

# FINITE DISTANCE PROBLEM ON THE MODULI OF NON-KÄHLER CALABI-YAU $\partial\bar{\partial}$ -THREEFOLDS

TSUNG-JU LEE

**ABSTRACT.** In this article, we study the finite distance problem with respect to the period-map metric on the moduli of non-Kähler Calabi–Yau  $\partial\bar{\partial}$ -threefolds via Hodge theory. We extended C.-L. Wang’s finite distance criterion for one-parameter degenerations to the present setting. As a byproduct, we also obtained a sufficient condition for a non-Kähler Calabi–Yau to support the  $\partial\bar{\partial}$ -lemma which generalizes the results by Friedman and Li. We also proved that the non-Kähler Calabi–Yau threefolds constructed by Hashimoto and Sano support the  $\partial\bar{\partial}$ -lemma.

## CONTENTS

0. Introduction	1
1. Preliminaries	4
2. The period-map metric on the moduli of CY $\partial\bar{\partial}$ -manifolds	11
3. The second Hodge–Riemann bilinear relation for finite distance degenerations	16
4. Proof of the $\partial\bar{\partial}$ -lemma for Hashimoto–Sano’s examples	27
References	35

## 0. INTRODUCTION

**0.1. Motivations.** Let  $Y$  be a three-dimensional Calabi–Yau (CY) manifold (for instance a quintic threefold in  $\mathbf{P}^4$ ) and  $C_1, \dots, C_r$  be disjoint smooth rational curves in  $Y$  whose normal bundle  $\mathcal{N}_{C_i/Y}$  is isomorphic to  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ . One can construct a contraction  $\pi: Y \rightarrow \bar{X}$  where  $\bar{X}$  is a singular threefold having  $r$  ordinary double points (ODPs)  $p_1, \dots, p_r$  whose pre-images under  $\pi$  are  $C_1, \dots, C_r$ . If one further assumes that  $[C_1], \dots, [C_r]$  span  $H_2(X; \mathbb{C})$  and there are  $m_1, \dots, m_r \in \mathbb{Q}$  such that

$$(0.1) \quad \sum_{i=1}^r m_i [C_i] = 0 \in H_2(X; \mathbb{C}) \text{ with } m_i \neq 0 \text{ for all } i,$$

the result of Friedman [Fri86] implies that  $\bar{X}$  is smoothable. Let  $X$  be a smoothing. We thus obtain a *conifold transition*  $X \nearrow Y$ ; a complex degeneration  $X \rightsquigarrow \bar{X}$  followed by a resolution  $Y \rightarrow \bar{X}$ . One checks that  $b_2(X) = 0$  and in particular,  $X$  is non-Kähler. The construction was

*Date:* May 1, 2024.

*2020 Mathematics Subject Classification.* 32Q25, 32G05, 14C30.

*Key words and phrases.* non-Kähler Calabi–Yau manifolds, the  $\partial\bar{\partial}$ -lemma, limiting mixed Hodge structures, period-map metrics.

firstly described by H. Clemens around 1985. Based on this, M. Reid speculated that there could be a single irreducible moduli space of possibly non-Kähler CY threefolds such that any CY threefold can be connected to another member in this moduli through conifold transitions; this is also known as Reid’s fantasy [Rei87]. Here, by a non-Kähler CY manifold we mean a compact complex manifold with trivial canonical bundle. From this perspective, one inevitably bumps into non-Kähler CY manifolds and it is of importance to investigate their moduli spaces.

In the Kähler regime, Yau’s theorem [Yau78], among other things, produces a canonical metric on the moduli space of *polarized* CY manifolds, called the *Weil–Petersson* (WP) metric. To explore the structure of moduli spaces, one may study the WP geometry on them. It is known that the WP metric is incomplete in general as Candelas *et al.* found a nodal CY threefold having finite WP distance [CGH90]. Later, C.-L. Wang gave a Hodge-theoretic criterion for finite distance fibers in the case of one-parameter degenerations [Wan97, Wan03] and proposed a finite distance conjecture for higher-dimensional bases. The author subsequently verified Wang’s conjecture up to a codimension two subset in the moduli space [Lee18].

Due to the absence of Kähler metrics, the standard tools such as Hodge theory do not apply when we study the moduli spaces of non-Kähler CY manifolds. Nevertheless, R. Friedman [Fri19] and C. Li [Li2202] showed that the smoothing  $X$  described in the previous paragraph supports the  $\partial\bar{\partial}$ -lemma, which allows us to define Hodge decomposition on their cohomology groups. Besides, it is proven by J. Fu, J. Li, and S.-T. Yau that such an  $X$  carries a balanced metric [FLY12]. Recall that a hermitian metric on a compact complex manifold of complex dimension  $n$  is called a *balanced metric* if its metric two form  $\omega$  satisfying the equation  $d\omega^{n-1} = 0$ . Based on Fu–Li–Yau’s balanced structure, T. Collins, S. Picard, and S.-T. Yau showed that the holomorphic tangent bundle of  $X$  admits a Hermitian–Yang–Mills connection [CPY24]. T. Collins, S. Gukov, S. Picard, and S.-T. Yau also constructed special Lagrangian spheres in  $X$  [CGPY23].

Non-Kähler CY  $\partial\bar{\partial}$ -manifolds have been extensively studied in the literature. Let  $W$  be a CY  $\partial\bar{\partial}$ -manifold of complex dimension  $n$ . D. Popovici proved that both the unobstructedness theorem and local Torelli theorem hold for  $W$  [Pop19]. Moreover, if the second Hodge–Riemann bilinear relation holds on  $H^n(W; \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(W)$ , we can then pullback the corresponding Fubini–Study metric on  $\mathbf{P}H^n(W; \mathbb{C})$  to the moduli space  $S$  via the local Torelli immersion  $S \rightarrow \mathbf{P}H^n(W; \mathbb{C})$ . Note that we do not need any metrics to define the pairing on the middle cohomology group. In this manner, we obtain the so-called *period-map metric* on  $S$  (cf. [Pop19]). Taking balanced metrics into account, D. Popovici also introduced several variants of Weil–Petersson metrics on the moduli of compact CY  $\partial\bar{\partial}$ -manifolds and studied the relationship among them [Pop19].

**0.2. Statements of the main results.** The aim of this note is to study the finite distance problem with respect to the period-map metric for one-parameter degenerations.

Let  $f: \mathcal{X} \rightarrow \Delta$  be a one-parameter degeneration of  $\partial\bar{\partial}$ -manifolds of complex dimension  $n$ . We may assume that  $f$  is a semi-stable family by a finite base change and blow-ups; these operations do not affect the finite distance property. Put  $E := f^{-1}(0)$  and let  $E_1, \dots, E_m$  be irreducible components of  $E$ . Suppose that any intersection of  $E_i$ ’s supports the  $\partial\bar{\partial}$ -lemma. In which case, following Steenbrink [Ste75], one can define the limiting mixed Hodge structure  $(\mathcal{F}_{\lim}^\bullet, \mathcal{W}(M)_\bullet, H^n(X; \mathbb{C}))$  where  $X$  is a general fiber. Let  $T$  be the monodromy operator on

$H^n(X; \mathbb{C})$  for  $f$  and  $N := \log T$  be the nilpotent operator. Then  $N$  is of type  $(-1, -1)$  with respect to the limiting mixed Hodge structure. Under another mild assumption

**Hypothesis A.** The nilpotent operator  $N$  induces an isomorphism between relevant quotients of the monodromy weight filtration, namely

$$N^k : \mathrm{Gr}_{n+k}^{\mathcal{W}(M)} H^n(X; \mathbb{C}) \rightarrow \mathrm{Gr}_{n-k}^{\mathcal{W}(M)} H^n(X; \mathbb{C})$$

is an *isomorphism* for each  $0 \leq k \leq n$ .

we will prove

**Theorem 0.1** (= Theorem 2.3). *Assume Hypothesis A. Then  $0 \in \Delta$  has infinite distance with respect to the period-map metric if  $N\mathcal{F}_{\lim}^n \neq 0$ .*

Note that the statement above indeed consists of two parts. First, since we do not know whether or not the second Hodge–Riemann bilinear relation holds on  $H^{n-1,1}(X)$ , we need to show that the period-map metric is an honest metric. Fortunately, this can be done when  $N\mathcal{F}_{\lim}^n \neq 0$ . After resolving this issue, we then compute the distance and prove that  $0 \in \Delta$  has infinite distance following C.-L. Wang’s approach [Wan97].

The other case  $N\mathcal{F}_{\lim}^n = 0$  is somewhat subtler. Consider a more stringent assumption.

**Hypothesis B.** There exists a cohomology class in  $H^2(E; \mathbb{R})$  whose restriction to  $E_i$  of  $E$  provides a Kähler class for  $E_i$ .

This tacitly asserts that each component of  $E$  is Kähler and in particular, it implies that any arbitrary intersections of  $E_i$ ’s are Kähler and hence the  $\partial\bar{\partial}$ -lemma holds on them.

Recall that a CY  $\partial\bar{\partial}$ -manifold  $X$  is *strict*, if  $H^d(X; \mathcal{O}_X) = 0$  for  $0 < d < n$ . We then have the following result for  $N\mathcal{F}_{\lim}^n = 0$ .

**Theorem 0.2** (= Proposition 3.1). *Under Hypothesis B, if the general fiber  $X$  of  $f$  is a strict smooth CY  $\partial\bar{\partial}$ -threefold and  $N\mathcal{F}_{\lim}^3 = 0$ , then the second Hodge–Riemann bilinear relation holds on  $H^{2,1}(X)$ . Consequently, the period-map metric is truly a metric.*

To prove the theorem, we shall follow C. Li’s approach in [Li2202]. Let us consider Deligne’s splitting  $I^{p,q}$  for Steenbrink’s limiting mixed Hodge structure on  $H^3(X; \mathbb{C})$ . Firstly, we construct a basis for each  $I^{p,q}$  and hence obtain a basis for  $\mathcal{F}_{\lim}^2$ . Secondly, we extend the proceeding basis to become a local frame for Deligne’s canonical extension. Finally we untwist the local frame to get a (multi-valued) frame for  $\mathcal{F}_t^2$  where  $\mathcal{F}^2$  is the holomorphic bundle over  $\Delta^*$  whose fiber over  $t$  is the vector space  $H^{3,0}(\mathcal{X}_t) \oplus H^{2,1}(\mathcal{X}_t)$ . We then check the positivity using the multi-valued frame.

**Corollary 0.3** (= Theorem 2.5). *In the situation of Theorem 0.2,  $0 \in \Delta$  has finite distance with respect to the period-map metric.*

As an application, we can use the multi-valued frame to obtain the following proposition, which slightly generalizes the results of R. Friedman [Fri19] and C. Li [Li2202].

**Proposition 0.4** (= Corollary 3.6). *Let  $g : \mathcal{X} \rightarrow \Delta$  be a semi-stable degeneration. We assume that the general fiber  $X$  of  $g$  has complex dimension 3 and satisfies the following conditions.*

- *The central fiber of  $f$  is at a finite distance with respect to the period-map metric;*

- $H^i(X; \mathcal{O}_X) = H^0(X; \Omega_X^j) = 0$  for  $1 \leq i, j \leq 2$ .

Then under Hypothesis A, the  $\partial\bar{\partial}$ -lemma holds on  $X$ .

Hashimoto and Sano constructed smooth non-Kähler CY threefolds with arbitrarily large  $b_2$  [HS23]. We may also apply the techniques here to show that

**Proposition 0.5** (= Theorem 4.4 and Theorem 4.5). *Let  $X$  be the smooth non-Kähler CY threefold constructed in [HS23]. Then  $X$  is a  $\partial\bar{\partial}$ -manifold. Moreover, the second Hodge–Riemann bilinear relation holds on  $H^{2,1}(X)$ .*

**Remark 0.1.** Some results presented in this manuscript were also obtained by K.-W. Chen independently [Che24], where the Hodge structure on the smoothings is carefully studied.

**Acknowledgment.** The author would like to thank Professor Chin-Lung Wang for his interest as well as many valuable conversations with the author on this project. He would like to thank Professor Shing-Tung Yau for his interest and comments on the manuscript. The author also wants to thank Chung-Ming Pan for inspiring discussions. Part of the results presented here was announced in the workshop “East Asian Symplectic Conference 2023” in October, 2023. In a communication with C.-L. Wang, the author was informed that K.-W. Chen has also obtained similar results in this direction independently. The author would also like to thank him for suggestions to improve the presentation.

## 1. PRELIMINARIES

**Notation.** Let  $X$  be a compact complex manifold. For a sheaf  $\mathcal{F}$  on  $X$ , the notation  $H^k(X; \mathcal{F})$  stands for the usual sheaf cohomology on  $X$ . For a bounded complex of  $\mathbb{C}$ -sheaves  $\mathcal{F}^\bullet$  on  $X$ , we denote by  $H^k(X; \mathcal{F}^\bullet)$  the  $k^{\text{th}}$  hypercohomology group.

**1.1. The  $\partial\bar{\partial}$ -lemma.** We begin with the statement of  $\partial\bar{\partial}$ -lemma. Let  $X$  be a compact complex manifold and  $\Omega_X^p$  be the sheaf of holomorphic  $p$ -forms on  $X$ . There is a complex of sheaves  $\Omega_X^\bullet$

$$(1.1) \quad 0 \rightarrow \Omega_X^0 \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_X^n \rightarrow 0, \quad n = \dim X,$$

which is quasi-isomorphic to the constant sheaf  $\mathbb{C}_X$  on  $X$ . The hypercohomology of  $\Omega_X^\bullet$  computes the usual cohomology (singular cohomology) of  $X$ .

The truncated complexes

$$F^p \Omega_X^\bullet := [0 \rightarrow \cdots \rightarrow 0 \rightarrow \Omega_X^p \rightarrow \cdots \rightarrow \Omega_X^n \rightarrow 0]$$

with  $\Omega_X^n$  at the  $n^{\text{th}}$  place defines a decreasing filtration on (1.1) on the level of complexes and they induce a decreasing filtration  $F^\bullet$  on the cohomology group  $H^k(X; \mathbb{C})$ ; specifically  $F^p H^k(X; \mathbb{C})$  is defined to be the image of the canonical map

$$H^k(X; F^p \Omega_X^\bullet) \rightarrow H^k(X; \Omega_X^\bullet) = H^k(X; \mathbb{C})$$

induced from  $F^p \Omega_X^\bullet \rightarrow \Omega_X^\bullet$ .

**Definition 1.1.** We say that the  $\partial\bar{\partial}$ -lemma holds on  $X$  or call  $X$  a  $\partial\bar{\partial}$ -manifold if for all  $p, q$  and for all d-closed  $(p, q)$ -form  $\eta \in A^{p,q}(X)$ , the following statement holds:

$$(*) \quad \eta \text{ is d-exact, } \partial\text{-exact, or } \bar{\partial}\text{-exact} \Leftrightarrow \eta = \partial\bar{\partial}\xi \text{ for some } \xi \in A^{p-1, q-1}(X).$$

**Remark 1.2.** It is known that the  $\partial\bar{\partial}$ -lemma is equivalent to the following statement. Let  $\eta \in A^k(X)$  be a both  $\partial$ -closed and  $\bar{\partial}$ -closed  $k$ -form. Then

$$\eta \text{ is d-exact} \Leftrightarrow \eta = \partial\bar{\partial}\xi \text{ for some } \xi \in A^{k-2}(X).$$

It is well-known ([DGMS77, Equation (5.21)] and [Del71, Equation (4.3.1)]) that the following two statements are equivalent.

- (i) (a) The Hodge-to-de Rham spectral sequence (a.k.a. the Frölicher spectral sequence) on  $X$  degenerates at  $E_1$ ;  
 (b) for all  $k$ , if  $F^\bullet$  denotes the corresponding filtration on  $H^k(X; \mathbb{C})$ , then  $F^\bullet$  and  $\bar{F}^\bullet$  are  $k$ -opposed; namely  $F^p \oplus \bar{F}^{k+1-p} = H^k(X; \mathbb{C})$ .
- (ii) The  $\partial\bar{\partial}$ -lemma holds on  $X$ .

For a compact complex manifold, consider the short exact sequence

$$(1.2) \quad 0 \rightarrow F^{p+1}\Omega_X^\bullet \rightarrow F^p\Omega_X^\bullet \rightarrow F^p\Omega_X^\bullet / F^{p+1}\Omega_X^\bullet \cong \Omega_X^p[-p] \rightarrow 0.$$

Taking hypercohomology, we obtain a long exact sequence

$$\cdots \rightarrow \mathbf{H}^k(X; F^{p+1}\Omega_X^\bullet) \rightarrow \mathbf{H}^k(X; F^p\Omega_X^\bullet) \rightarrow \mathbf{H}^{k-p}(X; \Omega_X^p) \rightarrow \mathbf{H}^{k+1}(X; F^{p+1}\Omega_X^\bullet) \rightarrow \cdots.$$

If  $X$  is a  $\partial\bar{\partial}$ -manifold, there exists a canonical injection

$$H^{p,q}(X) := H^q(X; \Omega_X^p) \rightarrow H^{p+q}(X; \mathbb{C})$$

from the Dolbeault cohomology to the de Rham cohomology. For a Dolbeault class  $[\theta] \in H^{p,q}(X)$ , we choose a  $\bar{\partial}$ -closed representative  $\theta \in A^{p,q}(X)$ . Now  $\partial\theta \in A^{p+1,q}(X)$  is d-closed and clearly  $\partial$ -exact. By assumption, there exists  $v \in A^{p,q-1}(X)$  such that  $\partial\theta = \partial\bar{\partial}v$ . So  $\theta - \bar{\partial}v$  is a d-closed representative of  $[\theta]$ .

