2\operatorname{Re}(\Omega) = \int d^2x d^4\theta \left(\delta\Phi_a \wedge \delta\Phi^a + \delta\Phi_{a'} \wedge \delta\Phi^{a'} + \delta\bar{\Phi}_{\bar{a}} \wedge \delta\bar{\Phi}^{\bar{a}} + \delta\bar{\Phi}_{\bar{a}'} \wedge \delta\bar{\Phi}^{\bar{a}'} \right). \quad (4.42)$$

The condition

$$\Phi_A = \frac{\partial K}{\partial \Phi^A} \quad (4.43)$$

defines a real Lagrangian submanifold with respect to $\operatorname{Re}(\Omega)$. The superfields constraints with their complex conjugate define a real isotropic submanifold. In analogy with Kähler case we can introduce introduce fermionic fields and define the corresponding generating functions so that the subspace defined by the superfield constraints is a real Lagrangian submanifold.

4.4 Geometric Structure of General Case

We now consider the target space of the of general sigma model with chiral, twisted chiral and semi-chiral superfields. We introduce local coordinates $Z^A = (z^a, w^{a'}, \zeta_L^n, \zeta_R^{n'})$ on the manifold M corresponding to the lowest components of the superfields $\varphi^A = (\phi^a, \chi^{a'}, X_L^n, X_R^{n'})$ together with dual coordinates $\pi_{Ln}, \pi_{Rn'}$, corresponding to the lowest components of the superfields $Y_{Ln}, Y_{Rn'}$, that are defined by

$$\pi_L = \frac{\partial K}{\partial \zeta_L}, \quad \pi_R = \frac{\partial K}{\partial \zeta_R}, \quad (4.44)$$

where $K(Z, \bar{Z})$ is the generalised Kähler potential. Then z, w, ζ_L, π_L are I_+ -holomorphic coordinates on M and $z, \bar{w}, \zeta_R, \pi_R$ are I_- -holomorphic coordinates on M . On the overlap

$U_\alpha \cap U_\beta$ of two patches on M , (4.1) gives the glueing conditions for the generalised Kähler potential in the two patches to be

$$K_\alpha - K_\beta = F_{\alpha\beta}^+(z, w, \zeta_L) + \bar{F}^+(\bar{z}, \bar{w}, \bar{\zeta}_L) + F_{\alpha\beta}^-(z, \bar{w}, \zeta_R) + \bar{F}_{\alpha\beta}^-(\bar{z}, w, \bar{\zeta}_R) . \quad (4.45)$$

Note that F^+ is I_+ -holomorphic and F^- is I_- -holomorphic.

This can be recast in terms of a product space of the kind considered in the previous subsections:

$$\mathcal{N} = M_+ \times M_- , \quad (4.46)$$

where $M_+ = (M, I_+)$ and $M_- = (M, I_-)$ and \mathcal{N} has complex structure $I = I_+ \times I_-$. Taking z, w, ζ_L, π_L as I_+ -holomorphic coordinates on M_+ and z', w', ζ_R, π_R as I_- -holomorphic coordinates on M_- , the manifold M with coordinates $z^a, w^{a'}, \zeta_R^n, \zeta_R^{n'}$ is obtained as the submanifold specified by

$$z' = z , \quad w' = \bar{w} , \quad \pi_L = \frac{\partial K}{\partial \zeta_L} , \quad \pi_R = \frac{\partial K}{\partial \zeta_R} . \quad (4.47)$$

Note the twist: z' is identified with z but w' is identified with the *complex conjugate* of w .

The space M_+ is foliated by leaves on which z and w are constant, with coordinates ζ_L, π_L on each leaf. Similarly, M_- is foliated by leaves on which z' and w' are constant, with coordinates ζ_R, π_R on each leaf. Introducing a relation $z \sim z', w \sim \bar{w}'$, then taking the quotient

$$\widehat{M} = \mathcal{N} / \sim \quad (4.48)$$

gives a space with coordinates $z, w, \zeta_L, \pi_L, \zeta_R, \pi_R$, which can be thought of as the original space M with ‘doubled’ leaves, which agrees with the space \widehat{M} from subsection 4.2 in the case in which there are no twisted chiral fields and hence no coordinates w . Much of the discussion from subsection 4.2 extends to this case.

We now construct a bundle \mathcal{Z} over \widehat{M} with fibre coordinates $p_a, r_{a'}$ and glueing conditions

$$p'_a = p_a + \frac{\partial}{\partial z^a} [F^+(z, w, \zeta_L) + F^-(z, \bar{w}, \zeta_R)] , \quad (4.49)$$

$$r'_{a'} = r_{a'} + \frac{\partial}{\partial w^{a'}} [F^+(z, w, \zeta_L) + \bar{F}^-(\bar{z}, w, \zeta_R)] . \quad (4.50)$$

It is important that these satisfy the cocycle condition in triple overlaps (see Appendix B) so that this is indeed a fibre bundle. With these glueing rules, the section of \mathcal{Z} defined by

$$p_a = \frac{\partial}{\partial z^a} K, \quad r_{a'} = \frac{\partial}{\partial w^{a'}} K \quad (4.51)$$

is a global section defining a submanifold.

This is then the geometric setting for the equations (3.10). Globally we are able to define only a real symplectic structure and there is no natural complex structure. Thus at best we can interpret the equations (3.10) as the defining intersection of two real Lagrangian submanifolds in the space of unconstrained superfields. In the next section we present an alternative formulation which does have a holomorphic structure.

5 Extended geometry

In the previous sections we discussed the symplectic interpretation of the equations (3.10). In certain special cases we have a superfield phase space equipped with a complex structure and a holomorphic symplectic form. However, this is not the case in the general situation. We seek here an extension of the target space that carries a holomorphic symplectic structure. We find a surprising reformulation that has an interesting geometry and is dictated by the superfield structure.

