

Data-Driven Min-Max MPC for Linear Systems: Robustness and Adaptation [★]

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Abstract

Data-driven controllers design is an important research problem, in particular when data is corrupted by the noise. In this paper, we propose a data-driven min-max model predictive control (MPC) scheme using noisy input-state data for unknown linear time-invariant (LTI) system. The unknown system matrices are characterized by a set-membership representation using the noisy input-state data. Leveraging this representation, we derive an upper bound on the worst-case cost and determine the corresponding optimal state-feedback control law through a semidefinite program (SDP). We prove that the resulting closed-loop system is robustly stabilized and satisfies the input and state constraints. Further, we propose an adaptive data-driven min-max MPC scheme which exploits additional online input-state data to improve closed-loop performance. Numerical examples show the effectiveness of the proposed methods.

Key words: data-based control, optimal controller synthesis for systems with uncertainties, model predictive control, adaptive control.

1 Introduction

Data-driven system analysis and control have received increasing interest in the recent years. Unlike the model-based control methods, which depend on prior knowledge of system models identified from measured data using system identification methods [1] or derived from first principles, data-driven control approaches design controllers directly from the available data. Several works have focused on designing controllers for unknown linear time-invariant (LTI) systems directly from noisy data [2–8], assuming assumptions on the noise such as energy bounds or instantaneous bounds. A matrix inequality can effectively characterize a set of LTI systems that explain the measured data, forming the basis of the data informativity framework [3, 9]. Within this framework, data-driven controller designs aim to find a controller that stabilizes all systems that are consistent with the data. Various controller design methods have

been proposed, including H_2 and H_∞ control [2, 4, 8], linear quadratic regulator approaches [5], and stabilization [2, 5, 6]. Nevertheless, the incorporation of constraints and controller design using noisy data remains largely unexplored in this framework. In this paper, we propose data-driven min-max model predictive control (MPC) schemes in this framework to design a controller that robustly stabilizes the system and handles ellipsoidal input and state constraints.

MPC is widely used due to its ability to handle constraints and consider performance criteria [10]. The fundamental concept of MPC is to solve an open-loop optimal control problem at each sampling time, which uses the system dynamics to predict future open-loop trajectories. Recently, data-driven MPC approaches have been studied, which directly use the measured input-output data to predict the future outputs [11–18]. This data-driven MPC framework is based on the Fundamental Lemma [19, 20], which states that for a controllable LTI system, all possible system trajectories can be parameterized in terms of linear combinations of time-shifts of one persistently exciting trajectory. This framework requires the availability of persistently exciting data, enabling the unique representation of the system from the data in the noise-free scenarios. In case of bounded output measurement noise, robust data-driven

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MPC schemes have been developed, which guarantee practical stability for the closed-loop system [12, 13]. These MPC schemes can be expanded via suitable constraint tightening to ensure robust state or output constraint satisfaction in the presence of bounded process or measurement noise [21, 22]. While these MPC schemes provide strong theoretical guarantees for guaranteeing constraint satisfaction of unknown systems based only on measured data, the employed constraint tightenings suffer from possibly large conservatism.

Min-max MPC can effectively address scenarios involving parametric uncertainty on the system dynamics and disturbance, as discussed in [23–29]. The basic idea of min-max MPC is to design control inputs that minimize the worst-case cost w.r.t. disturbances and/or parametric uncertainty in order to robustly stabilize the system. An especially popular approach is to employ linear matrix inequalities (LMIs) [30, 31] in the min-max MPC framework [23] to obtain a tractable state-feedback control law. This approach involves solving an LMI-based optimization problem at each time step that incorporates constraints and a description of the parametric uncertainty, thereby guaranteeing robust stability. The min-max MPC schemes typically require prior knowledge of the parametric uncertainty set, i.e., a known polytopic set [23, 26]. However, addressing min-max MPC schemes without prior knowledge of the parametric uncertainty set, relying solely on available data, remains an open challenge.

In this work, we propose a data-driven min-max MPC framework to control LTI systems with unknown system matrices and additive process noise using noisy input-state data. Our approach relies on a representation of the system matrices consistent with a sequence of noisy input-state data by using a quadratic matrix inequality [4, 7]. The scheme involves an infinite-horizon cost as well as ellipsoidal input and state constraints. It can be interpreted as a time-varying H_2 state-feedback controller design, analogous to the model-based min-max MPC scheme in [23]. We show that the proposed data-driven min-max MPC guarantees closed-loop recursive feasibility, constraint satisfaction and robust stability. Further, we propose an adaptive data-driven min-max MPC scheme that integrates online collected input-state data. Utilizing these online data reduces the parametric uncertainty on the system dynamics, thus the closed-loop performance resulting from the adaptive data-driven min-max MPC scheme improves. Numerical examples show that the proposed scheme ensures robust stability and constraint satisfaction in a less conservative fashion than the data-driven MPC based on the Fundamental Lemma [22].

We note that the recent works [32, 33] also propose data-driven MPC schemes for linear systems using ideas from the data informativity framework. However, in these papers, the data are assumed to be noise-free, contrary to

our framework which allows for process noise in the data. In [34], a data-driven MPC scheme employing noisy data is proposed, focusing on the H_∞ control objective. However, they assume an energy bound on the online noise, implying that the noise converges to zero for time approach infinity, and establish closed-loop stability accordingly. In contrast, we address a more practical scenario with instantaneous noise bound and establish robust stability for the closed-loop system. In the recent work [35], an online data-driven approach is proposed for iteratively learning controllers for systems with dynamic changes over time. The data considered is noise-free and constraints are not taken into account. Further, the literature contains various approaches on model-based adaptive MPC schemes [36–45]. Specifically, adaptive tube-based MPC schemes aim to construct prediction tubes for robust constraint satisfaction and incorporate model adaptation using set-membership estimation [36–41]. Model-based adaptive min-max MPC schemes update the parametric uncertainty set at each time step and design a MPC controller to robustly stabilize the uncertain systems [44, 45], which assumes prior knowledge of the parametric uncertainty set. In contrast, the proposed approach relies on an ellipsoidal uncertainty characterization based on the recent data-driven control literature [4, 7]. This allows to use LMI methods for the design of a state-feedback-based MPC scheme with robust stability guarantees.

The remainder of this paper is organized as follows. Section 2 introduces necessary preliminaries about the data-driven parameterization and the problem setup. In Section 3, we propose a data-driven min-max MPC problem with input and state constraints. We prove recursive feasibility, constraint satisfaction and robust stability for the closed-loop system. In Section 4, we consider an adaptive data-driven min-max MPC scheme which uses online data to reduce uncertainty and improve performance. We illustrate the advantage of the proposed schemes with numerical examples in Section 5. Finally, we conclude the paper in Section 6. Preliminary results on data-driven min-max MPC were presented in the conference paper [46]. The present paper extends [46] in multiple directions. First, [46] assumes that the online measurements used for feedback are noise-free, whereas we consider noise in both offline and online data. As a result, the theoretical analysis in the present paper is more involved and provides robust stability guarantees of a robust positive invariant (RPI) set around the origin. On the contrary, the noise-free setup in [46] allowed to prove exponential stability. Further, the present paper proposes an adaptive data-driven min-max MPC scheme which, as shown with a numerical example, can substantially reduce conservatism using online data.

Notation: Let $\mathbb{I}_{[a,b]}$ denote the set of integers in the interval $[a, b]$, $\mathbb{I}_{\geq 0}$ denote the set of nonnegative integers, and $\mathbb{I}_{\{a,+\infty\}}$ denote the set of integers larger than or equal to a . For a matrix P , we write $P \succ 0$ if P is positive definite

and $P \succeq 0$ if P is positive semi-definite. For a vector x and a matrix $P \succ 0$, we write $\|x\|_P = \sqrt{x^\top P x}$. For matrices A and B of compatible dimensions, we abbreviate ABA^\top to $AB \begin{bmatrix} \star \end{bmatrix}^\top$.

2 Preliminaries

In Section 2.1, the considered problem setup is introduced. In Section 2.2, we present the data-driven system parameterization used for the proposed min-max MPC approach.