To prove the injectivity, let  $[\theta] \in H^{p,q}(X)$  such that  $\theta - \bar{\partial}v$  is also d-exact. By assumption again, there exists  $u \in A^{p-1,q-1}(X)$  such that

$$\theta - \bar{\partial}v = \partial\bar{\partial}u.$$

Thus  $\theta = \bar{\partial}(v + \partial u)$  is  $\bar{\partial}$ -exact and hence  $[\theta] = 0 \in H^{p,q}(X)$ . It is also clear that

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

when we regard them as subspaces in  $H^{p+q}(X; \mathbb{C})$ . In particular,  $H^k(X; \mathbb{C})$  carries a weight  $k$  Hodge structure.

## 1.2. Calabi-Yau $\partial\bar{\partial}$ -manifolds.

**Definition 1.3.** Let  $X$  be a compact complex manifold of dimension  $n$ . We say that  $X$  is a *Calabi-Yau (CY)  $\partial\bar{\partial}$ -manifold* if  $X$  is a  $\partial\bar{\partial}$ -manifold whose canonical bundle  $\Omega_X^n$  is trivial.

Many theorems stated for CY manifolds (compact Kähler manifolds with trivial canonical bundle) also hold for CY  $\partial\bar{\partial}$ -manifold. For instance, we have the unobstructedness theorem, the Bogomolov-Tian-Todorov theorem for CY  $\partial\bar{\partial}$ -manifolds.

**Theorem 1.1** ([Pop19, Theorem 1.2]). *Let  $X$  be a compact CY  $\partial\bar{\partial}$ -manifold. Then  $X$  has unobstructed deformation, i.e. the Kuranishi space of  $X$  is smooth.*

We also have the following local Torelli theorem.

**Theorem 1.2** ([Pop19, Theorem 5.4]). *Let  $X$  be a compact CY  $\partial\bar{\partial}$ -manifold of complex dimension  $n$  and let  $\pi: \mathcal{X} \rightarrow S$  be its Kuranishi family. Then the associated period map*

$$\mathcal{P}: S \rightarrow D \subset \mathbf{PH}^n(X; \mathbb{C}), \quad s \mapsto H^{n,0}(\mathcal{X}_s; \mathbb{C})$$

*is a local holomorphic immersion.*

Let us recall a definition for the later use.

**Definition 1.4.** Let  $(X, \omega)$  be a balanced CY  $\partial\bar{\partial}$ -manifold. Given a Dolbeault cohomology class  $[\theta] \in H^{p,q}(X)$ , let  $\theta \in A^{p,q}(X)$  be a  $\Delta''_\omega$ -representative and  $v_{\min} \in A^{p,q-1}(X)$  such that  $v_{\min}$  is the solution to  $\partial\theta = \partial\bar{\partial}v$  with minimum  $L^2$  norm with respect to  $\omega$ . The d-closed form  $\theta_{\min} := \theta + \bar{\partial}v_{\min}$  is called the  $\omega$ -minimal d-closed representative of  $[\theta]$ .

**1.3. Deligne's canonical extension.** Let  $\mathcal{V} \rightarrow \Delta^*$  be a holomorphic vector bundle with a flat connection  $\nabla$  whose monodromy  $T$  is unipotent. Denote by

$$N := \log T = \log(I - (I - T)) = - \sum_{k=1}^{\infty} \frac{(I - T)^k}{k}.$$

the associated nilpotent operator so that  $T = e^N$ . Denote by

$$\pi: \mathfrak{h} \rightarrow \Delta^*, \quad z \mapsto t = e^{2\pi\sqrt{-1}z},$$

the universal cover. Then the pullback  $\pi^*\mathcal{V}$  becomes a trivial vector bundle. Let  $\{v_1, \dots, v_m\}$  be a flat trivialization for  $\pi^*\mathcal{V}$ . For a flat section  $v$  of  $\pi^*\mathcal{V}$ , we see that it satisfies the relation  $v(z+1) = Tv(z)$ . Therefore the twisted section

$$u := e^{-zN}v$$

is invariant under the translation  $z \mapsto z+1$ , for

$$u(z+1) = e^{-(z+1)N}v(z+1) = e^{-zN}e^{-N}Tv(z) = e^{-zN}v(z) = u(z).$$

It follows that  $u$  becomes a (non-flat) section of  $\mathcal{V}$  on  $\Delta^*$ . In this manner, we obtain a twisted section  $\{u_1, \dots, u_m\}$  for  $j_*\mathcal{V}$  where  $j: \Delta^* \rightarrow \Delta$  is the open inclusion. We can thus extend  $\mathcal{V}$  to a holomorphic vector bundle  $\bar{\mathcal{V}}$  on  $\Delta$  via the frame  $\{u_1, \dots, u_m\}$ . We compute

$$\nabla_{t\partial_t}u_k = \frac{1}{2\pi\sqrt{-1}}\nabla_{\partial_z}u_k = \frac{-Nu_k}{2\pi\sqrt{-1}}.$$

We obtain a logarithmic connection  $\bar{\nabla}$  on  $\bar{\mathcal{V}}$  with residue  $-N/2\pi\sqrt{-1}$  along  $t=0$ , i.e.

$$\bar{\nabla} = d - \frac{N}{2\pi\sqrt{-1}} \otimes \frac{dt}{t}.$$

In which case, the connection  $\nabla$  is extended to be a connection  $\bar{\nabla}$  on  $\bar{\mathcal{V}}$  with a logarithmic pole along  $t=0$ . The extension  $(\bar{\mathcal{V}}, \bar{\nabla})$  is called *Deligne's canonical extension*.

**1.4. Mixed Hodge structures and Deligne's splitting.** Let  $H_{\mathbb{R}}$  be a finite-dimensional real vector space. Put  $H := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ . Recall that a (real) mixed Hodge structure ( $\mathbb{R}$ -mixed Hodge structure) on  $H_{\mathbb{R}}$  consists of a pair of finite filtrations

$$\begin{aligned} \cdots \subset W_{k-1} \subset W_k \subset W_{k+1} \subset \cdots \\ \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \end{aligned}$$

such that

- (i)  $W$  is defined over  $\mathbb{R}$ ;
- (ii) for  $l \in \mathbb{Z}$ , the filtration  $F^{\bullet}$  induces a Hodge structure of weight  $l$  on the graded piece  $\text{Gr}_l^W H$ .

The increasing filtration  $W_{\bullet}$  is called the *weight filtration* whereas the decreasing filtration  $F^{\bullet}$  is referred as the *Hodge filtration*.

Recall that for a given  $\mathbb{R}$ -mixed Hodge structure  $(H, W_{\bullet}, F^{\bullet})$ , a *weak splitting* of  $(H, W_{\bullet}, F^{\bullet})$  is a decomposition of the vector space

$$(1.3) \quad H = \bigoplus_{p,q} J^{p,q}$$

such that

$$(1.4) \quad W_k^{\mathbb{C}} = \bigoplus_{p+q \leq k} J^{p,q} \text{ and } F^p = \bigoplus_{r \geq p} J_{\mathbb{C}}^{r,q}.$$

Notice that in (1.4) the notation  $W_k^{\mathbb{C}}$  stands for  $W_k \otimes_{\mathbb{R}} \mathbb{C}$ .

By a theorem of Deligne, any  $\mathbb{R}$ -mixed Hodge structure always admits such a splitting. This is called the *Deligne's splitting*. In fact, given a  $\mathbb{R}$ -mixed Hodge structure  $(H, W_{\bullet}, F^{\bullet})$ , Deligne's splitting is defined by

$$(1.5) \quad I^{p,q} := F^p \cap W_{p+q}^{\mathbb{C}} \cap \left( \overline{F^q} \cap W_{p+q}^{\mathbb{C}} + \sum_{j \geq 2} \overline{F^{q-j+1}} \cap W_{p+q-j}^{\mathbb{C}} \right).$$

One checks that they satisfy (1.4) and also the property

$$(1.6) \quad I^{p,q} = \overline{I^{q,p}} \quad \text{mod} \quad \bigoplus_{r < p, s < q} I^{r,s}.$$

For a proof, one can consult [CKS86, (2.13) Theorem].

**1.5. Steenbrink's limiting mixed Hodge structure.** We begin with the following setup.

- Let  $f: \mathcal{X} \rightarrow \Delta$  be a semi-stable model, i.e.  $\mathcal{X}$  is a complex manifold,  $f$  is smooth over the punctured disk  $\Delta^*$ , and the central fiber  $f^{-1}(0)$  is a simple normal crossing divisor.
- Let  $E_1, \dots, E_m$  be the irreducible components of  $E := f^{-1}(0)$ .
- For any subset  $I \subset \{1, \dots, m\}$ , we put

$$E_I := \bigcap_{i \in I} E_i \text{ and } E(k) := \bigsqcup_{|I|=k} E_I.$$

We have proper maps  $\iota_k: E(k) \rightarrow E$  for  $k \geq 1$ . We shall also set  $E(0) = E$ .

- For each  $I = \{i_1 < \dots < i_r\}$  and  $1 \leq j \leq r$ , put  $I_j := I \setminus \{i_j\}$ ; the subset of  $I$  with  $j^{\text{th}}$  element being omitted. Denote by  $\iota_{I,j}: E_I \rightarrow E_{I_j}$  the inclusion map. We have the restriction map  $\iota_{I,j}^*: H^k(E_{I_j}; \mathbb{C}) \rightarrow H^k(E_I; \mathbb{C})$  and the Gysin map  $(\iota_{I,j})_!: H^k(E_I; \mathbb{Z}) \rightarrow H^{k+2}(E_{I_j}; \mathbb{C})$ . Let  $\iota_{r,j} := \bigoplus_{|I|=r} \iota_{I,j}$  and put

$$\psi_r := \bigoplus_{j=1}^r (-1)^{j-1} \iota_{r,j}^* \text{ and } \phi_r := - \bigoplus_{j=1}^r (-1)^{j-1} (\iota_{r,j})_!.$$

They are refereed as an *alternating restriction map* and an *alternating Gysin map*. We shall skip the subscript  $r$  when no confusion occurs.

- Let  $i_t: \mathcal{X}_t \rightarrow \mathcal{X}$  be the inclusion of a fiber  $\mathcal{X}_t$ . The total space  $\mathcal{X}$  is homotopic to  $E$  by a deformation retraction  $r: \mathcal{X} \rightarrow E$ . Denote by  $r_t = r \circ i_t: \mathcal{X}_t \rightarrow E$ . The complex  $(Rr_t)_* i_t^* \mathbb{Z}_{\mathcal{X}}$  is called *the complex of nearby cocycles*.

The hypercohomology of the complex  $(Rr_t)_* i_t^* \mathbb{Z}_{\mathcal{X}}$  computes the cohomology of the nearby fiber. In fact, we have

$$\mathbf{H}^q((Rr_t)_* i_t^* \mathbb{Z}_{\mathcal{X}}) = \mathbf{H}^q(i_t^* \mathbb{Z}_{\mathcal{X}}) = \mathbf{H}^q(i_t^* \mathbb{Z}_{\mathcal{X}}) = \mathbf{H}^q(\mathcal{X}_t).$$

Nevertheless, we shall recall a concrete recipe to calculate the cohomology of the nearby fiber.

Let  $\pi: \mathfrak{h} \rightarrow \Delta^*$  be the universal cover. The fiber product  $\mathcal{X}_{\infty} := \mathcal{X} \times_{\Delta^*} \mathfrak{h}$  is called the *canonical fiber*. Note that  $\mathcal{X}_{\infty}$  is homotopic to any smooth fiber of  $f$ .

Look at the commutative diagram

$$\begin{array}{ccccccc} & & & k & & & \\ & & \nearrow & & \searrow & & \\ \mathcal{X}_{\infty} & \xrightarrow{p} & \mathcal{X}^* & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & E \\ \downarrow F & & \downarrow & & \downarrow f & & \downarrow \\ \mathfrak{h} & \xrightarrow{\pi} & \Delta^* & \longrightarrow & \Delta & \longleftarrow & \{0\}. \\ & & & \searrow & \nearrow & & \\ & & & \kappa & & & \end{array}$$

We can form the complex  $i^*(Rk)_* k^* \mathbb{Z}_{\mathcal{X}}$  which computes the cohomology of the nearby fiber.

**Definition 1.5.** Let  $\mathcal{K}^{\bullet} \in D^+(\mathcal{X}, A)$  be a bounded below complex of sheaves of  $A$ -modules on  $\mathcal{X}$ . We define the *nearby cocycle*, denoted by  $\Phi_f \mathcal{K}^{\bullet}$ , to be the complex

$$\Phi_f \mathcal{K}^{\bullet} := i^*(Rk)_* k^* \mathcal{K}^{\bullet} \in D^+(E, A).$$

There is a canonical morphism

$$i^* \mathcal{K}^{\bullet} \rightarrow \Phi_f \mathcal{K}^{\bullet}.$$

Let  $\phi_f \mathcal{K}^{\bullet}$  be the mapping cone the the morphism above. We obtain a triangle

$$i^* \mathcal{K}^{\bullet} \rightarrow \Phi_f \mathcal{K}^{\bullet} \rightarrow \phi_f \mathcal{K}^{\bullet} \xrightarrow{+1}$$

and  $\phi_f \mathcal{K}^{\bullet}$  is the complex of *vanishing cocycles*.



To define a mixed Hodge structure on  $\Phi_f \mathbb{C}_{\mathcal{X}}$ , we consider the *relative logarithmic de Rham complex*  $\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E)$ . Explicitly, it is defined as the cokernel of the morphism

$$\Omega_{\mathcal{X}}^{\bullet-1}(\log E) \xrightarrow{\wedge f^*(dt/t)} \Omega_{\mathcal{X}}^{\bullet}(\log E).$$

Let  $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$  be the ideal sheaf defining  $E$ . One can check that  $\mathcal{I}\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E)$  is stable under the differential. We thus obtain a complex  $\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E$ .

**Theorem 1.3.** *There exists a quasi-isomorphism in  $D^+(\mathbb{C}_E)$*

$$\Phi_f \mathbb{C}_{\mathcal{X}} \simeq \Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E.$$

Our goal is to define a mixed Hodge structure on  $\Phi_f \mathbb{C}_{\mathcal{X}}$  via  $\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E$  at the level of complexes. We need two pieces of information: (a) the Hodge filtration  $\mathcal{F}^{\bullet}$  and (b) the weight filtration  $\mathcal{W}_{\bullet}$ . For this purpose, we shall henceforth assume that  $E_I$  supports the  $\partial\bar{\partial}$ -lemma for each subset  $I$ .

1.5.1. *The Hodge filtration.* This part is easy; we may use the stupid filtration  $\mathcal{F}^{\bullet}$  on the complex  $\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E$ , i.e.

$$\mathcal{F}^k := \Omega_{\mathcal{X}/\Delta}^{\bullet \geq k}(\log E) \otimes \mathcal{O}_E.$$

By the  $\partial\bar{\partial}$ -lemma, this puts a Hodge filtration on the hypercohomology.

1.5.2. *The monodromy weight filtration.* There is a natural filtration  $\mathcal{W}_{\bullet}$  on  $\Omega_{\mathcal{X}}^{\bullet}(\log E)$

$$\mathcal{W}_p(\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E) := \text{The image of } \mathcal{W}_p \Omega_{\mathcal{X}}^{\bullet}(\log E)$$

under the canonical projection  $\Omega_{\mathcal{X}}^{\bullet}(\log E) \rightarrow \Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E$ .

Consider the double complex  $(\mathcal{D}^{\bullet, \bullet}, D_1, D_2)$  with

$$\begin{aligned} \mathcal{D}^{p,q} &:= \Omega_{\mathcal{X}}^{p+q+1}(\log E) \otimes \mathcal{O}_E / \mathcal{W}_p(\Omega_{\mathcal{X}}^{p+q+1}(\log E) \otimes \mathcal{O}_E) \\ &\quad \left( \cong \Omega_{\mathcal{X}}^{p+q+1}(\log E) / \mathcal{W}_p(\Omega_{\mathcal{X}}^{p+q+1}(\log E)) \text{ whenever } p \geq 0 \right) \end{aligned}$$

for  $p, q \geq 0$  with differentials

$$\begin{cases} D_1: \mathcal{D}^{p,q} \rightarrow \mathcal{D}^{p+1,q} & \text{via } \omega \mapsto (dt/t) \wedge \omega \\ D_2: \mathcal{D}^{p,q} \rightarrow \mathcal{D}^{p,q+1} & \text{via } \omega \mapsto d\omega. \end{cases}$$

Define  $\mu: \Omega_{\mathcal{X}/\Delta}^q(\log E) \otimes \mathcal{O}_E \rightarrow \mathcal{D}^{0,q}$  via

$$\omega \mapsto (-1)^q \frac{dt}{t} \wedge \omega \quad \text{mod } \mathcal{W}_0 \Omega_{\mathcal{X}}^{q+1}(\log E).$$

With this definition, we see that  $\mu$  induces a morphism of complexes

$$\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E \rightarrow \text{Tot}(\mathcal{D}^{\bullet, \bullet}).$$

Let us define an increasing filtration  $\mathcal{W}_{\bullet}$  on  $\mathcal{D}^{\bullet, \bullet}$  by

$$\mathcal{W}_r \mathcal{D}^{p,q} := \text{the image of } \mathcal{W}_{r+p+1}(\Omega_{\mathcal{X}}^{p+q+1}(\log E) \otimes \mathcal{O}_E) \text{ in } \mathcal{D}^{p,q}.$$

Also, let us define the Hodge filtration on  $\mathcal{D}^{p,q}$  via

$$\mathcal{F}^k(\text{Tot}(\mathcal{D}^{\bullet,\bullet})) := \bigoplus_p \bigoplus_{q \geq k} \mathcal{D}^{p,q}.$$

**Theorem 1.4.**  $\mu$  is a quasi-isomorphism between bi-filtered complexes

$$(\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet}) \text{ and } (\text{Tot}(\mathcal{D}^{\bullet,\bullet}), \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet}).$$

**Definition 1.6** (The monodromy weight filtration). We define another increasing filtration  $\mathcal{W}(M)_{\bullet}$  on the complex  $\mathcal{D}^{p,q}$  via

$$\mathcal{W}(M)_r \mathcal{D}^{p,q} := \text{the image of } \mathcal{W}_{r+2p+1}(\Omega_{\mathcal{X}}^{p+q+1}(\log E) \otimes \mathcal{O}_E) \text{ in } \mathcal{D}^{p,q}.$$

The filtration  $\mathcal{W}(M)_{\bullet}$  is called the *monodromy weight filtration*.