5.1 Dual superfield presentation

The (twisted) complex linear superfields can be re-expressed in terms of semi-chiral superfields. The complex linear constraints

$$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma_a = 0, \quad \bar{\mathbb{D}}_+ \mathbb{D}_- \Lambda_{a'} = 0 \quad (5.1)$$

are solved by

$$\Sigma_a = \bar{\mathbb{D}}_+ \Psi_a^+ + \bar{\mathbb{D}}_- \Psi_a^+, \quad \Lambda_{a'} = \bar{\mathbb{D}}_+ \Upsilon_{a'}^+ + \mathbb{D}_- \Upsilon_{a'}^- \quad (5.2)$$

for some unconstrained spinor superfields $\Psi_a^\pm, \Upsilon_{a'}^\pm$. Then (twisted) complex linear superfields can be rewritten as

$$\Sigma_a = U_{La} + U_{Ra}, \quad \Lambda_{a'} = V_{La'} + \bar{V}_{Ra'}, \quad (5.3)$$

where

$$U_{La} = \bar{\mathbb{D}}_+ \Psi_a^+, \quad U_{Ra} = \bar{\mathbb{D}}_- \Psi_a^+, \quad (5.4)$$

$$V_{La'} = \bar{\mathbb{D}}_+ \Upsilon_{a'}^+, \quad \bar{V}_{Ra'} = \mathbb{D}_- \Upsilon_{a'}^- \quad (5.5)$$

are semi-chiral superfields. As we shall see, formulating the theory in terms of the semi-chirals U_L, U_R, V_L, V_R instead of the linear superfields Σ, Λ is very helpful. Using this we can rewrite the equations (3.10) as follows

$$\begin{aligned} U_{La} + U_{Ra} &= \frac{\partial K}{\partial \phi^a}, & V_{La'} + \bar{V}_{Ra'} &= \frac{\partial K}{\partial \chi^{a'}}, & Y_{Ln} &= \frac{\partial K}{\partial X_L^n}, & Y_{Rn'} &= \frac{\partial K}{\partial X_R^{n'}} \\ \bar{\mathbb{D}}_+ U_{La} &= 0, & \bar{\mathbb{D}}_- U_{Ra} &= 0, & \bar{\mathbb{D}}_+ V_{La'} &= 0, & \mathbb{D}_- \bar{V}_{Ra'} &= 0, & \bar{\mathbb{D}}_+ Y_{Ln} &= 0, & \bar{\mathbb{D}}_- Y_{Rn'} &= 0 \\ \bar{\mathbb{D}}_\pm \phi^a &= 0, & \bar{\mathbb{D}}_+ \chi^{a'} &= \mathbb{D}_- \chi^{a'} = 0, & \bar{\mathbb{D}}_+ X_L^n &= 0, & \bar{\mathbb{D}}_- X_R^{n'} &= 0 \end{aligned} \quad (5.6)$$

Let us discuss the symmetries of these equations. From the glueing conditions (4.2),(4.3) we get the following glueing in terms of the new variables

$$(U_{La})_\beta = (U_{La})_\alpha + \frac{\partial}{\partial \phi^a} F_{\alpha\beta}^+(\phi, \chi, X_L) , \quad (5.7)$$

$$(U_{Ra})_\beta = (U_{Ra})_\alpha + \frac{\partial}{\partial \phi^a} F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) , \quad (5.8)$$

$$(V_{La'})_\beta = (V_{La'})_\alpha + \frac{\partial}{\partial \chi^{a'}} F_{\alpha\beta}^+(\phi, \chi, X_L) , \quad (5.9)$$

$$(\bar{V}_{Ra'})_\beta = (\bar{V}_{Ra'})_\alpha + \frac{\partial}{\partial \chi^{a'}} \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi, \bar{X}_R) , \quad (5.10)$$

whereas the transformations (4.6) for Σ and Λ become

$$\begin{aligned} U_L &\longrightarrow f_2(\phi)U_L + f_3(\phi, \chi, X_L, Y_L) , \\ U_R &\longrightarrow f_2(\phi)U_R + f_4(\phi, \bar{\chi}, X_R, Y_R) , \\ V_L &\longrightarrow f_6(\chi)V_L + f_7(\phi, \chi, X_L, Y_L) , \\ \bar{V}_R &\longrightarrow f_6(\chi)\bar{V}_R + f_8(\bar{\phi}, \chi, \bar{X}_R, \bar{Y}_R) . \end{aligned} \quad (5.11)$$

The glueing conditions and transformations for U_L, V_L are I_+ -holomorphic and those for U_R, V_R are I_- -holomorphic. This would be consistent with (U_L, V_L) being fibre coordinates for some holomorphic bundle over M_+ and (U_R, V_R) being fibre coordinates for some holomorphic bundle over M_- . Moreover, the transformations (5.11) are not the most general ones consistent with the chirality constraints and their special form is indicative of a role as fibre coordinates. However, this is not the correct picture. The crucial point is that (5.7)-(5.10) do not satisfy the cocycle conditions on triple intersections (see Appendix B) so that there is no interpretation of (U_L, V_L) or (U_R, V_R) as extra coordinates for an enlarged manifold. Instead, adding the extra variables gives rise to a more general structure. We discuss this further below.

5.2 Geometric Structure

In subsection 4.4, we considered the manifold $\mathcal{N} = M_+ \times M_-$ with complex structure $I = I_+ \times I_-$. The manifold M was recovered as the submanifold specified by (4.47). The space \widehat{M} with coordinates $z, w, \zeta_L, \pi_L, \zeta_R, \pi_R$ (obtained from the space M by doubling the leaves) was constructed as the quotient $\widehat{M} = \mathcal{N} / \sim$ of \widehat{M} by the relation $z \sim z', w \sim \bar{w}'$. We then constructed a bundle \mathcal{Z} over \widehat{M} with fibre coordinates $p_a, r_{a'}$ and glueing conditions (4.49), but this did not have a natural complex structure.

We now introduce variables $p_{La}, p_{Ra}, r_{La'}, r_{Ra'}$ corresponding to the lowest components of the superfields $U_{La}, U_{Ra}, V_{La'}, V_{Ra'}$. The relations (5.3) give

$$p = p_L + p_R , \quad r = r_L + \bar{r}_R . \quad (5.12)$$

It would be natural to attempt to generalise the previous constructions by interpreting p_L, r_L as fibre coordinates for some bundle E_+ over M_+ and p_R, r_R as fibre coordinates for some bundle E_- over M_- so that $E_+ \times E_-$ has a natural complex structure and the bundle \mathcal{Z} can be constructed from it by taking the quotient by \sim and using (5.12). We shall develop this picture below, finding a natural holomorphic setting for our equations. However, as we shall see, the E_\pm that arise in this way are *not* fibre bundles and are instead generalised spaces.