2.1 Problem Setup

In this paper, we consider an unknown discrete-time LTI system

$$x_{t+1} = A_s x_t + B_s u_t + \omega_t, \quad (1)$$

where $x_t \in \mathbb{R}^n$ denotes the state, $u_t \in \mathbb{R}^m$ denotes the input, and $\omega_t \in \mathbb{R}^n$ denotes the unknown noise for $t \in \mathbb{I}_{\geq 0}$. The matrices $A_s \in \mathbb{R}^{n \times n}$ and $B_s \in \mathbb{R}^{n \times m}$ are assumed to be unknown. The noise ω_t is assumed to satisfy the following assumption.

Assumption 1 For all $t \in \mathbb{I}_{\geq 0}$, the noise $\omega_t \in \mathbb{R}^n$ satisfies $\|\omega_t\|_2 \leq \epsilon$ for a known bound $\epsilon \geq 0$.

We define a sequence of offline input, noise and corresponding state of length T_f from the system (1), which are denoted in the matrices

$$\begin{aligned} U_f &:= \begin{bmatrix} u_0^f & u_1^f & \dots & u_{T_f-1}^f \end{bmatrix}, \\ W_f &:= \begin{bmatrix} \omega_0^f & \omega_1^f & \dots & \omega_{T_f-1}^f \end{bmatrix}, \\ X_f &:= \begin{bmatrix} x_0^f & x_1^f & \dots & x_{T_f}^f \end{bmatrix}. \end{aligned}$$

Throughout this paper, we assume that the offline input-state measurements U_f and X_f are available. The noise sequence W_f is unknown, but every element in W_f satisfies the bound in Assumption 1. In Section 3, the data-driven min-max MPC algorithm only based on the knowledge of offline input-state measurements (U_f, X_f) . Moreover, in Section 4, we employ online input-state measurements in addition to (U_f, X_f) to design an adaptive min-max MPC algorithm.

Our objective is to stabilize the origin for the unknown LTI system (1), while the closed-loop input and state satisfy given constraints. We consider the origin for simplicity but note that the results in this paper can be adapted for non-zero equilibria. In order to stabilize the origin, we define the following quadratic stage cost function

$$l(u, x) = \|u\|_R^2 + \|x\|_Q^2,$$

where $R, Q \succ 0$. We consider ellipsoidal constraints on the input and the state, i.e.,

$$\|u_t\|_{S_u} \leq 1, \forall t \in \mathbb{I}_{\geq 0}, \quad (2a)$$

$$\|x_t\|_{S_x} \leq 1, \forall t \in \mathbb{I}_{\geq 0}, \quad (2b)$$

where $S_u \succ 0$ and $S_x \succeq 0$.

2.2 Data-driven Parameterization

Given that the matrices A_s and B_s are unknown, the knowledge about the system relies on inferring information from input-state measurements. In this section, we introduce the employed data-driven parameterization method using offline input and state measurements.

First, we define the set of system matrices (A, B) consistent with the offline data $x_i^f, u_i^f, x_{i+1}^f, i \in \mathbb{I}_{[0, T_f-1]}$ by

$$\Sigma_i^f := \left\{ (A, B) : (1) \text{ holds for some } \omega_i^f \text{ satisfying } \|\omega_i^f\|_2 \leq \epsilon \right\}.$$

This set includes all system matrices for which there exists a noise realization satisfying Assumption 1 and the system dynamics (1). We proceed analogous to [2, 4, 7] to derive a data-driven parametrization of the system matrices. Using the system dynamics (1), the state x_i^f, x_{i+1}^f and input u_i^f satisfy the following equation

$$\omega_i^f = x_{i+1}^f - A_s x_i^f - B_s u_i^f.$$

Thus, the set Σ_i^f can be equivalently characterized by the following quadratic matrix inequality

$$\Sigma_i^f = \left\{ (A, B) : \begin{bmatrix} I & A & B \end{bmatrix} \begin{bmatrix} I & x_{i+1}^f \\ 0 & -x_i^f \\ 0 & -u_i^f \end{bmatrix} \begin{bmatrix} \epsilon^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \star \end{bmatrix}^\top \succeq 0 \right\}. \quad (3)$$

Furthermore, the set of system matrices consistent with the sequence of offline input-state measurements (U_f, X_f) is defined by

$$\mathcal{C}_f := \bigcap_{i=0}^{T_f-1} \Sigma_i^f.$$

We can characterize \mathcal{C}_f by the following quadratic matrix inequality [4, 7]

$$\mathcal{C}_f = \left\{ (A, B) : \begin{aligned} & \begin{bmatrix} I & A & B \end{bmatrix} \Pi_f(\tau) \begin{bmatrix} I & A & B \end{bmatrix}^\top \succeq 0, \\ & \forall \tau = (\tau_0, \dots, \tau_{T_f-1}), \tau_i \geq 0, i \in \mathbb{I}_{[0, T_f-1]} \end{aligned} \right\}, \quad (4)$$

where

$$\Pi_f(\tau) = \sum_{i=0}^{T_f-1} \tau_i \begin{bmatrix} I & x_{i+1}^f \\ 0 & -x_i^f \\ 0 & -u_i^f \end{bmatrix} \begin{bmatrix} \epsilon^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x_{i+1}^f \\ 0 & -x_i^f \\ 0 & -u_i^f \end{bmatrix}^\top. \quad (5)$$

We will later use the data-driven parameterization of \mathcal{C}_f in equation (4) to formulate the data-driven min-max MPC problem, thus designing a controller that robustly stabilizes all system with matrices in \mathcal{C}_f .

3 Data-driven Min-Max MPC

In this section, we present a data-driven min-max MPC problem with input and state constraints using offline input-state measurements. We restrict the optimization to state-feedback control laws, which allows to reformulate the data-driven min-max MPC problem as an SDP. We establish that the resulting MPC approach is recursively feasible and the closed-loop system is robustly stabilized and satisfies the input and state constraints.

3.1 Data-driven Min-Max MPC Problem

At time t , given offline input-state measurements (U_f, X_f) and an initial state x_t , the data-driven min-max MPC optimization problem is formulated as follows:

$$J_\infty^*(x_t) := \min_{\bar{u}(t)} \max_{(A,B) \in \mathcal{C}_f} \sum_{k=0}^{\infty} l(\bar{u}_k(t), \bar{x}_k(t)) \quad (6a)$$

$$\text{s.t. } \bar{x}_{k+1}(t) = A\bar{x}_k(t) + B\bar{u}_k(t), \quad (6b)$$

$$\bar{x}_0(t) = x_t. \quad (6c)$$

$$\|u_t\|_{S_u} \leq 1, \forall t \in \mathbb{I}_{\geq 0}, \quad (6d)$$

$$\|x_t\|_{S_x} \leq 1, \forall (A, B) \in \mathcal{C}_f, t \in \mathbb{I}_{\geq 0}, \quad (6e)$$

The objective function aims to minimize the worst-case value of the sum of infinite stage cost among all consistent system matrices in \mathcal{C}_f by adapting the control input $\bar{u}_k(t)$, $\forall k \in \mathbb{I}_{\geq 0}$. In the optimization problem, $\bar{x}_k(t)$ and $\bar{u}_k(t)$ are the predicted state and control input at time $t+k$ based on the measurement at time t . We use the nominal system dynamics $x_{t+1} = Ax_t + Bu_t$ with $(A, B) \in \mathcal{C}_f$ for future state prediction in constraint (6b). In constraint (6c), we initialize $\bar{x}_0(t)$ as the state measurement at time t . In constraints (6d) and (6e), the closed-loop input and state satisfy the ellipsoidal constraints in (2) for any system dynamics $x_{t+1} = Ax_t + Bu_t + \omega_t$ with $(A, B) \in \mathcal{C}_f$ and ω_t satisfying Assumption 1.

Remark 1 In the data-driven min-max MPC problem (6), we use the nominal system dynamics without the

noise ω_t to predict the future state. This approach avoids an additional maximization with regard to the noise in the min-max MPC problem. Even though the influence of the noise is not considered in the min-max problem, the proposed method guarantees robust stability for the closed-loop system in the presence of noise (cf. Section 3.3).

Remark 2 Similar to the existing LMI-based min-max MPC scheme in [23], we consider to minimize the worst-case value of infinite-horizon cost. The infinite-horizon cost allows us to reformulate the data-driven min-max MPC problem as an SDP. This reformulation fits well to the data-driven system parametrization described by (4), as detailed in Section 3.2.

