

Statistical Theory of Neutron-Induced Nuclear Fission and of Heavy-Ion Fusion

Hans A. Weidenmüller^{1,*}

¹Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany

(Dated: May 1, 2024)

For both reactions we use an approach similar to that of compound-nucleus reaction theory. For neutron-induced fission, we describe the compound system generated by absorption of the neutron and the nuclear system near the scission point as two statistically independent systems governed by random-matrix theory. The systems are connected either by a barrier penetration factor or by a set of transition states above the barrier. Each system is coupled to a different set of channels. An analogous model is used for heavy-ion fusion. Assuming that (seen from the entrance channel) the system on the other side of the barrier is in the regime of strongly overlapping resonances, we obtain for fixed spin and parity closed-form analytical expressions for the total probability for fission and for fusion. Parts of these expressions can be calculated reliably within existing compound-nucleus reaction theory. The remaining parts are the probabilities for passage through or over the barrier. These may be determined theoretically from the liquid-drop model or experimentally from total fission or fusion cross sections.

I. INTRODUCTION

The standard approach to neutron-induced nuclear fission is based on the liquid-drop model originally proposed by Bohr and Wheeler [1]. The capture of a thermal neutron leads to excitation energies of the compound nucleus close to or above the top of the fission barrier. The liquid drop deforms and eventually fissions. The deformation is described in terms of one or several collective variables. These must be able to pass the fission barrier. To that end, all or a good fraction of the excitation energy of the compound nucleus must be transferred to the collective variables. The barrier is passed via tunneling or via one or several transition states located on top of the barrier. On the other side of the fission barrier, the process is reversed, energy of collective motion is transferred back to the non-collective nuclear degrees of freedom. As it approaches the scission point, the nuclear system is again in a highly excited state as witnessed by neutron emission prior to fission. Most recently that time-dependent picture of the fission process was formulated theoretically in terms of the generator-coordinate method in Ref. [2] where numerous further references may be found. Extensive numerical work [3] shows good agreement with data.

Here we propose a radically different approach. In line with the standard approach to compound-nucleus reactions, we propose a statistical model for nuclear fission induced by neutrons, and for fusion of two heavy ions. The model makes use of the physical picture described in the previous paragraph. For fixed spin J and parity π , we describe both the compound system generated by absorption of the neutron and the nuclear system near the scission point as statistically independent systems governed by random-matrix theory. The two systems are connected either by a barrier penetration factor or by a set of transition states above the barrier. Each system is coupled to a different set of channels. We proceed analogously for heavy-ion fusion.

We implement that approach using the standard time-

independent formulation of nuclear reaction theory [4]. At fixed energy E and for fixed quantum numbers (J, π) , the elements of the scattering matrix $S_{1a,2b}(E)$ connect incident channels labeled $(1a)$ with outgoing channels $(2b)$ on the other side of the fission barrier. For a neutron-induced reaction, channel $(1a)$ carries the incident neutron and the target nucleus in its ground state, channels $(1a')$, $(1a'')$... account for neutrons inelastically scattered on the target nucleus, while channels $(2b)$, $(2b')$, ... carry pairs of scission fragments in their ground or excited states but also neutrons emitted from the nuclear system before it reaches the scission point. For a fusion reaction, channel $(2b)$ carries the two ions in their ground states while channels $(1a)$, $(1a')$, ... carry a neutron and the remaining nucleus in its ground state or any excited state or any other reaction products. The two sets of channels are linked by the Hamiltonian describing the intermediate nuclear system. The Hamiltonian consists of three parts. For a neutron-induced reaction, part (1) describes the compound system (system 1) formed by neutron absorption. Part (2) describes the nuclear system (system 2) near the scission point. The third part labeled (tr) describes either the transition through the barrier in terms of a matrix element for tunneling or the transition over the barrier in terms of a small number of transition states. For a fusion reaction, part (2) describes the nuclear system (system 2) reached by the merger of the two ions, part (1) describes the compound system (system 1) reached after transition through or over the barrier, and part (tr) the transition through or over the barrier. The model is obviously confined to situations where system 1 and system 2 can be clearly identified as separate entities, i.e., for excitation energies that are below or slightly above the height of the fission barrier. Without specific assumptions on the Hamiltonian, the elements $S_{1a,2b}(E)$ of the scattering matrix are well defined but cannot be worked out explicitly.