For M_+ we use the I_+ -holomorphic coordinates z, w, ζ_L, π_L and for each patch U_α we introduce additional variables p_L, r_L parameterising local fibres of a bundle V_α over U_α . Over intersections $U_\alpha \cap U_\beta$, we have, from (5.7),(5.9), the transition functions

$$(p_{La})_\beta = (p_{La})_\alpha + \frac{\partial}{\partial z^a} F_{\alpha\beta}^+(z, w, \zeta_L) , \quad (5.13)$$

$$(r_{La'})_\beta = (r_{La'})_\alpha + \frac{\partial}{\partial w^{a'}} F_{\alpha\beta}^+(z, w, \zeta_L) , \quad (5.14)$$

$$(\pi_{Ln})_\beta = (\pi_{Ln})_\alpha + \frac{\partial}{\partial \zeta_L^n} F_{\alpha\beta}^+(z, w, \zeta_L) \quad (5.15)$$

The V_α are then glued using the I_+ -holomorphic glueing relations (5.13),(5.14) to construct some ‘generalised space’ E_+ . Locally, $p_{La}, r_{La'}$ are fibre coordinates for a bundle V_α over a patch U_α . However, globally this does not extend to a bundle over M_+ , or indeed any picture in which these variables are coordinates of some manifold. This is because the transition functions for $p_{La}, r_{La'}$ do not satisfy the cocycle condition in triple overlaps (see Appendix B) and so these variables cannot be interpreted as coordinates on any bundle over M_+ . Then globally, E_+ is not a bundle and not a manifold.

We proceed formally and note that we can introduce a complex structure and symplectic structure on each V_α . These are invariant under the glueing relations and so extend to structures on E_+ . Then E_+ has a complex structure I_+ for which $z, w, p_L, r_L, \zeta_L, \pi_L$ are holomorphic variables, and an I_+ -holomorphic symplectic structure

$$\omega_+^{(2,0)} = d\pi_{Ln} \wedge d\zeta_L^n + dp_{La} \wedge dz^a + dr_{La'} \wedge dw^{a'} , \quad (5.16)$$

which is invariant under the glueing relations.

Similarly, we take p_R, r_R to be additional variables for a generalised space E_- over M_- . From (5.8),(5.10) we have the transition functions

$$(p_{Ra})_\beta = (p_{Ra})_\alpha + \frac{\partial}{\partial z'^a} F_{\alpha\beta}^-(z', w', \zeta_R) , \quad (5.17)$$

$$(r_{Ra'})_\beta = (r_{Ra'})_\alpha + \frac{\partial}{\partial w'^{a'}} F_{\alpha\beta}^-(z', w', \zeta_R) , \quad (5.18)$$

$$(\pi_{Rn'})_\beta = (\pi_{Rn'})_\alpha + \frac{\partial}{\partial \zeta_R^n} F_{\alpha\beta}^-(z', w', \zeta_R) . \quad (5.19)$$

This has simliar properties to E_+ , and for the same reasons E_- does not define a manifold. However, E_- has a complex structure I_- , for which $z', w', p_R, r_R, \zeta_R, \pi_R$ are holomorphic, and it has a I_- -holomorphic symplectic structure

$$\omega_-^{(2,0)} = d\pi_{Rn'} \wedge d\zeta_R^{n'} + dp_{Ra} \wedge dz'^a + dr_{Ra'} \wedge dw'^{a'} , \quad (5.20)$$

which is invariant under the glueing.

To understand the geometry of the construction, recall that for the general (2,2) sigma models the 3-form H represents an integral cohomology class in the quantum theory and is the curvature for a gerbe on M . The interplay between the gerbe strucure and the complex structures was explored in [10] where it was found that a rich system of holomorphic gerbes underpins the theory. In particular, the transition functions $F_{\alpha\beta}^+$ and $F_{\alpha\beta}^-$ defined in intersections do not satisfy the cocycle condition that $\delta F^\pm \equiv F_{\alpha\beta}^\pm + F_{\beta\gamma}^\pm + F_{\gamma\alpha}^\pm$ vanishes (or vanishes modulo 2π when appropriately normalised) but instead δF^+ and δF^- each provides a non-trivial map from triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ to $U(1)$ that defines a gerbe structure. Moreover, as F^+ is I_+ -holomorphic and F^- is I_- -holomorphic, these are holomorphic gerbes. See Appendix B for more details. In our construction, 1-forms $p_{La}, r_{La'}$ are introduced on each patch of M_+ and glued together using the gerbe transition functions through (5.13),(5.14), while $p_{Ra}, r_{Ra'}$ are introduced on each patch of M_- and glued using (5.17),(5.18). In fact, they are glued in the same way as certain prepotentials for holomorphic gerbe connections (see Appendix B). This can be viewed as a generalisation of the Donaldson construction, where (2.19) implies that p has the glueing relations of a holomorphic connection.

The advantage of this construction is its manifest holomorphic structure and its holomorphic symplectic structure. We introduce the space

$$\mathcal{X} = E_+ \times E_- , \quad (5.21)$$

which has a complex structure $I = I_+ \times I_-$ and a I -holomorphic symplectic structure $\omega^{(2,0)} = \omega_+^{(2,0)} + \omega_-^{(2,0)}$. The I -holomorphic coordinates \widehat{Z}^A and conjugate variables \widehat{P}_A are

$$\widehat{Z}^A = (z, w, z', w', \zeta_L, \zeta_R) , \quad \widehat{P}_A = (p_L, r_L, p_R, r_R, \pi_L, \pi_R) . \quad (5.22)$$

Then the I -holomorphic symplectic structure can be written as

$$\omega^{(2,0)} = d\widehat{P}_A \wedge d\widehat{Z}^A . \quad (5.23)$$

This space is an extended arena for our equations. In particular, our equations lead to the subspace of this given by (4.47) together with

$$p_L + p_R = \frac{\partial K}{\partial z} , \quad r_L + \bar{r}_R = \frac{\partial K}{\partial w} . \quad (5.24)$$

Recall that the manifold M is obtained as the submanifold of $M_+ \times M_-$ specified by (4.47). Then from \mathcal{X} we obtain a subspace in which over each point in this submanifold M we have fibre coordinates p_L, r_L, p_R, r_R . We now define

$$p = p_L + p_R, \quad r = r_L + \bar{r}_R. \quad (5.25)$$

and from (5.13),(5.14),(5.17),(5.18) the transition functions for p, r are

$$(p_a)_\beta = (p_a)_\alpha + \frac{\partial}{\partial z^a} \left(F_{\alpha\beta}^+ + F_{\alpha\beta}^- \right), \quad (5.26)$$

$$(r_{a'})_\beta = (r_{a'})_\alpha + \frac{\partial}{\partial w^{a'}} \left(F_{\alpha\beta}^+ + \bar{F}_{\alpha\beta}^- \right), \quad (5.27)$$

where the functions F^\pm have the following dependence on the coordinates:

$$F_{\alpha\beta}^+ = F_{\alpha\beta}^+(z, w, \zeta_L), \quad F_{\alpha\beta}^- = F_{\alpha\beta}^-(z, \bar{w}, \zeta_R). \quad (5.28)$$

These glueing relations do satisfy the cocycle condition in triple overlaps (see Appendix B) and so define a bundle over M , with coordinates z, w, ζ_L, ζ_R on M and fibre coordinates p, r . As the functions F^\pm are independent of π_L, π_R , this bundle in fact extends to a bundle $\widehat{\mathcal{Z}}$ over \widehat{M} with fibre coordinates p, r and coordinates $z, w, \zeta_L, \pi_L, \zeta_R, \pi_R$ on \widehat{M} .

