# CENTRAL ELEMENTS OF THE DEGENERATE QUANTUM GENERAL LINEAR GROUP 

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#### Abstract

We construct central elements of the degenerate quantum general linear group introduced by Cheng, Wang and Zhang [1]. In particular, we give an explicit formula for the quantum Casimir element. Our method is based on the explicit $L$ operators. Moreover, we construct a universal $L$ operator, which is a spectral parameter-dependent solution of the quantum Yang-Baxter equation in the tensor product of the degenerate quantum general linear group and the endomorphism ring of its natural representation. This construction leads to the FRT approach to the degenerate quantum general linear group.


## 1. Introduction

Recently, Cheng, Wang and Zhang [1] introduced a class of new Hopf algebras called degenerate quantum groups. This may be thought as a degenerate version of the usual Drinfeld-Jimbo quantum groups [3, 4]. The origin of this idea traces back to the work of Zachos, where he studied symmetry properties of wave functions of quantum mechanical systems under the action of quantum $\mathfrak{s l}_{2}$ at $\sqrt{-1}$, resulting in an Hopf algebra structure $\mathrm{U}_{q}\left(\mathfrak{s l}_{1,1}\right)$ defined by (2.11). In type $A$, the degenerate quantum group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is obtained from the Drinfeld-Jimbo quantum group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m+n}\right)$ by replacing the subalgebra $\mathrm{U}_{q}\left(\mathfrak{s l}_{2}\right)$ associated to the $(m+1)$-th node of the Dynkin diagram of $\mathfrak{g l}_{m+n}$ with Zachos' algebra $\mathrm{U}_{q}\left(\mathfrak{s l}_{1,1}\right)$. Additionally, the associated Serre relations are appropriately modified during this process. This construction can be generalised to degenerate quantum groups of types $B, C$ and $D$ [1].

Intriguingly, the classification of finite dimensional simple modules of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is the same [1. Theorem 4.1] as that for the quantum general linear supergroup $\mathrm{U}_{q}\left(\mathfrak{g l}_{m \mid n}\right)$ at generic $q$ [13]. This connection is particularly remarkable, as $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is merely an ordinary Hopf algebra containing no odd subspace. Unlike quantum supergroups, $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ does not arise as a deformation of the universal enveloping algebra of any Lie superalgebra. Possible connections between $\mathrm{U}_{q}\left(\mathfrak{g l}_{m \mid n}\right)$ and $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, and between $B, C$ and $D$ types of quantum supergroups and degenerated quantum groups of the corresponding types [1, §6.1], were discussed in [1, §6.3]. It indicates that degenerated quantum groups may provide a new way to study the theory of quantum supergroups.

In this paper, we are concerned with the centre of the degenerate quantum general linear group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, which was poorly understood. Characterising the centre will deepen

[^0]understanding of representation theory and algebraic structure of $\mathrm{U}_{q}\left(\mathfrak{g r}_{m, n}\right)$. This has close connections to soluble lattice models in statistical mechanics and topological invariants in low-dimensional topology [1].

As a first step towards understanding the centre, we construct central elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ using the method of Zhang, Gould and Bracken [12, 15, 16]. This approach provides a systematic technique for constructing an infinite family of central elements associated with any finite-dimensional $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$-module. The construction is based on a commutation condition derived from the $L$ operators $L^{ \pm}$of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. We provide explicit constructions of these $L$ operators and, in particular, derive an explicit formula for the quantum Casimir element. A similar method has been successfully applied in previous studies [2, [5, 6, where explicit generators and relations were constructed for the centre of the Drinfeld-Jimbo quantum group.

We construct a spectral parameter-dependent operator $L(x), x \in \mathbb{C}$, and prove that it is a solution of the quantum Yang-Baxter equation in $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}(V)$, where $V$ is the natural module of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. These types of solutions are useful for constructing integrable lattice models of quantum systems [10]. In the course of proof, we essentially utilise the RLL relations, which motivate us to propose the FRT approach to the $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ [9. The spectral parameter-dependent operator and the FRT approach have been thoroughly studied for the quantum general linear supergroup [12, 14].

In our forthcoming research, we will establish the Harish-Chandra isomorphism for the centre of the degenerate quantum group. Subsequently, using the explicit construction given in this paper, we aim to obtain an algebraic description of the centre in terms of generators and relations. Our explicit $L$ operators will also shed light on the expression of the universal $R$-matrix of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, which is usually difficult to construct.

The paper is organised as follows. In Section 2, we recall the definition of the degenerate quantum general linear group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ and prove some useful commutation relations. In Section 3, we first give a general method to construct central elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, and then use $L$ operators to construct an explicit infinite family of central elements. Our main results are given in Theorem 3.1 and Theorem 3.6. In Section 4, we construct the spectral parameter-dependent universal $L$ operator $L(x)$ and propose the FRT approach to degenerate quantum general linear groups.

## 2. Degenerate quantum general linear group

Let $\mathbb{C}$ be the complex field and $\mathbb{Z}_{+}$be the set of non-negative integers. Throughout the paper, we work over $\mathcal{K}=\mathbb{C}(q)$, the field of rational functions in the indeterminate $q$. We fix a pair of positive integers $m, n$. Let $I=\{1,2, \ldots, m+n\}$ and $I^{\prime}=I \backslash\{m+n\}$. Put $p=-q^{-1}$, and let $q_{a}=q$ if $a \leq m, q_{a}=p$ if $a>m$.
2.1. The degenerate quantum general linear group. We recall the definition from [1]. The degenerate quantum general linear group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is a unital associative algebra over $\mathcal{K}$ generated by the elements $e_{a}, f_{a}, K_{b}, K_{b}^{-1}, a \in I^{\prime}, b \in I$, subject to the following relations

$$
\begin{align*}
& K_{a} K_{a}^{-1}=1, \quad K_{a}^{ \pm 1} K_{b}^{ \pm 1}=K_{b}^{ \pm 1} K_{a}^{ \pm 1},  \tag{2.1}\\
& K_{a} e_{b} K_{a}^{-1}=q_{a}^{\delta_{a b}-\delta_{a, b+1}} e_{b},  \tag{2.2}\\
& K_{a} f_{b} K_{a}^{-1}=q_{a}^{-\delta_{a b}+\delta_{a, b+1}} f_{b},  \tag{2.3}\\
& e_{a} f_{b}-f_{b} e_{a}=\delta_{a b} \frac{k_{a}-k_{a}^{-1}}{q_{a}-q_{a}^{-1}}, \text { with } k_{a}=K_{a} K_{a+1}^{-1},  \tag{2.4}\\
& e_{a} e_{b}=e_{b} e_{a}, \quad f_{a} f_{b}=f_{b} f_{a}, \quad|a-b|>1,  \tag{2.5}\\
& e_{a}^{2} e_{a \pm 1}-\left(q_{a}+q_{a}^{-1}\right) e_{a} e_{a \pm 1} e_{a}+e_{a \pm 1} e_{a}^{2}=0, \quad a \neq m  \tag{2.6}\\
& f_{a}^{2} f_{a \pm 1}-\left(q_{a}+q_{a}^{-1}\right) f_{a} f_{a \pm 1} f_{a}+f_{a \pm 1} f_{a}^{2}=0, \quad a \neq m  \tag{2.7}\\
& e_{m}^{2}=f_{m}^{2}=0,  \tag{2.8}\\
& e_{m} E_{m-1, m+2}-E_{m-1, m+2} e_{m}=0,  \tag{2.9}\\
& f_{m} E_{m+2, m-1}-E_{m+2, m-1} f_{m}=0, \tag{2.10}
\end{align*}
$$

where $E_{m-1, m+2}$ and $E_{m+2, m-1}$ are defined by

$$
\begin{aligned}
& E_{m-1, m+2}:=E_{m-1, m+1} e_{m+1}-q_{m+1}^{-1} e_{m+1} E_{m-1, m+1} \\
& E_{m+2, m-1}:=f_{m+1} E_{m+1, m-1}-q_{m+1} E_{m+1, m-1} f_{m+1} \\
& E_{m-1, m+1}:=e_{m-1} e_{m+1}-q_{m}^{-1} e_{m} e_{m-1} \\
& E_{m+1, m-1}:=f_{m} f_{m-1}-q_{m} f_{m-1} f_{m}
\end{aligned}
$$

Denote by $\mathrm{U}_{q}\left(\mathfrak{s l}_{m, n}\right)$ the subalgebra of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ generated by $k_{a}^{ \pm 1}, e_{a}$, $f_{a}$ for all $a \in I^{\prime}$.
Note that if $m=1$ or $n=1$, relations (2.9) and (2.10) do not exist. For any $a \neq m$, the triple $e_{a}, f_{a}, k_{a}^{ \pm 1}$ generates a copy of $\mathrm{U}_{q}\left(\mathfrak{s l}_{2}\right)$. If $a=m$, the elements $k_{m}^{ \pm}, e_{m}, f_{m}$ generate a subalgebra isomorphic to $\mathrm{U}_{q}\left(\mathfrak{s l}_{1,1}\right)$ with relations

$$
\begin{align*}
& k_{m} k_{m}^{-1}=1, \quad k_{m} e_{m} k_{m}^{-1}=-e_{m}, \quad k_{m} f_{m} k_{m}^{-1}=-f_{m} \\
& e_{m} f_{m}-f_{m} e_{m}=\frac{k_{m}-k_{m}^{-1}}{q-q^{-1}}, \quad e_{m}^{2}=f_{m}^{2}=0 \tag{2.11}
\end{align*}
$$

This is Zachos' algebra [11], which looks similar to the quantum supergroup $\mathrm{U}_{q}\left(\mathfrak{s l}_{1 \mid 1}\right)$. However, despite the resemblance, Zachos' algebra is not a quantum deformation of any Lie algebra or Lie superalgebra [1, 11].