3.2 Reformulation based on LMIs

The data-driven min-max MPC problem (6) is intractable because of the min-max formulation and the constraints for all possible (A, B) within a set characterized by data. To effectively address problem (6) and derive a tractable solution, we limit our focus to find a state-feedback control law of the form $\bar{u}_k(t) = F_t^* \bar{x}_k(t)$, where $F_t^* \in \mathbb{R}^{m \times n}$. In the following, we formulate an SDP to derive an upper bound on the optimal cost over the set \mathcal{C}_f and to determine a state-feedback gain that minimizes this upper bound.

At time t , given offline input-state measurements (U_f, X_f) , an initial state $x_t \in \mathbb{R}^n$ and a constant $c > \lambda_{\min}(Q)$, the SDP is formulated as follows:

$$\text{minimize}_{\gamma > 0, H \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{m \times n}, \tau \in \mathbb{R}^{T_f}} \gamma \quad (7a)$$

$$\text{s.t. } \begin{bmatrix} 1 & x_t^\top \\ x_t & H \end{bmatrix} \succeq 0, \quad (7b)$$

$$\begin{bmatrix} \begin{bmatrix} -H + \frac{\gamma}{c} I & 0 \\ 0 & 0 \end{bmatrix} + \Pi_f(\tau) & \begin{bmatrix} 0 \\ H \\ L \end{bmatrix} \\ \begin{bmatrix} 0 & H & L^\top \\ 0 & & \end{bmatrix} & \begin{bmatrix} -H & \Phi^\top \\ \Phi & -\gamma I \end{bmatrix} \end{bmatrix} \prec 0, \quad (7c)$$

$$\tau = (\tau_0, \dots, \tau_{T_f-1}), \tau_i \geq 0, \forall i \in \mathbb{I}_{[0, T_f-1]}, \quad (7d)$$

$$\begin{bmatrix} H & L^\top \\ L & S_u^{-1} \end{bmatrix} \succeq 0, \quad (7e)$$

$$\begin{bmatrix} H & H \\ H & S_x^{-1} \end{bmatrix} \succeq 0. \quad (7f)$$

where $\Phi = \begin{bmatrix} M_R L \\ M_Q H \end{bmatrix}$, $M_R^\top M_R = R$ and $M_Q^\top M_Q = Q$.

The optimal solution of problem (7) at time t is denoted by $\gamma_t^*, H_t^*, L_t^*, \tau_t^*$. The corresponding optimal state-feedback gain is given by $F_t^* = L_t^*(H_t^*)^{-1}$.

We solve the SDP problem (7) in a receding-horizon manner to repeatedly find an optimal state-feedback gain, see Algorithm 1. In particular, at time t , we solve the optimization problem (7) and obtain the optimal state-feedback gain F_t^* . Only the first computed input $u_t = F_t^* x_t$ is implemented and, at time $t + 1$, we reiterate the described procedure.

Algorithm 1 Data-driven min-max MPC scheme.

- 1: **Input:** $U_f, X_f, Q, R, S_x, S_u, c$
 - 2: At time $t = 0$, measure state x_0
 - 3: Solve the problem (7)
 - 4: Apply the input $u_t = F_t^* x_t$
 - 5: Set $t = t + 1$, measure state x_t and go back to 3
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Remark 3 The constant c is required to prove robust stability for the resulting closed-loop system, as detailed in Section 3.3. At the initial time $t = 0$, we select c such that the optimization problem (7) is feasible for the initial state x_0 . The value of c remains constant in Algorithm 1 when $t \in \mathbb{I}_{[1, \infty)}$. This approach ensures convexity and recursive feasibility of the SDP problem (7), thereby simplifying computational burden. The precise choice of c influences the performance. As will be proved later in Theorem 2, $\|x_t\|_P^2$ is lower bounded by $\|x_t\|_Q^2$ and upper bounded by $c\|x_t\|^2$. Therefore, this requires $c \geq \lambda_{\min}(Q)$. A smaller value of c gives a smaller robust positive invariant (RPI) set to which the closed loop converges, as shown later in Theorem 2, but also a smaller feasible region and possibly larger optimal cost γ_t^* .

In the following theorem, we first neglect the input and state constraints (6d)-(6e) in the data-driven min-max MPC problem (6). We show that the optimal cost of problem (7) is an upper bound on the optimal cost of (6a)-(6c) using (7b)-(7d). Later in Theorem 2, we will show that Algorithm 1 ensures constraint satisfaction in closed loop.

Theorem 1 Given a state $x_t \in \mathbb{R}^n$ at time t and a constant $c > \lambda_{\min}(Q)$, suppose there exist γ, H, L, τ such that the LMIs (7b)-(7d) hold. Let $P = \gamma H^{-1}$. Then, the optimal cost of (6a)-(6c) is guaranteed to be at most $\|x_t\|_P^2$ and $\|x_t\|_P^2$ is upper bounded by γ , i.e.,

$$J_\infty^*(x_t) \leq \|x_t\|_P^2 \leq \gamma.$$

PROOF. Applying the Schur complement to the con-

straint (7c) twice yields the equivalent inequalities

$$\begin{bmatrix} -H + \frac{\gamma}{c}I & 0 \\ 0 & \begin{bmatrix} H \\ L \end{bmatrix} (H - \frac{1}{\gamma} \Phi^\top \Phi)^{-1} \begin{bmatrix} H \\ L \end{bmatrix}^\top \end{bmatrix} + \Pi_f(\tau) \prec 0, \quad (8a)$$

$$-H + \frac{1}{\gamma} \Phi^\top \Phi \prec 0. \quad (8b)$$

According to (4), given any τ satisfying constraint (7d), the inequality

$$\begin{bmatrix} I & A & B \end{bmatrix} \Pi_f(\tau) \begin{bmatrix} I & A & B \end{bmatrix}^\top \succeq 0 \quad (9)$$

holds for any $(A, B) \in \mathcal{C}_f$. Pre-multiplying (8a) with $\begin{bmatrix} I & A & B \end{bmatrix}$ and post-multiplying (8a) with $\begin{bmatrix} I & A & B \end{bmatrix}^\top$, the resulting inequality together with (9) imply that the following inequality must hold for any $(A, B) \in \mathcal{C}_f$

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} -H + \frac{\gamma}{c}I & 0 \\ 0 & \begin{bmatrix} H \\ L \end{bmatrix} (H - \frac{1}{\gamma} \Phi^\top \Phi)^{-1} \begin{bmatrix} H \\ L \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \prec 0. \quad (10)$$

This is equivalent to

$$(-H + \frac{\gamma}{c}I) + (AH + BL)(H - \frac{1}{\gamma} \Phi^\top \Phi)^{-1} (AH + BL)^\top \prec 0. \quad (11)$$

Using the Schur complement, (11) together with (8b) is equivalent to

$$\begin{bmatrix} -H + \frac{1}{\gamma} \Phi^\top \Phi & (AH + BL)^\top \\ (AH + BL) & -H + \frac{\gamma}{c}I \end{bmatrix} \prec 0. \quad (12)$$

Using the Schur complement again, (12) yields the equivalent inequality

$$(AH + BL)^\top (H - \frac{\gamma}{c}I)^{-1} (AH + BL) - H + \frac{1}{\gamma} \Phi^\top \Phi \prec 0, \quad (13a)$$

$$-H + \frac{\gamma}{c}I \prec 0. \quad (13b)$$

Let $P = \gamma H^{-1}$ and $F = LH^{-1}$. Using the Woodbury matrix identity [47], we have $\gamma[P + P(cI - P)^{-1}P]^{-1} = H - \frac{\gamma}{c}I$. Replacing $(H - \frac{\gamma}{c}I)^{-1}$ with $\gamma^{-1}[P + P(cI - P)^{-1}P]$ in the inequality (13a), multiplying both sides of the resulting inequality with P , and then dividing the resulting inequality by γ , we have

$$(A+BF)^\top [P + P(cI - P)^{-1}P] (A+BF) - P + Q + F^\top R F \prec 0. \quad (14)$$

Replacing H with γP^{-1} in (13b) and dividing by γ , we have $\frac{1}{c}I - P^{-1} \prec 0$. Multiplying with cP from left and right, we obtain

$$P - cI \prec 0. \quad (15)$$

Using the Schur complement, (14) and (15) yield the equivalent inequality

$$\begin{bmatrix} (A+BF)^\top P(A+BF) - P + Q + F^\top R F & (A+BF)^\top P \\ P(A+BF) & P - cI \end{bmatrix} \prec 0. \quad (16)$$