The space $\widehat{\mathcal{Z}}$ inherits a real symplectic structure from $\omega^{(2,0)}$ given by

$$\omega = dp_a \wedge dz^a + dr_{a'} \wedge d\bar{w}^{a'} + d\pi_{L_n} \wedge d\zeta_L^n + \pi_{R_n'} \wedge d\zeta_R^{n'} + \text{complex conjugate} \quad (5.29)$$

With the notation

$$Z^A = (z, w, \zeta_L, \zeta_R), \quad P_A = (p, r, \pi_L, \pi_R) \quad (5.30)$$

this symplectic structure can be written as

$$\omega = dP_A \wedge dZ^A + d\bar{P}^{\bar{A}} \wedge d\bar{Z}^{\bar{A}}. \quad (5.31)$$

However, \mathcal{Z} does not inherit a complex structure in general. In the special case in which there are no coordinates w, r this does inherit a complex structure from I for which $z, p, \zeta_L, \pi_L, \zeta_R, \pi_R$ are holomorphic coordinates; this is the case discussed in section 4.2, and similarly for the case with no coordinates z, p . In general, the symplectic structure ω of \mathcal{Z} is not holomorphic but descends from a holomorphic symplectic structure on \mathcal{N} .

The field equations specify the subspace of \mathcal{Z}

$$p_a = \frac{\partial K}{\partial z^a}, \quad r_{a'} = \frac{\partial K}{\partial w^{a'}}, \quad \pi_L = \frac{\partial K}{\partial \zeta_L}, \quad \pi_R = \frac{\partial K}{\partial \zeta_R}, \quad (5.32)$$

which is Lagrangian with respect to ω .

5.3 An Extended Formulation

We now consider supermaps to the extended space \mathcal{X}

$$\mathcal{Q} = \{ \Phi : \mathbb{R}^{2|4} \longrightarrow \mathcal{X} \} , \quad (5.33)$$

in order to interpret the superfield constraints as defining an isotropic submanifold. The coordinates on \mathcal{X} are $\widehat{Z}^A, \widehat{P}_A$ given by (5.22) with holomorphic symplectic structure $\omega^{(2,0)} = d\widehat{P}_A \wedge d\widehat{Z}^A$. The supermaps are unconstrained superfields $\widehat{\Phi}^A = \widehat{Z}^A(x, \theta, \bar{\theta})$, $\widehat{\Phi}_A = \widehat{P}_A(x, \theta, \bar{\theta})$ and this space has a holomorphic symplectic structure derived from (5.23), which is

$$\widehat{\Omega} = \int d^2x d^4\theta \delta\widehat{\Phi}_A \wedge \delta\widehat{\Phi}^A . \quad (5.34)$$

We now wish to formulate the superfield constraints as defining a submanifold of \mathcal{Q} . For $\zeta_L, \zeta_R, \pi_L, \pi_R, p_L, r_L, p_R, r_R$ we choose the constraints that correspond to identifying these supermaps with $X_L, X_R, Y_L, Y_R, U_L, V_L, U_R, V_R$ respectively. For z, z', w, w' we want constraints such that, on setting $z = z', w = \bar{w}'$, the supermap $z = z'$ corresponds to the chiral superfield ϕ and the supermap $w = \bar{w}'$ corresponds to the twisted chiral superfield χ . One way of doing this is to take z, z' to be chiral and w, w' to be twisted chiral. However, there is another possibility involving semi-chirals instead. We choose constraints that identify z, z', w, w' with semi-chiral fields Z, W :

$$z \sim Z_L , \quad z' \sim Z_R , \quad w \sim W_L , \quad w' \sim W_R . \quad (5.35)$$

Then $z = z'$ corresponds to

$$Z_L = Z_R \quad (5.36)$$

and the constraints on Z_L, Z_R imply that $\phi \equiv Z_L = Z_R$ is a chiral superfield. Similarly, $w = \bar{w}'$ corresponds to

$$W_L = \bar{W}_R \quad (5.37)$$

and the constraints on W_L, W_R imply that $\chi \equiv W_L = \bar{W}_R$ is a twisted chiral superfield.

This gives a formalism in which *all* superfields are semi-chiral. The superfields are

$$X_L, X_R, Y_L, Y_R, U_L, V_L, U_R, V_R, Z_L, Z_R, W_L, W_R \quad (5.38)$$

and the constraints are

$ \begin{aligned} U_{La} &= \frac{\partial \widehat{K}}{\partial Z_L^a} , & Z_R^a &= \frac{\partial \widehat{K}}{\partial U_{Ra}} , & V_{Ra'} &= \frac{\partial \widehat{K}}{\partial W_R^{a'}} , & W_L^{a'} &= \frac{\partial \widehat{K}}{\partial V_{La'}} , & Y_{Ln} &= \frac{\partial \widehat{K}}{\partial X_L^n} , & Y_{Rn'} &= \frac{\partial \widehat{K}}{\partial X_R^{n'}} \\ \bar{\mathbb{D}}_+ U_{La} &= 0 , & \bar{\mathbb{D}}_- U_{Ra} &= 0 , & \bar{\mathbb{D}}_+ V_{La'} &= 0 , & \bar{\mathbb{D}}_- V_{Ra'} &= 0 , & \bar{\mathbb{D}}_+ Y_{Ln} &= 0 , & \bar{\mathbb{D}}_- Y_{Rn'} &= 0 \\ \bar{\mathbb{D}}_+ X_L^n &= 0 , & \bar{\mathbb{D}}_- X_R^{n'} &= 0 , & \bar{\mathbb{D}}_+ Z_L^a &= 0 , & \bar{\mathbb{D}}_- Z_R^a &= 0 , & \bar{\mathbb{D}}_+ W_L^{a'} &= 0 , & \bar{\mathbb{D}}_- W_R^{a'} &= 0 \end{aligned} $	(5.39)
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where we have defined

$$\widehat{K} = K(Z_L, \bar{Z}_L, \bar{W}_R, W_R, X_L, \bar{X}_L, X_R, \bar{X}_R) - U_{Ra} Z_L^a - \bar{U}_{R\bar{a}} \bar{Z}_L^{\bar{a}} - V_{La'} \bar{W}_R^{a'} - \bar{V}_{L\bar{a}'} W_R^{\bar{a}'} \quad (5.40)$$

in terms of the generalised Kähler potential $K(\phi, \bar{\phi}, \chi, \bar{\chi}, X_L, \bar{X}_L, X_R, \bar{X}_R)$. One can easily check that the above equations are equivalent to the equations (5.6). The two last lines in (5.39) are the semi-chirality constraints and can be interpreted as defining a holomorphic isotropic submanifold of \mathcal{Q} with respect to $\widehat{\Omega}$. The first line in (5.39) defines a real Lagrangian submanifold of \mathcal{Q} with respect to $\text{Re}(\widehat{\Omega})$. The interesting new feature of this extended construction is that the chirality constraints for chiral and twisted chiral superfields appear as emergent constraints from the intersection of a Lagrangian submanifold with an isotropic one.