As with the usual Drinfeld-Jimbo quantum group, the degenerate quantum group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ has a Hopf algebra structure with a coproduct $\Delta: \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, a counit $\epsilon: \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \mathcal{K}$, and an antipode $S: \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, which are defined,
respectively, by

$$
\begin{gathered}
\Delta\left(e_{a}\right)=e_{a} \otimes k_{a}+1 \otimes e_{a}, \quad \Delta\left(f_{a}\right)=f_{a} \otimes 1+k_{a}^{-1} \otimes f_{a}, \quad \Delta\left(K_{b}\right)=K_{b} \otimes K_{b}, \\
\epsilon\left(e_{a}\right)=\epsilon\left(f_{a}\right)=0, \quad \epsilon\left(K_{b}\right)=1
\end{gathered}
$$

and

$$
\begin{equation*}
S\left(e_{a}\right)=-e_{a} k_{a}^{-1}, \quad S\left(f_{a}\right)=-k_{a} f_{a}, \quad S\left(K_{b}\right)=K_{b}^{-1} \tag{2.12}
\end{equation*}
$$

for all $a \in I^{\prime}, b \in I$. We define the the opposite coproduct $\Delta^{\prime}$ by

$$
\Delta^{\prime}:=P \Delta
$$

where $P\left(u_{1} \otimes u_{2}\right)=u_{2} \otimes u_{1}$ for any $u_{1}, u_{2} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$.
Let $V=\mathcal{K}^{m+n}$, and let $v_{a} \in V(a \in I)$ be the column vector with 1 at the $a$-th entry and 0 elsewhere. Let $e_{a b}(a, b \in I)$ be the matrix units such that $e_{a b} v_{c}=\delta_{b c} v_{a}$ for all $a, b, c \in I$. Then there is a natural representation $\pi: \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \operatorname{End}_{\mathcal{K}}(V)$ [1, Lemma 4.3], defined by

$$
\begin{equation*}
\pi\left(e_{a}\right)=e_{a, a+1}, \quad \pi\left(f_{a}\right)=e_{a+1, a}, \quad \pi\left(K_{b}\right)=1+\left(q_{b}-1\right) e_{b b}, \quad a \in I^{\prime}, b \in I \tag{2.13}
\end{equation*}
$$

For any $u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, we have

$$
u v_{a}=\sum_{b \in I} \pi(u)_{b a} v_{b}, \quad a \in I
$$

The classification of finite dimensional simple $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$-modules is essentially the same as that for the quantum general linear supergroup $\mathrm{U}_{q}\left(\mathfrak{g l}_{m \mid n}\right)$ (or $\mathfrak{g l}_{m \mid n}$ ) [1, Theorem 4.1], despite the fact that $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is not a quantum deformation of any Lie (super) algebra.
2.2. Commutation relations. For any $a, b \in I$, let

$$
E_{a, a+1}=\bar{E}_{a, a+1}=e_{a}, \quad E_{a+1, a}=\bar{E}_{a+1, a}=f_{a}
$$

We define recursively the following elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ :

$$
E_{a b}= \begin{cases}E_{a c} E_{c b}-q_{c}^{-1} E_{c b} E_{a c}, & a<c<b,  \tag{2.14}\\ E_{a c} E_{c b}-q_{c} E_{c b} E_{a c}, & b<c<a,\end{cases}
$$

and

$$
\bar{E}_{a b}= \begin{cases}\bar{E}_{a c} \bar{E}_{c b}-q_{c} \bar{E}_{c b} \bar{E}_{a c}, & a<c<b,  \tag{2.15}\\ \bar{E}_{a c} \bar{E}_{c b}-q_{c}^{-1} \bar{E}_{c b} \bar{E}_{a c}, & b<c<a\end{cases}
$$

We will give some commutation relations among these elements. These relations will be used in the construction of central elements and the universal $L$ operators of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$.

Lemma 2.1. For any $a \neq b \in I$, the following relations hold in $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ :

$$
\begin{align*}
& K_{c} E_{a b}=E_{a b} K_{c}, \quad c \neq a, b,  \tag{2.16}\\
& K_{a} E_{a b}=q_{a} E_{a b} K_{a}, \quad K_{b} E_{a b}=q_{b}^{-1} E_{a b} K_{b} \tag{2.17}
\end{align*}
$$

These relations similarly apply to $\bar{E}_{a b}$ mutatis mutandis.

Proof. We prove the first relation. We may assume that $a<b$, and the case $a>b$ can be treated similarly. By definition (2.14) $E_{a b}$ (resp. $E_{b a}$ ) is a linear combination of elements of the form $e_{i_{1}} e_{i_{2}} \cdots e_{i_{b-a}}$ (resp. $f_{i_{1}} f_{i_{2}} \cdots f_{i_{b-a}}$ ), where $i_{1}, i_{2}, \ldots, i_{b-a}$ form a permutation of $a, a+1, \ldots, b-1$. Assuming that $a<c<b$, we obtain $K_{c} e_{c-1}=q_{c}^{-1} e_{c-1} K_{c}, K_{c} e_{c}=q_{c} e_{c} K_{c}$ and $K_{c} e_{i}=e_{i} K_{c}$ for $i \neq c, c-1$. Then $K_{c} E_{a b}=E_{a b} K_{c}$ for $a<c<b$. Similarly, (2.16) holds for the cases $a<b<c$ and $c<a<b$. Relations in (2.17) can be proved similarly.

Lemma 2.2. We have the following relations:

$$
\begin{align*}
& {\left[e_{a}, E_{b a}\right]=-k_{a} E_{b, a+1}, \quad a+1<b}  \tag{2.18}\\
& {\left[f_{a}, E_{a b}\right]=E_{a+1, b} k_{a}^{-1}, \quad a+1<b}  \tag{2.19}\\
& {\left[e_{b}, E_{b+1, a}\right]=E_{b a} k_{b}^{-1}, \quad a<b}  \tag{2.20}\\
& {\left[f_{b}, E_{a, b+1}\right]=-k_{b} E_{a b}, \quad a<b,}  \tag{2.21}\\
& {\left[e_{a}, E_{b c}\right]=\left[f_{a}, E_{c b}\right]=0, \quad b<c,\{b, c\} \neq\{a, a+1\}} \tag{2.22}
\end{align*}
$$

Proof. Let us prove (2.18) first. For any $a+1<b$, we have

$$
\begin{aligned}
{\left[e_{a}, E_{b a}\right] } & =\left[e_{a}, E_{b, a+1} E_{a+1, a}-q_{a+1} E_{a+1, a} E_{b, a+1}\right] \\
& =\left[e_{a}, E_{b, a+1}\right] f_{a}+E_{b, a+1}\left[e_{a}, f_{a}\right]-q_{a+1}\left(\left[e_{a}, f_{a}\right] E_{b, a+1}+f_{a}\left[e_{a}, E_{b, a+1}\right]\right)
\end{aligned}
$$

We can prove by induction that $\left[e_{a}, E_{b, a+1}\right]=0$. It follows that

$$
\left[e_{a}, E_{b a}\right]=E_{b, a+1} \frac{k_{a}-k_{a}^{-1}}{q_{a}-q_{a}^{-1}}-q_{a+1} \frac{k_{a}-k_{a}^{-1}}{q_{a}-q_{a}^{-1}} E_{b, a+1}=-k_{a} E_{b, a+1}
$$

where in the last equation we have used Lemma 2.1. Relations (2.19), (2.20) and (2.21) can be proved similarly. The proof of $\left[f_{a}, E_{c b}\right]=0$ in (2.22) can be found in Lemma 3.8 [1], the first relation in (2.22) can be proved similarly.

Lemma 2.3. The following relations hold:

$$
\begin{align*}
& E_{c a}^{2}=0, \quad a \leq m<c,  \tag{2.23}\\
& {\left[E_{a b}, E_{c d}\right]=0, \quad b<a<d<c \text { or } d<b<a<c}  \tag{2.24}\\
& E_{a c} E_{b c}=q_{c} E_{b c} E_{a c}, \quad E_{c a} E_{c b}=q_{c} E_{c b} E_{c a}, \quad c<a<b,  \tag{2.25}\\
& E_{c a} E_{c b}=q_{c} E_{c b} E_{c a}, \quad E_{a c} E_{b c}=q_{c} E_{b c} E_{a c}, \quad a<b<c  \tag{2.26}\\
& {\left[E_{a b}, E_{c d}\right]=\left(q-q^{-1}\right) E_{c b} E_{a d}, \quad b<d<a<c \text { or } a<c<b<d,}  \tag{2.27}\\
& {\left[E_{a b}, E_{b c}\right]_{q_{b}^{-1}}=E_{a c}, \quad\left[E_{c b}, E_{b a}\right]_{q_{b}}=E_{c a} \quad a<b<c .} \tag{2.28}
\end{align*}
$$

Proof. The proof of relations (2.23), (2.24) and the first relations of (2.25) and (2.26), (2.27) can be found in [1, Lemma 3.9]. The second relations of (2.25) and (2.26) can be proved by using similar method as the first relations of (2.25) and (2.26), respectively. From the fact that $e_{a} e_{c}=e_{c} e_{a}$ and $f_{a} f_{c}=f_{c} f_{a}$ for $|a-c|>1$, we can easily prove (2.28).

Lemma 2.4. For $a>b$, we have

$$
S\left(E_{a b}\right)=-K_{a}^{-1} K_{b} \bar{E}_{a b}, \quad{ }_{5} S\left(E_{b a}\right)=-\bar{E}_{b a} K_{a} K_{b}^{-1} .
$$

Proof. We prove the first equation by using induction on $a-b$, and similarly for the second. If $a-b=1$, by (2.12) we have

$$
S\left(E_{a, a-1}\right)=-k_{a-1} f_{a-1}=-K_{a}^{-1} K_{a-1} \bar{E}_{a, a-1}
$$

In general, using definition (2.14) we obtain that

$$
\begin{aligned}
& S\left(E_{a b}\right)=S\left(E_{a-1, b}\right) S\left(E_{a, a-1}\right)-q_{a-1} S\left(E_{a, a-1}\right) S\left(E_{a-1, b}\right) \\
= & K_{a-1}^{-1} K_{b} \bar{E}_{a-1, b} K_{a}^{-1} K_{a-1} \bar{E}_{a, a-1}-q_{a-1} K_{a}^{-1} K_{a-1} \bar{E}_{a, a-1} K_{a-1}^{-1} K_{b} \bar{E}_{a-1, b} \\
= & q_{a-1}^{-1} K_{a}^{-1} K_{b} \bar{E}_{a-1, b} \bar{E}_{a, a-1}-K_{a}^{-1} K_{b} \bar{E}_{a, a-1} \bar{E}_{a-1, b} \\
= & -K_{a}^{-1} K_{b} \bar{E}_{a b},
\end{aligned}
$$

where the second equation follows from the induction hypothesis and the third equation is a consequence of Lemma 2.1.

## 3. Central elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$

3.1. A general construction. Although $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is not a quantum deformation of any existing Lie algebra or Lie superalgebra, we will show in this section that the method for constructing central elements for quantum (super) groups, as given in [2, 15], is applicable to $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. This is made possible due to its Hopf algebra structure.