This implies for any $(A, B) \in \mathcal{C}_f$, we have

$$(A + BF)^\top P(A + BF) - P + Q + F^\top R F \prec 0. \quad (17)$$

Multiplying left and right sides of (17) with x^\top and x , the following inequality holds for any $x \in \mathbb{R}^n$ and any $(A, B) \in \mathcal{C}_f$

$$x^\top (A + BF)^\top P(A + BF)x - x^\top P x \leq -x^\top (Q + F^\top R F)x. \quad (18)$$

The inequality (18) implies that the following inequality is satisfied for all states and inputs $\bar{x}_k(t), \bar{u}_k(t) = F\bar{x}_k(t), k \in \mathbb{I}_{\geq 0}$ predicted by the system dynamics (6b) with any $(A, B) \in \mathcal{C}_f$

$$\|\bar{x}_{k+1}(t)\|_P^2 - \|\bar{x}_k(t)\|_P^2 \leq -l(\bar{u}_k(t), \bar{x}_k(t)). \quad (19)$$

Summing the inequality (19) from $k = 0$ to $k = T - 1$ along an arbitrary trajectory, we obtain

$$\|\bar{x}_T(t)\|_P^2 - \|\bar{x}_0(t)\|_P^2 \leq -\sum_{k=0}^{T-1} l(\bar{u}_k(t), \bar{x}_k(t)). \quad (20)$$

Since $\|\bar{x}_T(t)\|_P^2 \geq 0$ and $\bar{x}_0(t) = x_t$, letting $T \rightarrow \infty$, we obtain

$$\sum_{k=0}^{\infty} l(\bar{u}_k(t), \bar{x}_k(t)) \leq \|x_t\|_P^2. \quad (21)$$

The inequality (21) holds for any $(A, B) \in \mathcal{C}_f$, it also holds for the worst-case value, i.e., we obtain

$$\max_{(A, B) \in \mathcal{C}_f} \sum_{k=0}^{\infty} l(\bar{u}_k(t), \bar{x}_k(t)) \leq \|x_t\|_P^2. \quad (22)$$

This provides an upper bound on the optimal cost of (6). Using the Schur complement, $\|x_t\|_P^2 \leq \gamma$ is equivalent to the inequality (6b).

In conclusion, given that (7b)-(7d) hold, we have thus shown that γ is an upper bound on the optimal cost of problem (6) without input and state constraints (6d)-(6e). \square

Remark 4 Theorem 1 derives an upper bound on the optimal cost of the data-driven min-max MPC problem

(6). Inequality (19) together with the convergence of $\bar{x}_k(t)$ to the origin as k tends to infinity allow us to show that $\|x_t\|_P^2$ serves as an upper bound on the infinite-horizon sum of stage costs of the nominal closed-loop system with any $(A, B) \in \mathcal{C}_f$. The optimal solution of (7) minimizes this upper bound and returns the corresponding state-feedback gain. However, it is important to note that this upper bound may not always be tight. The conservatism of this upper bound is due to the linear state-feedback form of the input as well as the quadratic choice of the upper bound. Reducing conservatism by considering more general state-feedback law and cost upper bounding functions is an interesting issue for future research.

Remark 5 The proof to derive the upper bound on the worst-case cost, i.e., (19)-(22), is inspired by the existing LMI-based min-max MPC approach in [23]. The difference is that [23] propose a model-based min-max MPC scheme where the parametric uncertainty set is a predefined polytope. On the other hand, in our case, \mathcal{C}_f is an ellipsoidal set characterized by offline input-state trajectory generated by the noisy system, requiring different technical tools for the convex reformulation.

Remark 6 In case of noise-free and persistently exciting data, i.e., $\epsilon = 0$ and $[U_f^\top, X_f^\top]^\top$ has full row rank, the data-driven min-max MPC problem (6) without the input and state constraints (6d)-(6e) reduces to a discrete-time linear quadratic regulator problem, as explored in [2-5]. In this case, the optimal state-feedback gain F_t^* remains constant and does not depend on the state x_t . However, in the presence of model uncertainty, even without the input and state constraints, employing a receding horizon algorithm and recalculating F at each sampling time shows significant performance improvement compared to using a static state-feedback control law. This is illustrated with a numerical example in Section 5.1.

3.3 Closed-loop Guarantees

In the following theorem, we first establish the recursive feasibility of the problem (7). Then, we use the optimal solution of problem (7) to define a Lyapunov function $V(x_t) = \|x_t\|_{P_t}^2$, and prove robust stability for the resulting closed-loop system with any $(A, B) \in \mathcal{C}_f$. Finally, we prove that the input and state constraints are satisfied for the closed-loop trajectory.

Theorem 2 Suppose Assumption 1 holds. If the optimization problem (7) is feasible at time $t = 0$, then

- i) the optimization problem (7) is feasible at any time $t \in \mathbb{I}_{[1, \infty)}$;
- ii) the set $\mathcal{E}_{RPI} := \{x \in \mathbb{R}^n : V(x) \leq \frac{c^2 \epsilon^2}{\lambda_{\min}(Q)}\}$ is robustly stabilized for the closed-loop system $x_{t+1} =$

- ($A + BF_t^*$) $x_t + \omega_t$ with any $(A, B) \in \mathcal{C}_f$;
 iii) the closed-loop trajectory of $x_{t+1} = (A + BF_t^*)x_t + \omega_t$ with any $(A, B) \in \mathcal{C}_f$ satisfies the constraints, i.e., $\|u_t\|_{S_u} \leq 1, \|x_t\|_{S_x} \leq 1$ for all $t \in \mathbb{I}_{\geq 0}$.

PROOF. The proof is composed of four parts. Part I proves the lower bound and upper bound on the Lyapunov function $V(x_t)$. Part II proves recursive feasibility of the problem (7). Part III establishes the robust stability of the set \mathcal{E}_{RPI} and Part IV shows that the input and state constraints are satisfied for the closed-loop trajectory.

Part I: First, we derive an upper bound and lower bound on $V(x_t)$. As we have shown in Theorem 1, $\|x_t\|_P^2$ with any feasible solution P of the LMIs (7b)-(7d) is an upper bound on the optimal cost of (6). Thus, $V(x_t) = \|x_t\|_{P_t^*}^2$ is an upper bound on the optimal cost of (6). Thus, we have

$$V(x_t) \geq l(u_t, x_t) \geq \|x_t\|_Q^2. \quad (23)$$

In the proof of Theorem 1, we have shown that any feasible solution of problem (7) satisfies the inequality (15). Thus, $V(x_t)$ is upper bounded by $c\|x_t\|_2^2$. Since $Q \succ 0$, we have $c\|x_t\|_2^2 \leq \frac{c}{\lambda_{\min}(Q)}\|x_t\|_Q^2$. Thus,

$$V(x_t) \leq c\|x_t\|_2^2 \leq \frac{c}{\lambda_{\min}(Q)}\|x_t\|_Q^2. \quad (24)$$

Part II: In the following, we prove recursive feasibility of the problem (7). Assuming problem (7) is feasible at time t , we have shown in the proof of Theorem 1 that the inequality (16) holds for γ_t^*, F_t^*, P_t^* and any $(A, B) \in \mathcal{C}_f$. Pre-multiplying (16) with $\begin{bmatrix} x_t^\top & \omega_t^\top \end{bmatrix}$ and post-multiplying (16) with $\begin{bmatrix} x_t^\top & \omega_t^\top \end{bmatrix}^\top$, we have

$$\begin{aligned} & [(A + BF_t^*)x_t + \omega_t]^\top P_t^* [(A + BF_t^*)x_t + \omega_t] - x_t^\top P_t^* x_t \\ & \leq -x_t^\top (Q + F_t^{*\top} R F_t^*) x_t + c\omega_t^\top \omega_t \end{aligned} \quad (25)$$

for any $x_t \in \mathbb{R}^n, \omega_t \in \mathbb{R}^n$ and $(A, B) \in \mathcal{C}_f$. Since $x_{t+1} = (A + BF_t^*)x_t + \omega_t$ and ω_t satisfies the Assumption 1, we have that

$$\begin{aligned} \|x_{t+1}\|_{P_t^*}^2 - \|x_t\|_{P_t^*}^2 & \leq -x_t^\top (Q + F_t^{*\top} R F_t^*) x_t + c\omega_t^\top \omega_t \\ & \leq -\|x_t\|_Q^2 + c\epsilon^2. \end{aligned} \quad (26)$$

By the upper bound on $V(x_t) = \|x_t\|_{P_t^*}^2$ in (24), the inequality (26) implies

$$\|x_{t+1}\|_{P_t^*}^2 - \|x_t\|_{P_t^*}^2 \leq -\frac{\lambda_{\min}(Q)}{c}\|x_t\|_{P_t^*}^2 + c\epsilon^2. \quad (27)$$

Subtracting $\frac{c^2\epsilon^2}{\lambda_{\min}(Q)}$ from both sides of the inequality (27) and adding $\|x_t\|_{P_t^*}^2$ on both sides, we have

$$\|x_{t+1}\|_{P_t^*}^2 - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)} \leq \left(1 - \frac{\lambda_{\min}(Q)}{c}\right) \left(\|x_t\|_{P_t^*}^2 - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}\right). \quad (28)$$

We define a set $\mathcal{E}_{ROA} := \{x \in \mathbb{R}^n : V(x) \leq \gamma_0^*\}$ and separate the state x_t into two cases to show recursive feasibility for the problem (7).