**Remark 1.7.** As we shall see, the punchline is that  $D_1$  becomes the zero map in the associated graded module with respect to  $\mathcal{W}(M)_{\bullet}$ .

Note that  $\mathcal{W}(M)_r \mathcal{D}^{p,q} = 0$  whenever  $r + 2p + 1 \leq p$ , i.e.  $r \leq -1 - p$ . Similarly, we have

$$\text{Gr}_r^{\mathcal{W}(M)}(\mathcal{D}^{p,q}) = \text{Gr}_{r+2p+1}^{\mathcal{W}} \Omega_{\mathcal{X}}^{p+q+1}(\log E) \cong (\iota_{r+2p+1})_* \Omega_{E(r+2p+1)}^{q-p-r}$$

and the last isomorphism holds when  $r + 2p \geq 0$ . In particular, this implies

$$\text{Gr}_r^{\mathcal{W}(M)}(\mathcal{D}^{p,q}) = 0$$

whenever  $r \geq q - p + 1 \geq -2p$  or  $r \leq -p - 1$ . Because of this shift, the morphism  $D_1$  becomes zero on the associated graded complex. The  $(p, q)$  term in the associated graded complex  $\text{Gr}_r^{\mathcal{W}(M)}(\text{Tot}(\mathcal{D}^{\bullet,\bullet}))$  becomes

$$\bigoplus_{-p \leq r \leq q-p} \text{Gr}_{r+2p+1}^{\mathcal{W}} \Omega_{\mathcal{X}}^{p+q+1}(\log E) \cong \bigoplus_{-p \leq r \leq q-p} (\iota_{r+2p+1})_* \Omega_{E(r+2p+1)}^{p+q}[-r-2p]$$

and the cohomology of  $(\text{Gr}_r^{\mathcal{W}(M)}(\text{Tot}(\mathcal{D}^{\bullet,\bullet})), D_1 + D_2 = D_2)$  is  $\mathbb{C}_{E(r+2p+1)}[-r-2p]$ , which computes the cohomology of the smooth variety  $E(r+2p+1)$  up to a shift. We remind the reader that the summation is taken over all non-negative integers  $p$  and  $q$  satisfying  $-p \leq r \leq q - p$ . It is now obvious that we can define the Hodge structure using the usual Hodge filtration on smooth varieties.

**Example 1.8** (Conifold transitions). Let  $X \nearrow Y$  be a conifold transition between Calabi–Yau threefolds, i.e.  $X$  and  $Y$  are smooth Calabi–Yau threefolds and there exist a complex degeneration  $X \rightsquigarrow \bar{X}$  to a conifold threefold and a small resolution  $Y \rightarrow \bar{X}$ . Let  $\{p_1, \dots, p_k\}$  be the set of singular points in  $\bar{X}$  and  $C_1, \dots, C_k$  be the rational curves in  $Y$  lying over  $p_1, \dots, p_k$ .

By Friedman’s result, the smoothing corresponds to a non-trivial relation

$$\sum_{i=1}^k m_i [C_i] = 0 \in H_2(Y, \mathbb{C}) \text{ with } m_i \neq 0 \text{ for all } i.$$

Let  $\pi: \mathcal{X}' \rightarrow \Delta$  be the complex degeneration. To obtain a semi-stable model, we first perform a degree two base change and then blow up all the singularities on the central fiber. Denote by

$f: \mathcal{X} \rightarrow \Delta$  the resulting semi-stable family.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ f \downarrow & & \downarrow \pi \\ \Delta & \longrightarrow & \Delta. \end{array}$$

The central fiber  $f^{-1}(0) = \tilde{Y} \cup \bigcup_{i=1}^r Q_i$  is a simple normal crossing divisor. Here

$$\tilde{Y} = \text{Bl}_{\sqcup_{i=1}^r C_i} Y \text{ and } Q_i \subset \mathbf{P}^4 \text{ is a smooth quadric hypersurface.}$$

Moreover, the exceptional divisor  $E_i = \tilde{Y} \cap Q_i$  is isomorphic to a quadric in  $\mathbf{P}^3$ .

We can compute the graded complex  $\text{Gr}_r^{\mathcal{W}(M)} \mathcal{D}^{p,q}$  in this case. We have

$$E(1) = \tilde{Y} \coprod \sqcup_{i=1}^r Q_i, E(2) = \coprod_{i=1}^r E_i, \text{ and } E(k) = \emptyset \text{ for } k \geq 3.$$

We wish to compute the group  $\text{Gr}_4^{\mathcal{W}(M)} \mathbf{H}^3(X, \mathbb{C})$  which is the cohomology of the complex

$$E_1^{-2,4} = \mathbf{H}^2(\text{Gr}_2^{\mathcal{W}(M)}) \rightarrow E_1^{-1,4} = \mathbf{H}^3(\text{Gr}_1^{\mathcal{W}(M)}) \rightarrow E_1^{0,4} = \mathbf{H}^4(\text{Gr}_0^{\mathcal{W}(M)}).$$

A similar calculation shows that

$$\text{Gr}_1^{\mathcal{W}(M)} \cong \mathbb{C}_{E(2)}[-1], \text{ and } \text{Gr}_0^{\mathcal{W}(M)} \cong \mathbb{C}_{E(1)}$$

and the sequence is transformed into

$$0 \rightarrow \mathbf{H}^2(E(2), \mathbb{C}) = \bigoplus_{i=1}^r \mathbf{H}^2(E_i, \mathbb{C}) \rightarrow \mathbf{H}^4(E(1), \mathbb{C}) = \mathbf{H}^4(\tilde{Y}, \mathbb{C}) \oplus \bigoplus_{i=1}^r \mathbf{H}^4(Q_i, \mathbb{C}).$$

This is dual to the sequence in [LLW18, Equation (1.4)].

**Theorem 1.5.** *The triple  $(\text{Tot}(\mathcal{D}^{\bullet,\bullet}), \mathcal{W}(M)_{\bullet}, \mathcal{F}^{\bullet})$  forms a mixed Hodge structure. Consequently, by Theorem 1.4, this puts a mixed Hodge structure on  $\Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E) \otimes \mathcal{O}_E \sim \Phi_f \mathbb{C}_{\mathcal{X}}$ .*

**Corollary 1.6.** *The cohomology of the derived pushforward*

$$\mathbf{R}f_* \Omega_{\mathcal{X}/\Delta}^{\bullet}(\log E)$$

*is a locally free sheaf.*

**Corollary 1.7.** *The sheaf*

$$R^q f_* \Omega_{\mathcal{X}/\Delta}^p(\log E)$$

*is locally free. Consequently,  $\mathbf{R}f_* \mathcal{F}^k$  gives a locally free extension of the Hodge bundles.*

## 2. THE PERIOD-MAP METRIC ON THE MODULI OF CY $\partial\bar{\partial}$ -MANIFOLDS

**2.1. Weil–Petersson metric and the period-map metric.** For a compact Kähler CY manifold  $(X, \omega)$  of complex dimension  $n$ , it is known that its Kuranishi space  $S$  is smooth. Let  $\mathcal{X} \rightarrow S$  be a local universal deformation of  $X$  with a fixed Kähler class  $[\omega] \in \mathbf{H}^2(X; \mathbb{Z})$ . There exists a canonical metric on  $S$ , called the Weil–Petersson metric. Indeed, Yau’s theorem [Yau78] provides us with a Ricci flat metric  $g_s$  on  $\mathcal{X}_s$  whose Kähler class equals  $[\omega]$ . Let  $\rho: T_s S \rightarrow \mathbf{H}^1(\mathcal{X}_s, T\mathcal{X}_s)$

be the Kodaira–Spencer map at  $s \in S$ . Identify  $H^1(\mathcal{X}_s, T\mathcal{X}_s)$  with  $\mathbb{H}^{0,1}(T\mathcal{X}_s)$  (the  $T\mathcal{X}_s$ -valued harmonic  $(0,1)$ -forms w.r.t.  $g_s$ ). For  $\lambda, \theta \in T_s S$ , one can check that the pairing

$$G_s(\lambda, \theta) := \int_{\mathcal{X}_s} \langle \rho(\lambda), \rho(\theta) \rangle_s$$

is positive definite and hence defines a metric on  $S$ . Here  $\langle -, - \rangle_s$  is the product on  $\mathbb{H}^{0,1}(T\mathcal{X}_s)$  induced by  $g_s$ . The metric  $G$  is called the *Weil–Petersson metric*; this is a Kähler metric on  $S$ .

Denote by  $Q: H^n(X; \mathbb{C}) \times H^n(X; \mathbb{C}) \rightarrow \mathbb{C}$  the topological pairing

$$(2.1) \quad Q([\alpha], [\beta]) = (-1)^{n(n-1)/2} \int_X \alpha \wedge \beta.$$

From Hodge theory, the sesquilinear form

$$(2.2) \quad H(u, v) := Q(Cu, \bar{v})$$

defines a positive definite hermitian pairing on  $H_{\text{prim}}^n(X; \mathbb{C})$ . Here  $C$  is the Weil operator:

$$Cu = \sqrt{-1}^{p-q} u \quad \text{for } u \in H^{p,q}(X).$$

As observed by Bogomolov and Tian, the Kähler potential of  $G$  is given by the formula

$$-\log \tilde{Q}(\Omega_s, \bar{\Omega}_s) = -\log \sqrt{-1}^n \left( \int_{\mathcal{X}_s} \Omega_s \wedge \bar{\Omega}_s \right)$$

and the Kähler form for  $G$  can be calculated by

$$-\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \tilde{Q}(\Omega_s, \bar{\Omega}_s) = \frac{\sqrt{-1}}{2} \sum_{i,j} -\partial_{t_i} \partial_{\bar{t}_j} \log \tilde{Q}(\Omega_s, \bar{\Omega}_s) dt_i \wedge d\bar{t}_j.$$

In other words, the Weil–Petersson metric is nothing but the pullback of the Fubini–Study metric induced by  $H$  via the local Torelli immersion

$$S \rightarrow \mathbf{PH}_{\text{prim}}^n(X; \mathbb{C}).$$

**Remark 2.1.** The local universal complex deformation of  $X$  is isomorphic to an open set  $U$  in  $H^1(X; TX)$ . Put

$$H^1(X; TX)_{[\omega]} := \{[\theta] \in H^1(X; TX) \mid [\theta \lrcorner \omega] = 0 \text{ in } H^{0,2}(X)\}$$

and set  $U_{[\omega]} := U \cap H^1(X; TX)_{[\omega]}$ . Then  $U_{[\omega]}$  consists of local deformations of  $X$  preserving the polarization  $[\omega] \in H^2(X; \mathbb{Z})$ .

If  $X$  is a strict Calabi–Yau, i.e.  $H^d(X; \mathcal{O}_X) = 0$  for  $0 < d < n$ , then we particularly have

$$H^1(X; TX)_{[\omega]} = H^1(X; TX)$$

when  $n \geq 3$ . Moreover, we have  $H^1(X; TX) \cong H^1(X; \Omega_X^{n-1}) = H_{\text{prim}}^1(X; \Omega_X^{n-1})$ , because

$$([\theta] \lrcorner \Omega) \wedge \omega = \pm \Omega \wedge ([\theta] \lrcorner \omega) = 0.$$

Said differently, the sesquilinear pairing  $H$  in (2.2) defines a hermitian metric on  $H^n(X; \mathbb{C})$ . In this case, thanks to the formula for the Kähler potential, the Weil–Petersson metric is independent of the choice of the polarization  $[\omega]$ .

For a non-Kähler CY  $\partial\bar{\partial}$ -manifold  $X$ , we do not have the notion of primitive cohomology and even in the strict case, we do not know whether or not the second Hodge–Riemann bilinear relation holds on  $H^{n-1,1}(X)$ . However, we can still consider the “potential function”

$$-\partial\bar{\partial}\log\tilde{Q}(\Omega_s, \bar{\Omega}_s)$$

and consider the associated two form

$$(2.3) \quad -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\tilde{Q}(\Omega_s, \bar{\Omega}_s) = \frac{\sqrt{-1}}{2}\sum_{i,j} -\partial_{t_i}\partial_{\bar{t}_j}\log\tilde{Q}(\Omega_s, \bar{\Omega}_s)dt_i\wedge d\bar{t}_j.$$

If the matrix  $(-\partial_{t_i}\partial_{\bar{t}_j}\log\tilde{Q}(\Omega_s, \bar{\Omega}_s))_{i,j=1}^n$  on the right hand side of (2.3) happens to be semi-positive definite, then we obtain a (pseudo) metric near  $s \in S$ . In this case, following [Pop19], we will call it the *period-map metric*.

The purpose of this section is to study the positivity of the matrix  $(-\partial_{t_i}\partial_{\bar{t}_j}\log\tilde{Q}(\Omega_s, \bar{\Omega}_s))_{i,j=1}^n$  and also give a criterion about when the second Hodge–Riemann bilinear relation holds on  $H^{n-1,1}(X)$  and study the finite distance problem on the moduli space. We will be mainly interested in the case when  $X$  arises as a small smoothing of a simple normal crossing variety, i.e.  $X$  is a general fiber of a semi-stable one-parameter degeneration  $\mathcal{X} \rightarrow \Delta$  over a small disc.

For a compact CY  $\partial\bar{\partial}$ -manifold  $(X, \omega)$ , due to the presence of balanced metrics, one can define a Weil–Petersson metric on the local universal deformation  $S$  of  $(X, \omega)$  polarized by a fixed class  $[\omega^{n-1}]$ . Given two Dolbeault cohomology classes

$$[\theta], [\eta] \in H^1(X; TX)_{[\omega^{n-1}]} := \{[\theta] \in H^1(X; TX) \mid [\theta \lrcorner \omega^{n-1}] = 0 \text{ in } H^{n-2,n}(X)\},$$

following [Pop19, §5.2], one can define an inner product

$$G_{WP}([\theta], [\eta]) := \frac{\langle\langle \theta \lrcorner \Omega, \eta \lrcorner \Omega \rangle\rangle_\omega}{\tilde{Q}(\Omega, \bar{\Omega})}$$

where the representatives  $\theta$  and  $\eta$  are chosen such that  $\theta \lrcorner \Omega$  and  $\eta \lrcorner \Omega$  are both  $\omega$ -minimal and  $d$ -closed under the Calabi–Yau isomorphism

$$\lrcorner \Omega: H^1(X; TX) \xrightarrow{\cong} H^{n-1,1}(X)$$

and  $\langle\langle -, - \rangle\rangle_\omega$  is the  $L^2$  inner product induced by  $\omega$ . This induces a metric on  $S$ ; it is called a Weil–Petersson metric in [Pop19].

From now on, we assume Hypothesis A is valid; that is, the nilpotent operator  $N$  induces an isomorphism between relevant quotients of the monodromy weight filtration:

$$N^k: \text{Gr}_{n+k}^{\mathcal{W}(M)} H^n(X; \mathbb{C}) \rightarrow \text{Gr}_{n-k}^{\mathcal{W}(M)} H^n(X; \mathbb{C})$$

is an *isomorphism* for each  $0 \leq k \leq n$ .

We also remark that the Hypothesis A implies that the monodromy filtration  $\mathcal{W}(M)$  on  $H^n(X; \mathbb{C})$  is identical to the filtration induced by the nilpotent operator  $N$ .

**2.2. An infinite distance criterion via Hodge theory.** Let  $f: \mathcal{X} \rightarrow \Delta$  be a one-parameter degeneration of CY  $\partial\bar{\partial}$ -manifolds. We also assume that  $f$  is a semi-stable model and every irreducible component in  $E = f^{-1}(0)$  is a  $\partial\bar{\partial}$ -manifold. There exists a holomorphic function  $\mathbf{a}(t)$  such that the multi-valued function (considered as a function on  $\Delta^*$ )

$$\Omega_z := e^{zN} \mathbf{a}(t)$$

is a local section of the bundle  $\mathcal{F}^n$  over  $\Delta^*$ . The metric two-form for the period-map “metric” is given by the formula

$$-\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}).$$

Here  $\tilde{Q}(-, -) := (\sqrt{-1})^n Q(-, -)$  and  $Q(-, -)$  is the topological pairing (cf. (2.1)) on the relevant fiber. Moreover, since  $Q(T\alpha, T\beta) = Q(\alpha, \beta)$ , the nilpotent operator  $N$  is an infinitesimal isometry with respect to  $Q$ , i.e.  $Q(N\alpha, \beta) + Q(\alpha, N\beta) = 0$ . In particular, since

$$0 = Q(\mathcal{F}_t^n, \mathcal{F}_t^1) = Q(e^{-zN} \mathcal{F}_t^n, e^{-zN} \mathcal{F}_t^1),$$

by taking the limit  $\text{Im} z \rightarrow \infty$ , we have  $Q(\mathcal{F}_{\text{lim}}^n, \mathcal{F}_{\text{lim}}^1) = 0$ . Since  $X$  is Calabi–Yau, we have  $\dim \mathcal{F}_{\text{lim}}^n = 1$ . The non-degeneracy of  $Q(-, -)$  implies that  $Q(\mathcal{F}_{\text{lim}}^n, \overline{\mathcal{F}_{\text{lim}}^n}) \neq 0$ .