To start with, let $M$ be an arbitrary finite dimensional $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$-module, and let $\zeta$ : $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \operatorname{End}_{\mathcal{K}}(M)$ be the corresponding representation of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. Denote by $\operatorname{Tr}_{2}$ the partial trace on the second tensor factor of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(M)$, i.e.,

$$
\operatorname{Tr}_{2}(u \otimes A)=\operatorname{Tr}(A) u, \quad \forall u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right), A \in \operatorname{End}_{\mathcal{K}}(M)
$$

where $\operatorname{Tr}$ denotes the usual trace.
An important ingredient in the construction is the element $K_{2 \rho}$. In the context of quantum (super) groups, $2 \rho$ denotes the sum of positive roots and $K_{2 \rho}$ is merely a product of $K_{\alpha}$ over all positive roots $\alpha$. However, when dealing with $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, it is crucial to define $K_{2 \rho}$ properly to ensure consistency with the antipode $S$. Recall from [1, Lemma 5.1] that $K_{2 \rho} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is defined by

$$
K_{2 \rho}= \begin{cases}K_{2 \rho}^{\prime}, & \text { if } m+n \text { is even }  \tag{3.1}\\ K_{2 \rho}^{\prime} K^{\prime}, & \text { if } m+n \text { is odd }\end{cases}
$$

where

$$
K_{2 \rho}^{\prime}=\prod_{a=1}^{m} K_{a}^{m-n+1-2 a} \prod_{b=1}^{n} K_{m+b}^{m+n+1-2 b}, \quad K^{\prime}=\prod_{a=1}^{m} K_{a} \prod_{b=1}^{n} K_{m+b}^{-1}
$$

This invertible element satisfies

$$
S^{2}(u)=K_{2 \rho} u K_{2 \rho}^{-1}, \quad \forall u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right),
$$

which is analogous to the usual quantum group case. In particular, it is straightforward to check the following useful relations:

$$
\begin{align*}
K_{2 \rho}^{\prime} e_{a} K_{2 \rho}^{\prime-1} & =k_{a} e_{a} k_{a}^{-1}=q_{a}^{2} e_{a}, \quad \forall a \neq m, \\
K^{\prime} e_{a} K^{\prime-1} & =e_{a}, \quad \forall a \neq m,  \tag{3.2}\\
K_{2 \rho} e_{m} K_{2 \rho}^{-1} & =k_{m} e_{m} k_{m}^{-1}=-e_{m} .
\end{align*}
$$

Now we construct central elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ as follows.
Theorem 3.1. Let $\zeta: \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \rightarrow \operatorname{End}_{\mathcal{K}}(M)$ be a representation, and let $\Gamma_{M} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes$ $\operatorname{End}_{\mathcal{K}}(M)$ be an element satisfying

$$
\left[\Gamma_{M},(1 \otimes \zeta) \Delta(u)\right]=0, \quad \forall u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)
$$

Then the elements

$$
C_{k}=\operatorname{Tr}_{2}\left(\left(1 \otimes \zeta\left(K_{2 \rho}\right)\right) \Gamma_{M}^{k}\right), \quad k \geq 1
$$

belong to the centre of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$.
Proof. For any $k \geq 1$, it suffices to show that $C_{k}$ commutes with the generators $e_{a}, f_{a}, K_{b}^{ \pm 1}$, $a \in I^{\prime}, b \in I$. Let $\Gamma_{M}^{k}=\sum_{i} x_{i} \otimes \zeta\left(y_{i}\right)$ be a finite sum, where $x_{i}, y_{i} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. Then for any $a \in I^{\prime}$, we have $\left[\Gamma_{M}^{k},(1 \otimes \zeta) \Delta\left(e_{a}\right)\right]=0$. It follows that

$$
\begin{aligned}
0 & =\operatorname{Tr}_{2}\left(\left(1 \otimes \zeta\left(k_{a}^{-1} K_{2 \rho}\right)\right)\left[\Gamma_{M}^{k},(\operatorname{id} \otimes \zeta) \Delta\left(e_{a}\right)\right]\right) \\
& =\sum_{i} \operatorname{Tr}_{2}\left(\left(1 \otimes \zeta\left(k_{a}^{-1} K_{2 \rho}\right)\right)\left[x_{i} \otimes \zeta\left(y_{i}\right), e_{a} \otimes \zeta\left(k_{a}\right)+1 \otimes \pi\left(e_{a}\right)\right]\right) \\
& =\sum_{i}\left(x_{i} e_{a}-e_{a} x_{i}\right) \operatorname{Tr}\left(\zeta\left(K_{2 \rho} y_{i}\right)\right)+\sum_{i} x_{i}\left(\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} y_{i} e_{a}\right)\right)-\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} e_{a} y_{i}\right)\right)\right) \\
& =\left[C_{k}, e_{a}\right]+\sum_{i} x_{i}\left(\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} y_{i} e_{a}\right)\right)-\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} e_{a} y_{i}\right)\right)\right)
\end{aligned}
$$

We will show that the sum in the last equation equals zero. This implies that $\left[C_{k}, e_{a}\right]=0$, proving that $C_{k}$ commutes with $e_{a}$. Note that

$$
\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} y_{b} e_{a}\right)\right)=\operatorname{Tr}\left(\zeta\left(e_{a} k_{a}^{-1} K_{2 \rho} y_{b}\right)\right)= \begin{cases}-\operatorname{Tr}\left(\zeta\left(k_{m}^{-1} e_{m} K_{2 \rho} y_{b}\right)\right), & \text { if } a=m \\ q_{a}^{2} \operatorname{Tr}\left(\zeta\left(k_{a}^{-1} e_{a} K_{2 \rho} y_{b}\right)\right), & \text { if } a \neq m\end{cases}
$$

If $a=m$, by the last equation of (3.2) we have $e_{m} K_{2 \rho}=-e_{m} K_{2 \rho}$, whence

$$
\operatorname{Tr}\left(\zeta\left(k_{m}^{-1} K_{2 \rho} y_{b} e_{m}\right)\right)=-\operatorname{Tr}\left(\zeta\left(k_{m}^{-1} e_{m} K_{2 \rho} y_{b}\right)\right)=\operatorname{Tr}\left(\zeta\left(k_{m}^{-1} K_{2 \rho} e_{m} y_{b}\right)\right)
$$

Therefore, we have $\left[C_{k}, e_{m}\right]=0$. If $a \neq m$, then

$$
e_{a} K_{2 \rho}= \begin{cases}e_{a} K_{2 \rho}^{\prime}=q_{a}^{-2} K_{2 \rho}^{\prime}=q_{a}^{-2} K_{2 \rho}, & \text { if } m+n \text { is even }, \\ e_{a} K_{2 \rho}^{\prime} K^{\prime}=q_{a}^{-2} K_{2 \rho}^{\prime} K^{\prime}=q_{a}^{-2} K_{2 \rho}, & \text { if } m+n \text { is odd }, \\ 7 & \end{cases}
$$

where we have used relations in (3.2). It follows that

$$
\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} y_{b} e_{a}\right)\right)=q_{a}^{2} \operatorname{Tr}\left(\zeta\left(k_{a}^{-1} e_{a} K_{2 \rho} y_{b}\right)\right)=\operatorname{Tr}\left(\zeta\left(k_{a}^{-1} K_{2 \rho} e_{a} y_{b}\right)\right)
$$

Thus, we obtain $\left[C_{k}, e_{a}\right]=0$ for $a \neq m$. Similarly, one can prove that $\left[C_{k}, f_{a}\right]=\left[C_{k}, K_{b}\right]=0$ for $1 \leq a \leq m+n-1$ and $1 \leq b \leq m+n$. Therefore, the elements $C_{k}$ are central in $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$.
3.2. Explicit formulae for central elements. We will construct an explicit element $\Gamma_{M}$ which satisfies the condition in Theorem [3.1, In this way, we obtain corresponding central elements $C_{k}$ for all $k \geq 1$. In the following, we are concerned with $M=V=\mathcal{K}^{m+n}$, the natural representation of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ as given in (2.13).

We define the following two elements of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V)$ :

$$
\begin{align*}
L^{+} & =\sum_{a \in I} K_{a} \otimes e_{a a}+\left(q-q^{-1}\right) \sum_{a<b} K_{b} E_{a b} \otimes e_{b a},  \tag{3.3}\\
L^{-} & =\sum_{a \in I} K_{a}^{-1} \otimes e_{a a}-\left(q-q^{-1}\right) \sum_{a<b} E_{b a} K_{b}^{-1} \otimes e_{a b} . \tag{3.4}
\end{align*}
$$

The element $L^{-}$has the inverse given as follows.
Lemma 3.2. Recall that $S$ is the antipode of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ given by (2.12). Then

$$
\begin{equation*}
\left(L^{-}\right)^{-1}=(S \otimes 1)\left(L^{-}\right)=\sum_{a \in I} K_{a} \otimes e_{a a}+\left(q-q^{-1}\right) \sum_{a<b} K_{a} \bar{E}_{b a} \otimes e_{a b} \tag{3.5}
\end{equation*}
$$

Proof. Using Lemma 2.4, we obtain

$$
\begin{aligned}
(S \otimes 1)\left(L^{-}\right) & =\sum_{a \in I} K_{a} \otimes e_{a a}-\left(q-q^{-1}\right) \sum_{a<b} S\left(E_{b a} K_{b}^{-1}\right) \otimes e_{a b} \\
& =\sum_{a \in I} K_{a} \otimes e_{a a}+\left(q-q^{-1}\right) \sum_{a<b} K_{a} \bar{E}_{b a} \otimes e_{a b} .
\end{aligned}
$$

This proves the second equation. Next we check directly that $(S \otimes 1)\left(L^{-}\right)$is the inverse of $L^{-}$. Note that

$$
L^{-}\left((S \otimes 1)\left(L^{-}\right)\right)=1 \otimes 1+\left(q-q^{-1}\right) \sum_{a<b}\left(\bar{E}_{b a}-E_{b a}-\left(q-q^{-1}\right) \sum_{c=a+1}^{b-1} E_{c a} \bar{E}_{b c}\right) \otimes e_{a b} .
$$

We claim that

$$
\bar{E}_{b a}=E_{b a}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-1} E_{c a} \bar{E}_{b c}, \quad a<b
$$

and hence $L^{-}\left((S \otimes 1)\left(L^{-}\right)\right)=1 \otimes 1$. To prove this claim, we use induction on $b-a$. If $b=a+1$, then $\bar{E}_{b a}=E_{b a}=f_{a}$ holds true by definition. Assume that the claim holds for
$\bar{E}_{b-1, a}$ for $b-a>1$. Then we obtain

$$
\begin{aligned}
& \bar{E}_{b a}=\bar{E}_{b, b-1} \bar{E}_{b-1, a}-q_{b-1}^{-1} \bar{E}_{b-1, a} \bar{E}_{b, b-1} \\
= & E_{b, b-1}\left(E_{b-1, a}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-2} E_{c a} \bar{E}_{b-1, c}\right)-q_{b-1}^{-1}\left(E_{b-1, a}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-2} E_{c a} \bar{E}_{b-1, c}\right) E_{b, b-1} \\
= & E_{b, b-1} E_{b-1, a}-q_{b-1}^{-1} E_{b, b-1} E_{b-1, a}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-2} E_{c a}\left(E_{b, b-1} \bar{E}_{b-1, c}-q_{b-1}^{-1} \bar{E}_{b-1, c} E_{b, b-1}\right) \\
= & E_{b a}+\left(q_{b-1}-q_{b-1}^{-1}\right) E_{b-1, a} E_{b, b-1}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-2} E_{c a} \bar{E}_{b c} \\
= & E_{b a}+\left(q-q^{-1}\right) \sum_{c=a+1}^{b-1} E_{c a} \bar{E}_{b c} .
\end{aligned}
$$

Therefore, the claim holds. Similarly, one can prove that $\left((S \otimes 1)\left(L^{-}\right)\right) L^{-}=1 \otimes 1$, and hence $(S \otimes 1)\left(L^{-}\right)$is the inverse of $L^{-}$.