Case I: The state is outside the robust positive invariant (RPI) set, i.e., $x_t \in \mathcal{E}_{ROA} \setminus \mathcal{E}_{RPI}$. As $c > \lambda_{\min}(Q)$, we derive $1 > 1 - \frac{\lambda_{\min}(Q)}{c} > 0$. By the definition of \mathcal{E}_{RPI} and \mathcal{E}_{ROA} , we have $V(x_t) = \|x_t\|_{P_t^*}^2 \in (\frac{c^2\epsilon^2}{\lambda_{\min}(Q)}, \gamma_0]$. Thus, the inequality (28) implies

$$\|x_{t+1}\|_{P_t^*}^2 - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)} \leq \|x_t\|_{P_t^*}^2 - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}.$$

We further derive

$$\|x_{t+1}\|_{P_t^*}^2 \leq \|x_t\|_{P_t^*}^2 \leq \gamma_t^*. \quad (29)$$

The only constraint in the problem (7) that depends explicitly on the measured state x_t is the inequality (7b). The inequality (29) implies that the feasible solution of the optimization problem (7) at time t is also feasible at time $t + 1$. This argument can be continued to establish feasibility for any time $t \in \mathbb{I}_{[1, \infty)}$.

Case II: The state is in the RPI set, i.e., $x_t \in \mathcal{E}_{RPI}$. By the definition of \mathcal{E}_{RPI} , we have $V(x_t) = \|x_t\|_{P_t^*}^2 \leq \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}$. Plugging $V(x_t) \leq \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}$ into the inequality (28), we derive

$$\|x_{t+1}\|_{P_t^*}^2 - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)} \leq 0. \quad (30)$$

Therefore, $\gamma'_{t+1} = \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}, H'_{t+1} = H_t^*, L'_{t+1} = L_t^*, \tau'_{t+1} = \tau_t^*$ is a feasible solution of problem (7) at time $t + 1$.

Part III: Now we prove robust stability of the set \mathcal{E}_{RPI} . Since $P_{t+1}^* = \gamma_{t+1}^*(H_{t+1}^*)^{-1}$ is the optimal solution of problem (7) at time $t + 1$ while P_t^* is a feasible solution, we have

$$V(x_{t+1}) = \|x_{t+1}\|_{P_{t+1}^*}^2 \leq \|x_{t+1}\|_{P_t^*}^2. \quad (31)$$

When the state is outside the RPI set, i.e., $x_t \in \mathcal{E}_{ROA} \setminus \mathcal{E}_{RPI}$, we derive the following inequality using (28) and (31)

$$V(x_{t+1}) - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)} \leq \left(1 - \frac{\lambda_{\min}(Q)}{c}\right) \left[V(x_t) - \frac{c^2\epsilon^2}{\lambda_{\min}(Q)}\right]. \quad (32)$$

Since $1 > 1 - \frac{\lambda_{\min}(Q)}{c} > 0$, the inequality (32) implies that all states in \mathcal{E}_{ROA}^c converge exponentially to the set \mathcal{E}_{RPI} . When the state is in the RPI set, i.e., $x_t \in \mathcal{E}_{RPI}$, we have

$$V(x_{t+1}) \leq \|x_{t+1}\|_{P_t^*}^2 \leq \frac{c^2 \epsilon^2}{\lambda_{\min}(Q)}, \quad (33)$$

which implies that the state x_{t+1} stay inside the RPI set \mathcal{E}_{RPI} . Thus, the set \mathcal{E}_{RPI} is robustly stabilized for the closed-loop system $x_{t+1} = (A + BF_t^*)x_t + \omega_t$ with any $(A, B) \in \mathcal{C}_f$.

Part IV: Finally, we prove that the input and state constraints are satisfied for any closed-loop trajectory with $(A, B) \in \mathcal{C}_f$.

By constraint (7b), we have $x_t \in \mathcal{E}_t = \{x \in \mathbb{R}^n : \|x\|_{P_t^*}^2 \leq \gamma_t^*\}$. Given that the input is in a state-feedback form, we can write the input constraint (2a) as

$$\max_{t \in \mathbb{N}} \|u_t\|_{S_u}^2 = \max_{t \in \mathbb{N}} \|F_t^* x_t\|_{S_u}^2 \leq 1.$$

For any state $x_t \in \mathcal{E}_t$, the state $x_{t+1} = (A + BF_t^*)x_t + \omega_t$ lies inside the set \mathcal{E}_t at the next time step. Thus, the input constraint (6d) can be written as

$$\max_{t \in \mathbb{N}} \|F_t^* x_t\|_{S_u}^2 \leq \max_{x \in \mathcal{E}_t} \|F_t^* x\|_{S_u}^2 \leq 1. \quad (34)$$

The inequality (34) holds if

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^\top \begin{bmatrix} -F_t^{*\top} S_u F_t^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0,$$

holds for all x such that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^\top \begin{bmatrix} -P_t^* & 0 \\ 0 & \gamma_t^* \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0.$$

Using the S-procedure [31], if there exists $\lambda_t \geq 0$ such that

$$\begin{bmatrix} -F_t^{*\top} S_u F_t^* & 0 \\ 0 & 1 \end{bmatrix} - \lambda_t \begin{bmatrix} -P_t^* & 0 \\ 0 & \gamma_t^* \end{bmatrix} \succeq 0, \quad (35)$$

then the input constraint (6d) must be satisfied. The inequality (35) holds iff the following inequalities hold

$$\lambda_t P_t^* - F_t^{*\top} S_u F_t^* \succeq 0, \quad (36a)$$

$$1 - \lambda_t \gamma_t^* \geq 0. \quad (36b)$$

Without loss of generality, we choose the multiplier to be $\lambda_t = \frac{1}{\gamma_t^*}$. Multiplying both sides of (36a) with H_t^* , the inequality (36a) is equivalent to

$$H_t^* - L_t^{*\top} S_u L_t^* \succeq 0. \quad (37)$$

Using the Schur complement, the inequality (37) is equivalent to

$$\begin{bmatrix} H_t^* & L_t^{*\top} \\ L_t^* & S_u^{-1} \end{bmatrix} \succeq 0. \quad (38)$$

Similarly, the state constraint (6e) can be written as

$$\max_{t \in \mathbb{N}} \|x_t\|_{S_x}^2 \leq \max_{x \in \mathcal{E}_t} \|x\|_{S_x}^2 \leq 1.$$

Thus, the state constraint (6e) holds if $x^\top S_x x \leq 1$ holds for all x such that $x^\top P_t^* x \leq \gamma_t^*$. The statement holds if $S_x \preceq (\gamma_t^*)^{-1} P_t^*$. Multiplying H_t^* from left and right, we have

$$H_t^* S_x H_t^* \preceq H_t^*. \quad (39)$$

Using the Schur complement, the inequality (39) is equivalent to (7f). Thus, if problem (7) is feasible at time t , then the input and state constraints are satisfied for the closed-loop system at time t . Since the problem (7) is recursively feasible, the closed-loop system $x_{t+1} = (A + BF_t^*)x_t + \omega_t$ with any $(A, B) \in \mathcal{C}_f$ satisfies the input and state constraints (2). \square

Remark 7 Theorem 2 shows that if the optimization problem (7) is feasible at initial time $t = 0$, then the closed-loop trajectory converges robustly and exponentially to the RPI set \mathcal{E}_{RPI} for the closed-loop system with any $(A, B) \in \mathcal{C}_f$. The idea is to construct a Lyapunov function V satisfying the inequality (26). The size of the RPI set depends on the optimal solution of problem (7) at time $t = 0$, the pre-chosen constant c and the noise bound ϵ . Furthermore, Theorem 2 shows that the input and state constraints are satisfied for the closed-loop system. This is achieved via a Lyapunov function sublevel set \mathcal{E}_t inside the constraints that contains the state. Since $(A_s, B_s) \in \mathcal{C}_f$, the closed-loop trajectory of (1) is robustly stabilized and satisfy the input and state constraints.

