Hidden Symmetries of Power-Law Inflation

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A scalar field with an exponential potential in FLRW universe admits the exact solution. We uncover the hidden symmetries behind the system by utilising the Eisenhart lift of field theories. We find that a conformal Killing vector field in the field space exists only for a particular combination of exponential functions which includes a single exponential potential. This implies the existence of additional conserved quantity and explains the integrability of the system.

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I. INTRODUCTION

Power-law inflation [1] is a model of inflation in which the scale factor grows as a power-law of cosmic time instead of exponential of time. It is described by a scalar field with an exponential potential of the form $V(\phi) = V_0 e^{-\lambda\phi}$ (in the reduced Planck units $8\pi G = 1$). A remarkable property of the model is that it admits the exact solutions of equations of motion of scalar factor and the scalar field and hence is integrable. This fact suggests the existence of symmetries of the system.

An important step toward revealing the hidden symmetries of power-law inflation was taken in [2] where the Noether symmetry for the Lagrangian was studied. It was found there that such a symmetry exists only for $\lambda = \sqrt{6}/2$. However, to our knowledge, no symmetries for general λ have been found. Although power-law inflation is ruled out as a model of inflation by the Planck data [3], exponential potentials appear ubiquitously from the perspective of higher dimensional theories such as string theory and/or by changing the frame (Jordan vs. Einstein), and the study of symmetries of such a system may be interesting in its own right.

In this paper, we find symmetries for power-law inflation by applying the Eisenhart lift of scalar field theory introduced in [4] to the system of a scalar field in Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. The equations of motion of the scale factor and the scalar field are then reduced to geodesic equations for null geodesics in lifted field space. Thus, finding the conserved quantities of the system is reduced to finding the conserved quantities along null geodesics. We find that a nontrivial conformal Killing vector field exists for a particular combination of exponential functions which includes a single exponential potential with general λ: the "hidden symmetry" of power-law inflation is revealed.

The paper is organized as follows. After reviewing Eisenhart lift for the scalar fields in Sec. II, the system of a homogeneous scalar field in FLRW universe is lifted, and the existence a conformal Killing vector field for a particular combination of exponential functions and the relation to the previous work are discussed in Sec. III. Sec. IV is devoted to summary. In Appendix A, the explicit solutions of the equations of motion are constructed for a single exponential potential.

Our convention of the metric signature is $(-,+,+,+)$ and we use the units of $8\pi G = c = 1$.

II. EISENHART LIFT OF RIEMANNIAN TYPE FOR A PARTICLE AND FOR SCALAR FIELDS

A. Eisenhart Lift for Scalar Fields

Eisenhart showed that the classical motion of a particle under the influence of a potential is equivalent to a geodesic of a higher dimensional Riemannian manifold with one extra coordinate [5] (see [6, 7] for a review). Hence, finding the conserved quantities of the system is reduced to finding the conserved quantities for geodesics.

The Eisenhart lift is extended to scalar field theories by [4]. Consider the system of n scalar fields in a four dimensional spacetime (with the metric $g_{\mu\nu}$)

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}k_{IJ}(\boldsymbol{\phi})\partial_\mu\phi^I\partial_\nu\phi^J - V(\boldsymbol{\phi})\right), \tag{1}$$

where the first term is the Einstein -Hilbert term (in units of $8\pi G = 1$) and $I = 1, \dots, n$ is the field space index and $k_{IJ}(\boldsymbol{\phi})$ is the scalar field space metric, while μ is the spacetime index. The Einstein equation and the equation of motion of ϕ^I are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k_{IJ}\partial_\mu\phi^I\partial_\nu\phi^J - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}k_{IJ}\partial_\alpha\phi^I\partial_\beta\phi^J + 2V\right) \tag{2}$$

$$\Box\phi^I + \Gamma^I_{JK}g^{\mu\nu}\partial_\mu\phi^J\partial_\nu\phi^K - k^{IJ}\partial_J V = 0\,, \tag{3}$$

where Γ^I_{JK} is the Christoffel symbol constructed from the field space metric k_{IJ}.