Recall that $\pi$ is the natural representation of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$. The elements $L^{ \pm}$have the following important property.

Lemma 3.3. For any $u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, we have

$$
\begin{equation*}
L^{ \pm}(1 \otimes \pi)(\Delta(u))=(1 \otimes \pi)\left(\Delta^{\prime}(u)\right) L^{ \pm} \tag{3.6}
\end{equation*}
$$

Proof. We introduce the following elements:

$$
\begin{aligned}
& \widetilde{L}^{+}:=1 \otimes 1+\left(q-q^{-1}\right) \sum_{a<b} E_{a b} \otimes e_{b a}, \\
& \widetilde{L}^{-}:=1 \otimes 1-\left(q-q^{-1}\right) \sum_{a<b} E_{b a} \otimes e_{a b},
\end{aligned}
$$

which are related to $L^{ \pm}$by

$$
L^{+}=\left(\sum_{a \in I} K_{a} \otimes e_{a a}\right) \widetilde{L}^{+}, \quad L^{-}=\widetilde{L}^{-}\left(\sum_{a \in I} K_{a}^{-1} \otimes e_{a a}\right) .
$$

To prove (3.6), it suffices to show the following equations:

$$
\begin{align*}
\widetilde{L}^{+}\left(E_{c, c+1} \otimes \pi\left(k_{c}\right)+1 \otimes e_{c, c+1}\right) & =\left(E_{c, c+1} \otimes \pi\left(k_{c}^{-1}\right)+1 \otimes e_{c, c+1}\right) \widetilde{L}^{+},  \tag{3.7}\\
\widetilde{L}^{+}\left(E_{c+1, c} \otimes 1+k_{c}^{-1} \otimes e_{c+1, c}\right) & =\left(E_{c+1, c} \otimes 1+k_{c} \otimes e_{c+1, c}\right) \widetilde{L}^{+},  \tag{3.8}\\
\widetilde{L}^{-}\left(E_{c, c+1} \otimes 1+k_{c}^{-1} \otimes e_{c, c+1}\right) & =\left(E_{c, c+1} \otimes 1+k_{c} \otimes e_{c, c+1}\right) \widetilde{L}^{-},  \tag{3.9}\\
\widetilde{L}^{-}\left(E_{c+1, c} \otimes \pi\left(k_{c}\right)+1 \otimes e_{c+1, c}\right) & =\left(E_{c+1, c} \otimes \pi\left(k_{c}^{-1}\right)+1 \otimes e_{c+1, c}\right) \widetilde{L}^{-} . \tag{3.10}
\end{align*}
$$

We will only prove (3.7) and (3.8), the other two follow similarly. Consider (3.7), and define

$$
\begin{aligned}
& A_{1}=\widetilde{L}^{+}\left(E_{c, c+1} \otimes \pi\left(k_{c}\right)\right)-\left(E_{c, c+1} \otimes \pi\left(k_{c}^{-1}\right)\right) \widetilde{L}^{+} \\
& A_{2}=\left[\widetilde{L}^{+},\left(1 \otimes e_{c, c+1}\right)\right] .
\end{aligned}
$$

Then (3.7) is equivalent to $A_{1}+A_{2}=0$. Note that

$$
A_{1}=E_{c, c+1} \otimes\left(\pi\left(k_{c}\right)-\pi\left(k_{c}^{-1}\right)\right)+\left(q-q^{-1}\right) \sum_{a<b} A_{a b}
$$

where

$$
A_{a b}=E_{a b} E_{c, c+1} \otimes e_{b a} \pi\left(k_{c}\right)-E_{c, c+1} E_{a b} \otimes \pi\left(k_{c}^{-1}\right) e_{b a}
$$

By (2.13), we have

$$
\begin{aligned}
e_{b a} \pi\left(k_{c}\right) & =e_{b a}+\left(q_{c}-1\right) \delta_{a c} e_{b c}+\left(q_{c+1}^{-1}-1\right) \delta_{a, c+1} e_{b, c+1}, \\
\pi\left(k_{c}^{-1}\right) e_{b a} & =e_{b a}+\left(q_{c}^{-1}-1\right) \delta_{b c} e_{c a}+\left(q_{c+1}-1\right) \delta_{b, c+1} e_{c+1, a} .
\end{aligned}
$$

It follows that

$$
\sum_{a<b} A_{a b}=\sum_{a<c, b=c, c+1} A_{a b}+\sum_{b>c+1, a=c, c+1} A_{a b}+\sum_{a=c, b=c+1} A_{a b} .
$$

Applying (2.14) and Lemma 2.3, we obtain

$$
\begin{aligned}
\sum_{a<c, b=c, c+1} A_{a b}= & \sum_{a<c}\left(E_{a c} E_{c, c+1} \otimes e_{c a}-E_{c, c+1} E_{a c} \otimes q_{c}^{-1} e_{c a}\right) \\
& +\sum_{a<c}\left(E_{a, c+1} E_{c, c+1} \otimes e_{c+1, a}-E_{c, c+1} E_{a, c+1} \otimes q_{c+1} e_{c+1, a}\right) \\
= & \sum_{a<c} E_{a, c+1} \otimes e_{c a}
\end{aligned}
$$

Similarly, one can deduce that

$$
\begin{aligned}
\sum_{b>c+1, a=c, c+1} A_{a b} & =-\sum_{b>c+1} E_{c b} \otimes e_{b, c+1} \\
\sum_{a=c, b=c+1} A_{a b} & =\left(q_{c}-q_{c+1}\right) E_{c, c+1}^{2} \otimes e_{c+1, c}=0
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
A_{1} & =E_{c, c+1} \otimes\left(\pi\left(k_{c}\right)-\pi\left(k_{c}^{-1}\right)\right)+\left(q-q^{-1}\right) \sum_{a<b} A_{a b} \\
& =\left(q-q^{-1}\right) E_{c, c+1} \otimes\left(e_{c c}-e_{c+1, c+1}\right)+\left(q-q^{-1}\right)\left(\sum_{a<c} E_{a, c+1} \otimes e_{c a}-\sum_{b>c+1} E_{c b} \otimes e_{b, c+1}\right) .
\end{aligned}
$$

On the other hand, we have

$$
A_{2}=\left[\widetilde{L}^{+},\left(1 \otimes e_{c, c+1}\right)\right]=\left(q-q^{-1}\right)\left(\sum_{\substack{b>c \\ 10}} E_{c b} \otimes e_{b, c+1}-\sum_{a<c} E_{a, c+1} \otimes e_{c a}\right) .
$$

It is clear that $A_{1}+A_{2}=0$, proving (3.7).
We proceed to prove (3.8). Define

$$
B_{1}=\left[\widetilde{L}^{+}, E_{c+1, c} \otimes 1\right]=-\left(q-q^{-1}\right) \sum_{a<b}\left[E_{c+1, c}, E_{a b}\right] \otimes e_{b a}
$$

Applying Lemma 2.2, we obtain

$$
\begin{aligned}
B_{1} & =-\left(q-q^{-1}\right)\left(\left[E_{c+1, c}, E_{c, c+1}\right] \otimes e_{c+1, c}+\sum_{a<c}\left[E_{c+1, c}, E_{a, c+1}\right] \otimes e_{c+1, a}+\sum_{b>c+1}\left[E_{c+1, c},, E_{c b}\right] \otimes e_{b c}\right) \\
& =-\left(q-q^{-1}\right)\left(-\frac{k_{c}-k_{c}^{-1}}{q_{c}-q_{c}^{-1}} \otimes e_{c+1, c}+\sum_{a<c}-k_{c} E_{a c} \otimes e_{c+1, a}+\sum_{b>c+1} E_{c+1, b} k_{c}^{-1} \otimes e_{b c}\right) \\
& =\left(k_{c}-k_{c}^{-1}\right) \otimes e_{c+1, c}+\left(q-q^{-1}\right)\left(\sum_{a<c} k_{c} E_{a c} \otimes e_{c+1, a}-\sum_{b>c+1} E_{c+1, b} k_{c}^{-1} \otimes e_{b c}\right) .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
B_{2} & =\widetilde{L}^{+}\left(k_{c}^{-1} \otimes e_{c+1, c}\right)-\left(k_{c} \otimes e_{c+1, c}\right) \widetilde{L}^{+} \\
& =\left(k_{c}^{-1}-k_{c}\right) \otimes e_{c+1, c}+\left(q-q^{-1}\right)\left(-\sum_{a<c} k_{c} E_{a c} \otimes e_{c+1, a}+\sum_{b>c+1} E_{c+1, b} k_{c}^{-1} \otimes e_{b c}\right) .
\end{aligned}
$$

Clearly, $B_{1}+B_{2}=0$, which is equivalent to (3.8). This finishes the proof of the lemma.
Now we define the following element $\Gamma_{V} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V)$ associated to the natural representation $V$ :

$$
\begin{equation*}
\Gamma_{V}:=\left(L^{-}\right)^{-1} L^{+} . \tag{3.11}
\end{equation*}
$$

Lemma 3.4. For any $u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, we have

$$
\left[\Gamma_{V},(1 \otimes \pi)(\Delta(u))\right]=0
$$

Proof. Using Lemma 3.3, we have

$$
\left(L^{-}\right)^{-1} L^{+}(1 \otimes \pi)(\Delta(u))=\left(L^{-}\right)^{-1}(1 \otimes \pi)\left(\Delta^{\prime}(u)\right) L^{+}=(1 \otimes \pi)(\Delta(u))\left(L^{-}\right)^{-1} L^{+}
$$

This completes the proof.
Combining Lemma 3.4 and Theorem 3.1, we obtain explicit central elements $C_{k}$ of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ associated to $V$ for any $k \geq 1$.