Remark 8 The length of the offline data influences the close-loop system performance. A longer offline data sequence may lead to smaller volume of the set \mathcal{C}_f , hence leading to a better closed-loop performance. However, this leads to a higher computational complexity. As the length of offline input-state measurements T_f increases, the computational complexity of problem (7) increases due to the decision variable $\tau \in \mathbb{R}^{T_f}$. It is possible to use different choices of multipliers which reduce the computational complexity at the cost of additional conservatism, see [4, Section V.B]. For example, when imposing the additional condition $\tau_i = \tau, \forall i \in \mathbb{I}_{[1, T_f]}$ for some $\tau \geq 0$, the number of decision variables of problem (7) is independent of the data length.

4 Adaptive Data-driven Min-Max MPC

The data-driven min-max MPC scheme proposed in the previous section employs only an offline input-state data

sequence to characterize consistent systems matrices for prediction. During online operation, the collection of additional online data can improve performance especially when offline data are inadequate. To this end, in the present section, we propose an adaptive data-driven min-max MPC scheme using both offline and online data.

4.1 Adaptive Data-driven Min-Max MPC Problem

The online input-state measurements collected in the closed loop from initial time 0 until time t are denoted by

$$\begin{aligned} U_t &:= [u_0 \ u_1 \ \dots \ u_{t-1}], \\ X_t &:= [x_0 \ x_1 \ \dots \ x_t]. \end{aligned}$$

The set of system matrices (A, B) consistent with the online data x_t, u_t, x_{t+1} is defined by

$$\Sigma_t^o := \{(A, B) : (1) \text{ holds for some } \omega_t \text{ satisfying } \|\omega_t\|_2 \leq \epsilon\}.$$

which can be characterized using the same manner as in equation (3). The set of system matrices consistent with the offline input-state sequence (U_f, X_f) and the online input-state sequence (U_t, X_t) is updated recursively by

$$\mathcal{C}_t = \begin{cases} \mathcal{C}_{t-1} \cap \Sigma_{t-1}^o, & t \in \mathbb{I}_{[1, \infty)} \\ \mathcal{C}_f, & t = 0 \end{cases} \quad (40)$$

where \mathcal{C}_f is characterized by (4). The set \mathcal{C}_t can also be written as

$$\mathcal{C}_t = \bigcap_{i=1}^{t-1} \Sigma_i^o \cap \bigcap_{i=0}^{T_f-1} \Sigma_i^f.$$

We can characterize \mathcal{C}_t by the following inequality

$$\mathcal{C}_t = \left\{ (A, B) : \begin{aligned} & [I \ A \ B] (\Pi_f(\tau) + \Pi_o(\delta)) [I \ A \ B]^\top \succeq 0, \\ & \forall \tau = (\tau_0, \dots, \tau_{T_f-1}), \tau_i \geq 0, i \in \mathbb{I}_{[0, T_f-1]} \\ & \forall \delta = (\delta_0, \dots, \delta_{t-1}), \delta_i \geq 0, i \in \mathbb{I}_{[0, t-1]} \end{aligned} \right\}, \quad (41)$$

where $\Pi_f(\tau)$ is defined as in equation (5) using the offline input-state sequence and $\Pi_o(\delta)$ is defined by

$$\Pi_o(\delta) = \sum_{i=0}^{t-1} \delta_i \begin{bmatrix} I & x_{i+1}^o \\ 0 & -x_i^o \\ 0 & -u_i^o \end{bmatrix} \begin{bmatrix} \epsilon^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x_{i+1}^o \\ 0 & -x_i^o \\ 0 & -u_i^o \end{bmatrix}^\top. \quad (42)$$

Given the offline input-state sequence (U_f, X_f) of length T_f , the online input-state sequence (U_t, X_t) , the current

state x_t , the adaptive data-driven min-max MPC optimization problem is formulated as follows:

$$J_\infty^*(x_t) := \min_{\bar{u}(t)} \max_{(A, B) \in \mathcal{C}_t} \sum_{k=0}^{\infty} l(\bar{u}_k(t), \bar{x}_k(t)) \quad (43a)$$

$$\text{s.t. } \bar{x}_{k+1}(t) = A\bar{x}_k(t) + B\bar{u}_k(t), \quad (43b)$$

$$\bar{x}_0(t) = x_t, \quad (43c)$$

$$\|u_t\|_{S_u} \leq 1, \forall t \in \mathbb{I}_{\geq 0}, \quad (43d)$$

$$\|x_t\|_{S_x} \leq 1, \forall (A, B) \in \mathcal{C}_t, t \in \mathbb{I}_{\geq 0}, \quad (43e)$$

Different from the data-driven min-max MPC problem (6), the objective function (43a) is a minimization of the worst-case cost over all consistent system matrices in \mathcal{C}_t . The set \mathcal{C}_t is recursively updated using the input-state measurements collected online by (40). The state prediction, the initial state constraint and the input and state constraints remain the same as the data-driven min-max MPC problem.

4.2 Reformulation based on LMIs

To derive a tractable solution, we consider a state feedback control law $\bar{u}_k(t) = F_t \bar{x}_k(t)$ in problem (43), where $F_t \in \mathbb{R}^{m \times n}$ is the optimized state-feedback gain at time t . In the following, we formulate an SDP to derive a state-feedback control law that minimizes an upper bound on the optimal cost of (43) employing analogous techniques as in Section 3.

At time $t \in \mathbb{I}_{[1, \infty)}$, given offline input-state measurements (U_f, X_f) and online input-state measurements (U_t, X_t) , an initial state $x_t \in \mathbb{R}^n$ and a constant $c > \lambda_{\min}(Q)$, the SDP is formulated as follows:

$$\begin{aligned} & \text{minimize} && \gamma \\ & \gamma > 0, H \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{m \times n}, \tau \in \mathbb{R}^{T_f}, \delta \in \mathbb{R}^t \end{aligned} \quad (44a)$$

$$\text{s.t. (7b), (7e) and (7f) hold,} \quad (44b)$$

$$\begin{bmatrix} \begin{bmatrix} -H + \frac{\gamma}{c} I & 0 \\ 0 & 0 \end{bmatrix} + \Pi_f(\tau) + \Pi_o(\delta) & \begin{bmatrix} 0 \\ H \\ L \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & H & L^\top \end{bmatrix} & -H & \Phi^\top \\ 0 & \Phi & -\gamma I \end{bmatrix} \prec 0, \quad (44c)$$

$$\tau = (\tau_0, \dots, \tau_{T_f-1}), \tau_i \geq 0, \forall i \in \mathbb{I}_{[0, T_f-1]}, \quad (44d)$$

$$\delta = (\delta_0, \dots, \delta_{t-1}), \delta_i \geq 0, \forall i \in \mathbb{I}_{[0, t-1]}. \quad (44e)$$

The optimal solution of the optimization problem (44) at time t is denoted by $\gamma_t^*, H_t^*, L_t^*, \tau_t^*, \delta_t^*$, providing the optimal state-feedback gain $F_t^* = L_t^*(H_t^*)^{-1}$.

Similar to Theorem 1, given the state x_t , we can show that the optimal cost of problem (43) is guaranteed to be at most $\|x_t\|_P^2$ with $P = \gamma(H)^{-1}$ and $\|x_t\|_P^2$ is

upper bounded by γ if (7b) and (44c)-(44d) hold for $\gamma, H, L, \tau, \delta$. Therefore, problem (44) minimizes an upper bound on the optimal cost of the adaptive data-driven min-max MPC problem (43).