[4] observed that the same dynamics can be described by the following Eisenhart lift through the introduction of the fictitious vector field B^μ

$$I_R = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}k_{IJ}(\boldsymbol{\phi})\partial_\mu\phi^I\partial_\nu\phi^J + \frac{1}{4V(\boldsymbol{\phi})}\left(\nabla_\mu B^\mu\right)^2\right)\,. \tag{4}$$

The Einstein equation and the equations of motion of ϕ^I and B^μ are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k_{IJ}\partial_\mu\phi^I\partial_\nu\phi^J - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}k_{IJ}\partial_\alpha\phi^I\partial_\beta\phi^J\right) + 2B_{(\mu}\partial_{\nu)}\pi_B - g_{\mu\nu}B^\alpha\partial_\alpha\pi_B - g_{\mu\nu}V\pi_B^2 \tag{5}$$

$$\Box\phi^I + \Gamma^I_{JK}g^{\mu\nu}\partial_\mu\phi^J\partial_\nu\phi^K - \pi_B^2 k^{IJ}\partial_J V = 0\,, \tag{6}$$

$$\partial_\mu\pi_B = 0\,, \tag{7}$$

where

$$\pi_B = \frac{\nabla_\mu B^\mu}{2V}\,. \tag{8}$$

From Eq. (7), π_B is a constant. Plugging this into Eq. (5) and Eq. (6) and setting $\pi_B = 1$ reproduces Eq. (2) and Eq. (3).

III. INTEGRABLE COSMOLOGY: SCALAR FIELD IN FLRW UNIVERSE

We apply the formalism of the Eisenhart lift of scalar fields to the system of a single scalar field in Friedmann-Lemaitre-Robertson-Walker (FLRW) universe [8].

A. Field Space Metric and the Equations of Motion

We consider a flat FLRW universe $g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2dt^2 + a(t)^2d\boldsymbol{x}^2$ where $N(t)$ is the lapse function and $a(t)$ is the scale factor. Assuming that a scalar field ϕ and a vector field B^μ are homogeneous, the lifted system Eq. (4) reduces to the particle system whose Lagrangian is

$$\mathcal{L} = -\frac{3a}{N}\dot{a}^2 + \frac{a^3}{2N}\dot{\phi}^2 + \frac{1}{4Na^3V}\dot{\chi}^2 \equiv \frac{1}{2}G_{AB}\dot{\phi}^A\dot{\phi}^B\,, \tag{9}$$

where $\chi \equiv Na^3B^0$, $\phi^A = (a, \phi, \chi)$, and the field space metric G_{AB} is given by

$$G_{AB} = \begin{pmatrix} -\frac{6a}{N} & & \\ & \frac{a^3}{N} & \\ & & \frac{1}{2Na^3V} \end{pmatrix}\,. \tag{10}$$

In terms of the conjugate momenta, $p_a = -6a\dot{a}/N, p_\phi = a^3\dot{\phi}/N, p_\chi = \frac{\dot{\chi}}{2Na^3V}$ which coincides with π_B in Eq. (8), the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}G^{AB}p_Ap_B = N\left(-\frac{1}{12a}p_a^2 + \frac{1}{2a^3}p_\phi^2 + a^3V(\phi)p_\chi^2\right)\,, \tag{11}$$

and variation of $\mathcal{H}$ with respect to N yields the Hamiltonian constraint

$$H = \frac{1}{2}\left(-\frac{p_a^2}{6a} + \frac{p_\phi^2}{a^3} + 2a^3V(\phi)p_\chi^2\right) = 0\,. \tag{12}$$

Therefore, p_A is a null vector. Henceforth, we set $N = 1$.

The equations of motion of ϕ^A are null geodesic equations of G_{AB} and are derived from the canonical equations of motion, $\dot{\phi}^A = \frac{\partial H}{\partial p_A}, \dot{p}_A = -\frac{\partial H}{\partial \phi^A}$. Along with the Hamiltonian constraint Eq. (12), they are given by

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\phi}^2 - V p_\chi^2\right), \tag{13}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V' p_\chi^2 = 0, \tag{14}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V p_\chi^2\right), \tag{15}$$

where p_χ is a constant, and as shown in II A, setting $p_\chi = 1$ reproduces the Einstein equations and the equation of motion of ϕ in a flat FLRW universe.

B. Conformal Killing Vectors

We look for constants of motion in the field space with the metric G_{AB} (10).

One immediately finds that the vector field $\partial/\partial\chi$ is a Killing vector field since the metric components do not depend on χ. Moreover, in the special case when the potential V is constant, the metric components do not depend on ϕ and additional Killing vector field $\partial/\partial\phi$ arises.

We are interested in whether other conformal Killing vector fields ξ_A exist for some potential $V(\phi)$ so that $\xi^A p_A$ is a constant of motion. This is accomplished by solving the conformal Killing equations $\nabla_{(A}\xi_{B)} = f G_{AB}$.