Example 3.5. Let $\mathrm{U}_{q}\left(\mathfrak{g l}_{1,1}\right)$ be the degenerate quantum group generated by $K_{1}^{ \pm}, K_{2}^{ \pm}, E=$ $E_{12}$ and $F=E_{21}$, subject to the following relations:

$$
\begin{aligned}
& K_{1} E K_{1}^{-1}=q E, \quad K_{2} E K_{2}^{-1}=-q E, \\
& K_{1} F K_{1}^{-1}=q^{-1} F, \quad K_{2} F K_{2}^{-1}=-q^{-1} F, \\
& E F-F E=\frac{K_{1} K_{2}^{-1}-K_{1}^{-1} K_{2}}{q-q^{-1}},
\end{aligned}
$$

$$
E^{2}=F^{2}=0 .
$$

Then using the element $\Gamma_{V}$ defined by (3.11) and Theorem 3.1] we obtain a central element

$$
C_{1}=q^{-1} K_{1}^{2}-q^{-1} K_{2}^{2}-\left(q-q^{-1}\right)^{2} K_{1} K_{2} F E .
$$

This is analogous to the quantum Casimir element of $\mathrm{U}_{q}\left(\mathfrak{g l}_{2}\right)$.
To obtain a more concise formula for the central elements, we introduce a variant of $\Gamma_{V}$ defined by

$$
\Gamma:=\frac{\Gamma_{V}-1 \otimes 1}{q-q^{-1}} .
$$

This still satisfies the commutation relation in Lemma 3.4 and will be used to construct the central elements as described below.

We introduce some notation. Define the following elements:

$$
\begin{align*}
& X_{a b}=K_{b} E_{a b}, \quad X_{b a}=K_{a} \bar{E}_{b a}, \quad a<b, \\
& X_{a a}=\frac{K_{a}-1}{q-q^{-1}}, \quad a \in I . \tag{3.12}
\end{align*}
$$

Then we can rewrite $L^{+}$and $\left(L^{-}\right)^{-1}$ defined by (3.3) and (3.5) as

$$
\begin{aligned}
& L^{+}=1 \otimes 1+\left(q-q^{-1}\right) \sum_{a \leq b} X_{a b} \otimes e_{b a}, \\
& \left(L^{-}\right)^{-1}=1 \otimes 1+\left(q-q^{-1}\right) \sum_{a \geq b} X_{a b} \otimes e_{b a} .
\end{aligned}
$$

It follows that

$$
\Gamma=\frac{\left(L^{-}\right)^{-1} L^{+}-1 \otimes 1}{q-q^{-1}}=\sum_{a, b \in I} X_{a b} \otimes e_{b a}+\left(q-q^{-1}\right) \sum_{a \geq b, a \geq c} X_{a b} X_{c a} \otimes e_{b c} .
$$

The following is a consequence of Lemma 3.4 and Theorem 3.1.
Theorem 3.6. For $k \in \mathbb{Z}_{+}$, the elements

$$
C_{k}=\operatorname{Tr}_{2}\left(\left(1 \otimes \pi\left(K_{2 \rho}\right)\right)\left(\sum_{a, b \in I} X_{a b} \otimes e_{b a}+\left(q-q^{-1}\right) \sum_{a \geq b, a \geq c} X_{a b} X_{c a} \otimes e_{b c}\right)^{k}\right)
$$

lie in the centre of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, where $\pi$ is the natural representation defined by (2.13), and $K_{2 \rho}$ and $X_{a b}$ are defined by (3.1) and (3.12), respectively.

Using Theorem 3.6, it is straightforward to check that the central element $C_{1}$ associated to $\Gamma$ is given by the following formula: if $m+n$ is even,

$$
\begin{aligned}
C_{1}= & \sum_{a=1}^{m} q_{a}^{m-n+1-2 a} \frac{K_{a}-1}{q-q^{-1}}+\sum_{b=m+1}^{m+n} q_{b}^{3 m+n+1-2 b} \frac{K_{b}-1}{q-q^{-1}} \\
& +\left(q-q^{-1}\right)\left(\sum_{a \geq b, b \leq m} q_{b}^{m-n+1-2 b} K_{b} \bar{E}_{a b} K_{a} E_{b a}+\sum_{a \geq b, b>m} q_{b}^{3 m+n+1-2 b} K_{b} \bar{E}_{a b} K_{a} E_{b a}\right),
\end{aligned}
$$

and if $m+n$ is odd, then

$$
\begin{aligned}
C_{1}= & \sum_{a=1}^{m} q_{a}^{m-n+2-2 a} \frac{K_{a}-1}{q-q^{-1}}+\sum_{b=m+1}^{m+n} q_{b}^{3 m+n-2 b} \frac{K_{b}-1}{q-q^{-1}} \\
& +\left(q-q^{-1}\right)\left(\sum_{a \geq b, b \leq m} q_{b}^{m-n+2-2 b} K_{b} \bar{E}_{a b} K_{a} E_{b a}+\sum_{a \geq b, b>m} q_{b}^{3 m+n-2 b} K_{b} \bar{E}_{a b} K_{a} E_{b a}\right) .
\end{aligned}
$$

We call element $C_{1}$ the quantum Casimir element of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$.

## 4. A universal $L$ operator and the FRT approach

The aim of this section is to construct a spectral parameter-dependent solution which satisfies the quantum Yang-Baxter equation in $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V) \otimes \operatorname{End}_{\mathcal{K}}(V)$. As a byproduct, we propose the FRT approach to the degenerate quantum general linear group using the RTT relation. This is analogous to the classic work [9] in the context of quantum groups.
4.1. The main result. Recall that the $R$-matrix associated to the natural representation $V$ of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ is an invertible element $R \in \operatorname{End}_{\mathcal{K}}(V \otimes V)$ satisfying the quantum Yang-Baxter equation in $\operatorname{End}_{\mathcal{K}}(V \otimes V \otimes V)$ :

$$
R_{12} R_{13} R_{23}=R_{23} R_{13} R_{12}
$$

This element $R$ has been constructed explicitly in [1].
Our construction of $L(x)$ will make use of the $R$-matrix and the $L$ operators $L^{ \pm}$. Recall from [1] that

$$
\begin{equation*}
R=1 \otimes 1+\sum_{a \in I}\left(q_{a}-1\right) e_{a a} \otimes e_{a a}+\left(q-q^{-1}\right) \sum_{a<b} e_{a b} \otimes e_{b a} \in \operatorname{End}_{\mathcal{K}}(V \otimes V) \tag{4.1}
\end{equation*}
$$

Let $v_{a}, a \in I$ be the standard basis of $V$. Then it is clear that

$$
R\left(v_{a} \otimes v_{b}\right)= \begin{cases}v_{a} \otimes v_{b}, & a<b  \tag{4.2}\\ q_{a} v_{a} \otimes v_{a}, & a=b \\ v_{a} \otimes v_{b}+\left(q-q^{-1}\right) v_{b} \otimes v_{a}, & a>b\end{cases}
$$

Let $T$ be the linear permutation operator on $V \otimes V$ such that $T(u \otimes v)=v \otimes u$ for $u, v \in V$, and let $R^{-T}:=T\left(R^{-1}\right) T$. Then we have

$$
R^{-T}=1 \otimes 1+\sum_{a \in I}\left(q_{a}^{-1}-1\right) e_{a a} \otimes e_{a a}-\left(q-q^{-1}\right) \sum_{a<b} e_{b a} \otimes e_{a b}
$$

For any $x \in \mathbb{C}$, we define the $R$-matrix $R(x) \in \operatorname{End}_{\mathcal{K}}(V \otimes V)$ by

$$
R(x):=x R-x^{-1} R^{-T} .
$$

Similarly, recalling the elements $L^{ \pm}$given in (3.3) and (3.4), we introduce

$$
\begin{equation*}
L(x):=x L^{+}-x_{13}^{-1} L^{-}, \quad x \in \mathbb{C} . \tag{4.3}
\end{equation*}
$$

The element $L(x)$ belongs to $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V)$ and is referred to as a universal $L$ operator [12]. Note that $(\pi \otimes 1) L(x)=R(x)$.

The following is the main result of this section.

Theorem 4.1. Let $L(x)$ be as defined in (4.3). Then $L(x)$ satisfies the quantum YangBaxter equation in $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V) \otimes \operatorname{End}_{\mathcal{K}}(V)$ :

$$
L_{12}(x) L_{13}(x y) R_{23}(y)=R_{23}(y) L_{13}(x y) L_{12}(x), \quad x, y \in \mathbb{C} .
$$

4.2. Proof of Theorem 4.1. To prove Theorem 4.1, we expand the quantum Yang-Baxter equation with parameters $x, y$ and compare coefficients of $x^{i} y^{j}$. It turns out that we only need to prove the following proposition.