Following [Wan97], we now introduce a special class of functions on the universal cover  $\mathfrak{h} \rightarrow \Delta^*$ ,  $z \mapsto t = e^{2\pi\sqrt{-1}z}$ . Let us write  $z = x + \sqrt{-1}y$ . Then  $t = e^{2\pi\sqrt{-1}x - 2\pi y}$ . The function  $t$  has the property that its derivatives has exponential decay as  $y \rightarrow \infty$ . More specifically, for each  $(r, s) \in (\mathbb{Z}_{\geq 0})^2$

$$|\partial_x^r \partial_y^s(t)| \leq C(r, s) \cdot e^{-\pi y}$$

for some positive constant  $C(r, s)$  and  $y \gg 0$ .

**Notation.** We use the symbol  $\mathbf{h}$  to denote the class of smooth functions on  $\mathfrak{h}$  having the property as above, i.e. all of its derivatives has exponential decay as  $y \rightarrow \infty$ . The notation  $f \in \mathbf{h}$  means  $f$  is such a function. It is clear that  $\mathbf{h}$  is closed under addition and multiplication.

Consider the power series expansion

$$\mathbf{a}(t) = a_0 + a_1 t + \cdots.$$

Let  $d = \min\{k \in \mathbb{N} \mid N^{k+1} a_0 = 0\}$ .

**Lemma 2.1.** *We have*

$$\tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}) = p(y) + h$$

where  $p(y)$  is a polynomial of degree  $d$  and  $h \in \mathbf{h}$ . Moreover, the leading coefficient of  $p(y)$  is given by  $\tilde{Q}(a_0, N^d \bar{a}_0) \neq 0$ .

*Proof.* Note that

$$\tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}) = \tilde{Q}(e^{(z-\bar{z})N} \mathbf{a}(t), \overline{\mathbf{a}(t)}) = \tilde{Q}(e^{2\sqrt{-1}yN} \mathbf{a}(t), \overline{\mathbf{a}(t)}).$$

Now the function

$$\begin{aligned} y^m t^l \bar{t}^k &= y^m e^{2\pi\sqrt{-1}lz} e^{-2\pi\sqrt{-1}k\bar{z}} \\ &= y^m e^{2\pi\sqrt{-1}lx - 2\pi ny} e^{-2\pi\sqrt{-1}kx - 2\pi ky} \end{aligned}$$

belongs to  $\mathbf{h}$  for any  $m \geq 0$  as long as  $(l, k) \neq (0, 0)$ . Therefore, using the power series expansion, we obtain

$$\tilde{Q}(e^{2\sqrt{-1}yN} \mathbf{a}(t), \overline{\mathbf{a}(t)}) = \tilde{Q}(e^{2\sqrt{-1}yN} a_0, \bar{a}_0) + h = p(y) + h$$

as desired. The polynomial  $p(y)$  has degree  $d$  and the leading coefficient is clearly equal to  $\tilde{Q}(a_0, N^d \bar{a}_0)$ ; this is non-zero since  $a_0 \in \mathcal{F}_{\lim}^n$ ,  $N^d \bar{a}_0 \notin \mathcal{F}_{\lim}^1$ , and  $\dim \mathcal{F}_{\lim}^0 / \mathcal{F}_{\lim}^1 = 1$ .  $\square$

**Proposition 2.2.** *If  $d > 0$ , we have*

$$-\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}) = \frac{d}{y^2} + h.$$

*In particular, this implies that the period-map “metric” is really a metric for  $y \gg 0$ .*

*Proof.* Let us write

$$\tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}) = p(y) + h.$$

Then for any non-negative integer  $d$ , we have

$$\begin{aligned} -\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)}) &= -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log(p(y) + h) \\ &= \frac{d}{y^2} + h. \end{aligned}$$

Here we slightly abuse the notation; we use the same  $h$  to denote different functions in  $\mathbf{h}$ . Since  $h$  decays exponentially as  $y \rightarrow \infty$ , we conclude that for  $d > 0$

$$\frac{d}{y^2} + h > 0, \text{ for } y \gg 0.$$

This shows that  $-\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN} \mathbf{a}(t), \overline{e^{zN} \mathbf{a}(t)})$  is positive when  $y \gg 0$ .  $\square$

Because of the proposition, it makes sense to talk about the distance near  $0 \in \Delta$ . In fact, in the present situation, we have

**Theorem 2.3.** *If  $d > 0$ , then  $0 \in \Delta$  has infinite distance w.r.t. the period-map metric.*

*Proof.* By the proceeding proposition, for any curve  $\gamma$  in  $\mathbf{h}$  towards  $\infty$ , we see that

$$\ell(\gamma) \geq \int_{y=c}^{\infty} \sqrt{\frac{d-\varepsilon}{y^2}} dy = \int_{y=c}^{\infty} \frac{\sqrt{d-r}}{y} dy = \infty.$$

$\square$

**Theorem 2.4.** *For a one-parameter degeneration  $\mathcal{X} \rightarrow \Delta$  of compact CY  $\partial\bar{\partial}$ -manifolds, if  $d > 0$ , then  $0 \in \Delta$  has infinite distance w.r.t.  $G_{WP}$ .*

*Proof.* This follows from the proceeding theorem and the comparison [Pop19, Corollary 5.11].  $\square$

**2.3. A finite distance criterion via Hodge theory and polarization.** For  $d = 0$ . Then we have  $Na_0 = 0$  and  $\tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)}) \in \mathbf{h}$ . Therefore,

$$\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)}) \in \mathbf{h}.$$

However, we do not know whether or not this is truly a metric, that is, if

$$-\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)})$$

is positive (even non-negative). From the calculation in [Pop19, §5.3], we learn that

$$(2.4) \quad -\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)}) > 0 \iff -\tilde{Q}(\theta \lrcorner \Omega_z, \overline{\theta \lrcorner \Omega_z}) > 0.$$

Here  $\theta$  is a representative of the Dolbeault cohomology class  $[\theta] \in H^{0,1}(\mathcal{X}_t, T\mathcal{X}_t)$  associated to the deformation of  $\mathcal{X}_t$  in the given family  $f: \mathcal{X} \rightarrow \Delta$  such that the interior product  $\theta \lrcorner \Omega_z$  is a d-closed  $(n-1, 1)$ -form. The right hand side in (2.4) is valid if the second Hodge–Riemann bilinear relation on  $H^{n-1,1}(\mathcal{X}_t; \mathbb{C})$  holds. Assuming this, we immediately have the following.

**Theorem 2.5.** *If  $d = 0$ , then  $0 \in \Delta$  has finite distance with respect to the period-map metric.*

*Proof.* It suffices to find a curve having finite distance. One can use the curve  $\gamma(t) := (c, t + R)$  in  $\mathfrak{h}$  for a fixed constant  $c$  and  $R > 0$ . Indeed, we can compute

$$\begin{aligned} \ell(\gamma) &= \int_{y=R}^{\infty} \sqrt{-\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)})} |dz| \\ &= \int_{y=R}^{\infty} \sqrt{-\partial_z \partial_{\bar{z}} \log \tilde{Q}(e^{zN}\mathbf{a}(t), \overline{e^{zN}\mathbf{a}(t)})} dy \\ &\leq \int_{y=R}^{\infty} \varepsilon e^{-\delta y/2} dy < \infty. \end{aligned}$$

Hence  $0 \in \Delta$  has finite distance with respect to the period-map metric.  $\square$

In the next section, we shall focus on the situation  $n = 3$  and discuss when the second Hodge–Riemann bilinear relation holds on  $H^{2,1}(\mathcal{X}_t; \mathbb{C})$  for finite distance degenerations.

### 3. THE SECOND HODGE–RIEMANN BILINEAR RELATION FOR FINITE DISTANCE DEGENERATIONS

In this section, Hypothesis A remains assumed. In what follows, we restrict ourselves to the threefold case, i.e.  $n = 3$ . Put  $X = \mathcal{X}_t$ . We also assume that  $X$  is CY in the strict sense, i.e.

$$H^1(X; \mathcal{O}_X) = H^2(X; \mathcal{O}_X) = 0.$$

This implies  $H^1(X; \mathbb{C}) \cong H^5(X; \mathbb{C}) = 0$ . For instance, these conditions are fulfilled if  $X$  is simply connected. The strictness also implies  $H^2(X; \mathbb{C}) = H^{1,1}(X)$  and the sequence

$$(3.1) \quad E_1^{0,1} = H^1(E(1); \mathbb{C}) \rightarrow E_1^{1,1} = H^1(E(2); \mathbb{C}) \rightarrow E_1^{2,1} = H^1(E(3); \mathbb{C})$$

is exact at the middle.

We remark that for  $H^3(X; \mathbb{C})$  and any  $r$ , the quotient  $\mathrm{Gr}_{3+r}^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  is equal to the cohomology of the complex

$$(3.2) E_1^{-r-1, 3+r} = \mathbf{H}^2(\mathrm{Gr}_{r+1}^{\mathcal{W}(M)}) \rightarrow E_1^{-r, 3+r} = \mathbf{H}^3(\mathrm{Gr}_r^{\mathcal{W}(M)}) \rightarrow E_1^{-r+1, 3+r} = \mathbf{H}^4(\mathrm{Gr}_{r-1}^{\mathcal{W}(M)})$$



where the corresponding morphisms are both alternating Gysin maps, both alternating restriction maps, or an alternating Gysin map composed with an alternating restriction map.

To discuss positivity, we recall a result of Fujisawa. In [Fuj14, §7], Fujisawa defined for a log deformation  $E \rightarrow *$  a *trace morphism*

$$\mathrm{Tr}: \mathbf{H}^{2n}(E; \mathcal{K}_{\mathbb{C}}) \rightarrow \mathbb{C}$$

and introduced a pairing on  $\mathbf{H}^q(E; \mathcal{K}_{\mathbb{C}}) \otimes \mathbf{H}^{2n-q}(E; \mathcal{K}_{\mathbb{C}})$  by means of a *weak cohomological mixed Hodge complex*  $\mathcal{K}$  on  $E$ . Here  $n$  is the complex dimension of  $E$ . The hypercohomology of  $\mathcal{K}_{\mathbb{C}}$  also computes the cohomology of the nearby fiber. However, unlike the relative logarithmic de Rham complex, the complex  $\mathcal{K}$  carries a natural multiplicative structure and hence together with the trace morphism, it induces a pairing between their cohomology groups.

Specializing to  $q = n$  and noting that  $\mathbf{H}^n(E; \mathcal{K}_{\mathbb{C}}) \cong H^n(X; \mathbb{C})$ , we obtain a pairing

$$\langle -, - \rangle: H^n(X; \mathbb{C}) \otimes H^n(X; \mathbb{C}) \rightarrow \mathbb{C}.$$

The upshot is that the pairing descends to a pairing between quotients

$$\langle -, - \rangle: \mathrm{Gr}_{n+r}^{\mathcal{W}(M)} H^n(X; \mathbb{C}) \otimes \mathrm{Gr}_{n-r}^{\mathcal{W}(M)} H^n(X; \mathbb{C}) \rightarrow \mathbb{C}$$

and it is given by a sum (up to a factor) of the usual cohomological pairing on  $E(k)$ 's. For the later use, let us give the precise formula for the case  $n = 3$  and  $r = 0, 1$  and the explanation will be given later in a moment.

**3.1.** For  $r = 0$ , the above sequence (3.2) becomes

$$(3.3) \quad H^1(E(2); \mathbb{C}) \xrightarrow{a} H^3(E(1); \mathbb{C}) \oplus H^1(E(3); \mathbb{C}) \xrightarrow{b} H^3(E(2); \mathbb{C})$$

whose cohomology gives  $\mathrm{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ . For  $[\alpha, \beta]$  and  $[\gamma, \delta] \in \mathrm{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ , the pairing is defined as

$$(3.4) \quad \langle [\alpha, \beta], [\gamma, \delta] \rangle = (2\pi\sqrt{-1})^3 \left( \frac{(-1)^3}{(2\pi\sqrt{-1})^3} \int_{E(1)} \alpha \cup \gamma + \frac{(-1)^3}{(2\pi\sqrt{-1})} \int_{E(3)} \beta \cup \delta \right)$$

where  $(\bullet, \bullet)$  is a representative of  $[\bullet, \bullet]$  inside  $\ker(b)$ . Observe that the pairing is well-defined. If  $(\alpha, \beta) \in \mathrm{Im}(a)$ , then  $(\alpha, \beta) = (\phi(x), \psi(x))$  for some  $x \in H^1(E(2); \mathbb{C})$ . (See §1.5 for the notations.) On one hand, we calculate

$$\int_{E(1)} \alpha \cup \gamma = \int_{E(1)} \phi(x) \cup \gamma = -(2\pi\sqrt{-1}) \int_{E(2)} x \cup \psi(\gamma).$$

On the other hand, we have

$$\int_{E(3)} \beta \cup \delta = \int_{E(3)} \psi(x) \cup \delta = -\frac{1}{2\pi\sqrt{-1}} \int_{E(2)} x \cup \phi(\delta).$$

The equation (3.4) becomes

$$(2\pi\sqrt{-1})^3 \left( -\frac{(-1)^3}{(2\pi\sqrt{-1})^2} \int_{E(2)} x \cup \psi(\gamma) - \frac{(-1)^3}{(2\pi\sqrt{-1})^2} \int_{E(3)} x \cup \phi(\delta) \right) = 0$$

since  $\psi(\gamma) + \phi(\delta) = 0$ . The remaining cases can be checked in a similar fashion.

**3.2.** For  $r = 1$ , the quotient  $\mathrm{Gr}_4^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C})$  is given by the cohomology of

$$\mathrm{H}^0(E(3); \mathbb{C}) \rightarrow \mathrm{H}^2(E(2); \mathbb{C}) \oplus \mathrm{H}^0(E(4); \mathbb{C}) \rightarrow \mathrm{H}^4(E(1); \mathbb{C}) \oplus \mathrm{H}^2(E(3); \mathbb{C})$$

while for  $r = -1$ , the quotient  $\mathrm{Gr}_2^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C})$  is given by the cohomology of

$$\mathrm{H}^2(E(1); \mathbb{C}) \oplus \mathrm{H}^0(E(3); \mathbb{C}) \rightarrow \mathrm{H}^2(E(2); \mathbb{C}) \oplus \mathrm{H}^0(E(4); \mathbb{C}) \rightarrow \mathrm{H}^2(E(3); \mathbb{C}).$$

For  $[\alpha, \beta] \in \mathrm{Gr}_4^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C})$  and  $[\gamma, \delta] \in \mathrm{Gr}_2^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C})$  the pairing is given by

$$(3.5) \quad \langle [\alpha, \beta], [\gamma, \delta] \rangle = (2\pi\sqrt{-1})^3 \left( \frac{(-1)^3}{(2\pi\sqrt{-1})^2} \int_{E(2)} \alpha \cup \gamma + \frac{(-1)^3}{(2\pi\sqrt{-1})^0} \int_{E(4)} \beta \cup \delta \right)$$

where  $(\bullet, \bullet)$  is a lifting of  $[\bullet, \bullet]$  in the corresponding vector spaces. One can check again that this is well-defined.

Now recall that the family is a finite distance degeneration, i.e.  $Na_0 = 0$ . This in turns implies  $N^2 = 0$  on  $\mathrm{H}^3(X; \mathbb{C})$  and consequently  $\mathrm{Gr}_1^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C}) = 0$ ; namely the following sequence is exact

$$(3.6) \quad E_1^{1,1} = \mathrm{H}^1(E(2); \mathbb{C}) \rightarrow E_1^{2,1} = \mathrm{H}^1(E(3); \mathbb{C}) \rightarrow E_1^{3,1} = \mathrm{H}^1(E(4); \mathbb{C}) = 0.$$

Recall that a one-parameter semi-stable degeneration  $f: \mathcal{X} \rightarrow \Delta$  satisfying Hypothesis B is called a *one-parameter Kähler degeneration* in [PS08]. We remark that Hypothesis B also implies Hypothesis A by [PS08, Theorem 11.40].

Let us explain how the pairings (3.4) and (3.5) are related to the topological pairing on  $X$ . Under Hypothesis B, we have the Clemens–Schmid exact sequence (cf. [PS08, Corollary 11.44])

$$\cdots \rightarrow \mathrm{H}_{2n+2-m}(E; \mathbb{C}) \rightarrow \mathrm{H}^m(E; \mathbb{C}) \xrightarrow{i} \mathrm{H}^m(X; \mathbb{C}) \xrightarrow{N} \mathrm{H}^m(X; \mathbb{C}) \rightarrow \mathrm{H}_{2n-m}(E; \mathbb{C}) \rightarrow \cdots.$$

We can put a mixed Hodge structure on  $\mathrm{H}^k(E; \mathbb{C})$  via the Mayer–Vietoris resolution and we shall denote by  $W$  the corresponding weight filtration. This is an exact sequence of mixed Hodge structures whose morphisms are of type  $(n+1, n+1)$ ,  $(0, 0)$ ,  $(-1, -1)$ , and  $(-n, -n)$  respectively. The map  $N$  is the nilpotent operator while  $i$  is the map induced from the inclusion  $X \subset \mathcal{X}$  and the deformation retraction  $E \rightarrow \mathcal{X}$ .