The situation changes, and an explicit expression for the probability of the transitions $(1a) \rightarrow (2b)$ and $(2b) \rightarrow (1a)$ can be derived, if it is assumed that the Hamiltonian for part (1) describing the compound nucleus and the Hamiltonian for part (2) describing the nuclear system near the scission point (or the point of merger of the two ions) are statistically independent members of the GOE, the time-reversal invariant

*Electronic address: haw@mpi-hd.mpg.de

Gaussian Orthogonal Ensemble of Random Matrices [5]. Use of the GOE implies that in the vicinity of the actual energy of the reaction, the local spectral fluctuation properties of the nuclear system coincide with those of the GOE. That same assumption is actually used for the nuclear Hamiltonian in the statistical theory of compound-nucleus reactions. For the assumption to hold, the excitation energy in the compound nucleus (system 1) and in the scissioning system (system 2) must be larger than several MeV. With neutron binding energies in the region of several MeV that condition is always met for neutron-induced reactions. We consider heavy-ion induced fission for energies where the condition is likewise met. Then the success of the statistical theory of compound-nucleus reactions [6, 7] supports the hypothesis for system 1. It is natural to extend the hypothesis to system 2, the nuclear system near the scission point (or point of merger). The compound nucleus created by absorption of the incident neutron (system 1) and the nuclear system near the scission point (system 2) are sufficiently different physical systems (characterized by very different deformation) to justify the assumption that their spectral fluctuations at energy E are statistically independent.

Under these assumptions, the matrix element $S_{1a,2b}(E)$ turns into a stochastic process that depends upon the statistically independent random Hamiltonians H_1 for system 1 and H_2 for system 2. For the explicit calculation of the average transition probability $P_{1a,2b}(E) = \langle |S_{1a,2b}(E)|^2 \rangle$ (the angular brackets denote the average over both H_1 and H_2) we consider the following cases, distinguished by the excitation energies of system 1 and system 2. Case (a): The excitation energy of system 1 is in the regime of isolated compound-nucleus resonances (that case corresponds to neutron energies of up to 10 keV or so); case (b): the excitation energy of system 1 is in the regime of strongly overlapping compound-nucleus resonances (the number of channels strongly coupled to system 1 is large compared to unity); case (α): the excitation energy of system 2 is in the regime of isolated resonances; case (β): the excitation energy of system 2 is in the regime of strongly overlapping resonances (the number of channels strongly coupled to system 2 is large compared to unity). Which combination [(a α) or (a β) or (b α) or (b β)] of these cases is realized depends, of course, on the energy of the reaction and on the binding energies of the nuclei involved. As shown in Refs. [8, 12] it is possible to derive a closed-form expression for $P_{ab}(E)$ for three out these four cases with the exception of the case (a α). In case (b β) the resulting expression is completely explicit. In cases (a β) and (b α) the expression involves the threefold integral familiar from the theory of compound-nucleus reactions [9]. Case (a α) can only be treated by numerical simulation.

Scission is frequently preceded by the emission of one or several neutrons. In that case the index (2b) of the element $S_{1a,2b}(E)$ of the scattering matrix refers to the channels that carry the first emitted neutron, and $P_{1a,2b}(E)$ gives only the probability for that process to happen and does not give any information on the actual scission process. To the best of our knowledge it is not possible to address successive multiple particle emission in the framework of time-independent scattering theory. It is, thus, not possible to predict in our framework

what happens after the first neutron has been emitted. We turn that seeming disadvantage into an advantage by arguing that no matter how many neutrons are emitted prior to scission, channel (2 b) surely leads to scission eventually. We accordingly interpret $P_{\text{fission},1a}(E) = \sum_b P_{1a,2b}(E)$ as the total fission probability for the reaction starting in channel (1a) and focus attention on that quantity. Correspondingly, we interpret $P_{\text{fusion},2b}(E) = \sum_a P_{1a,2b}(E)$ as the total fusion probability for the reaction starting in channel (2 b).

In summary we calculate the average total fission probability $P_{\text{fission},1a}(E)$ and the average total fusion probability $P_{\text{fusion},2b}(E)$ for a system of fixed spin J and parity π and under the proviso that the resonances in system 1 or system 2 or both, strongly overlap. Technically, the average is over the distribution of the matrix elements of H_1 and of H_2 . Physically, the average corresponds to an average over a spectral domain containing a large number of resonances. Multiplying $P_{\text{fission},1a}(E)$ and $P_{\text{fusion},2b}(E)$ with appropriate geometrical and kinematical factors and summing over J and π one obtains the total average reaction cross section for neutron-induced fission and for heavy-ion fusion, respectively. That step is standard.