We solve the SDP problem (44) in a receding horizon manner, see Algorithm 2. At time $t = 0$, we solve the optimization problem (7) and implement the first computed input $u_t = F_t^* x_t$. At the next sampling time $t + 1$, we measure the state x_{t+1} . With the collection of on-line input-state data x_t, u_t, x_{t+1} , we adapt the set of consistent system matrices \mathcal{C}_t by (41). A new optimization variable $\delta_t \geq 0$ is introduced to the optimization problem (44) and the constraint (44e) is updated. Additionally, we update the constraint (44c) by incorporating the collected online input-state measurements x_t, u_t, x_{t+1} and variable δ_t into $\Pi_o(\delta)$. Then we solve the problem (44) and iterate the above procedure.

Algorithm 2 Adaptive data-driven min-max MPC scheme.

- 1: **Input:** $(U_f, X_f), Q, R, S_x, S_u, c$
 - 2: At time $t = 0$, measure state x_0
 - 3: Solve the problem (44)
 - 4: Apply the input $u_t = F_t^* x_t$
 - 5: Set $t = t + 1$ and measure state x_t
 - 6: Update the constraints (44c) and (44e)
 - 7: Go back to 4
-

In the following theorem, we show recursive feasibility of the problem (44) and robust stability for the closed-loop system (1) resulting from the adaptive data-driven min-max MPC scheme.

Theorem 3 Suppose Assumption 1 holds. If the optimization problem (7) is feasible at time $t = 0$, then

- i) the optimization problem (44) is feasible at any time $t \in \mathbb{I}_{[1, \infty)}$;
- ii) the set $\mathcal{E}_{RPI} := \{x \in \mathbb{R}^n : V(x) \leq \frac{c^2 \epsilon^2}{\lambda_{\min}(Q)}\}$ is robustly stabilized for the closed-loop system (1) resulting from Algorithm 2;
- iii) the closed-loop trajectory of (1) resulting from Algorithm 2 satisfies the constraints, i.e., $\|u_t\|_{S_u} \leq 1, \|x_t\|_{S_x} \leq 1$ for all $t \in \mathbb{I}_{\geq 0}$.

PROOF. The proof is similar to that of Theorem 2; hence, we only provide a sketch. The difference is that a new optimization variable is introduced to the problem (44) at each time step. To prove recursive feasibility, suppose the optimal solution of the problem (44) at time t is $\gamma_t^*, H_t^*, L_t^*, \tau_t^*, \delta_t^*$. When the state is outside the RPI set, i.e., $x_t \in \mathcal{E}_{ROA} \setminus \mathcal{E}_{RPI}$, we define a candidate solution of problem (44) at time $t + 1$ as follows:

$$\gamma'_{t+1} = \gamma_t^*, H'_{t+1} = H_t^*, L'_{t+1} = L_t^*, \tau'_{t+1} = \tau_t^*, \delta'_{t+1} = [\delta_t^*, 0].$$

When the state is inside the RPI set, i.e., $x_t \in \mathcal{E}_{RPI}$, we define a candidate solution of problem (44) at time $t + 1$ as follows:

$$\gamma'_{t+1} = \frac{c^2 \epsilon^2}{\lambda_{\min}(Q)}, H'_{t+1} = H_t^*, L'_{t+1} = L_t^*, \tau'_{t+1} = \tau_t^*, \delta'_{t+1} = [\delta_t^*, 0].$$

Constraint (44c) is trivially satisfied with the defined candidate solution in these two cases.

The satisfaction of (7b) and (44c)-(44e) implies that (16) is satisfied for any $(A, B) \in \mathcal{C}_t$. The set \mathcal{C}_t is recursively updated over time. Based on its definition, the uncertainty set \mathcal{C}_t satisfies $(A_s, B_s) \in \mathcal{C}_{t+1} \subseteq \mathcal{C}_t$ for all $t \in \mathbb{N}$. As t approaches infinity, we can establish that (16) holds for any $(A, B) \in \mathcal{C}_t$, where the true system matrices always lie within this set. Using (16), we can show robust stability and constraint satisfaction for the closed-loop system (1) following the same steps as Theorem 2. \square

Remark 9 As the online input-state measurements are collected and \mathcal{C}_t is recursively updated, the uncertainty due to the set \mathcal{C}_t never increased and may possibly decrease. Thus, the closed-loop performance resulting from the adaptive data-driven min-max MPC scheme is no worse than the approach in Section 3.

Remark 10 The adaptive data-driven min-max MPC scheme directly incorporates new data to the SDP problem (44), which increases the computational complexity due to the introduction of a new optimization variable δ_t in problem (44) at each time step. One possible method to reduce the computational complexity is to stop adding new data when the closed-loop performance is satisfactory. Alternatively, one can recursively compute an outer approximation of the set \mathcal{C}_t using the previous uncertainty set along with the newly collected data at time t . Possible methods for computing an outer approximation have been investigated in [7, 48].

5 Simulation

In this section, we demonstrate the effectiveness of the proposed data-driven min-max MPC schemes through two numerical examples. First, we implement the proposed MPC schemes in Sections 3 and 4 on a continuous stirred-tank reactor (CSTR). Both schemes have good closed-loop performance and the adaptive scheme shows performance improvement compared to the approach in Sections 3. Second, we compare the proposed approach with the robust data-driven MPC scheme based on the Fundamental Lemma from [22]. Simulation results illustrate that our scheme is less conservative and achieves better closed-loop performance.

5.1 Implementation of the Proposed MPC Schemes

We consider the linearization of the nonlinear CSTR consider in [49], which is linearized at $[0.9831, 0.3918]^\top$ and discretized with a sampling time 0.5 seconds. The linearized system is given by

$$x_{t+1} = \begin{bmatrix} 0.9749 & -0.0135 \\ 0.0004 & 0.9888 \end{bmatrix} x_t + 10^{-4} \cdot \begin{bmatrix} 0.041 \\ 5.934 \end{bmatrix} u_t + \omega_t, \quad (45)$$

where the noise satisfies Assumption 1, i.e., $\omega_t \in \{\omega \in \mathbb{R}^2 : \|\omega\|_2 \leq 10^{-3}\}$. The system matrices are unknown, but an offline input-state trajectory (U_f, X_f) of length $T_f = 15$ is available, where the input $u \in U_f$ is chosen uniformly from the unit interval $[-10, 10]$. Moreover, the input and state constraints are given by $\|u_t\|_{S_u} \leq 1$ and $\|x_t\|_{S_x} \leq 1$, where $S_u = 0.01I, S_x = 100I$. We choose the weighting matrices of the stage cost function by $Q = 0.01I, R = 10^{-4}$. The initial state is given as $x_0 = [0.04, -0.04]^\top$. We apply the proposed MPC schemes in Sections 3 and 4 to the system (45), respectively.

Figure 1 illustrates the closed-loop input and state trajectories resulting from the application of the proposed data-driven min-max MPC schemes and the static state-feedback control law. The static state-feedback gain is computed at time $t = 0$ of the data-driven min-max MPC scheme in Section 3, as explained in Remark 6. For all three approaches, closed-loop state trajectories converge to a neighborhood of the origin. Note that the input and state constraints are satisfied in the closed-loop operation. The inputs does not get close to the constraint boundary, which can be attributed to the conservatism due to two factors: first, we only optimize over state-feedback inputs, and second, the reformulation of the constraints may be conservative. Comparison of the closed-loop input trajectories reveals that the state and input trajectory of the proposed schemes converges to a region closer to the origin compared with the static state-feedback control law. Table 1 presents the sum of closed-loop stage costs over all 300 iterations and the average computational time per iteration for the proposed data-driven min-max MPC schemes and the static state-feedback control law. The closed-loop cost of the adaptive data-driven min-max MPC scheme is smaller than the other two, while the average computational time per iteration is larger than the scheme in Section 3.