Specializing to  $n = m = 3$ , we obtain

$$0 \rightarrow \mathrm{H}_5(E; \mathbb{C}) \rightarrow \mathrm{H}^3(E; \mathbb{C}) \xrightarrow{i} \mathrm{H}^3(X; \mathbb{C}) \xrightarrow{N} \mathrm{H}^3(X; \mathbb{C}) \rightarrow \mathrm{H}_3(E; \mathbb{C}) \rightarrow \cdots.$$

Look at the graded piece

$$0 \rightarrow \mathrm{Gr}_5^W \mathrm{H}_5(E; \mathbb{C}) \rightarrow \mathrm{Gr}_3^W \mathrm{H}^3(E; \mathbb{C}) \rightarrow \mathrm{Gr}_3^{\mathcal{W}(M)} \mathrm{H}^3(X; \mathbb{C}) \xrightarrow{N} 0.$$

Here the last 0 comes from the assumption that  $\mathcal{X} \rightarrow \Delta$  has finite distance. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^3(E(1); \mathbb{C}) & \xrightarrow{\phi} & \mathrm{H}^3(E(2); \mathbb{C}) & & \\ \downarrow & & \downarrow & & \\ \mathrm{H}^1(E(2); \mathbb{C}) & \xrightarrow{a} & \mathrm{H}^3(E(1); \mathbb{C}) \oplus \mathrm{H}^1(E(3); \mathbb{C}) & \xrightarrow{b} & \mathrm{H}^3(E(2); \mathbb{C}). \end{array}$$

We then obtain a map

$$\ker(\phi) \rightarrow \ker(b) \rightarrow \ker(b)/\text{im}(a)$$

which gives the surjection in the Clemens–Schmid exact sequence

$$\text{Gr}_3^W H^3(E; \mathbb{C}) \rightarrow \text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C}).$$

We have a pairing (3.4) on  $\ker(b)$ . From the proceeding discussion, for any  $[\alpha_i, \beta_i] \in \ker(b)$ , we can find an element  $\xi_i \in \ker(\phi)$  such that  $[\xi_i, 0] = [\alpha_i, \beta_i]$  by the surjectivity. It follows that

$$(3.7) \quad \langle [\alpha_1, \beta_1], [\alpha_2, \beta_2] \rangle = \langle [\xi_1, 0], [\xi_2, 0] \rangle = (-1)^3 \int_{E(1)} \xi_1 \cup \xi_2 = (-1)^3 \int_E \xi_1 \cup \xi_2.$$

In the last integral, we think of  $\xi_i$  as an element in  $H^3(E; \mathbb{C})$  and use the fact that  $H_6(E; \mathbb{Z}) = H_6(E(1); \mathbb{Z})$ . Together with the deformation retraction  $\mathcal{X} \supset E$  and noticing that  $X$  and  $E$  are homologous, the last integral can be identified with the topological pairing on  $X$ , i.e.,

$$(-1)^3 \int_E \xi_1 \cup \xi_2 = (-1)^3 \int_X i(\xi_1) \cup i(\xi_2),$$

and this gives (3.4) up to twist.

Similarly, we have an isomorphism

$$\text{Gr}_2^W H^3(E; \mathbb{C}) \rightarrow \text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C})$$

from the Clemens–Schmid exact sequence; this is deduced from the commutative diagram

$$\begin{array}{ccccc} H^2(E(1); \mathbb{C}) & \xrightarrow{\phi} & H^2(E(2); \mathbb{C}) & \longrightarrow & H^2(E(3); \mathbb{C}) \\ \downarrow & & \downarrow & & \parallel \\ H^2(E(1); \mathbb{C}) \oplus H^0(E(3); \mathbb{C}) & \xrightarrow{c} & H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C}) & \xrightarrow{d} & H^2(E(3); \mathbb{C}). \end{array}$$

We can construct a long exact sequence involving the map  $\phi$  in the top row as follows. Consider the short exact sequence

$$0 \rightarrow A^\bullet(E(2))[-1] \rightarrow A^\bullet(E(1)) \oplus A^{\bullet-1}(E(2)) \rightarrow A^\bullet(E(1)) \rightarrow 0$$

where the second map is given by  $\eta \rightarrow (0, \eta)$  while the third one is the projection. The differential  $D$  on the complex  $A^\bullet(E(1)) \oplus A^{\bullet-1}(E(2))$  is given by the formula

$$D(\omega, \eta) := (d\omega, \phi(\omega) - d\eta).$$

One checks  $D^2 = 0$ . The cohomology group is denoted by  $H^k(E(1); E(2))$ . We have an induced long exact sequence

$$\cdots \rightarrow H^{k-1}(E(2)) \rightarrow H^k(E(1); E(2)) \rightarrow H^k(E(1)) \xrightarrow{\varepsilon} H^k(E(2)) \rightarrow \cdots$$

whose connecting homomorphism  $\varepsilon$  is equal to  $\phi$ . We have an injection

$$\text{Gr}_2^W H^3(E; \mathbb{C}) \hookrightarrow H^2(E(2))/\text{im}(\phi) \hookrightarrow H^3(E(1); E(2)).$$

A diagram chasing shows that under the inclusion, we have

$$\text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \cong \text{Gr}_2^W H^3(E; \mathbb{C}) \hookrightarrow H_c^3(U; \mathbb{C})$$

where  $U = E \setminus E_{\text{sing}}$ . Consider the dual diagram

$$\begin{array}{ccccc}
 H_2(E(3); \mathbb{C}) & \xrightarrow{\Psi'} & H_2(E(2); \mathbb{C}) & \xrightarrow{\Psi} & H_2(E(1); \mathbb{C}) \\
 \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\
 H^0(E(3); \mathbb{C}) & \longrightarrow & H^2(E(2); \mathbb{C}) & \xrightarrow{\Psi} & H^4(E(1); \mathbb{C}) \\
 \parallel & & \uparrow & & \uparrow \\
 H^0(E(3); \mathbb{C}) & \longrightarrow & H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C}) & \longrightarrow & H^4(E(1); \mathbb{C}) \oplus H^2(E(3); \mathbb{C}).
 \end{array}$$

Write  $\Psi = (\Psi_1, \dots, \Psi_m)$ . If  $(\sigma_I) \in H_2(E(2); \mathbb{C})$  belongs to  $\ker(\Psi)$ , then  $\Psi_j(\sigma_I)$  is a coboundary, i.e. there exists a 3-chain  $\Gamma_j$  such that  $\partial \Gamma_j = \Psi_j(\sigma_I)$ . Let us denote by  $\mathcal{S}$  the collection of chains constructed as above, i.e.

$$\mathcal{S} = \{\Gamma := (\Gamma_1, \dots, \Gamma_m) \mid \Gamma_i \in H_3(E_i, \cup_{j \neq i} E_{ij}; \mathbb{C}), \partial \Gamma_i = \Psi_i(\sigma_I) \text{ for all } (\sigma_I) \in \ker(\Psi)\}.$$

Note that for each  $\Gamma \in \mathcal{S}$  the chains  $\Gamma_i$ 's can be glued into a 3-cycle in  $E$  along their boundaries.

Recall that

$$\Psi = \bigoplus_{j=1}^2 (-1)^{j-1} (\iota_{2,j})_* : H_q(E(2); \mathbb{C}) \rightarrow H_q(E(1); \mathbb{C})$$

is defined on the level of chains. Combined with Lefschetz duality, it gives rise to a map

$$\ker(\Psi) \rightarrow \mathcal{S} \subseteq \bigoplus_{i=1}^m H_3(E_i, \cup_{j \neq i} E_{ij}; \mathbb{C}) \subseteq H^3(U; \mathbb{C})$$

an induces an injection

$$\text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \cong \ker(\Psi) / \text{im}(\Psi') \rightarrow \mathcal{S}.$$

Consequently, the induced pairing

$$\langle -, - \rangle : \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \times \text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \rightarrow \mathbb{C}$$

can be computed via the usual pairing between relative cohomology and relative homology, or the usual Poincaré pairing  $H^3(U) \times H_c^3(U) \rightarrow H_c^6(U) \cong H_0(U) \cong \mathbb{C}$  on  $U$ . Here the isomorphism  $H_c^6(U) \cong \mathbb{C}$  comes from integration over  $U$ . This gives (3.5) up to twist.

**Remark 3.1.** We remark that the hermitian pairing

$$\sqrt{-1} \langle u, N_{\text{st}} \bar{v} \rangle$$

is *negative definite* on  $\text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ , as one can easily derive from the formula (3.5). Here  $N_{\text{st}}$  is the nilpotent operator from Steenbrink's theory; it can be regarded as the induced map from the canonical map

$$\mathcal{D}^{p,q} \rightarrow \mathcal{D}^{p+1,q-1} \text{ via } \omega \mapsto \bar{\omega}.$$

We note that  $N = -N_{\text{st}}$ . In fact, recall that the trivialization of  $\mathcal{X}_\infty \rightarrow \mathfrak{h}$  induces a diffeomorphism

$$\mathcal{X}_{\infty,0} \xrightarrow{\simeq} \mathcal{X}_{\infty,1}$$

and hence a monodromy operator  $T_{\text{st}}: H^k(\mathcal{X}_{\infty,1}; \mathbb{C}) \rightarrow H^k(\mathcal{X}_{\infty,0}; \mathbb{C})$ . We have  $T = (T_{\text{st}}^{-1})^*$  where  $T$  is the monodromy operator described in §1.3. Consequently, we infer that the hermitian pairing

$$\sqrt{-1} \langle u, N\bar{v} \rangle$$

becomes *positive definite*.

**Proposition 3.1.** *Assuming the Hypothesis B, then the quotient  $\text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  is equipped with a polarized Hodge structure.*

*Proof.* Since  $H^5(X; \mathbb{C}) = 0$ , we have  $\text{Gr}_5^{\mathcal{W}(M)} H^5(X; \mathbb{C}) = 0$ , i.e. the sequence

$$H^3(E(2); \mathbb{C}) \rightarrow H^5(E(1); \mathbb{C}) \rightarrow 0$$

is exact at the middle.

Let  $\omega_I$  be the Kähler form on  $E_I$  coming from the restriction of the universal class in  $H^2(E; \mathbb{R})$  under Hypothesis B. Consider the commutative diagram

$$(3.8) \quad \begin{array}{ccc} H^1(E(2); \mathbb{C}) & \xrightarrow{a} & H^3(E(1); \mathbb{C}) \oplus H^1(E(3); \mathbb{C}) \\ \downarrow & & \downarrow L \\ H^3(E(2); \mathbb{C}) & \longrightarrow & H^5(E(1); \mathbb{C}). \end{array}$$

In this diagram,

- the top row is from the spectral sequence computing  $\text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ ;
- the bottom row is from the spectral sequence computing  $\text{Gr}_5^{\mathcal{W}(M)} H^5(X; \mathbb{C})$ ;
- the vertical maps are give by a product of Lefschetz operators using  $\omega_I$ .

From the discussion above, the bottom arrow is surjective. Moreover, the left vertical arrow is an isomorphism. Thus

$$L|_{\text{Im}(a)}: \text{Im}(a) \rightarrow H^5(E(1); \mathbb{C})$$

is also a surjection. In particular, if

$$[\gamma, \delta] \in \text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$$

and  $(\gamma, \delta) \in H^3(E(1); \mathbb{C}) \oplus H^1(E(3); \mathbb{C})$  is any representative, we can use elements in  $\text{Im}(a)$  to modify  $(\gamma, \delta)$  such that  $L(\gamma, \delta) = 0$ , i.e.

$$\gamma \in H_{\text{prim}}^3(E(1); \mathbb{C}).$$

Therefore the quotient  $\text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  carries a polarized Hodge structure. □

The associated sesquilinear pairing  $\langle Cu, \bar{v} \rangle$  (here  $C$  is the Weil operator) between the quotients  $\text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  and  $\text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  is positive definite as long as the surface  $E(2)$  is compact Kähler and the statement

$$(*) \quad "[\alpha, \beta] \in \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \Rightarrow \alpha \in H_{\text{prim}}^2(E(2); \mathbb{C})"$$

holds. Fortunately, we have

**Proposition 3.2.** *Under the Hypothesis B, if*

$$[\alpha, \beta] \in \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C})$$

*then there exists a representative  $(\alpha, \beta)$  such that  $\alpha \in H_{\text{prim}}^2(E(2); \mathbb{C})$ .*

*Proof.* Since  $H^5(X; \mathbb{C}) = 0$ , we have  $\text{Gr}_4^{\mathcal{W}(M)} H^5(X; \mathbb{C}) = 0$ , i.e. the sequence

$$H^4(E(1); \mathbb{C}) \oplus H^2(E(3); \mathbb{C}) \rightarrow H^4(E(2); \mathbb{C}) \rightarrow 0$$

is exact at the middle.

Let  $\omega_I$  be the Kähler form on  $E_I$  coming from the restriction of the universal class in  $H^2(E; \mathbb{R})$  under Hypothesis B. Consider the commutative diagram

$$(3.9) \quad \begin{array}{ccc} H^2(E(1); \mathbb{C}) \oplus H^0(E(3); \mathbb{C}) & \xrightarrow{a} & H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C}) \\ \downarrow & & \downarrow L \\ H^4(E(1); \mathbb{C}) \oplus H^2(E(3); \mathbb{C}) & \longrightarrow & H^4(E(2); \mathbb{C}). \end{array}$$

In this diagram,

- the top row is from the spectral sequence computing  $\text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ ;
- the bottom row is from the spectral sequence computing  $\text{Gr}_4^{\mathcal{W}(M)} H^5(X; \mathbb{C})$ ;
- the vertical maps are give by a product of Lefschetz operators using  $\omega_I$ .

From the discussion above, the bottom arrow is surjective. Moreover, the left vertical arrow is an isomorphism. Thus

$$L|_{\text{Im}(a)} : \text{Im}(a) \rightarrow H^4(E(2); \mathbb{C})$$

is also a surjection. In particular, if

$$[\gamma, \delta] \in \text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C})$$

and  $(\gamma, \delta) \in H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C})$  is any representative, we can use elements in  $\text{Im}(a)$  to modify  $(\gamma, \delta)$  such that  $L(\gamma, \delta) = 0$ , i.e.

$$\gamma \in H_{\text{prim}}^2(E(2); \mathbb{C}).$$

Next, under the Hypothesis A, that is, the identity map

$$\begin{array}{ccc} H^0(E(3); \mathbb{C}) & \xrightarrow{c} & H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C}) \xrightarrow{d} H^4(E(1); \mathbb{C}) \oplus H^2(E(3); \mathbb{C}) \\ & & \downarrow \text{id} \\ H^2(E(1); \mathbb{C}) \oplus H^0(E(3); \mathbb{C}) & \xrightarrow{a} & H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C}) \xrightarrow{b} H^2(E(3); \mathbb{C}) \end{array}$$

induces an isomorphism between  $\ker(d)/\text{im}(c)$  and  $\ker(b)/\text{im}(a)$ , it follows that  $\text{im}(a) \subseteq \text{im}(c)$ . For  $[\alpha, \beta] \in \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ , if  $(\alpha, \beta) \in H^2(E(2); \mathbb{C}) \oplus H^0(E(4); \mathbb{C})$  is a representative, we can use elements in  $\text{im}(a) \subseteq \text{im}(c)$  to adjust  $(\alpha, \beta)$  so that  $\alpha \in H_{\text{prim}}^2(E(2); \mathbb{C})$  as desired.  $\square$

Now we can state our main result in this section.

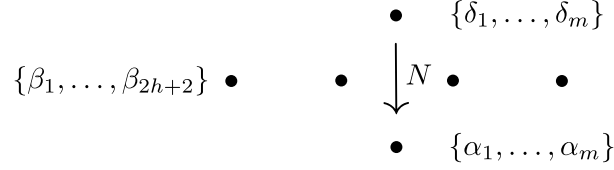


FIGURE 1. The limiting mixed Hodge diamond for the middle cohomology group  $H^3(X; \mathbb{C})$  and the bases for the corresponding subspaces  $I^{p,q}$ .

**Theorem 3.3.** *Let  $\mathcal{X} \rightarrow \Delta$  be a finite distance and semi-stable degeneration of smooth CY  $\partial\bar{\partial}$ -threefolds. Let  $E$  be the central fiber and  $X$  be a general fiber sufficiently close to  $E$ . Under Hypothesis B, the Hodge structure on  $H^3(X; \mathbb{C})$  is polarized.*

The rest of this section is devoted to demonstrate Theorem 3.3. We shall closely follow Li's idea in [Li2202]. Consider Deligne's splitting  $I^{p,q}$  for Steenbrink's limiting mixed Hodge structure on  $H^3(X; \mathbb{C})$  (cf. Theorem 1.5). Firstly, we construct a basis for each  $I^{p,q}$  and hence obtain a basis for  $\mathcal{F}_{\text{lim}}^2$ . Secondly, we extend the proceeding basis to become a local frame for Deligne's canonical extension. Finally we untwist the local frame by  $e^{zN}$  to get a (multi-valued) frame for  $\mathcal{F}_t^2$ . We then check the positivity using the multi-valued frame.