The ambiguities (4.1) in the definition of K are now realised as holomorphic diffeomorphisms

$$\begin{aligned} U_L &\rightarrow U_L + \partial_{Z_L} F^+(Z_L, W_L, X_L) , \\ V_L &\rightarrow V_L + \partial_{W_L} F^+(Z_L, W_L, X_L) , \\ Y_L &\rightarrow Y_L + \partial_{X_L} F^+(Z_L, W_L, X_L) , \\ V_R &\rightarrow V_R + \partial_{W_R} F^-(Z_R, W_R, X_R) , \\ U_R &\rightarrow U_R + \partial_{Z_R} F^-(Z_R, W_R, X_R) , \\ Y_R &\rightarrow Y_R + \partial_{X_R} F^-(Z_R, W_R, X_R) , \end{aligned} \quad (5.41)$$

which obviously respect the corresponding chirality constraints.

6 Doubly extended space and superfields

In the previous section we constructed an extension of the original superfield phase space that is equipped with a holomorphic symplectic form. The equations of motion and constraints are equivalent to the intersection of a real Lagrangian submanifold with a holomorphic isotropic submanifold. In our construction of \mathcal{X} , the set of coordinates z, w were ‘quadrupled’ to the set $z, z', w, w, p_L, p_R, r_L, r_R$ while the leaf coordinates ζ_L, ζ_R were doubled to $\zeta_L, \zeta_R, \pi_L, \pi_R$. In this section, we present an alternative formulation in which all coordinates are quadrupled. Here we present the superfield formulation and proceed formally. The present construction could be related to the double groupoid picture outlined in [14].

6.1 Model with semi-chiral superfields only

Let us start with the discussion of the case in which there are only semi-chiral fields. We follow the discussion and notation from subsection 4.1. We double the construction given

there and define a complex manifold

$$\mathcal{Z} \times \mathcal{Z} = M_+ \times M_- \times M_+ \times M_- \quad (6.1)$$

with the holomorphic symplectic structure

$$\omega^{(2,0)} = dP_A \wedge dZ^A + dP_A^* \wedge dZ^{*A} = d\pi_{L_n} \wedge d\zeta_L^n + d\pi_{R_{n'}} \wedge d\zeta_R^{n'} + d\pi_{L_n}^* \wedge d\zeta_L^{*n} + d\pi_{R_{n'}}^* \wedge d\zeta_R^{*n'} , \quad (6.2)$$

where the coordinates with a star are the coordinates for the second \mathcal{Z} . Doubling the coordinates also doubles the corresponding superfields. We use the same notation, adding a star for the superfields taking values in the second \mathcal{Z} . For a given generalized Kähler potential $K(X_L, \bar{X}_L, X_R, \bar{X}_R)$ we define the following real generating function

$$\widehat{K} = K(X_L, \bar{X}_L, X_R, \bar{X}_R) - X_L^n Y_{L_n}^* - \bar{X}_L^{\bar{n}} \bar{Y}_{L_{\bar{n}}}^* - X_R^{n'} Y_{R_{n'}}^* - \bar{X}_R^{\bar{n}'} \bar{Y}_{R_{\bar{n}'}}^* , \quad (6.3)$$

which depends on exactly half of the fields. We observe that \widehat{K} defines the submanifold specified by the following equations

$$\begin{aligned} Y_{L_n} &= \frac{\partial \widehat{K}}{\partial X_L^n} = \frac{\partial K}{\partial X_L^n} - Y_{L_n}^* , \\ X_L^{*n} &= \frac{\partial \widehat{K}}{\partial Y_{L_n}^*} = X_L^n , \\ Y_{R_{n'}} &= \frac{\partial \widehat{K}}{\partial X_R^{n'}} = \frac{\partial K}{\partial X_R^{n'}} - Y_{R_{n'}}^* , \\ X_R^{*n'} &= \frac{\partial \widehat{K}}{\partial Y_{R_{n'}}^*} = X_R^{n'} . \end{aligned} \quad (6.4)$$

If we make the shifts $Y_L \rightarrow Y_L + Y_L^*$ and $Y_R \rightarrow Y_R + Y_R^*$ we recover the equations (4.7).

The space of unconstrained superfields

$$\{\Phi : \mathbb{R}^{2|4} \longrightarrow \mathcal{Z} \times \mathcal{Z}\} , \quad (6.5)$$

is equipped with the holomorphic symplectic form

$$\Omega = \int d^2x d^4\theta \left(\delta\Phi_A \wedge \delta\Phi^A + \delta\Phi_A^* \wedge \delta\Phi^{*A} \right) . \quad (6.6)$$

The semi-chiral constraints on the superfields correspond to a holomorphic isotropic submanifold with respect to Ω . On the other hand, the equations (6.4) correspond to a real Lagrangian submanifold with respect to $\text{Re}(\Omega)$. Thus we conclude that the original equations of motion and constraints can be interpreted as the intersection of these two submanifolds.