Proposition 4.2. Maintain the notation above. We have

$$
L_{12}^{ \pm} L_{13}^{ \pm} R_{23}=R_{23} L_{13}^{ \pm} L_{12}^{ \pm}, \quad L_{12}^{ \pm} L_{13}^{\mp} R_{23}=R_{23} L_{13}^{\mp} L_{12}^{ \pm} .
$$

Proof. The proof of these equations involves considering many similar cases. To illustrate the method, we will prove the first relation as an example. We establish $L_{12}^{+} L_{13}^{+} R_{23}=R_{23} L_{13}^{+} L_{12}^{+}$ by acting on the basis vectors $1 \otimes v_{c} \otimes v_{d}(c, d \in I)$ of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \otimes \operatorname{End}_{\mathcal{K}}(V) \otimes \operatorname{End}_{\mathcal{K}}(V)$. The proof is then divided into the following three claims.
Claim 1: $L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{c}\right)=R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{c}\right)$ for all $c \in I$.
On the right hand side, we have

$$
\begin{aligned}
& R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{c}\right)=R_{23} L_{13}^{+}\left(K_{c} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} \otimes v_{b} \otimes v_{c}\right) \\
= & R_{23}\left(K_{c}^{2} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{b} \otimes v_{c}\right. \\
& \left.+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} K_{c} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{c<b, c<a} K_{b} E_{c b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}\right) \\
= & q_{c} K_{c}^{2} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{b} \otimes v_{c} \\
& +\left(q-q^{-1}\right)^{2} \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} K_{c} \otimes v_{c} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} \sum_{c<a} q_{a} K_{a} E_{c a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{3} \sum_{c<a<b} K_{a} E_{c a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2}\left(\sum_{c<b<a}+\sum_{c<a<b}\right) K_{b} E_{c b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

On the left hand side, we have

$$
\begin{aligned}
& L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{c}\right)=q_{c} L_{12}^{+} L_{13}^{+}\left(1 \otimes v_{c} \otimes v_{c}\right) \\
= & q_{c} L_{12}^{+}\left(K_{c} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} K_{c} \otimes v_{c} \otimes v_{b}\right) \\
= & q_{c} K_{c}^{2} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) q_{c} \sum_{c<b} K_{b} E_{c b} K_{c} \otimes v_{b} \otimes v_{c}+\left(q-q^{-1}\right) q_{c} \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{c} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} q_{c} \sum_{c<a} K_{a} E_{c a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{2} q_{c} \sum_{c<a<b} K_{a} E_{c a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} q_{c} \sum_{c<b<a} K_{a} E_{c a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

These six terms in the above expression are denoted by $S_{i}, 1 \leq i \leq 6$, respectively. Applying Lemma 2.1, we obtain

$$
\begin{aligned}
S_{2} & =\left(q-q^{-1}\right) \sum_{c<b} K_{b} K_{c} E_{c b} \otimes v_{b} \otimes v_{c}, \\
S_{3} & =\left(q-q^{-1}\right)\left(q-q^{-1}+q_{c}^{-1}\right) \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{c} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{c<b} K_{c} K_{b} E_{c b} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} K_{c} \otimes v_{c} \otimes v_{b},
\end{aligned}
$$

where in the expression for $S_{3}$ we have used the fact that $q-q^{-1}=q_{c}-q_{c}^{-1}$ for any $c \in I$. Let us consider $S_{4}$. If $c \geq m$, then $q_{c}=q_{a}$ for any $a>c$. If $a \geq m>c$, then $E_{a c}^{2}=0$ by Lemma 2.3. Therefore, we have

$$
\begin{aligned}
S_{4} & =\left(q-q^{-1}\right)^{2} \sum_{m \leq c<a} q_{a} K_{a} E_{c a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{2} q_{c} \sum_{c<m \leq a} q_{a}^{-1} K_{a} E_{c a}^{2} K_{a} \otimes v_{a} \otimes v_{a} \\
& =\sum_{c<a}\left(q-q^{-1}\right)^{2} q_{a} K_{a} E_{c a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a} .
\end{aligned}
$$

Applying Lemma 2.3 again, we obtain

$$
\begin{aligned}
S_{5} & =\left(q-q^{-1}\right)^{2}\left(q_{c}^{-1}+q-q^{-1}\right) \sum_{c<a<b} K_{a} E_{c a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{c<a<b} K_{b} E_{c b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{c<a<b} K_{a} E_{c a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b}, \\
S_{6} & =\left(q-q^{-1}\right)^{2} \sum_{c<b<a} K_{b} E_{c b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

Adding up $S_{i}$ for $1 \leq i \leq 6$, we obtain the expression for $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{c}\right)$ obtained earlier.
Claim 2: $L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)=R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$ for all $c<d$.
On the right hand side, we have

$$
R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)
$$

$$
\begin{aligned}
= & R_{23}\left(K_{d} K_{c} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{d<b} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b}\right. \\
& \left.+\left(q-q^{-1}\right) \sum_{c<b} K_{d} K_{b} E_{c b} \otimes v_{b} \otimes v_{d}+\left(q-q^{-1}\right)^{2} \sum_{d<b, c<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}\right) \\
= & K_{d} K_{c} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{d<b} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) q_{d} K_{d}^{2} E_{c d} \otimes v_{d} \otimes v_{d} \\
& +\left(q-q^{-1}\right)\left(\sum_{c<b<d}+\sum_{b>d}\right) K_{d} K_{b} E_{c b} \otimes v_{b} \otimes v_{d}+\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{d} K_{b} E_{c b} \otimes v_{d} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} \sum_{b>d} q_{b} K_{b} E_{d b} K_{b} E_{c b} \otimes v_{b} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{b} E_{d b} K_{d} E_{c d} \otimes v_{d} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2}\left(\sum_{d<a<b}+\sum_{c<a<d<b}\right) K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} \sum_{d<b<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{d<b<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{b} \otimes v_{a}
\end{aligned}
$$

where in the last equation we have partitioned the sum $\sum_{c<b}$ into $\sum_{b=d}+\sum_{c<b<d}+\sum_{b>d}$ and similarly split $\sum_{d<b, c<a}$ into $\sum_{d<a=b}+\sum_{a=d<b}+\sum_{d<a<b}+\sum_{c<a<d<b}+\sum_{d<b<a}$.

Considering the left hand side, we have

$$
\begin{aligned}
& L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)=L_{12}^{+} L_{13}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right) \\
= & L_{12}^{+}\left(K_{d} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{b>d} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b}\right) \\
= & K_{c} K_{d} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) K_{c} \sum_{b>d} K_{b} E_{d b} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) \sum_{b>c} K_{b} E_{c b} K_{d} \otimes v_{b} \otimes v_{d} \\
& +\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{b} E_{c b} K_{b} E_{d b} \otimes v_{b} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{d} E_{c d} K_{b} E_{d b} \otimes v_{d} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2}\left(\sum_{d<b<a}+\sum_{c<a<d<b}+\sum_{d<a<b}\right) K_{a} E_{c a} K_{b} E_{d b} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

We denote the eight terms appearing in the above expression by $S_{i}^{\prime}, 1 \leq i \leq 8$, respectively.
Now we compare two expressions for $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$ and $L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)$. Notice that $S_{1}^{\prime}$ and $S_{2}^{\prime}$ have appeared in the expression for $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$. Splitting the sum $\sum_{b>c}$ into $\sum_{b=d}+\sum_{c<b<d}+\sum_{b>d}$, we obtain

$$
S_{3}^{\prime}=\left(q-q^{-1}\right) q_{d} K_{d}^{2} E_{c d} \otimes v_{d} \otimes v_{d}+\left(q-q^{-1}\right)\left(\sum_{c<b<d}+\sum_{b>d}\right) K_{d} K_{b} E_{c b} \otimes v_{b} \otimes v_{d}
$$

Using Lemma 2.1 and Lemma 2.3, we have

$$
\begin{aligned}
S_{4}^{\prime} & =\left(q-q^{-1}\right)^{2} \sum_{b>d} q_{b} K_{b}^{2} E_{c b} E_{d b} \otimes v_{b} \otimes v_{b}=\left(q-q^{-1}\right)^{2} \sum_{b>d} q_{b}^{2} K_{b}^{2} E_{d b} E_{c b} \otimes v_{b} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{b>d} q_{b} K_{b} E_{d b} K_{b} E_{c b} \otimes v_{b} \otimes v_{b}
\end{aligned}
$$

$$
\begin{aligned}
S_{5}^{\prime} & =\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{d} K_{b} E_{c d} E_{d b} \otimes v_{d} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{d} K_{b}\left(q_{d}^{-1} E_{d b} E_{c d}+E_{c b}\right) \otimes v_{d} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{b} E_{d b} K_{d} E_{c d} \otimes v_{d} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{d} K_{b} E_{c b} \otimes v_{d} \otimes v_{b}, \\
S_{6}^{\prime} & =\left(q-q^{-1}\right)^{2} \sum_{d<b<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}, \\
S_{7}^{\prime} & =\left(q-q^{-1}\right)^{2} \sum_{c<a<d<b} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}, \\
S_{8}^{\prime} & =\left(q-q^{-1}\right)^{2} \sum_{d<a<b} K_{a} K_{b} E_{c a} E_{d b} \otimes v_{a} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{d<a<b} K_{a} K_{b}\left(E_{d b} E_{c a}+\left(q-q^{-1}\right) E_{d a} E_{c b}\right) \otimes v_{a} \otimes v_{b} \\
& =\left(q-q^{-1}\right)^{2} \sum_{d<a<b} K_{a} K_{b} E_{d b} E_{c a} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{d<a<b} K_{a} E_{d a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

Summing up the expressions for $S_{i}^{\prime}, 1 \leq i \leq 8$, we obtain $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$ as desired.
Finally, we need to verify the following.
Claim 3: $L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)=R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$ for all $c>d$.
On the right hand side, we have

$$
\begin{aligned}
& R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)=R_{23} L_{13}^{+}\left(K_{c} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{c<b} K_{b} E_{c b} \otimes v_{b} \otimes v_{d}\right) \\
= & R_{23}\left(K_{d} K_{c} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{d<b} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) \sum_{c<b} K_{d} K_{b} E_{c b} \otimes v_{b} \otimes v_{d}\right. \\
& \left.+\left(q-q^{-1}\right)^{2} \sum_{d<b, c<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}\right) \\
= & K_{d} K_{c} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) K_{d} K_{c} \otimes v_{d} \otimes v_{c}+\left(q-q^{-1}\right) q_{c} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c} \\
& +\left(q-q^{-1}\right) \sum_{b>c} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right) \sum_{d<b<c} K_{b} E_{d b} K_{c} \otimes v_{c} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} \sum_{d<b<c} K_{b} E_{d b} K_{c} \otimes v_{b} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{b>c} K_{d} K_{b} E_{c b} \otimes v_{b} \otimes v_{d} \\
& +\left(q-q^{-1}\right)^{2} \sum_{b>c} K_{d} K_{b} E_{c b} \otimes v_{d} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{b>c} q_{b} K_{b} E_{d b} K_{b} E_{c b} \otimes v_{b} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{c} E_{d c} K_{a} E_{c a} \otimes v_{a} \otimes v_{c}+\left(q-q^{-1}\right)^{3} \sum_{a>c} K_{c} E_{d c} K_{a} E_{c a} \otimes v_{c} \otimes v_{a} \\
& +\left(q-q^{-1}\right)^{2}\left(\sum_{c<a<b}+\sum_{c<b<a}\right) K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{c<b<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{b} \otimes v_{a}
\end{aligned}
$$

$$
+\left(q-q^{-1}\right)^{2} \sum_{d<b<c<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{d<b<c<a} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{b} \otimes v_{a}
$$