**3.3. A construction of the basis for the canonical extension.** Let the notations be as before. We construct a basis for  $\mathcal{F}_{\text{lim}}^2$  as follows. Recall that we have

$$(3.10) \quad W_3 = W_2 \oplus \left( \bigoplus_{p=0}^3 I^{p,3-p} \right).$$

Notice that both  $W_2$  and  $\bigoplus_{p=0}^3 I^{p,3-p}$  are complex vector spaces with real structure coming from the complex conjugation on  $H$ . Indeed, by

$$(3.11) \quad I^{p,q} = \overline{I^{q,p}} \quad \text{mod} \quad \bigoplus_{0 \leq r < p, 0 \leq s < q} I^{r,s},$$

it follows that  $W_2 = I^{1,1} = \overline{I^{1,1}} = \overline{W_2}$  and  $V = \overline{V}$  for  $V := \bigoplus_{p=0}^3 I^{p,3-p}$ . Consequently, we can pick a *real* basis  $\{\alpha_1, \dots, \alpha_m\}$  for  $W_2$  and  $\{\beta_1, \dots, \beta_{h+1}, \beta_{h+2}, \dots, \beta_{2h+2}\}$  for  $V$ . Now consider  $I^{1,1} \oplus I^{2,2}$ . This is again a complex vector space with real structure by (1.6) and therefore it also admits a real basis. We may extend the basis  $\{\alpha, \dots, \alpha_m\}$  to become a real basis  $\{\alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_m\}$  for  $I^{1,1} \oplus I^{2,2}$ . Also, there are complex numbers  $x_{kl} \in \mathbb{C}$  with  $1 \leq k, l \leq m$  such that

$$(3.12) \quad \delta_k := \gamma_k - \sum_{l=1}^m x_{kl} \alpha_l \in I^{2,2}.$$

Under the Hypothesis A, we may further assume that  $N(\delta_k) = N(\gamma_k) = \alpha_k$ .

**Notation.** Let us fix the notation for the later use.

- Let  $\{u_1, \dots, u_{h+1}\}$  be a basis for  $I^{3,0} \oplus I^{2,1}$ . We can write  $u_p = \sum_{i=1}^{2h+2} b_p^i \beta_i$ .
- Let  $\{v_1 = \delta_1, \dots, v_m = \delta_m\}$  be a basis for  $I^{2,2}$ .

Since  $e^{-zN}\mathcal{F}^2$  is a holomorphic subbundle in Deligne's canonical extension, we may extend

$$(3.13) \quad \{u_1, \dots, u_{h+1}, v_1, \dots, v_m\}$$

to become a local frame for  $e^{-zN}\mathcal{F}^2$  over  $\Delta$ . Explicitly, we can write

$$(3.14) \quad \begin{aligned} u_p(t) &= \sum_{i=1}^m A_p^i(t) \alpha_i + \sum_{i=1}^{2h+2} B_p^i(t) \beta_i + \sum_{i=1}^m C_p^i(t) \delta_i, \\ v_q(t) &= \sum_{i=1}^m D_q^i(t) \alpha_i + \sum_{i=1}^{2h+2} E_q^i(t) \beta_i + \sum_{i=1}^m F_q^i(t) \delta_i. \end{aligned}$$

In the above expression, all the coefficients are *holomorphic*.

In terms of the real basis  $\{\alpha, \dots, \alpha_m, \beta_1, \dots, \beta_{2h+2}, \gamma_1, \dots, \gamma_m\}$ , we can write

$$(3.15) \quad \begin{aligned} u_p(t) &= \sum_{i=1}^m \left( A_p^i(t) - \sum_{k=1}^m C_p^k(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h+2} B_p^i(t) \beta_i + \sum_{i=1}^m C_p^i(t) \gamma_i, \\ v_q(t) &= \sum_{i=1}^m \left( D_q^i(t) - \sum_{k=1}^m F_q^k(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h+2} E_q^i(t) \beta_i + \sum_{i=1}^m F_q^i(t) \gamma_i. \end{aligned}$$

Also, these functions  $A_p^i(t), \dots, F_q^i(t)$  satisfy the properties

- (1)  $A_p^i(t) \rightarrow 0$  and  $C_p^i(t) \rightarrow 0$  as  $t \rightarrow 0$ ;
- (2)  $B_p^i(t) \rightarrow b_p^i$  as  $t \rightarrow 0$ ;
- (3)  $D_q^i(t) \rightarrow 0$  and  $E_q^i(t) \rightarrow 0$  as  $t \rightarrow 0$ ;
- (4)  $F_q^i(t) \rightarrow \delta_q^i$ , the Kronecker delta, as  $t \rightarrow 0$ ;

Let  $\mathfrak{h} \rightarrow \Delta^*$  via  $z \mapsto t = \exp(2\pi\sqrt{-1}z)$  be the universal cover. It is not hard to see that

**Lemma 3.4.** *We have  $A_p^i, C_p^i, D_q^i, E_q^i \in \mathbf{h}$ . Moreover,  $F_q^i \in \mathbf{h}$  if  $i \neq q$ .*

To obtain a frame of  $\mathcal{F}_t^2$  for  $|t| \ll 1$ , we need to “untwist” the frame  $\{u_p(t), v_r(t) \mid p = 1, \dots, h+1 \text{ and } r = 1, \dots, m\}$  and consider the multi-valued sections

$$(3.16) \quad \begin{aligned} u'_p(t) &:= u_p(t) + zNu_p(t), \\ v'_r(t) &:= v_r(t) + zNv_r(t). \end{aligned}$$

Then  $\{u'_1(t), \dots, u'_{h+1}(t), v'_1(t), \dots, v'_m(t)\}$  is a (multi-valued) frame for  $\mathcal{F}_t^2$ . Explicitly,

$$(3.17) \quad \begin{aligned} u'_p(z) &= \sum_{i=1}^m \left( A_p^i(t) - \sum_{k=1}^m C_p^k(t) x_{ki} + zC_p^i(t) \right) \alpha_i + \sum_{i=1}^{2h+2} B_p^i(t) \beta_i + \sum_{i=1}^m C_p^i(t) \gamma_i \\ v'_r(z) &= \sum_{i=1}^m \left( D_r^i(t) - \sum_{k=1}^m F_r^k(t) x_{ki} + zF_r^i(t) \right) \alpha_i + \sum_{i=1}^{2h+2} E_r^i(t) \beta_i + \sum_{i=1}^m F_r^i(t) \gamma_i. \end{aligned}$$

We may and will further assume that  $z$  lies in a bounded vertical strip, i.e.  $\operatorname{Re}(z) \in [0, 1]$ .

Now we are ready to prove Theorem 3.3.



*Proof of Theorem 3.3.* For CY  $\partial\bar{\partial}$ -threefolds,  $H^{3,0}(X)$  is always polarized by the sesquilinear pairing  $\tilde{Q}(\bullet, \bar{\bullet})$ . Thus it suffices to check the second Hodge–Riemann bilinear relation on  $H^{2,1}(X)$ ; in other words, we have to check

$$\langle \sqrt{-1}u, \bar{u} \rangle := -\sqrt{-1} \int_X u \cup \bar{u} \geq 0 \text{ for } u \in H^{2,1}(X)$$

and the equality holds if and only if  $u = 0$ .

Given a degeneration  $\mathcal{X} \rightarrow \Delta$  as above, we constructed a basis for  $\mathcal{F}_t$  with  $t \neq 0$ . Thus it suffices to show that the hermitian matrix with respect to the basis  $\{u'_i(z), v'_j(z)\}$  is positive definite, i.e. the matrix

$$(3.18) \quad \begin{bmatrix} M(u'_p(z), \bar{u}'_q(z)) & M(u'_p(z), \bar{v}'_s(z)) \\ M(v'_r(z), \bar{u}'_q(z)) & M(v'_r(z), \bar{v}'_s(z)) \end{bmatrix}$$

is positive definite for  $\text{Im} z \gg 0$ . Here  $1 \leq p, q \leq h+1$ ,  $1 \leq r, s \leq m$ , and

$$M(u, v) := \langle \sqrt{-1}u, v \rangle = -\sqrt{-1} \int_X u \cup v.$$

Now under our hypothesis, we know that the limit

$$a_{p,\bar{q}}^\infty := \lim_{\text{Im} z \rightarrow \infty} M(u'_p(z), \bar{u}'_q(z))$$

exists for all  $1 \leq p, q \leq h+1$ . Moreover, the matrix  $(a_{p,\bar{q}}^\infty)_{1 \leq p, q \leq h+1}$  is positive definite since  $\text{Gr}_3^{\mathcal{W}(M)} H^3(X; \mathbb{C})$  carries a polarized Hodge structure. Consequently, the matrix

$$(a_{p\bar{q}}(z) = M(u'_p(z), \bar{u}'_q(z)))_{1 \leq p, q \leq h+1}$$

is positive definite when  $\text{Im}(z) \gg 0$ . On the other hand, we have

$$Q(\alpha_r, \alpha_s) = Q(\alpha_i, \beta_j) = 0 \text{ for any } r, s, i, j.$$

Also, since  $\delta_k = \gamma_k - \sum_{l=1}^m x_{kl} \alpha_l$ , from  $Q(\delta_r, N\delta_s) = Q(\delta_r, \alpha_s) = Q(\gamma_r, \alpha_s)$  and

$$Q(\gamma_r, \alpha_s) = Q(\gamma_r, N\gamma_s) = -Q(N\gamma_s, \gamma_r) = Q(-N\gamma_s, \gamma_r) = Q(\gamma_s, N\gamma_r) = Q(\gamma_s, \alpha_r),$$

we see that the real symmetric matrix  $(q_{rs} = Q(\delta_r, \alpha_s))_{1 \leq r, s \leq m}$  is positive definite, because the sesquilinear form  $Q(u, N\bar{v})$  (where  $u, v \in \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C})$ ) is positive definite.

We compute

$$M(v'_r(z), \bar{v}'_s(z)) = 2\text{Im}(z)q_{rs} + P_{r\bar{s}} + H_{r\bar{s}}(z)$$

for some constant  $P_{r\bar{s}}$  and some function  $H_{r\bar{s}}(z) \in \mathbf{h}$ . It follows that the matrix

$$(d_{r\bar{s}}(z) = M(v'_r(z), \bar{v}'_s(z)) = 2\text{Im}(z)(q_{rs}) + (P_{r\bar{s}}) + (H_{r\bar{s}}(z)))$$

is also positive definite when  $\text{Im}(z) \gg 0$ .

Put  $b_{p\bar{s}}(z) = M(u'_p(z), \bar{v}'_s(z))$  and  $c_{r\bar{q}}(z) = M(v'_r(z), \bar{u}'_q(z))$ . Then

$$(3.18) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \sim \begin{bmatrix} A(z) - C(z)D(z)^{-1}B(z) & \mathbf{0}_{(h+1) \times m} \\ \mathbf{0}_{m \times (h+1)} & D(z) \end{bmatrix}.$$

Let us investigate the inverse  $D(z)^{-1}$ . Write

$$D(z) = 2\text{Im}(z) \left( Q + \frac{P + H(z)}{2\text{Im}(z)} \right) \text{ with } Q = (q_{r\bar{s}}).$$

Since  $Q$  is invertible and  $(2\operatorname{Im}(z))^{-1}(P+H(z)) \rightarrow 0$  as  $\operatorname{Im}(z) \rightarrow \infty$ , we conclude that

$$D(z)^{-1} = \frac{1}{2\operatorname{Im}(z)} (Q^{-1} + G(z))$$

for some matrix  $G(z)$  with  $G(z) \rightarrow \mathbf{0}_{m \times m}$  as  $\operatorname{Im}(z) \rightarrow \infty$ . Notice that the entries in  $C(z)$  and  $D(z)$  are all bounded. We infer that

$$A(z) - C(z)D(z)^{-1}B(z) \rightarrow (a_{p\bar{q}}^\infty) \text{ as } \operatorname{Im}(z) \rightarrow \infty.$$

In particular,  $A(z) - C(z)D(z)^{-1}B(z)$  is positive definite for  $\operatorname{Im}(z) \gg 0$ . This shows that

$$\begin{bmatrix} M(u'_p(z), \bar{u}'_q(z)) & M(u'_p(z), \bar{v}'_s(z)) \\ M(v'_r(z), \bar{u}'_q(z)) & M(v'_r(z), \bar{v}'_s(z)) \end{bmatrix}$$

is positive definite whenever  $\operatorname{Im}(z) \gg 0$ . The proof is completed.  $\square$

**3.4. The  $\partial\bar{\partial}$ -lemma for finite distance degenerations.** We have constructed a basis for  $\mathcal{F}_t^2$  with  $t \neq 0$  in (3.17). It turns out that we are able to use this basis to prove that the  $\partial\bar{\partial}$ -lemma holds for small smoothings.

Let  $f: \mathcal{X} \rightarrow \Delta$  be a semi-stable degeneration and  $E := f^{-1}(0)$  be the central fiber as before.

**Proposition 3.5.** *Assuming Hypothesis A, then  $\mathcal{F}_t^2 \cap \overline{\mathcal{F}_t^2} = (0)$  for  $t$  small.*

*Proof.* To prove that  $\mathcal{F}_t^2 \cap \overline{\mathcal{F}_t^2} = (0)$ , it now suffices to show that

$$(3.19) \quad \bigwedge_{i=1}^{h+1} u'_i(t) \wedge \overline{u'_i(t)} \wedge \bigwedge_{i=1}^m v'_i(t) \wedge \overline{v'_i(t)} \neq 0$$

where  $u'_i(t)$  and  $v'_i(t)$  are given in (3.17). More accurately, using the fact  $N(\alpha_i) = N(\beta_i) = 0$  and  $N(\delta_i) = N(\gamma_i) = \alpha_i$ , we get

$$\begin{aligned} u'_p(t) &= \sum_{i=1}^m \left( A_p^i(t) - \sum_{k=1}^m C_p^k(t)x_{ki} + zC_p^i(t) \right) \alpha_i + \sum_{i=1}^{2h+2} B_p^i(t)\beta_i + \sum_{i=1}^m C_p^i(t)\gamma_i \\ v'_q(t) &= \sum_{i=1}^m \left( D_q^i(t) - \sum_{k=1}^m F_q^k(t)x_{ki} + zF_q^i(t) \right) \alpha_i + \sum_{i=1}^{2h+2} E_q^i(t)\beta_i + \sum_{i=1}^m F_q^i(t)\gamma_i. \end{aligned}$$

Using Lemma 3.4, we conclude

$$(3.20) \quad \bigwedge_{i=1}^{h+1} u'_i(t) \wedge \overline{u'_i(t)} = c \cdot \beta_1 \wedge \cdots \wedge \beta_{2h+2} + \sum_J \phi^J(z) \omega_J.$$

In the above displayed equation,  $c$  is a non-zero constant,  $\omega_J$  is a  $2h+2$  form and  $\phi^J \in \mathbf{h}$ . The constant  $c$  is non-zero follows from the fact that  $\{u_1, \dots, u_{h+1}\}$  is a basis for  $I^{3,0} \oplus I^{2,1}$  and  $I^{0,3} = \overline{I^{3,0}}$  and  $I^{1,2} = \overline{I^{2,1}}$ .

Observe that by the construction of our basis we have for each  $p$  and  $i$

$$(3.21) \quad F_q^i(t) = \delta_q^i + G_q^i(t)$$

for some holomorphic function  $G_p^i(t)$  with  $G_p^i \in \mathbf{h}$ . Now we can re-write

$$v'_q(t) = (z - x_{qq})\alpha_q + \sum_{i=1}^m \phi_q^i(z)\alpha_i + \sum_{i=1}^{2h+2} E_q^i(t)\beta_i + \sum_{i=1}^m \psi_q^i(z)\gamma_i + \gamma_q$$

for some functions  $\phi_q^i(z), \psi_q^i(z) \in \mathbf{h}$ . Therefore, we have for each  $q$

$$(3.22) \quad v'_q(t) \wedge \overline{v'_q(t)} = (z - \bar{z} - x_{qq} + \bar{x}_{qq})\alpha_q \wedge \gamma_q + \sum_J \phi_q^J(z)\omega_J.$$

Here  $\omega_J$  are two forms other than  $\alpha_q \wedge \gamma_q$  and  $\phi_q^J \in \mathbf{h}$ .

Taking (3.22) and (3.20) into accounts, we see that

$$(3.23) \quad \bigwedge_{i=1}^{h+1} u'_i(t) \wedge \overline{u'_i(t)} \wedge \bigwedge_{i=1}^m v'_i(t) \wedge \overline{v'_i(t)} \\ = \left( \prod_{i=1}^m (2\sqrt{-1}\operatorname{Im}(z) - 2\sqrt{-1}\operatorname{Im}(x_{ii})) + \phi(z) \right) \bigwedge_{i=1}^m \alpha_i \wedge \bigwedge_{i=1}^{2h+2} \beta_i \wedge \bigwedge_{i=1}^m \gamma_i$$

with  $\phi \in \mathbf{h}$  and hence we conclude that it is non-zero as long as  $\operatorname{Im}(z) \gg 0$ .  $\square$

**Corollary 3.6.** *Let  $f: \mathcal{X} \rightarrow \Delta$  be as above. We assume that the general fiber  $X$  has complex dimension 3 and satisfies the following conditions.*

- The central fiber of  $f$  is at a finite distance with respect to the perid-map metric;
- $H^i(X; \mathcal{O}_X) = H^0(X; \Omega_X^j) = 0$  for  $1 \leq i, j \leq 2$ .

Then the  $\partial\bar{\partial}$ -lemma holds on  $X$ .

*Proof.* This follows immediately from [Fri19, Corollary 1.6]. Indeed, the Hodge-de Rham spectral sequence degenerates at  $E_1$  in the present situation (cf. [PS08, Corollary 11.24]).  $\square$

#### 4. PROOF OF THE $\partial\bar{\partial}$ -LEMMA FOR HASHIMOTO-SANO'S EXAMPLES

We can use the basis similar to (3.17) to prove that the smooth non-Kähler Calabi-Yau threefolds constructed by Hashimoto and Sano also support the  $\partial\bar{\partial}$ -lemma. We will explain the details in this section.