6.2 General case

Formally, the general case can be viewed as a combination of the constructions from subsections 5.3 and 6.1. For a generalised Kähler potential $K(\phi, \bar{\phi}, \chi, \bar{\chi}, X_L, \bar{X}_L, X_R, \bar{X}_R)$ we define

$$\begin{aligned} \widehat{K} = & K(Z_L, \bar{Z}_L, \bar{W}_R, W_R, X_L, \bar{X}_L, X_R, \bar{X}_R) - U_{Ra} Z_L^a - \bar{U}_{R\bar{a}} \bar{Z}_L^{\bar{a}} - V_{La'} \bar{W}_R^{a'} - \bar{V}_{L\bar{a}'} W_R^{\bar{a}'} \\ & - X_L^n Y_{Ln}^* - \bar{X}_L^{\bar{n}} \bar{Y}_{L\bar{n}}^* - X_R^{n'} Y_{Rn'}^* - \bar{X}_R^{\bar{n}'} \bar{Y}_{R\bar{n}'}^* . \end{aligned} \quad (6.7)$$

Then the equations of motion can be written as follows

$$\begin{aligned} U_{La} = \frac{\partial \widehat{K}}{\partial Z_L^a} , \quad Z_R^a = \frac{\partial \widehat{K}}{\partial U_{Ra}} , \quad V_{Ra'} = \frac{\partial \widehat{K}}{\partial W_R^{a'}} , \quad W_L^{a'} = \frac{\partial \widehat{K}}{\partial V_{La'}} , \\ X_L^{*n} = \frac{\partial \widehat{K}}{\partial Y_{Ln}^*} , \quad X_R^{*n'} = \frac{\partial \widehat{K}}{\partial Y_{Rn'}^*} , \quad Y_{Ln} = \frac{\partial \widehat{K}}{\partial X_L^n} , \quad Y_{Rn'} = \frac{\partial \widehat{K}}{\partial X_R^{n'}} , \end{aligned} \quad (6.8)$$

provided that we impose the semi-chiral conditions on all fields. All local geometry and holomorphic symplectic structures follow from those in subsections 5.3 and 6.1. In this bigger space we can interpret the equations of motion as the intersection of a holomorphic isotropic submanifold (imposing the semi-chiral conditions on the superfields) and the real Lagrangian submanifold defined by \widehat{K} . As previously discussed, the ambiguities in K are now realized as diffeomorphisms in this bigger space.

7 Summary and outlook

We have taken a fresh look at generalised Kähler geometry motivated by the $N = (2, 2)$ superfield formulation of the corresponding sigma model. Instead of looking at the action, we have developed a formulation with a doubled target space, and constraining to a Lagrangian submanifold implies the field equations. This might be viewed as a kind of superspace Hamiltonian formalism.

Somewhat surprisingly, alternative superfield formulations leads us to quadruple some or all of the superfields so as to elegantly embed the original equations of motion in a bigger framework. We saw that in some situations our glueing conditions do not satisfy the cocycle condition on triple intersections so that the space is not properly a manifold, but formally these spaces appear to have well-defined complex and symplectic structures.

One of the motivations of this work was to try find a physicists' explanation of the construction of [8]. It is interesting to also compare with the PhD thesis by Yucong Jiang

[14] where bigger spaces associated with generalised Kähler manifold are constructed. There are interesting formal similarities and it would be interesting to understand the relation between these constructions in more detail.

One drawback of our discussion is that it assumes the regularity of the Poisson foliations appearing in the generalised Kähler geometry. We need to impose this requirement in order to use the original superfield formalism involving different kind of fields (chiral, twisted chiral and semi-chiral) globally. However, in general this is not the case: generalised Kähler geometry can exhibit type change so that different regions of the space will have different foliation structures [6]. In Section 6 we reformulated the equations of motion in terms of semi-chiral fields only with a quadrupled target space. In this formulation the chirality conditions are emergent, arising from imposing a real Lagrangian condition via the real generating function \widehat{K} defined on the bigger space. We can imagine a situation in which the changes in \widehat{K} lead to changes in the superspace constraints (corresponding to type change). That is, a formulation in such an extended space could provide a global description even for target spaces in which there is type change. We find this observation exciting and plan to study it further elsewhere.

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A Appendix: $N = (1, 1)$ and $N = (2, 2)$ superspace conventions

Here we summarise briefly our superspace conventions. We follow the conventions used in [15] and in [4]. For a detailed introduction to superfield formalism, see [9].

We work in 2-dimensional Minkowski space and use two-component spinors ψ^α which carry a spinor index $\alpha = (+, -)$. The spinor indices are raised and lowered using the anti-symmetric charge conjugation matrix $C_{\alpha\beta}$ with

$$C_{+-} = i, \quad C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta}, \quad (\text{A.1})$$

so that

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta C_{\beta\alpha}, \quad \psi_\pm = \mp i\psi^\mp. \quad (\text{A.2})$$

The Lorentz-invariant spinor inner product is

$$C_{\alpha\beta}\psi^\alpha\chi^\beta = \psi_\alpha\chi^\alpha = i\psi^+\chi^- - i\psi^-\chi^+ \quad (\text{A.3})$$

and its complex conjugate is

$$(C_{\alpha\beta}\psi^\alpha\chi^\beta)^* = -i(\psi^+)^*(\chi^-)^* + i(\psi^-)^*(\chi^+)^* = C_{\alpha\beta}(\chi^\alpha)^*(\psi^\beta)^* \quad (\text{A.4})$$

(where we have assumed that both ψ and χ are Grassman odd).

The $N = (1, 1)$ superspace $\mathbb{R}^{2|2}$ has an even sector which is two-dimensional Minkowski space with coordinates $x^{\alpha\beta}$ which we represent as (symmetric traceless) bispinors, and we use the notation $x^{++} = x^{++}, x^{--} = x^{--}$. The corresponding even derivative is written as $\partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}$. The Grassmann (odd) coordinates θ^α are real (Majorana) two component spinors. On $\mathbb{R}^{2|2}$ we define two odd first order differential operators D_α and Q_α as follows

$$D_+ = \partial_+ + i\theta^+\partial_{++}, \quad D_- = \partial_- + i\theta^-\partial_{--}, \quad (\text{A.5})$$

$$Q_+ = i\partial_+ + \theta^+\partial_{++}, \quad Q_- = i\partial_- + \theta^-\partial_{--}. \quad (\text{A.6})$$

They satisfy the algebra

$$D_+^2 = i\partial_{++}, \quad D_-^2 = i\partial_{--}, \quad \{D_+, D_-\} = 0, \quad (\text{A.7})$$

$$Q_+^2 = i\partial_{++}, \quad Q_-^2 = i\partial_{--}, \quad \{Q_+, Q_-\} = 0, \quad (\text{A.8})$$

with

$$\{Q_\alpha, D_\beta\} = 0. \quad (\text{A.9})$$

In the supersymmetry literature, the D_α are referred to as spinorial covariant derivatives and the Q_α are referred to as the supersymmetry generators. On the space of maps

$$\{\mathbb{R}^{2|2} \longrightarrow M\} \quad (\text{A.10})$$

Q_α and D_α can be viewed as odd vector fields satisfying the algebra presented above.