In particular, we find that for the potential given by

$$V(\phi) = V_0 \left(c_1 e^{\alpha\phi} + c_2 e^{-\alpha\phi}\right)^{-2+\frac{\sqrt{6}}{\alpha}} \tag{16}$$

where V_0, c_1, c_2 and α are constants, there exists a conformal Killing vector field

$$\xi_{(1)} = -a^{-\sqrt{6}\alpha+1}\left(c_1 e^{\alpha\phi} - c_2 e^{-\alpha\phi}\right)\frac{\partial}{\partial a} + \sqrt{6} a^{-\sqrt{6}\alpha}\left(c_1 e^{\alpha\phi} + c_2 e^{-\alpha\phi}\right)\frac{\partial}{\partial\phi}. \tag{17}$$

$\xi_{(1)}$ satisfies the conformal Killing equation

$$\nabla_{(A}\xi_{(1)B)} = f G_{AB} \tag{18}$$

with

$$f = \frac{1}{3}\nabla_A \xi^A_{(1)} = \sqrt{6}\left(\alpha - \frac{\sqrt{6}}{4}\right) a^{-\sqrt{6}\alpha}\left(c_1 e^{\alpha\phi} - c_2 e^{-\alpha\phi}\right). \tag{19}$$

There also exists a Killing vector field $\xi_{(2)} = \partial/\partial\chi$, and $\xi_{(1)}$ and $\xi_{(2)}$ commute, $[\xi_{(1)}, \xi_{(2)}] = 0$. Therefore, together with the Hamiltonian constraint Eq. (12), we have three linearly independent constants of motion for the system with three degrees of freedom, and the system is completely integrable (in the sense of Liouville). This is the main result of this paper.

It is interesting to note that for $c_1 = 0$ or $c_2 = 0$ $V(\phi)$ becomes

$$V(\phi) = V_0 \exp\left[\pm\left(\sqrt{6} - 2\alpha\right)\phi\right], \tag{20}$$

which includes general exponential potential since α is an arbitrary constant. It is also interesting to note that in this case the Cotton tensor vanishes and the field space is conformally flat. The explicit solutions of the equations of motion for this potential are constructed in Appendix A.

Note also that if $\alpha = \frac{\sqrt{6}}{4}$, then $f = 0$ and the conformal Killing vector field becomes a Killing vector field found by [2]. Moreover, the potential Eq. (16) reduces to

$$V(\phi) = V_0 \left(c_1 e^{\frac{\sqrt{6}}{4}\phi} + c_2 e^{-\frac{\sqrt{6}}{4}\phi}\right)^2, \tag{21}$$

which coincides with that found by [2] where the integrable cosmological models are studied by searching for the Noether symmetry for the Lagrangian.

IV. SUMMARY

We applied the Eisenhart lift for scalar fields to a scalar field in a flat FLRW universe and found that a conformal Killing vector field exists for a particular combination of exponential functions (16) which includes the potential for power-law inflation model. When restricting $\alpha = \sqrt{6}/4$, the conformal Killing vector and the scalar potential reduce to those found in the literature.

The existence of the conformal Killing vector implies the existence of additional conserved quantity and explains why the scalar field as well as the scale factor can be solved exactly for an exponential potential in FLRW universe.

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Appendix A: Construction of the Solutions

Given three constants of motion, $H, C = \xi^A_{(1)} p_A$ and p_χ, we can construct the solutions of the equations of motion by follow the proof of the Liouville theorem given in [9]. For simplicity, we consider the special case with $c_1 = 0$ and $c_2 = 1$ so that

$$V(\phi) = V_0 e^{-\left(\sqrt{6}-2\alpha\right)\phi} \equiv V_0 e^{-\lambda\phi}\,, \tag{A1}$$

By setting $p_\chi = 1$ we reduce the degrees of freedom, and the two constants of motion are

$$\begin{aligned} C &= a^{-\sqrt{6}\alpha+1} e^{-\alpha\phi} p_a + \sqrt{6} a^{-\sqrt{6}\alpha} e^{-\alpha\phi} p_\phi\,, & \text{(A2)}\\ H &= \frac{1}{2}\left(-\frac{p_a^2}{6a} + \frac{p_\phi^2}{a^3} + 2a^3 V(\phi)\right) = 0\,. & \text{(A3)} \end{aligned}$$