The model we use is a variant of a general statistical approach to transition-state theory, based upon a suggestion in Refs. [10, 11] and worked out in Refs. [8, 12]. To make the paper reasonably self-contained, we define the model in Section II. We formulate the statistical assumptions in Section III. Results derived in Refs. [8, 12] are collected, applied, and discussed in the following Sections. The reader who is not interested in technical aspects is advised to skip Sections II and III.

II. MODEL

We confine ourselves to states of fixed spin and parity. We use the notation of Ref. [12] throughout. In the time-reversal invariant Hamiltonian H , system 1 and system 2 are coupled either by the matrix element for tunneling through the barrier, or by a set of k transition states right above the barrier. In matrix form, the tunneling Hamiltonian for the first case is

$$H = \begin{pmatrix} H_1 & V \\ V & H_2 \end{pmatrix}. \quad (1)$$

The time-reversal invariant Hamiltonian for transition over the barrier is given by

$$H = \begin{pmatrix} H_1 & V_1 & 0 \\ V_1^T & H_{\text{tr}} & V_2^T \\ 0 & V_2 & H_2 \end{pmatrix}. \quad (2)$$

In both Eqs. (1, 2), H_1 (H_2) denotes the real and symmetric Hamiltonian matrix governing system 1 (system 2, respectively), acting in Hilbert space 1 (in Hilbert space 2, respectively). Both Hilbert spaces have dimension N . In Eq. (1) the matrix V has rank one and carries the tunneling matrix element \mathcal{V} . In Eq. (2) the real and symmetric Hamiltonian matrix

H_{tr} acts in transition space and has dimension k . In what follows, the matrix indices μ, μ' denote states in Hilbert space 1 and run from 1 to N . The matrix indices ν, ν' denote states in Hilbert space 2. For the model of Eq. (1), ν, ν' run from $N+1$ to $2N$. For the model of Eq. (2), these indices run from $N+k+1$ to $2N+k$. The states in transition space are labeled m, n . These indices run from $N+1$ to $N+k$.

In its most general form, the rank-one coupling matrix in Eq. (1) is written as

$$V_{\mu\nu} = (O_1)_{\mu N} \mathcal{V}(O_2)_{(N+1)\nu}. \quad (3)$$

Here O_1 and O_2 are two arbitrary N -dimensional orthogonal matrices in space 1 (space 2, respectively). The parameter \mathcal{V} measures the strength of the tunneling process. In Eq. (2), the coupling matrices V_1 and V_2 are real and have N rows and k columns each. The upper index T denotes the transpose. In their most general form, V_1 and V_2 can be written as [12]

$$\begin{aligned} (V_1)_{\mu m} &= \sum_{m'} (O_1)_{\mu m'} \mathcal{V}_{1,m'} (O_{\text{tr},1})_{mm'}, \\ (V_2)_{\nu m} &= \sum_{m'} (O_2)_{\nu m'} \mathcal{V}_{2,m'} (O_{\text{tr},2})_{mm'}. \end{aligned} \quad (4)$$

Here O_1 (O_2) are N -dimensional orthogonal matrices in space 1 (in space 2, respectively) while $O_{\text{tr},1}$ and $O_{\text{tr},2}$ are orthogonal matrices of dimension k in transition space. The $2k$ real parameters $\mathcal{V}_{1,m'}$ and $\mathcal{V}_{2,m'}$ determine the strengths of the couplings.

The scattering problem mentioned in Section I is defined by coupling the states in space 1 to open channels labeled $(1a, 1a', 1a'', \dots)$ via real matrix elements $W_{1,a\mu}$, those in space 2 to open channels labeled $(2b, 2b', 2b'', \dots)$ via real matrix elements $W_{2,b\nu}$. The relations

$$\begin{aligned} \sum_{\mu} W_{1,a\mu} W_{1,a'\mu} &= \delta_{aa'} v_{1a}^2, \\ \sum_{\nu} W_{2,b\nu} W_{2,b'\nu} &= \delta_{bb'} v_{2b}^2 \end{aligned} \quad (5)$$