We proceed to compute the left hand side:

$$
\begin{aligned}
& L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)=L_{12}^{+} L_{13}^{+}\left(1 \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) 1 \otimes v_{d} \otimes v_{c}\right) \\
& =L_{12}^{+}\left(K_{d} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) K_{c} \otimes v_{d} \otimes v_{c}+\left(q-q^{-1}\right) \sum_{b>d} K_{b} E_{d b} \otimes v_{c} \otimes v_{b}\right. \\
& \left.+\left(q-q^{-1}\right)^{2} \sum_{b>c} K_{b} E_{c b} \otimes v_{d} \otimes v_{b}\right) \\
& =K_{c} K_{d} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{b>c} K_{b} E_{c b} K_{d} \otimes v_{b} \otimes v_{d}+\left(q-q^{-1}\right) K_{d} K_{c} \otimes v_{d} \otimes v_{c} \\
& +\left(q-q^{-1}\right)^{2} \sum_{b>d} K_{b} E_{d b} K_{c} \otimes v_{b} \otimes v_{c}+\left(q-q^{-1}\right)^{2} K_{d} \sum_{b>c} K_{b} E_{c b} \otimes v_{d} \otimes v_{b} \\
& +\left(q-q^{-1}\right) K_{c} \sum_{a>c, b>d} K_{b} E_{d b} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right)^{2} K_{a} E_{c a} \sum_{b>d} K_{b} E_{d b} \otimes v_{a} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{3} \sum_{a>d, b>c} K_{a} E_{d a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} \\
& =K_{c} K_{d} \otimes v_{c} \otimes v_{d}+\left(q-q^{-1}\right) \sum_{b>c} K_{b} E_{c b} K_{d} \otimes v_{b} \otimes v_{d}+\left(q-q^{-1}\right) K_{d} K_{c} \otimes v_{d} \otimes v_{c} \\
& +\left(q-q^{-1}\right)^{2} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right)^{2}\left(\sum_{a>c}+\sum_{d<a<c}\right) K_{a} E_{d a} K_{c} \otimes v_{a} \otimes v_{c} \\
& +\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{d} K_{b} E_{c a} \otimes v_{d} \otimes v_{a}+\left(q-q^{-1}\right) K_{c}^{2} E_{d c} \otimes v_{c} \otimes v_{c} \\
& +\left(q-q^{-1}\right)\left(\sum_{b>c}+\sum_{d<b<c}\right) K_{c} K_{b} E_{d b} \otimes v_{c} \otimes v_{b}+\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{c a} K_{a} E_{d a} \otimes v_{a} \otimes v_{a} \\
& +\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{c a} K_{c} E_{d c} \otimes v_{a} \otimes v_{c}+\left(q-q^{-1}\right)^{2}\left(\sum_{c<b<a}+\sum_{c<a<b}+\sum_{d<b<c<a}\right) K_{a} E_{c a} K_{b} E_{d b} \otimes v_{a} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{3} \sum_{a>c} K_{a} E_{d a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{3} \sum_{b>c} K_{c} E_{d c} K_{b} E_{c b} \otimes v_{c} \otimes v_{b} \\
& +\left(q-q^{-1}\right)^{3}\left(\sum_{c<a<b}+\sum_{c<b<a}+\sum_{d<a<c<b}\right) K_{a} E_{d a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

It remains to compare the expressions for both sides. In the expression for $L_{12}^{+} L_{13}^{+} R_{23}(1 \otimes$ $v_{c} \otimes v_{d}$ ), we observe

$$
\begin{aligned}
& \left(q-q^{-1}\right)^{2} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) K_{c}^{2} E_{d c} \otimes v_{c} \otimes v_{c} \\
= & \left(q-q^{-1}\right)^{2} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c}+\left(q-q^{-1}\right) q_{c}^{-1} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c} \\
= & \left(q-q^{-1}\right) q_{c} K_{c} E_{d c} K_{c} \otimes v_{c} \otimes v_{c},
\end{aligned}
$$

where the last equation follows from the substitution $q_{c}^{-1}=q_{c}-\left(q-q^{-1}\right)$. Note that the resulting expression is a term of $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$. This method will be continued for further comparison. Applying Lemma 2.1 and Lemma [2.3, we have

$$
\begin{aligned}
& \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{d a} K_{c} \otimes v_{a} \otimes v_{c}+\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{c a} K_{c} E_{d c} \otimes v_{a} \otimes v_{c} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a}\left(E_{d c} E_{c a}-q_{c}^{-1} E_{c a} E_{d c}\right) K_{c} \otimes v_{a} \otimes v_{c}+\left(q-q^{-1}\right)^{2} q_{c}^{-1} \sum_{a>c} K_{a} E_{c a} E_{d c} K_{c} \otimes v_{a} \otimes v_{c} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{d c} E_{c a} K_{c} \otimes v_{a} \otimes v_{c} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{c} E_{d c} K_{a} E_{c a} \otimes v_{a} \otimes v_{c} .
\end{aligned}
$$

Using (2.16) and (2.28), we deduce

$$
\begin{aligned}
& \left(q-q^{-1}\right)^{2}\left(\sum_{d<b<c<a}+\sum_{b>a>c}\right) K_{a} E_{c a} K_{b} E_{d b} \otimes v_{a} \otimes v_{b} \\
= & \left(q-q^{-1}\right)^{2}\left(\sum_{d<b<c<a}+\sum_{b>a>c}\right) K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

Using (2.16) and (2.27), we arrive at

$$
\begin{aligned}
& \left(q-q^{-1}\right)^{2} \sum_{a>b>c} K_{a} E_{c a} K_{b} E_{d b} \otimes v_{a} \otimes v_{b}+\left(q-q^{-1}\right)^{3} \sum_{a>b>c} K_{a} E_{d a} K_{b} E_{c b} \otimes v_{a} \otimes v_{b} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>b>c}\left(K_{a} K_{b} E_{c a} E_{d b}+\left(q-q^{-1}\right) K_{a} K_{b} E_{d a} E_{c b}\right) \otimes v_{a} \otimes v_{b} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>b>c} K_{b} K_{a}\left(E_{c a} E_{d b}+\left(q-q^{-1}\right) E_{d a} E_{c b}\right) \otimes v_{a} \otimes v_{b} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>b>c} K_{b} E_{d b} K_{a} E_{c a} \otimes v_{a} \otimes v_{b} .
\end{aligned}
$$

From (2.17), (2.26), and $q-q^{-1}+q_{a}^{-1}=q_{a}$ for $a \in I$, we have

$$
\begin{aligned}
& \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{c a} K_{a} E_{d a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{3} \sum_{a>c} K_{a} E_{d a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a}\left(q_{a}^{-1} E_{c a} E_{d a} K_{a}+\left(q-q^{-1}\right) E_{d a} K_{a} E_{c a}\right) \otimes v_{a} \otimes v_{a} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a}\left(q_{a}^{-1} E_{d a} K_{a} E_{c a}+\left(q-q^{-1}\right) E_{d a} K_{a} E_{c a}\right) \otimes v_{a} \otimes v_{a} \\
= & \left(q-q^{-1}\right)^{2} \sum_{a>c}\left(q_{a}-1\right) K_{a} E_{d a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a}+\left(q-q^{-1}\right)^{2} \sum_{a>c} K_{a} E_{d a} K_{a} E_{c a} \otimes v_{a} \otimes v_{a} .
\end{aligned}
$$

Now the remaining terms in $L_{12}^{+} L_{13}^{+} R_{23}\left(1 \otimes v_{c} \otimes v_{d}\right)$ and $R_{23} L_{13}^{+} L_{12}^{+}\left(1 \otimes v_{c} \otimes v_{d}\right)$ coincide. Combining the results together, we complete the proof of Claim 3.

We are in a position to prove Theorem 4.1.
Proof of Theorem 4.1. Expanding the quantum Yang-Baxter equation in the theorem and comparing the coefficients of $x^{i} y^{j}$ for $i, j \geq 0$, we obtain the following equations:

$$
\begin{aligned}
& L_{12}^{ \pm} L_{13}^{ \pm} R_{23}=R_{23} L_{13}^{ \pm} L_{12}^{ \pm}, \quad L_{12}^{ \pm} L_{13}^{\mp} R_{23}=R_{23} L_{13}^{\mp} L_{12}^{ \pm}, \quad L_{12}^{ \pm} L_{13}^{ \pm} R_{23}^{-T}=R_{23}^{-T} L_{13}^{ \pm} L_{12}^{ \pm}, \\
& L_{12}^{+} L_{13}^{-} R_{23}-L_{12}^{-} L_{13}^{+} R_{23}^{-T}=R_{23} L_{13}^{-} L_{12}^{+}-R_{23}^{-T} L_{13}^{+} L_{12}^{-} .
\end{aligned}
$$

It can be readily verified that these equations follow from the four key equations given in Proposition 4.2.
4.3. The FRT approach. Motivated by Proposition4.2, we propose the FRT approach to the degenerate quantum group $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ using the RTT relation. Similar to the quantum group case [9], we conjecture that this approach will provide an equivalent formulation of degenerate quantum general linear groups and can be generalised to other types. Our formalism will be guided by ideas from [14].