Next we discuss the $N = (2, 2)$ superspace $\mathbb{R}^{2|4}$. The bosonic coordinates are $x^{\alpha\beta}$ as before, but now the grassmann variables are *complex* spinors θ^α with complex conjugates $\bar{\theta}^\alpha$ and odd derivatives $(\partial_\alpha, \bar{\partial}_\alpha)$. We define two sets of first order odd differential operators $(\mathbb{D}_\alpha, \bar{\mathbb{D}}_\alpha)$ and $(\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha)$ by

$$\mathbb{D}_+ = \partial_+ + \frac{i}{2}\bar{\theta}^+\partial_{++}, \quad \bar{\mathbb{D}}_+ = \bar{\partial}_+ + \frac{i}{2}\theta^+\partial_{++}, \quad (\text{A.11})$$

$$\mathbb{Q}_+ = i\partial_+ + \frac{1}{2}\bar{\theta}^+\partial_{++}, \quad \bar{\mathbb{Q}}_+ = i\bar{\partial}_+ + \frac{1}{2}\theta^+\partial_{++}, \quad (\text{A.12})$$

for the + sector, with analogous formulae for the – sector. The algebra is given by the following relations

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = i\partial_{++}, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = i\partial_{--}, \quad \{\mathbb{D}_\alpha, \mathbb{D}_\beta\} = 0, \quad \{\bar{\mathbb{D}}_\alpha, \bar{\mathbb{D}}_\beta\} = 0, \quad (\text{A.13})$$

$$\{\mathbb{Q}_+, \bar{\mathbb{Q}}_+\} = i\partial_{++}, \quad \{\mathbb{Q}_-, \bar{\mathbb{Q}}_-\} = i\partial_{--}, \quad \{\mathbb{Q}_\alpha, \mathbb{Q}_\beta\} = 0, \quad \{\bar{\mathbb{Q}}_\alpha, \bar{\mathbb{Q}}_\beta\} = 0, \quad (\text{A.14})$$

where all $\mathbb{Q}/\bar{\mathbb{Q}}$ -operators anti commute with all $\mathbb{D}/\bar{\mathbb{D}}$ -operators. Again, in the supersymmetry literature, $\mathbb{D}_\alpha, \bar{\mathbb{D}}_\alpha$ are referred to as spinorial covariant derivatives and $\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha$ are referred to as supersymmetry generators. On the space of maps

$$\{\mathbb{R}^{2|4} \longrightarrow M\} \quad (\text{A.15})$$

$\mathbb{D}_\alpha, \bar{\mathbb{D}}_\alpha$ and $\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha$ can be interpreted as odd vector fields with the algebra given above.

Here we have a new feature, we can consider maps which are annihilated by specific nilpotent combinations of \mathbb{D} 's and $\bar{\mathbb{D}}$'s so that the supersymmetry algebra generated by $\mathbb{Q}, \bar{\mathbb{Q}}$ acts on the subspace that is constrained in this way. That is, the superfields constrained in this way furnish a representation of the supersymmetry algebra generated by $\mathbb{Q}, \bar{\mathbb{Q}}$ and $\frac{\partial}{\partial x^{\alpha\beta}}$. Below we summarise the constraints, that we use throughout the paper

$\bar{\mathbb{D}}_\pm \phi = 0$	chiral superfield
$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Sigma = 0$	complex linear superfield
$\bar{\mathbb{D}}_+ \chi = 0, \quad \mathbb{D}_- \chi = 0$	twisted chiral fields
$\bar{\mathbb{D}}_+ \mathbb{D}_- \Lambda = 0$	twisted complex linear superfield
$\bar{\mathbb{D}}_+ \bar{X}_L = 0$ (or $\bar{\mathbb{D}}_+ \bar{Y}_L = 0$)	left semichiral superfield
$\bar{\mathbb{D}}_- X_R = 0$ (or $\bar{\mathbb{D}}_- Y_R = 0$)	right semichiral superfield

Each of these constraints has a general solution in terms of unconstrained complex superfields. These are as follows:

$$\begin{aligned} \phi &= \bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- \Phi \\ \Sigma &= \bar{\mathbb{D}}_+ \Phi_1 + \bar{\mathbb{D}}_- \Phi_2 \\ \chi &= \bar{\mathbb{D}}_+ \mathbb{D}_- \Phi \\ \Lambda &= \bar{\mathbb{D}}_+ \Phi_1 + \mathbb{D}_- \Phi_2 \\ X_L &= \bar{\mathbb{D}}_+ \Phi \\ X_R &= \bar{\mathbb{D}}_- \Phi \end{aligned} \quad (\text{A.16})$$

Here the Φ 's are unrestricted complex superfields.

The constrains on the superfields allow the following list of field redefinitions which are

compatible with the constraints

$$\begin{aligned}
\phi &\longrightarrow f_1(\phi) , \\
\Sigma &\longrightarrow f_2(\phi)\Sigma + f_3(\phi, \chi, X_L) + f_4(\phi, \bar{\chi}, X_R) , \\
\chi &\longrightarrow f_5(\chi) , \\
\Lambda &\longrightarrow f_6(\chi)\Lambda + f_7(\phi, \chi, X_L) + f_8(\bar{\phi}, \chi, \bar{X}_R) , \\
X_L &\longrightarrow f_9(\phi, \chi, X_L) , \\
X_R &\longrightarrow f_{10}(\phi, \bar{\chi}, X_R) ,
\end{aligned} \tag{A.17}$$

(together with their complex conjugates) and here f_i are arbitrary functions. In the context of non-linear sigma models the lowest components of these superfields are interpreted as coordinates on the target space M and thus we get the restrictions on the diffeomorphisms that are compatible with the constraints, and these in turn are related to geometric structures on M .

B Appendix: Generalised Kähler geometry, gerbes and glueings

In this appendix we review the relevant facts about generalised Kähler geometry and gerbes. We follow closely [10] and point out some properties which were not discussed in [10] but which are important for this paper. We consider a smooth manifold M with an open cover $\{U_\alpha\}$ such that all open sets and finite intersections thereof are contractible.