rule out direct scattering processes $(1a) \rightarrow (1a')$ and $(2b) \rightarrow (2b')$ without intermediary formation of the compound systems 1 or 2, respectively. The factor v_{1a}^2 (v_{2b}^2) measures the strength of the coupling of system 1 to channel $(1a)$ (of system 2 to channel $(2b)$, respectively). To account for multiple backscattering processes $(1a) \leftrightarrow$ system 1 and $(2b) \leftrightarrow$ system 2, we define the width matrices

$$\begin{aligned} (\Gamma_1)_{\mu\mu'} &= 2\pi \sum_a W_{1,a\mu} W_{1,a\mu'}, \\ (\Gamma_2)_{\nu\nu'} &= 2\pi \sum_b W_{2,b\nu} W_{2,b\nu'}. \end{aligned} \quad (6)$$

These matrices are part of the total width matrix Γ . For Eq. (1), we define

$$\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \quad (7)$$

while for Eq. (2) we use

$$\Gamma = \begin{pmatrix} \Gamma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma_2 \end{pmatrix}. \quad (8)$$

With these definitions, the elements $S_{1a,2b}(E)$ of the scattering matrix for the reaction $(1a) \rightarrow (2b)$ at energy E are given by [4]

$$S_{1a,2b}(E) = -2i\pi \sum_{\mu\nu} W_{1,a\mu} D_{\mu\nu}^{-1}(E) W_{2,b\nu}. \quad (9)$$

Here $D^{-1}(E)$ is the propagator matrix with inverse

$$D(E) = E\mathbf{1} - H + (i/2)\Gamma, \quad (10)$$

and $\mathbf{1}$ is the unit matrix in the Hilbert space of the Hamiltonian H defined in Eqs. (1) or (2).

III. STATISTICAL ASSUMPTIONS

As mentioned in the Introduction, the elements (9) of the scattering matrix, although well defined, cannot be worked out analytically without further simplifying assumptions on the Hamiltonian H . We assume that the excitation energies of system 1 and of system 2 are both sufficiently large to justify a statistical treatment. We accordingly assume that the matrices H_1 and H_2 are statistically independent members of the Gaussian Orthogonal Ensemble (GOE) of random matrices. The elements are zero-centered real Gaussian random variables with second moments

$$\begin{aligned} \langle (H_1)_{\mu_1\mu'_1} (H_1)_{\mu_2\mu'_2} \rangle &= \frac{\lambda^2}{N} (\delta_{\mu_1\mu_2} \delta_{\mu'_1\mu'_2} + \delta_{\mu_1\mu'_2} \delta_{\mu'_1\mu_2}), \\ \langle (H_2)_{\nu_1\nu'_1} (H_2)_{\nu_2\nu'_2} \rangle &= \frac{\lambda^2}{N} (\delta_{\nu_1\nu_2} \delta_{\nu'_1\nu'_2} + \delta_{\nu_1\nu'_2} \delta_{\nu'_1\nu_2}) \end{aligned} \quad (11)$$

The angular brackets denote the ensemble average. The parameter λ defines the ranges of the two spectra. We eventually consider the limit $N \rightarrow \infty$, keeping λ and k fixed. We assume that the k eigenvalues E_m of H_{tr} are all located near the centers of the spectra of H_1 and H_2 .

In Refs. [8, 12] it is shown that \mathcal{V} in Eq. (3) must be small in magnitude compared to λ , and that the same must hold for all m' of the parameters $\mathcal{V}_{1,m'}$ and $\mathcal{V}_{2,m'}$ in Eqs. (4). These bounds imply that the coupling matrix V in Eq. (1) and the coupling matrices V_1 and V_2 in Eq. (2) connect mostly those eigenstates of H_1 and of H_2 to H_{tr} that lie within or close to the spectrum of H_{tr} . It is only in that range of energies (which we assume to be small compared to λ) that the statistics of eigenvalues and eigenfunctions of H_1 and of H_2 defined via Eqs. (11) are actually needed. In other words, we use Eqs. (11) for a quantification of the *local* spectral fluctuation properties of H_1 and H_2 only, not for a quantitative parametrization of their overall spectra. That is in line with the use of random-matrix theory in the statistical theory of nuclear reactions [6].