Recall from [8] that an ideal $\mathfrak{a}$ of an Hopf algebra $A$ is called cofinite if $\operatorname{dim} A / \mathfrak{a}$ is finite. Define the finite dual $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)^{\circ}$ of $\left.\mathrm{U}_{\left(\mathfrak{g l}_{m, n}\right)}\right)$ by

$$
\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)^{\circ}:=\left\{f \in\left(\mathrm{U}_{q}\left(\mathfrak{g}_{m, n}\right)\right)^{*} \mid \operatorname{Ker} f \text { contains a cofinite ideal } \mathfrak{a} \text { of } \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)\right\}
$$

It follows from the standard Hopf algebra theory that $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)^{\circ}$ is a Hopf algebra.
For any $a, b \in I$, we define the matrix element $t_{a b} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)^{\circ}$ by

$$
\left\langle t_{a b}, u\right\rangle:=\pi(u)_{a b}, \quad u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right),
$$

where $\pi$ is the natural representation defined by (2.13). Let $\mathcal{M}_{m, n}$ be the unital subalgebra of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)^{\circ}$ generated by $t_{a b}, a, b \in I$. The multiplication on $\mathcal{M}_{m, n}$ is given by

$$
\begin{equation*}
\left\langle t t^{\prime}, u\right\rangle=\sum_{(u)}\left\langle t \otimes t^{\prime}, u_{(1)} \otimes u_{(2)}\right\rangle=\sum_{(u)}\left\langle t, u_{(1)}\right\rangle\left\langle t^{\prime}, u_{(2)}\right\rangle, \tag{4.4}
\end{equation*}
$$

for any $t, t^{\prime} \in \mathcal{M}_{m, n}, u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, where we have used Sweedler's notation $\Delta(u)=$ $\sum_{(u)} u_{(1)} \otimes u_{(2)}$. The algebra $\mathcal{M}_{m, n}$ has a bialgebra structure with comultiplication $\Delta$ and counit $\epsilon$ given by

$$
\begin{equation*}
\Delta\left(t_{a b}\right)=\sum_{c \in I} t_{a c} \otimes t_{c b}, \quad \epsilon\left(t_{a b}\right)=\delta_{a b}, \quad t_{a b} \in \mathcal{M}_{m, n}, \quad a, b \in I \tag{4.5}
\end{equation*}
$$

Applying $\pi \otimes 1$ to both sides of relation (3.6) for $L^{+}$, we obtain

$$
\begin{equation*}
R(\pi \otimes \pi) \Delta(u)=(\pi \otimes \pi) \Delta^{\prime}(u) R, \quad u \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right) \tag{4.6}
\end{equation*}
$$

This leads to the following relations.

Lemma 4.3. The algebra $\mathcal{M}_{m, n}$ is generated by the matrix elements $t_{a b}, a, b \in I$ with the following relations:

$$
\begin{aligned}
& \left(t_{a b}\right)^{2}=0, \quad a \leq m<b \text { or } b \leq m<a \\
& t_{a c} t_{b c}=q_{c} t_{b c} t_{a c}, \quad a>b, \\
& t_{a b} t_{a c}=q_{a} t_{a c} t_{a b}, \quad b>c \\
& t_{a c} t_{b d}=t_{b d} t_{a c}, \quad a>b, c<d, \\
& t_{a c} t_{b d}=t_{b d} t_{a c}+\left(q-q^{-1}\right) t_{b c} t_{a d}, \quad a>b, c>d .
\end{aligned}
$$

Proof. Let $R=\sum_{a, b, c, d \in I} R_{b d}^{a c} e_{a b} \otimes e_{c d}$. By (4.1), the nonzero entries of $R$ are

$$
R_{a b}^{a b}=1, a \neq b, \quad R_{a a}^{a a}=q_{a}, \quad R_{b a}^{a b}=q-q^{-1}, a<b
$$

The left hand side of (4.6) can be written as

$$
\begin{aligned}
R(\pi \otimes \pi) \Delta(u) & =\sum_{a, a^{\prime}, b, b^{\prime}, c, d \in I} \sum_{(u)}\left(R_{a^{\prime} b^{b^{\prime}}}^{a b} e_{a a^{\prime}} \otimes e_{b b^{\prime}}\right)\left(\pi\left(u_{(1)}\right)_{a^{\prime} c} e_{a^{\prime} c}\right) \otimes\left(\pi\left(u_{(2)}\right)_{b^{\prime} d} e_{b^{\prime} d}\right) \\
& =\sum_{a, a^{\prime}, b, b^{\prime}, c, d \in I} R_{a^{\prime} b^{\prime}}^{a b}\left(\sum_{(u)} \pi\left(u_{(1)}\right)_{a^{\prime} c} \pi\left(u_{(2)}\right)_{b^{\prime} d}\right) e_{a c} \otimes e_{b d} \\
& =\sum_{a, a^{\prime}, b, b^{\prime}, c, d \in I} R_{a^{\prime} b^{\prime}}^{a b}\left\langle t_{a^{\prime} c} t_{b^{\prime} d}, u\right\rangle e_{a c} \otimes e_{b d} .
\end{aligned}
$$

Similarly, the right hand side of (4.6) has the expression

$$
(\pi \otimes \pi) \Delta^{\prime}(u) R=\sum_{a, a^{\prime}, b, b^{\prime}, c, d \in I} R_{c d}^{a^{\prime} b^{\prime}}\left\langle t_{b b^{\prime}} t_{a a^{\prime}}, u\right\rangle e_{a c} \otimes e_{b d}
$$

There, we obtain

$$
\sum_{a^{\prime}, b^{\prime} \in I} R_{a^{\prime} b^{\prime}}^{a b} t_{a^{\prime} c} t_{b^{\prime} d}=\sum_{a^{\prime}, b^{\prime} \in I} t_{b b^{\prime}} t_{a a^{\prime}} R_{c d}^{a^{\prime} b^{\prime}}, \quad a, b, c, d \in I
$$

from which the relations stated in the lemma follow.
Remark 4.4. Let $T_{1}=\sum_{a, b \in I} e_{a b} \otimes t_{a b}$ and $T_{2}=\sum_{a, b \in I} t_{a b} \otimes e_{a b}$. The relations in Lemma 4.3 can be written concisely as

$$
R T_{1} T_{2}=T_{2} T_{1} R
$$

Remark 4.5. Lemma 4.3 can be regarded as a sign-free version of [17, Lemma 2.6], which deals with the quantum general linear supergroup (see also [14]). This comparison reveals an interesting connection between degenerate quantum groups and quantum supergroups, suggesting that the study of one object may offer insights into the other.

Recall from (3.3) and (3.4) the $L$ operators $L^{ \pm}$. We may write them as follows:

$$
L^{+}=\sum_{a, b \in I} \ell_{a b}^{+} \otimes e_{a b} \quad \text { and } \quad L^{-}=\sum_{a, b \in I} \ell_{a b}^{-} \otimes e_{a b}
$$

where $\ell_{a b}^{ \pm} \in \mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ for all $a, b \in I$. Noting that $(\pi \otimes 1) L^{+}=R,(\pi \otimes 1) L^{-}=R^{-T}$, and $R=\sum_{a, b, c, d \in I} R_{b d}^{a c} e_{a b} \otimes e_{c d}$, we have

$$
\left\langle t_{a b}, \ell_{c d}^{+}\right\rangle=R_{b d}^{a c} \quad \text { and } \quad\left\langle t_{a b}, \ell_{c d}^{-}\right\rangle=\left(R^{-1}\right)_{d b}^{c a}, \quad a, b, c, d \in I .
$$

Let $\mathrm{U}(R)$ denote the unital subalgebra of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ generated by the elements $\ell_{a b}^{ \pm}$for all $a, b \in I$. In view of Proposition 4.2, those generators satisfy the following relations:

$$
\begin{equation*}
L_{1}^{ \pm} L_{2}^{ \pm} R=R L_{2}^{ \pm} L_{1}^{ \pm}, \quad L_{1}^{-} L_{2}^{+} R=R L_{2}^{+} L_{1}^{-}, \tag{4.7}
\end{equation*}
$$

where $L_{1}^{ \pm}=\sum_{a, b \in I} e_{a b} \otimes \ell_{a b}^{ \pm}$and $L_{2}^{ \pm}=\sum_{a, b \in I} \ell_{a b}^{ \pm} \otimes e_{a b}$. More explicitly, the relations among generators $\ell_{a b}^{+}$can be written as follows:

$$
\begin{aligned}
& \left(\ell_{a b}^{+}\right)^{2}=0, \quad a \leq m<b \text { or } b \leq m<a, \\
& \ell_{a c}^{+} \ell_{b c}^{+}=q_{c} \ell_{b c}^{+} \ell_{a c}^{+}, \quad a<b, \\
& \ell_{a b}^{+} \ell_{a c}^{+}=q_{a} \ell_{a c}^{+} \ell_{a b}^{+}, \quad b<c, \\
& \ell_{a c}^{+} \ell_{b d}^{+}=\ell_{b d}^{+} \ell_{a c}^{+}, \quad a>b, c<d, \\
& \ell_{a c}^{+} \ell_{b d}^{+}=\ell_{b d}^{+} \ell_{a c}^{+}+\left(q-q^{-1}\right) \ell_{b c}^{+} \ell_{a d}^{+}, \quad a<b, c<d .
\end{aligned}
$$

The same relations hold for all elements $\ell_{a b}^{-}$, and the commutation relations between $\ell_{a b}^{+}$and $\ell_{c d}^{-}$for all $a, b, c, d \in I$ can be derived explicitly in a similar manner.

The subalgebra $\mathrm{U}(R)$ of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ has a bialgebra structure with comultiplication and counit defined as follows:

$$
\Delta\left(\ell_{a b}^{ \pm}\right)=\sum_{c \in I} \ell_{a c}^{ \pm} \otimes \ell_{c b}^{ \pm}, \quad \epsilon\left(\ell_{a b}^{ \pm}\right)=\delta_{a b}, \quad a, b \in I
$$

Conjecture 4.6. The equality $\mathrm{U}(R)=\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$ holds, and consequently, $\mathrm{U}(R)$ inherits a Hopf algebra structure.

The proof of this conjecture requires deep understanding of structural properties of $\mathrm{U}_{q}\left(\mathfrak{g l}_{m, n}\right)$, which will be studied systematically in a subsequent paper. The following verifies the simplest case of this conjecture.

Example 4.7. Consider $\mathrm{U}_{q}\left(\mathfrak{g l}_{1,1}\right)$ with the presentation in terms of $K_{1}^{ \pm 1}, K_{2}^{ \pm}, E=E_{12}$ and $F=E_{21}$ given in Example 3.5. The $R$-matrix of the natural representation is given by (4.1) as

$$
R=q e_{11} \otimes e_{11}+e_{12} \otimes e_{12}+q^{-1} e_{22} \otimes e_{22}+e_{21} \otimes e_{21}+\left(q-q^{-1}\right) e_{12} \otimes e_{21}
$$

By definition (3.3) and (3.4), we have the following generators of $\mathrm{U}(R)$ :

$$
\begin{aligned}
& \ell_{11}^{+}=K_{1}, \quad \ell_{12}^{+}=0, \quad \ell_{21}^{+}=K_{2} E_{12}, \quad \ell_{22}^{+}=K_{2}, \\
& \ell_{11}^{-}=K_{1}^{-1}, \quad \ell_{12}^{-}=E_{21} K_{2}^{-1}, \quad \ell_{21}^{-}=0, \quad \ell_{22}^{-}=K_{2}^{-1} .
\end{aligned}
$$

These generators satisfy relations given in (4.7), and clearly generate the whole algebra $\mathrm{U}_{q}\left(\mathfrak{g l}_{1,1}\right)$.

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