These equations are solved for p_a and p_ϕ

$$\begin{aligned} p_a &= \frac{C}{2} a^{\sqrt{6}\alpha-1} e^{\alpha\phi} + \frac{6V_0}{C} a^{5-\sqrt{6}\alpha} e^{(\alpha-\sqrt{6})\phi} \equiv f_a(a,\phi,C)\,, & \text{(A4)}\\ p_\phi &= \frac{C}{2\sqrt{6}} a^{\sqrt{6}\alpha} e^{\alpha\phi} - \frac{\sqrt{6}V_0}{C} a^{6-\sqrt{6}\alpha} e^{(\alpha-\sqrt{6})\phi} \equiv f_p(a,\phi,C)\,. & \text{(A5)} \end{aligned}$$

If we can introduce new canonical coordinates and momenta (Q_1, Q_2, P_1, P_2) by a canonical transformation such that $P_1 = C$ and $P_2 = H$, then the canonical equations motion become $\dot{Q}_1 = \partial H/\partial P_1 = 0, \dot{Q}_2 = \partial H/\partial P_2 = 1$ and the solutions are trivially given by $Q_1 = \text{const.}$ and $Q_2 = t + \text{const.}$ which involve two additional constants.

In fact, such a canonical transformation is provided by the following generating function

$$S(a,\phi,C) = \int f_a(a,\phi,C)da + f_\phi(a,\phi,C)d\phi\,. \tag{A6}$$

Then, the new coordinates are given by $(i = 1, 2)$

$$Q_i = \frac{\partial S}{\partial P_i} = \int \left(\frac{\partial f_a}{\partial P_i} da + \frac{\partial f_\phi}{\partial P_i} d\phi\right)\,. \tag{A7}$$

Moreover, from the definition of the new momenta, $C = P_1$ and $H = P_2$, we have the following relations

$$\begin{pmatrix} \frac{\partial C}{\partial p_a} & \frac{\partial C}{\partial p_\phi} \\ \frac{\partial H}{\partial p_a} & \frac{\partial H}{\partial p_\phi} \end{pmatrix} \begin{pmatrix} \frac{\partial f_a}{\partial P_1} & \frac{\partial f_a}{\partial P_2} \\ \frac{\partial f_\phi}{\partial P_1} & \frac{\partial f_\phi}{\partial P_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{A8}$$

Then, $\frac{\partial f_a}{\partial P_i}$ and $\frac{\partial f_\phi}{\partial P_i}$ in the integrand can be expressed as functions of a, ϕ, p_a and p_ϕ. In concretely Q_1 and Q_2 are written by

$$Q_1 = \int \frac{1}{\frac{\partial C}{\partial p_a}\frac{\partial H}{\partial p_\phi} - \frac{\partial C}{\partial p_\phi}\frac{\partial H}{\partial p_a}} \left(\frac{\partial H}{\partial p_\phi} da - \frac{\partial H}{\partial p_a} d\phi\right) \tag{A9}$$

$$Q_2 = \int \frac{1}{\frac{\partial C}{\partial p_a}\frac{\partial H}{\partial p_\phi} - \frac{\partial C}{\partial p_\phi}\frac{\partial H}{\partial p_a}} \left(-\frac{\partial C}{\partial p_\phi} da + \frac{\partial C}{\partial p_a} d\phi\right), \tag{A10}$$

and using Eq. (A2) and Eq. (A3), we obtain

$$Q_1 = \frac{12V_0}{(\sqrt{6}\alpha - 6)C^2} a^{6-\sqrt{6}\alpha} e^{(\alpha-\sqrt{6})\phi} + \frac{1}{\sqrt{6}\alpha} a^{\sqrt{6}\alpha} e^{\alpha\phi} = \text{const.}, \tag{A11}$$

$$Q_2 = \frac{6}{\alpha(2\alpha - \sqrt{6})C} a^{3-\sqrt{6}\alpha} e^{-\alpha\phi} = t + \text{const.}. \tag{A12}$$

We seek a particular solution by setting constants zero so that $Q_1 = 0$ and $Q_2 = t$. Then, from Eq. (A11), $e^\phi \propto a^{\sqrt{6}-2\alpha}$ and putting this into Eq. (A12), we find

$$a \propto t^{\frac{2}{(\sqrt{6}-2\alpha)^2}} = t^{\frac{2}{\lambda^2}}, \tag{A13}$$

$$\phi = \frac{2}{\sqrt{6} - 2\alpha} \ln t + \text{const.} = \frac{2}{\lambda} \ln t + \text{const.}, \tag{A14}$$

thus reproducing the well-known solutions for power-law inflation [1].

More detailed accounts will be given elsewhere [10].

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