The average transition probability discussed in Section IV below is parametrized in terms of transmission coefficients

$T_{1,a}$ and $T_{2,b}$. These auxiliary parameters are familiar from the theory of compound-nucleus reactions [9]. They are defined for two scattering problems obtained by putting $\mathcal{V} = 0$ in Eq. (1) (or, equivalently, by putting $V_1 = 0 = V_2$ in Eq. (2)), i.e., for the case without any coupling of system 1 and system 2. The associated matrices for backscattering from channel (1a) to channel (1a') and for backscattering from channel (2b) to channel (2b') are

$$\begin{aligned} (S_1)_{1a,1a'}(E) &= \delta_{aa'} - 2i\pi \sum_{\mu\mu'} W_{1,a\mu} D_1^{-1}{}_{\mu\mu'} W_{1,a'\mu'} , \\ (S_2)_{2b,2b'}(E) &= \delta_{bb'} - 2i\pi \sum_{\nu\nu'} W_{2,b\nu} D_2^{-1}{}_{\nu\nu'} W_{2,b'\nu'} , \end{aligned} \quad (12)$$

where

$$\begin{aligned} (D_1)_{\mu\mu'}(E) &= E\delta_{\mu\mu'} - (H_1)_{\mu\mu'} + (i/2)(\Gamma_1)_{\mu\mu'} , \\ (D_2)_{\nu\nu'}(E) &= E\delta_{\nu\nu'} - (H_2)_{\nu\nu'} + (i/2)(\Gamma_2)_{\nu\nu'} , \end{aligned} \quad (13)$$

and where H_1 and H_2 both are members of the GOE and obey the statistical assumptions (11). The matrix $S_1(E)$ ($S_2(E)$) is equal to the scattering matrix for compound-nucleus scattering [9] where the compound nucleus is described by the GOE Hamiltonian H_1 (H_2 , respectively). The transmission coefficients $T_{1,a}$ and $T_{2,b}$ are defined by

$$\begin{aligned} T_{1,a} &= 1 - |\langle (S_1)_{1a,1a}(E) \rangle|^2 , \\ T_{2,b} &= 1 - |\langle (S_2)_{2b,2b}(E) \rangle|^2 . \end{aligned} \quad (14)$$

The angular brackets denote the average over H_1 (over H_2 , respectively). These coefficients measure the absorption of flux from the channel to the compound system or, equivalently, the emission of flux from the compound system to the channel in question.

The case of strongly overlapping resonances mentioned in Section I is formally defined for system 1 (for system 2) by the inequality $\sum_a T_{1,a} \gg 1$ ($\sum_b T_{2,b} \gg 1$, respectively). If either of these conditions holds, the results given below are obtained in terms of an asymptotic expansion in inverse powers of $\sum_a T_{1,a}$ ($\sum_b T_{2,b}$, respectively) where only terms of leading order are kept. For the scattering matrices $S_1(E)$ and $S_2(E)$ that approximation gives [13] for $a \neq a'$ and for $b \neq b'$

$$\begin{aligned} \langle |(S_1)_{1a,1a'}(E)|^2 \rangle &= T_{1,a} \mathcal{T}_{1,a'} , \\ \langle |(S_2)_{2b,2b'}(E)|^2 \rangle &= T_{2,b} \mathcal{T}_{2,b'} . \end{aligned} \quad (15)$$

The factors

$$\mathcal{T}_{1,a'} = \frac{T_{1,a'}}{\sum_{a''} T_{1,a''}} , \quad \mathcal{T}_{2,b'} = \frac{T_{2,b'}}{\sum_{a''} T_{2,b''}} \quad (16)$$

give the relative probability for decay of the compound nucleus into channel (1a') (channel (2b'), respectively).

IV. TOTAL TRANSITION PROBABILITY FOR FISSION AND FUSION

With $S_{1a,2b}(E)$ given by Eq. (9), the average transition probability $P_{1a,2b}(E)$ at energy E is given by

$$P_{1a,2b}(E) = \langle |S_{1a,2b}(E)|^2 \rangle . \quad (17)$$

The average is over the matrix elements of H_1 and of H_2 . The symmetry of $S_{1a,2b}(E)$ implies $P_{1a,2b}(E) = P_{2b,1a}(E)$.

It may be instructive to mention why it is not possible to perform the average in Eq. (17) analytically without additional simplifying assumptions. The simpler problem of calculating $\langle |S_{1a,1a'}(E)|^2 \rangle$ in Eq. (15) analytically without resorting to the approximation $\sum_{a'} T_{1,a'} \gg 1$ was solved [9] with the help of the supersymmetry technique. Inspection of Ref. [9] shows that for two coupled GOE's, the number of variables in the supersymmetry approach is so large as to defy an analytical solution. Therefore, the explicit expressions for the transition probability through or over a barrier derived in Refs. [8, 12] are obtained with the help of further simplifying assumptions.