Table 1

Closed-loop cost and average computation time.

	cost	time (s)
static feedback	0.0068	-
scheme in Sec.3	0.0059	0.0501
scheme in Sec.4	0.0040	0.1212

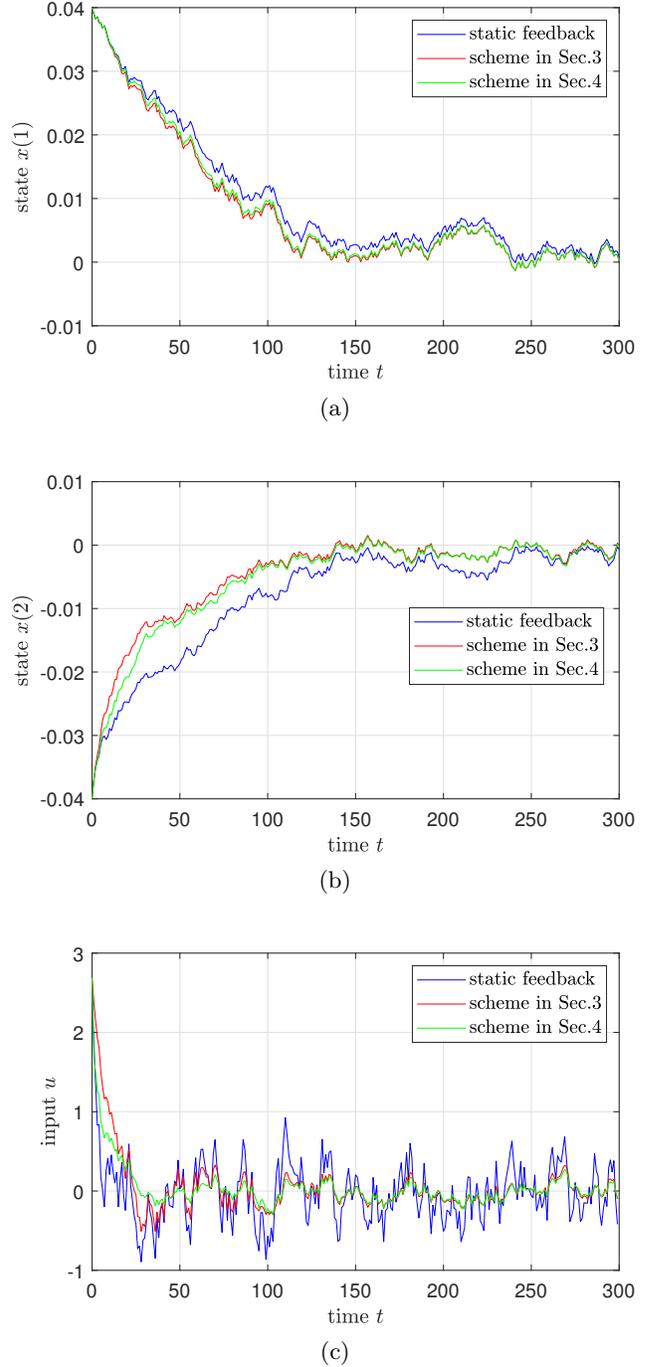


Fig. 1. Closed loop trajectories under the proposed data-driven min-max MPC schemes: (a) Closed-loop state $x(1)$. (b) Closed-loop state $x(2)$. (c) Closed-loop input u .

5.2 Comparison to the Data-driven MPC Schemes in literature

In this section, we contrast our data-driven min-max MPC scheme with the data-driven MPC scheme from [22], which relies on the Fundamental Lemma and

includes a constraint tightening guaranteeing robust constraint satisfaction. While our approach incorporates ellipsoidal constraints, [22] employs hypercube constraints. To facilitate a comparative analysis, we implement both schemes on a scalar system

$$x_{t+1} = 1.1x_t + 0.5u_t + \omega_t, \quad (46)$$

where the noise satisfies $\omega_t \in \{\omega \in \mathbb{R} : \|\omega\|_2 \leq \epsilon\}$ with $\epsilon = 10^{-4}$. The input and state constraints are $|u_t| \leq 2$ and $|x_t| \leq 2$. An input-state trajectory (U_f, X_f) of length $T_f = 20$ is available. The weighting matrices of the stage cost function are $Q = 1, R = 0.1$. The initial state is given as $x_0 = -1$.

We apply the proposed data-driven min-max MPC scheme and the data-driven MPC scheme in [22]. While our approach accounts for process noise in the system described by (46), [22] focuses on measurement noise. To translate the bound on process noise into a bound on measurement noise as required for [22], we use the fact that process noise bounded by ϵ results in measurement noise bounded by $\sum_{i=0}^{k-1} A_s^i \epsilon$ at time k . Figure 2 illustrates the closed-loop input and state trajectories resulting from the application of both schemes. The input and state trajectories from both schemes converges to a neighborhood of the origin and satisfy the input and state constraints. The sum of closed-loop stage costs over all 20 iterations for the proposed data-driven min-max MPC scheme is 9.58% lower than that for data-driven MPC scheme in [22].

We now increase the bound on the noise and implement both schemes as explained above. When ϵ approaches 0.0005, the approach from [22] becomes infeasible. In contrast, the proposed MPC scheme remains feasible and robustly stabilizes the system for ϵ up to 0.02. This result shows that our proposed data-driven min-max MPC scheme exhibits less conservatism compared to the approach in [22], allowing for stability and constraint satisfaction guarantees with higher noise levels. Further, as shown in Section 4, it allows to employ online data in order to improve closed-loop performance, which is not easily possible for the approach from [22].

6 Conclusion

In this paper, we present a data-driven min-max MPC scheme that uses noisy input-state data to design state-feedback controllers for unknown LTI systems. We reformulate the data-driven min-max MPC problem with ellipsoidal input and state constraints as an SDP. A receding-horizon algorithm is proposed to repeatedly solve the SDP at each time step and obtain a state-feedback gain. We establish that the proposed scheme guarantees closed-loop recursive feasibility, constraint satisfaction and robust stability for any systems consistent with the noisy input-state data. Furthermore, we

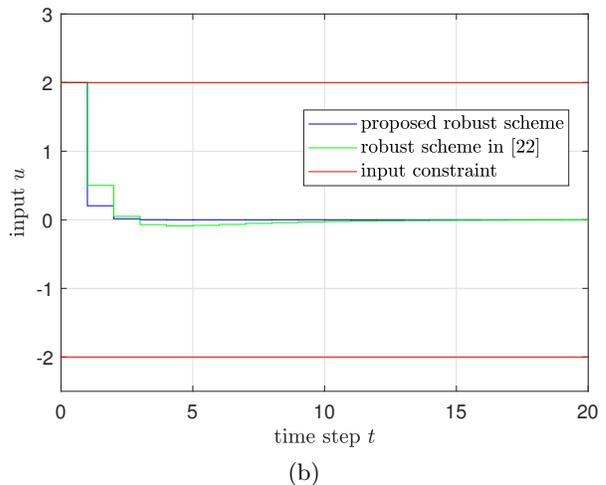
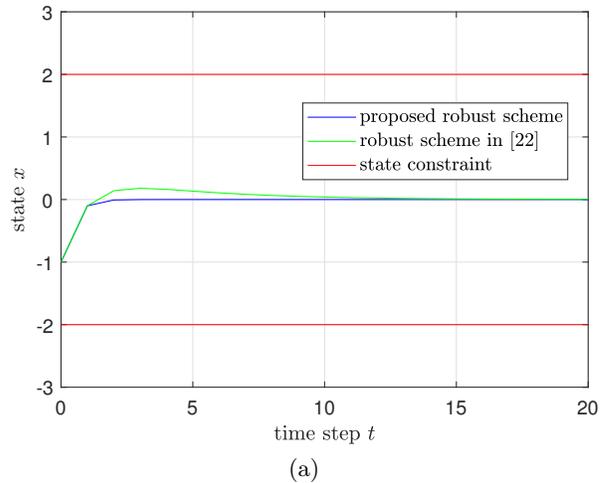


Fig. 2. Closed loop trajectories under the proposed data-driven min-max MPC scheme and the robust constraint tightening data-driven MPC scheme in [22]: (a) Closed-loop state x . (b) Closed-loop input u .

propose an adaptive data-driven min-max MPC scheme that employing online collected input-state data to improve closed-loop performance when the offline data are insufficient. We establish that the resulting closed-loop trajectory satisfies the input and state constraint and is robustly stabilized. Two numerical examples show that the adaptive scheme indeed improve the closed-loop performance compared with the robust scheme, and our proposed data-driven min-max MPC scheme exhibits less conservatism than the robust constraint tightening MPC scheme in the literature. In the future, we plan to investigate the data-driven min-max MPC scheme using noisy input-output data. Further, extending our results to nonlinear systems is another interesting direction.

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