A set of transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$ with $(f_{\alpha\beta} = -f_{\beta\alpha})$ from the intersection of any two patches $U_\alpha \cap U_\beta$ to \mathbb{R} define a real line bundle if they satisfy the cocycle condition on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ that

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \tag{B.1}$$

while if they satisfy this modulo 2π then the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U(1)$ with $g_{\alpha\beta} = e^{if_{\alpha\beta}}$ satisfy the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \tag{B.2}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and so define a complex line bundle. A gerbe is a set of maps on triple intersections $h_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow \mathbb{R}$ satisfying

$$h_{\beta\gamma\delta} + h_{\delta\gamma\alpha} + h_{\alpha\beta\delta} + h_{\beta\alpha\gamma} = 0 \tag{B.3}$$

modulo 2π so that $k_{\alpha\beta\gamma} = e^{ih_{\alpha\beta\gamma}}$ are maps $k_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1)$ satisfying

$$k_{\beta\gamma\delta}k_{\delta\gamma\alpha}k_{\alpha\beta\delta}k_{\beta\alpha\gamma} = 1 \tag{B.4}$$

We follow the setup and the notation of [10]. On the intersection of two patches $U_\alpha \cap U_\beta$ the generalized real Kähler potential has the glueing

$$K_\alpha - K_\beta = F_{\alpha\beta}^+(\phi, \chi, X_L) + \bar{F}^+(\bar{\phi}, \bar{\chi}, \bar{X}_L) + F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) + \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi, \bar{X}_R) . \quad (\text{B.5})$$

The functions F^\pm are defined up to the following ambiguities

$$F_{\alpha\beta}^+(\phi, \chi, X_L) \rightarrow F_{\alpha\beta}^+(\phi, \chi, X_L) + \rho_{\alpha\beta}(\phi) + \sigma_{\alpha\beta}(\chi) , \quad (\text{B.6})$$

$$F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) \rightarrow F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) - \rho_{\alpha\beta}(\phi) - \bar{\sigma}_{\alpha\beta}(\bar{\chi}) \quad (\text{B.7})$$

and they can be chosen such that $F_{\alpha\beta}^\pm = -F_{\beta\alpha}^\pm$. Moreover they satisfy the following conditions on the triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$

$$F_{\alpha\beta}^+(\phi, \chi, X_L) + F_{\beta\gamma}^+(\phi, \chi, X_L) + F_{\gamma\alpha}^+(\phi, \chi, X_L) = ic_{\alpha\beta\gamma}(\phi) - ib_{\alpha\beta\gamma}(\chi) , \quad (\text{B.8})$$

$$F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) + F_{\beta\gamma}^-(\phi, \bar{\chi}, X_R) + F_{\gamma\alpha}^-(\phi, \bar{\chi}, X_R) = -ic_{\alpha\beta\gamma}(\phi) - i\bar{b}_{\alpha\beta\gamma}(\bar{\chi}) \quad (\text{B.9})$$

for some functions $c_{\alpha\beta\gamma}(\phi), b_{\alpha\beta\gamma}(\chi)$. On quadruple intersections $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$c_{\beta\gamma\delta}(\phi) + c_{\delta\gamma\alpha}(\phi) + c_{\alpha\beta\delta}(\phi) + c_{\beta\alpha\gamma}(\phi) = \frac{i}{4}d_{\alpha\beta\gamma\delta} , \quad (\text{B.10})$$

$$b_{\beta\gamma\delta}(\chi) + b_{\delta\gamma\alpha}(\chi) + b_{\alpha\beta\delta}(\chi) + b_{\beta\alpha\gamma}(\chi) = \frac{i}{4}d_{\alpha\beta\gamma\delta} , \quad (\text{B.11})$$

where $d_{\alpha\beta\gamma\delta}$ is constant. If the 3-form $H/2\pi$ is quantised, i.e. if it represents an integral cohomology class, then $d_{\alpha\beta\gamma\delta} \in 2\pi\mathbb{Z}$ so that $e^{4c_{\alpha\beta\gamma}}$ and $e^{4b_{\alpha\beta\gamma}}$ each satisfy (B.4) and so each defines a gerbe, and since these are holomorphic functions, they are holomorphic gerbes. In our discussion here of the classical theory, the quantisation condition on H does not play a role.

Using the above properties we can construct different objects which satisfy the cocycle condition on the triple intersection. For example we have the following relation on $U_\alpha \cap U_\beta \cap U_\gamma$

$$\begin{aligned} \frac{\partial}{\partial\phi^a} \left(F_{\alpha\beta}^+(\phi, \chi, X_L) + F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) \right) + \frac{\partial}{\partial\phi^a} \left(F_{\beta\gamma}^+(\phi, \chi, X_L) + F_{\beta\gamma}^-(\phi, \bar{\chi}, X_R) \right) \\ + \frac{\partial}{\partial\phi^a} \left(F_{\gamma\alpha}^+(\phi, \chi, X_L) + F_{\gamma\alpha}^-(\phi, \bar{\chi}, X_R) \right) = 0 . \end{aligned} \quad (\text{B.12})$$

Analogously, the following objects satisfy the cocycle conditions on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$

$$\frac{\partial}{\partial\chi^{a'}} \left(F_{\alpha\beta}^+(\phi, \chi, X_L) + \bar{F}_{\alpha\beta}^-(\bar{\phi}, \chi, \bar{X}_R) \right) , \quad (\text{B.13})$$

$$\frac{\partial}{\partial X_L^n} F_{\alpha\beta}^+(\phi, \chi, X_L) , \quad (\text{B.14})$$

$$\frac{\partial}{\partial X_R^{n'}} F_{\alpha\beta}^-(\phi, \bar{\chi}, X_R) . \quad (\text{B.15})$$

Thus we can use them to construct different affine bundles.

Next we turn to the geometrical setting of the construction of section 5. Let $\partial_{\pm}, \bar{\partial}_{\pm}$ be the Dolbeaut exterior derivatives for the complex structure I_{\pm} . Suppose in each U_{α} there is a $(1,0)$ form X_{α} with

$$X_{\alpha} = X_{\alpha a} dz^a + X_{\alpha a'} dr^{a'} + X_{\alpha n} d\zeta^n \quad (\text{B.16})$$

with glueing relations

$$X_{\beta} = X_{\alpha} + \partial_+ F_{\alpha\beta}^+ \quad (\text{B.17})$$

These then satisfy (B.8),(B.10),(B.11) and so see the gerbe structure. Defining $b_{\alpha} = \bar{\partial}_+ X_{\alpha}$, we have

$$b_{\beta} = b_{\alpha} \quad (\text{B.18})$$

so b is a globally defined $(1,1)$ form and

$$h \equiv \bar{\partial}_+ b_{\alpha} \quad (\text{B.19})$$

is zero, so that b_{α} is a holomorphic gerbe connection which is flat ($h = 0$), with prepotential X_{α} . Now the components $(X_{\alpha a}, X_{\alpha a'}, X_{\alpha n})$ have exactly the same glueing relations as p_L, r_L, π_L from (5.13),(5.14), i.e. p_L, r_L, π_L can be viewed as the components of the prepotential for a flat holomorphic gerbe connection.

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