Let us review the construction of non-Kähler Calabi-Yau threefolds given in [HS23] and record the data needed later. Given a positive integer  $a > 0$ , Hashimoto and Sano constructed a smooth non-Kähler Calabi-Yau threefold  $X(a)$  as a smoothing of a certain simple normal crossing variety  $X_0(a)$ . Let us now review their construction and recall some necessary details. Fix a positive integer  $a$ . Let  $X_2 := \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  and  $S$  be a very general hypersurface of degree  $(2, 2, 2)$  in  $X_2$ . Then  $S$  is a smooth K3 surface with Picard number 3. Indeed,

$$(4.1) \quad \operatorname{Pic}(S) \cong \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}h_3$$

where  $h_i$  is the pullback of the line bundle  $\mathcal{O}_{\mathbf{P}^1}(1)$  via the projection to the  $i^{\text{th}}$  factor.

Let  $S \rightarrow \mathbf{P}^1$  be the projection to the first factor and let  $C_1, \dots, C_a$  be distinct smooth fibers, i.e.  $C_k \in |\mathcal{O}_S(1, 0, 0)|$ . Finally, let  $X_1$  be a blow-up of  $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  along the curves  $C_1, \dots, C_a$  and  $C$ , where  $C$  is a smooth curve in the linear system

$$(4.2) \quad |\mathcal{O}_S(16a^2 - a + 4, 4 - 8a, 4 + 8a)|.$$

Note that the blow-up morphism  $\mu : X_1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  induces an isomorphism between  $S$  and its proper transform, which will be denoted by  $S_1$ .

The projection  $\pi_{ij} : S \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  to the  $i^{\text{th}}$  and  $j^{\text{th}}$  factors is a double cover and induces an order two automorphism  $\iota_{ij}$  on  $S$ . Denote by  $\iota := \iota_{12} \circ \iota_{13}$  and  $\iota^a$  be its  $a^{\text{th}}$  power. Put

$$(4.3) \quad \iota_a := \iota^a \circ \mu|_{S_1} : S_1 \rightarrow S.$$

We can construct the pushout of  $X_1$  and  $X_2$  by the automorphism

$$(4.4) \quad \iota_a : S_1 \rightarrow S$$

where  $S_1 \subset X_1$  and  $S \subset X_2$  are viewed as *subvarieties*. The action of  $\iota_a$  on  $\text{Pic}(S)$  is given by the matrix (under the ordered basis  $\{h_1, h_2, h_3\}$ )

$$(4.5) \quad \begin{bmatrix} 1 & 4a^2 - 2a & 2a^2 + 2a \\ 0 & 1 - 2a & -2a \\ 0 & 2a & 1 + 2a \end{bmatrix}.$$

The pushout is a SNC variety consisting of two components whose intersection is  $S$ . Let us denote the pushout by  $X_0$ . It is proven that  $X_0$  admits a semistable smoothing, that is, there exists a one parameter family  $\mathcal{X} \rightarrow \Delta$  such that

- the central fiber is isomorphic to  $X_0$ ;
- the total space  $\mathcal{X}$  is smooth;
- the general fiber is smooth.

Let  $X \equiv X(a)$  be the smooth fiber. Using the semistable model, one can construct a limiting mixed Hodge structure on  $H^k(X; \mathbb{C})$  via the relative logarithmic de Rham complex. It is known that the Hodge–de Rham spectral sequence degenerates at  $E_1$  page (cf. [PS08]). In [HS23], Hashimoto and Sano investigated the variety  $X$  in great detail and proved several results concerning  $X$ . We summarize them as follows.

**Proposition 4.1.**  *$X$  is simply-connected with  $b_1(X) = 0$  and  $b_2(X) = a + 3$ . In addition, we have the following vanishing*

$$(4.6) \quad H^1(X; \mathcal{O}_X) = H^0(X; \Omega_X) = H^2(X; \mathcal{O}_X) = H^0(X; \Omega_X^2) = 0.$$

Furthermore,  $X$  is non-projective and in particular,  $X$  is non-Kähler.

The following lemma will be useful later.

**Lemma 4.2.** *The image of the pullback*

$$(4.7) \quad (\iota_a^{-1})^* : H^2(X_1; \mathbb{C}) \rightarrow H^2(S; \mathbb{C})$$

is contained in  $\mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}h_3$ .

*Proof.* This follows from the construction.

Indeed,  $X_1$  is the blow-up of  $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  along the curves  $C_1, \dots, C_a, C$ . Let  $Y_{k+1} := \text{Bl}_{C_k} Y_k$  with  $Y_1 = X$  and  $E_k$  be the exceptional divisor. Let  $T_{k+1}$  be the proper transform of  $T_k$  with  $T_1 = S$ . We have

$$(4.8) \quad E_k \cap T_{k+1} \cong C_k \subset T_k$$

under the blow-up morphism. Also in the last blow-up  $X_1 = \text{Bl}_C Y_{k+1}$ , we have

$$(4.9) \quad E \cap S_1 \in |\mathcal{O}_{S_1}(D)|$$

where  $D$  is the restriction of the divisor  $(16a^2 - a + 4, 4 - 8a, 4 + 8a)$  under the morphism

$$(4.10) \quad S_1 \xrightarrow{\cong} T_{a+1} \xrightarrow{\cong} T_a \xrightarrow{\cong} \cdots \xrightarrow{\cong} T_1 = S \subset X$$

and  $S_1 \subset X_1$  is the proper transform of  $T_{a+1}$ . The stated result follows since  $H^2(X_1; \mathbb{C})$  is generated by the pullback of  $h_1, h_2, h_3$  and  $E_1, \dots, E_a, E$ .  $\square$

**4.1. The limiting mixed Hodge structure.** In this subsection, we will explicitly compute Steenbrink's limiting mixed Hodge structure on the middle cohomology of  $X$ .

We have the spectral sequence induced by the “monodromy weight filtration”  $\mathcal{W}(M)$  whose  $E_1$  page is given by

$$(4.11) \quad E_1^{-r, k+r} = \mathbf{H}^k(\text{Gr}_r^{\mathcal{W}(M)} \text{Tot}(\mathcal{D}^{\bullet, \bullet})) \Rightarrow \text{Gr}_{k+r}^{\mathcal{W}(M)} \mathbf{H}^k(\text{Tot}(\mathcal{D}^{\bullet, \bullet})).$$

Moreover, the spectral sequence degenerates at  $E_2$  page.

Using the residue homomorphisms, one can compute the graded complexes

- $\text{Gr}_{-1}^{\mathcal{W}(M)} \text{Tot}(\mathcal{D}^{\bullet, \bullet}) \simeq \mathbb{C}_S[-1];$
- $\text{Gr}_0^{\mathcal{W}(M)} \text{Tot}(\mathcal{D}^{\bullet, \bullet}) \simeq \mathbb{C}_{X_1} \oplus \mathbb{C}_{X_2};$
- $\text{Gr}_1^{\mathcal{W}(M)} \text{Tot}(\mathcal{D}^{\bullet, \bullet}) \simeq \mathbb{C}_S[-1]$
- $\text{Gr}_r^{\mathcal{W}(M)} \text{Tot}(\mathcal{D}^{\bullet, \bullet}) = 0$  for  $r \neq -1, 0, 1$ .

Here the notation “ $\simeq$ ” refers to “is quasi-isomorphic to.” Notice that the hypercohomology  $\mathbf{H}^k(\text{Tot}(\mathcal{D}^{\bullet, \bullet}))$  of the total complex computes  $H^k(X; \mathbb{C})$ .

**Lemma 4.3.** *We have*

$$(4.12) \quad \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \cong \ker(H^2(S; \mathbb{C}) \rightarrow H^4(X_1; \mathbb{C}) \oplus H^4(X_2; \mathbb{C}))$$

and

$$(4.13) \quad \text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \cong \text{coker}(H^2(X_1; \mathbb{C}) \oplus H^2(X_2; \mathbb{C}) \rightarrow H^2(S; \mathbb{C}))$$

where the arrow in (4.12) is induced by Gysin maps and the one in (4.13) is induced from the restrictions  $\iota_a: S \rightarrow X_1$  and  $S \subset X_2$ . Moreover, the monodromy operator

$$(4.14) \quad N: \text{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \rightarrow \text{Gr}_2^{\mathcal{W}(M)} H^3(X; \mathbb{C})$$

is precisely the morphism induced from the identity on  $H^2(S; \mathbb{C})$  and it is an isomorphism in the present situation, i.e. it satisfies Hypothesis A.

*Proof.* The  $E_2$ -term  $E_2^{-r, m+r}$  is the cohomology of

$$(4.15) \quad E_1^{-r-1, m+r} \rightarrow E_1^{-r, m+r} \rightarrow E_1^{-r+1, m+r}.$$

Put  $m = 3$  and  $r = 1$ . We obtain the sequence

$$(4.16) \quad E_1^{-2, 4} = 0 \rightarrow E_1^{-1, 4} = H^2(S; \mathbb{C}) \rightarrow E_1^{0, 4} = H^4(X_1; \mathbb{C}) \oplus H^4(X_2; \mathbb{C}).$$

This implies the first isomorphism. Similarly, by putting  $m = 3$  and  $r = -1$ , we obtain

$$(4.17) \quad E_1^{0, 2} = H^2(X_1; \mathbb{C}) \oplus H^2(X_2; \mathbb{C}) \rightarrow E_1^{1, 2} = H^2(S; \mathbb{C}) \rightarrow E_1^{2, 2} = 0$$

which implies the second statement. Now since

$$(4.18) \quad \mathrm{Gr}_4^{\mathcal{W}(M)} H^3(X; \mathbb{C}) \text{ and } \mathrm{Gr}_2^{\mathcal{W}(M)} H^2(X; \mathbb{C})$$

have the same dimension, it suffices to show that  $N$  is an injection.

Look at the following diagram

$$(4.19) \quad \begin{array}{ccc} & H^2(S; \mathbb{C}) & \xrightarrow{\varphi^\vee} H^4(X_1; \mathbb{C}) \oplus H^4(X_2; \mathbb{C}) \\ & \downarrow \mathrm{id} & \\ H^2(X_1; \mathbb{C}) \oplus H^2(X_2; \mathbb{C}) & \xrightarrow{\varphi} & H^2(S; \mathbb{C}) \end{array}$$

where  $\varphi$  is the usual signed restriction and  $\varphi^\vee$  is the signed Gysin map. It is now sufficient to prove that if  $a \in \ker(\varphi^\vee)$  and  $a \in \mathrm{im}(\varphi)$ , then  $a = 0$ . Write  $a = \varphi(b)$  for some  $b \in H^2(X_1; \mathbb{C}) \oplus H^2(X_2; \mathbb{C})$ . By Lemma 4.2, we may assume that  $b \in H^2(X_2; \mathbb{C})$ . The conditions imply that  $\varphi^\vee \varphi(b) = 0$ . But we have

$$(4.20) \quad \varphi^\vee \varphi(b) = b \cup (2h_1 + 2h_2 + 2h_3), \quad b \in H^2(X_2; \mathbb{C})$$

and one checks this is an injection (indeed, it is an isomorphism). Therefore,  $b = 0$  and hence  $a = 0$  as desired.  $\square$

**4.2. A construction of the basis for the canonical extension.** Let the notations be as before. We construct a basis for  $F^2$  as follows. Recall that in the present situation we have

$$(4.21) \quad W_2 = I^{2,0} \oplus I^{1,1} \oplus I^{0,2},$$

$$(4.22) \quad W_3 = W_2 \oplus I^{3,0} \oplus I^{2,1} \oplus I^{1,2} \oplus I^{0,3},$$

$$(4.23) \quad W_4 = W_3 \oplus I^{3,1} \oplus I^{2,2} \oplus I^{1,3}.$$

Note that from (1.6), both  $I^{2,0} \oplus I^{0,2}$  and  $I^{1,1}$  are complex vector spaces with real structure. So is  $\bigoplus_{p=0}^3 I^{p,3-p}$ . Since both  $X_1$  and  $X_2$  in Hashimoto–Sano’s construction are rational, we have  $\dim_{\mathbb{C}} I^{3,1} = 1$  and  $I^{3,0} = 0$ . Consequently, we can pick a *real* basis  $\{\varepsilon_1, \varepsilon_2\}$  for  $I^{2,0} \oplus I^{0,2}$  and  $\{\alpha_1, \dots, \alpha_m\}$  for  $I^{1,1}$ . Moreover, there are complex numbers  $w_{kl}$  with  $1 \leq k, l \leq 2$  such that

$$(4.24) \quad \xi_1 := \sum_{l=1}^2 w_{1l} \varepsilon_l \in I^{2,0} \text{ and } \xi_2 := \sum_{l=1}^2 w_{2l} \varepsilon_l \in I^{0,2}.$$

We may also assume that  $\bar{\xi}_1 = \xi_2$ , i.e.

$$(4.25) \quad w_{11} = \bar{w}_{21} \text{ and } w_{12} = \bar{w}_{22}.$$

Since  $I^{2,1} \oplus I^{1,2}$  is also a complex vector space with real structure, we can pick a real basis  $\{\beta_1, \dots, \beta_{2h}\}$  for it.

Next let us consider  $I^{1,1} \oplus I^{2,2}$ . This is again a complex vector spaces with real structure by (1.6) and therefore it also admits a real basis. We may extend the basis  $\{\alpha, \dots, \alpha_m\}$  to become a real basis  $\{\alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_m\}$  for  $I^{1,1} \oplus I^{2,2}$  as before. Also, there are complex numbers  $x_{kl} \in \mathbb{C}$  with  $1 \leq k, l \leq m$  such that

$$(4.26) \quad \delta_k := \gamma_k - \sum_{l=1}^m x_{kl} \alpha_l \in I^{2,2}.$$

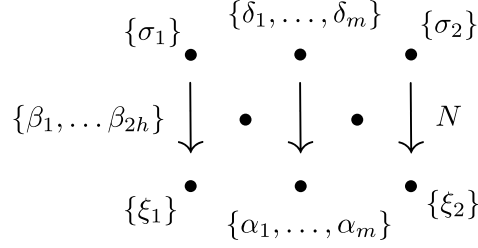


FIGURE 2. The limiting mixed Hodge diamond for  $H^3(X; \mathbb{C})$  and the bases for the corresponding subspaces  $I^{p,q}$ .

We may further assume that  $N(\delta_k) = N(\gamma_k) = \alpha_k$  because  $N$  is an isomorphism on the relevant grade piece by Lemma 4.3.

Finally, consider  $I^{3,1} \oplus I^{1,3} \oplus I^{2,0} \oplus I^{0,2}$ . Again by (1.6), this is a complex vector space (of dimension four) with real structure. We may extend  $\{\varepsilon_1, \varepsilon_2\}$  to a real basis  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$  for it. We may assume that  $N(\varepsilon_3) = \varepsilon_1$  and  $N(\varepsilon_4) = \varepsilon_2$ . Again there are complex numbers  $y_{kl}$  with  $k = 1, 2$  and  $1 \leq l \leq 4$  such that

$$\sigma_1 := \sum_{l=1}^4 y_{1l} \varepsilon_l \in I^{3,1} \text{ and } \sigma_2 := \sum_{l=1}^4 y_{2l} \varepsilon_l \in I^{1,3}$$

Under our assumption, we have

$$(4.27) \quad N(\sigma_1) = y_{13} \varepsilon_1 + y_{14} \varepsilon_2 \in I^{2,0} \text{ and } N(\sigma_2) = y_{23} \varepsilon_1 + y_{24} \varepsilon_2 \in I^{0,2}.$$

After rescaling, we may further assume  $y_{13} = w_{11}$ ,  $y_{14} = w_{12}$ ,  $y_{23} = w_{21}$ , and  $y_{24} = w_{22}$ , i.e.  $N(\sigma_1) = \xi_1$  and  $N(\sigma_2) = \xi_2$ .

**4.3. Notation.** Let us fix the notation for the later use.

- Let  $\{u_1, \dots, u_h\}$  be a basis for  $I^{2,1}$ . We can write  $u_p = \sum_{i=1}^{2h} b_p^i \beta_i$ .
- Let  $\{f_1 := \sigma_1\}$  be a basis for  $I^{3,1}$ .
- Let  $\{g_1 := \xi_1\}$  be a basis for  $I^{2,0}$ .
- Let  $\{v_1 = \delta_1, \dots, v_m = \delta_m\}$  be a basis for  $I^{2,2}$ .

Since  $\mathcal{F}^2$  is a holomorphic subbundle in Deligne's canonical extension, we may extend

$$(4.28) \quad \{u_1, \dots, u_h, v_1, \dots, v_m, f_1, g_1\}$$

to become a local frame for  $\mathcal{F}^2$  over  $\Delta$ . Explicitly, we can write

$$\begin{aligned} u_p(t) &= \sum_{i=1}^m U_p^{\alpha_i}(t) \alpha_i + \sum_{i=1}^{2h} U_p^{\beta_i}(t) \beta_i + \sum_{i=1}^m U_p^{\delta_i}(t) \delta_i + \sum_{i=1}^2 U_p^{\sigma_i}(t) \sigma_i + \sum_{i=1}^2 U_p^{\xi_i}(t) \xi_i, \\ v_q(t) &= \sum_{i=1}^m V_p^{\alpha_i}(t) \alpha_i + \sum_{i=1}^{2h} V_p^{\beta_i}(t) \beta_i + \sum_{i=1}^m V_p^{\delta_i}(t) \delta_i + \sum_{i=1}^2 V_p^{\sigma_i}(t) \sigma_i + \sum_{i=1}^2 V_p^{\xi_i}(t) \xi_i, \\ f_1(t) &= \sum_{i=1}^m F_1^{\alpha_i}(t) \alpha_i + \sum_{i=1}^{2h} F_1^{\beta_i}(t) \beta_i + \sum_{i=1}^m F_1^{\delta_i}(t) \delta_i + \sum_{i=1}^2 F_1^{\sigma_i}(t) \sigma_i + \sum_{i=1}^2 F_1^{\xi_i}(t) \xi_i, \\ g_1(t) &= \sum_{i=1}^m G_1^{\alpha_i}(t) \alpha_i + \sum_{i=1}^{2h} G_1^{\beta_i}(t) \beta_i + \sum_{i=1}^m G_1^{\delta_i}(t) \delta_i + \sum_{i=1}^2 G_1^{\sigma_i}(t) \sigma_i + \sum_{i=1}^2 G_1^{\xi_i}(t) \xi_i. \end{aligned}$$

In the above expression, the functions  $U, V, F, G$  are all *holomorphic*. In terms of the real basis  $\{\alpha, \dots, \alpha_m, \beta_1, \dots, \beta_{2h}, \gamma_1, \dots, \gamma_m, \varepsilon_1, \dots, \varepsilon_4\}$ , we have

$$\begin{aligned} u_p(t) &= \sum_{i=1}^m \left( U_p^{\alpha_i}(t) - \sum_{k=1}^m U_p^{\delta_k}(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h} U_p^{\beta_i}(t) \beta_i + \sum_{i=1}^m U_p^{\delta_i}(t) \gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij} U_p^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} U_p^{\xi_i}(t) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij} U_p^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ v_p(t) &= \sum_{i=1}^m \left( V_p^{\alpha_i}(t) - \sum_{k=1}^m V_p^{\delta_k}(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h} V_p^{\beta_i}(t) \beta_i + \sum_{i=1}^m V_p^{\delta_i}(t) \gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij} V_p^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} V_p^{\xi_i}(t) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij} V_p^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ f_1(t) &= \sum_{i=1}^m \left( F_1^{\alpha_i}(t) - \sum_{k=1}^m F_1^{\delta_k}(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h} F_1^{\beta_i}(t) \beta_i + \sum_{i=1}^m F_1^{\delta_i}(t) \gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij} F_1^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} F_1^{\xi_i}(t) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij} F_1^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ g_1(t) &= \sum_{i=1}^m \left( G_1^{\alpha_i}(t) - \sum_{k=1}^m G_1^{\delta_k}(t) x_{ki} \right) \alpha_i + \sum_{i=1}^{2h} G_1^{\beta_i}(t) \beta_i + \sum_{i=1}^m G_1^{\delta_i}(t) \gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij} G_1^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} G_1^{\xi_i}(t) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij} G_1^{\sigma_i}(t) \right) \varepsilon_{j+2}. \end{aligned}$$

Also, these functions satisfy the properties

- (1)  $U_p^{\alpha_i}, U_p^{\beta_i}, U_p^{\sigma_i}, U_p^{\xi_i} \in \mathbf{h}$  and  $B_p^{\beta_i}(t) \rightarrow b_p^i$  as  $t \rightarrow 0$ ;
- (2)  $V_q^{\alpha_i}, V_q^{\beta_i}, V_q^{\sigma_i}, V_q^{\xi_i} \in \mathbf{h}$  and  $V_q^{\delta_i}(t) \rightarrow \delta_q^i$  as  $t \rightarrow 0$ ;
- (3)  $F_1^{\alpha_i}, F_1^{\beta_i}, F_1^{\delta_i}, F_1^{\xi_i} \in \mathbf{h}$  and  $F_1^{\sigma_i}(t) \rightarrow \delta_1^i$  as  $t \rightarrow 0$ ;
- (4)  $G_1^{\alpha_i}, G_1^{\beta_i}, G_1^{\delta_i}, G_1^{\sigma_i} \in \mathbf{h}$  and  $G_1^{\xi_i}(t) \rightarrow \delta_1^i$  as  $t \rightarrow 0$ .



**4.4. Main calculations.** As before, let  $\mathfrak{h} \rightarrow \Delta^*$  be the universal cover. In order to get a frame of  $\mathcal{F}_t^2$  for  $t \neq 0$ , we “untwist” the frame  $\{u_1(t), \dots, u_h(t), v_1(t), \dots, v_m(t), f_1(t), g_1(t)\}$  and obtain the multi-valued sections

$$(4.29) \quad \begin{aligned} u'_p(t) &:= u_p(t) + zNu_p(t), \\ v'_q(t) &:= v_q(t) + zNv_q(t), \\ f'_1(t) &:= f_1(t) + zNf_1(t), \\ g'_1(t) &:= g_1(t) + zNg_1(t). \end{aligned}$$

Then  $\{u'_1(t), \dots, u'_h(t), v'_1(t), \dots, v'_m(t), f'_1(t), g'_1(t)\}$  is a frame for  $\mathcal{F}_t^2$  for  $t \neq 0$ . We may and will further assume that  $z$  lies in a bounded vertical strip, i.e.  $\operatorname{Re}(z) \in [0, 1]$ .

To prove that  $\mathcal{F}_t^2 \cap \overline{\mathcal{F}_t^2}$ , it now suffices to show that

$$(4.30) \quad \bigwedge_{i=1}^h u'_i(t) \wedge \overline{u'_i(t)} \wedge \bigwedge_{i=1}^m v'_i(t) \wedge \overline{v'_i(t)} \wedge f'_1(t) \wedge \overline{f'_1(t)} \wedge g'_1(t) \wedge \overline{g'_1(t)} \neq 0.$$

More accurately, using the fact  $N(\alpha_i) = N(\beta_i) = N(\xi_i) = 0$ ,  $N(\delta_i) = N(\gamma_i) = \alpha_i$ , and  $N(\sigma_i) = \xi_i$ , we get

$$\begin{aligned} u'_p(t) &= \sum_{i=1}^m \left( U_p^{\alpha_i}(t) - \sum_{k=1}^m U_p^{\delta_k}(t)x_{ki} + zU_p^{\delta_i}(t) \right) \alpha_i + \sum_{i=1}^{2h} U_p^{\beta_i}(t)\beta_i + \sum_{i=1}^m U_p^{\delta_i}(t)\gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij}U_p^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} \left( U_p^{\xi_i}(t) + zU_p^{\sigma_i}(t) \right) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij}U_p^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ v'_p(t) &= \sum_{i=1}^m \left( V_p^{\alpha_i}(t) - \sum_{k=1}^m V_p^{\delta_k}(t)x_{ki} + zV_p^{\delta_i}(t) \right) \alpha_i + \sum_{i=1}^{2h} V_p^{\beta_i}(t)\beta_i + \sum_{i=1}^m V_p^{\delta_i}(t)\gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij}V_p^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} \left( V_p^{\xi_i}(t) + zV_p^{\sigma_i}(t) \right) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij}V_p^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ f'_1(t) &= \sum_{i=1}^m \left( F_1^{\alpha_i}(t) - \sum_{k=1}^m F_1^{\delta_k}(t)x_{ki} + zF_1^{\delta_i}(t) \right) \alpha_i + \sum_{i=1}^{2h} F_1^{\beta_i}(t)\beta_i + \sum_{i=1}^m F_1^{\delta_i}(t)\gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij}F_1^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} \left( F_1^{\xi_i}(t) + zF_1^{\sigma_i}(t) \right) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij}F_1^{\sigma_i}(t) \right) \varepsilon_{j+2}, \\ g'_1(t) &= \sum_{i=1}^m \left( G_1^{\alpha_i}(t) - \sum_{k=1}^m G_1^{\delta_k}(t)x_{ki} + zG_1^{\delta_i}(t) \right) \alpha_i + \sum_{i=1}^{2h} G_1^{\beta_i}(t)\beta_i + \sum_{i=1}^m G_1^{\delta_i}(t)\gamma_i \\ &\quad + \sum_{j=1}^2 \left( \sum_{i=1}^2 y_{ij}G_1^{\sigma_i}(t) + \sum_{i=1}^2 w_{ij} \left( G_1^{\xi_i}(t) + zG_1^{\sigma_i}(t) \right) \right) \varepsilon_j + \sum_{j=1}^2 \left( \sum_{i=1}^2 w_{ij}G_1^{\sigma_i}(t) \right) \varepsilon_{j+2}. \end{aligned}$$

Using the items (1)–(4), we conclude that

$$(4.31) \quad \bigwedge_{i=1}^h u'_i(t) \wedge \overline{u'_i(t)} = c \cdot \beta_1 \wedge \dots \wedge \beta_{2h} + \sum_J \phi^J(z) \omega_J.$$

In the displayed equation above,  $c$  is a non-zero constant,  $\omega_J$  is a  $2h$  form and  $\phi^J \in \mathbf{h}$ . The constant  $c$  is non-zero follows from the fact that  $\{u_1, \dots, u_h\}$  is a basis for  $I^{2,1}$  and  $I^{1,2} = \overline{I^{2,1}}$ .

Observe that by the construction of our basis we have for each  $p$  and  $i$

$$(4.32) \quad V_q^{\delta_i}(t) = \delta_q^i + \tilde{V}_q^i(t)$$

for some holomorphic function  $\tilde{V}_q^i(t)$  with  $\tilde{V}_q^i \in \mathbf{h}$ . Now we can re-write

$$v'_q(t) = (z - x_{qq})\alpha_q + \sum_{i=1}^m \phi_q^i(z)\alpha_i + \sum_{i=1}^{2h} V_q^{\beta_i}(t)\beta_i + \sum_{i=1}^m \psi_q^i(z)\gamma_i + \gamma_q + \sum_{i=1}^4 \theta_q^i(z)\varepsilon_i$$

for some functions  $\phi_q^i, \psi_q^i, \theta_q^i \in \mathbf{h}$ . Therefore, we have for each  $q$

$$(4.33) \quad v'_q(t) \wedge \overline{v'_q(t)} = (z - \bar{z} - x_{qq} + \bar{x}_{qq})\alpha_q \wedge \gamma_q + \sum_J \phi_q^J(z)\omega_J.$$

Here  $\omega_J$  are two forms other than  $\alpha_q \wedge \gamma_q$  and  $\phi_q^J \in \mathbf{h}$ .

Next, let us compute

$$(4.34) \quad f'_1(t) \wedge \overline{f'_1(t)} \wedge g'_1(t) \wedge \overline{g'_1(t)}.$$

Using the same trick, we conclude that

$$(4.35) \quad f'_1(t) \wedge \overline{f'_1(t)} \wedge g'_1(t) \wedge \overline{g'_1(t)} = (w_{11}\bar{w}_{12} - w_{12}\bar{w}_{11})\varepsilon_1 \wedge \dots \wedge \varepsilon_4 + \sum_I \psi^I \omega_I$$

where  $\omega_I$  is a four form and  $\psi^I \in \mathbf{h}$ .

Finally, we observe that

$$(4.36) \quad w_{11}\bar{w}_{12} - w_{12}\bar{w}_{11} = w_{11}w_{22} - w_{12}w_{21} \neq 0$$

since  $\{\xi_1, \xi_2\}$  is a basis for  $I^{2,0} \oplus I^{0,2}$ .

Taking all the estimates above into accounts, we see that

$$\begin{aligned} & \bigwedge_{i=1}^h u'_i(t) \wedge \overline{u'_i(t)} \wedge \bigwedge_{i=1}^m v'_i(t) \wedge \overline{v'_i(t)} \wedge f'_1(t) \wedge \overline{f'_1(t)} \wedge g'_1(t) \wedge \overline{g'_1(t)} \\ &= (w_{11}\bar{w}_{12} - w_{12}\bar{w}_{11}) \left( \prod_{i=1}^m (2\sqrt{-1} \operatorname{Im}(z) - 2\sqrt{-1} \operatorname{Im}(x_{ii})) + \phi(z) \right) \Omega \end{aligned}$$

where

$$\Omega = \bigwedge_{i=1}^4 \varepsilon_i \wedge \bigwedge_{i=1}^m \alpha_i \wedge \bigwedge_{i=1}^{2h} \beta_i \wedge \bigwedge_{i=1}^m \gamma_i$$

and  $\phi \in \mathbf{h}$ . Therefore we conclude that it is non-zero as long as  $\operatorname{Im}(z) \gg 0$ .

**Theorem 4.4.** *Let notations be as above. For  $t \in \Delta^*$  with  $|t| \ll 1$ , we have*

$$(4.37) \quad \mathcal{F}_t^2 \cap \overline{\mathcal{F}_t^2} = (0).$$

Hence the  $\partial\bar{\partial}$ -lemma holds on the non-Kähler Calabi–Yau threefolds constructed by Hashimoto and Sano in [HS23].

*Proof.* The first statement  $\mathcal{F}_t^2 \cap \overline{\mathcal{F}_t^2} = (0)$  follows from our previous calculation, while the second one follows from [Fri19, Corollary 1.6].  $\square$

**Theorem 4.5.** *The Hodge structure on  $H^3(X; \mathbb{C})$  is polarized by  $Q(C-, -)$ .*

*Proof.* This follows from the argument of Theorem 3.3. From the formula (3.4) and (3.5) which is true without any assumption on the shape of the limiting Hodge diamond, the second Hodge–Riemann bilinear relation still holds on  $H^{2,1}(X)$ .  $\square$

## REFERENCES

- [CGH90] P. Candelas, P. Green, and Tristan Hübsch, *Rolling among Calabi–Yau vacua*, Nuclear Physics B **330** (1990), no. 1, 49–102. [↑2](#)
- [CGPY23] Tristan C. Collins, Sergei Gukov, Sebastien Picard, and Shing-Tung Yau, *Special Lagrangian cycles and Calabi–Yau transitions*, Comm. Math. Phys. **401** (2023), no. 1, 769–802. MR4604907 [↑2](#)
- [Che24] Kuan-Wen Chen, *On the Hodge structure of global smoothings of normal crossing varieties*, in preparation (2024). [↑4](#)
- [CKS86] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. **123** (1986), 457–535. [↑7](#)
- [CPY24] Tristan Collins, Sebastien Picard, and Shing-Tung Yau, *Stability of the tangent bundle through conifold transitions*, Comm. Pure Appl. Math. **77** (2024), no. 1, 284–371. MR4666627 [↑2](#)
- [Del71] Pierre Deligne, *Théorie de Hodge. II*, Institut des Hautes Études Scientifiques. Publications Mathématiques **40** (1971), 5–57. MR498551 [↑5](#)
- [DGMS77] P. Deligne, F. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk **32** (1977), no. 3(195), 119–152, 247. Translated from the English by Ju. I. Manin. MR0460700 [↑5](#)
- [FLY12] Jixiang Fu, Jun Li, and Shing-Tung Yau, *Balanced metrics on non-Kähler Calabi–Yau threefolds*, Journal of Differential Geometry **90** (2012), no. 1, 81–129. MR2891478 [↑2](#)
- [Fri19] Robert Friedman, *The  $\partial\bar{\partial}$ -lemma for general Clemens manifolds*, Pure Appl. Math. Q. **15** (2019), no. 4, 1001–1028. MR4085665 [↑2](#), [3](#), [27](#), [34](#)
- [Fri86] ———, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986), no. 4, 671–689. MR848512 [↑1](#)
- [Fuj14] Taro Fujisawa, *Polarizations on limiting mixed Hodge structures*, Journal of Singularities **8** (2014), 146–193. MR3395244 [↑17](#)
- [HS23] Kenji Hashimoto and Taro Sano, *Examples of non-Kähler Calabi–Yau 3-folds with arbitrarily large  $b_2$* , Geom. Topol. **27** (2023), no. 1, 131–152. MR4584262 [↑4](#), [27](#), [28](#), [34](#)
- [Lee18] Tsung-Ju Lee, *A Hodge theoretic criterion for finite Weil–Petersson degenerations over a higher dimensional base*, Mathematical Research Letters **25** (2018), no. 2, 617–647. MR3826838 [↑2](#)
- [Li2202] Chi Li, *Polarized Hodge structures for Clemens manifolds* (202202), available at [2202.10353](#). [↑2](#), [3](#), [23](#)
- [LLW18] Yuan-Pin Lee, Hui-Wen Lin, and Chin-Lung Wang, *Towards  $A + B$  theory in conifold transitions for Calabi–Yau threefolds* **110** (2018), no. 3, 495–541. MR3880232 [↑11](#)
- [Pop19] Dan Popovici, *Holomorphic deformations of balanced Calabi–Yau  $\partial\bar{\partial}$ -manifolds*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 2, 673–728. MR3978322 [↑2](#), [5](#), [6](#), [13](#), [15](#), [16](#)
- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625 [↑18](#), [27](#), [28](#)
- [Rei87] Miles Reid, *The moduli space of 3-folds with  $K = 0$  may nevertheless be irreducible*, Mathematische Annalen **278** (1987), no. 1-4, 329–334. MR909231 [↑2](#)
- [Ste75] Joseph Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1975/76), no. 3, 229–257. MR429885 [↑2](#)
- [Wan03] Chin-Lung Wang, *Quasi-Hodge metrics and canonical singularities*, Math. Res. Lett. **10** (2003), no. 1, 57–70. MR1960124 [↑2](#)

- [Wan97] ———, *On the incompleteness of the Weil–Petersson metric along degenerations of Calabi–Yau manifolds*, Math. Res. Lett. **4** (1997), no. 1, 157–171. MR1432818 ↑[2](#), [3](#), [14](#)
- [Yau78] Shing Tung Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I*, Commun. Pure Appl. Math. **31** (1978), no. 3, 339–411. MR480350 ↑[2](#), [11](#)

*Email address:* `tsungju@gs.ncku.edu.tw`

DEPARTMENT OF MATHEMATICS, NATIONAL CHENG KUNG UNIVERSITY, NO. 1, DAXUE RD., EAST DISTRICT, TAINAN 70101, TAIWAN.