As mentioned in the Introduction we focus attention here on cases where at least in one of the two systems the compound resonances overlap strongly. The technical aspects of that assumption are defined at the end of Section III. Even within that assumption, the resulting expressions cannot be derived without use of the supersymmetry technique [8]. Results derived on that basis in Refs. [8, 12] are used in Sections V and VI.

We recall that the total average probability for neutron-induced fission and that for heavy-ion fusion are given, respectively, by

$$\begin{aligned} P_{\text{fission},1a}(E) &= \sum_b P_{1a,2b}(E) \\ P_{\text{fusion},2b}(E) &= \sum_a P_{1a,2b}(E) . \end{aligned} \quad (18)$$

V. NEUTRON-INDUCED FISSION

Depending on the energy of the incident neutron we distinguish the cases where the excitation energy of the compound system (system 1) is far below, below but close to, or above the height of the fission barrier. In each of these cases we further distinguish the case where the resonances in the compound system (system 1) do not overlap or overlap weakly and the case where they overlap strongly. In all cases we assume that the excitation energy of system 2 is in the regime of strongly overlapping resonances.

A. Excitation energy far below the height of the fission barrier

The tunneling matrix element \mathcal{V} is so small in magnitude that lowest-order perturbation theory in \mathcal{V} suffices. We use Eq. (34) of Ref. [8] and the fact that for strongly overlapping resonances in system 2 the last factor in that equation is equal to $\mathcal{T}_{2,b}$ defined in Eq. (16). With $\sum_b \mathcal{T}_{2,b} = 1$ the first of Eqs. (18) gives

$$\begin{aligned} P_{\text{fission},1a}(E) &= (\mathcal{V}/\lambda)^2 \\ &\times \langle |\sqrt{2\pi\lambda} \sum_{\mu} W_{1,a\mu} [(E - H_1 + (i/2)\Gamma_1)^{-1}]_{\mu N}|^2 \rangle . \end{aligned} \quad (19)$$

Here and in what follows we interpret the dimensionless factor $(\mathcal{V}/\lambda)^2$ as the energy-dependent probability $P_{\text{tun}}(E)$ for tunneling through the fission barrier and write

$$(\mathcal{V}/\lambda)^2 = P_{\text{tun}}(E). \quad (20)$$

Then

$$P_{\text{fission},1a}(E) = P_{\text{tun}}(E) \times \left\langle \left| \sqrt{2\pi\lambda} \sum_{\mu} W_{1,a\mu} [(E - H_1 + (i/2)\Gamma_1)^{-1}]_{\mu N} \right|^2 \right\rangle. \quad (21)$$

The factor in angular brackets in Eq. (21) is given in Eq. (A4) of Ref. [8]. That expression involves the threefold integral well known from the theory of compound-nucleus reactions [9]. For the case of overlapping resonances in system 1, that factor is equal to $\mathcal{T}_{1,a}$, and Eq. (21) becomes

$$P_{\text{fission},1a}(E) = \mathcal{T}_{1,a} P_{\text{tun}}(E). \quad (22)$$

For first-order perturbation theory to apply, i.e., for Eq. (21) to hold, $P_{\text{tun}}(E)$ must be less than 0.1 or so because the correction term is of order $1/(1 + P_{\text{tun}}(E))^2$, see Eq. (23).

B. Excitation energy close to but below the height of the fission barrier

If $P_{\text{tun}}(E)$ in Eq. (20) obeys $0.1 \leq P_{\text{tun}}(E) < 1$ an analytical solution for $P_{\text{fission},1a}(E)$ is available only if the resonances in the compound system (system 1) strongly overlap. Then Eqs. (31) and (32) of Ref. [8] give

$$P_{\text{fission},1a}(E) = \mathcal{T}_{1,a} \frac{P_{\text{tun}}(E)}{(1 + P_{\text{tun}}(E))^2}. \quad (23)$$

The term in the denominator accounts for repeated tunneling back and forth through the barrier.

C. Passage over the barrier via a set of transition states

Again, an analytical expression is available only if the resonances in the compound system (system 1) strongly overlap. Then, Eqs. (25), (30), and (16) of Ref. [12] give

$$P_{\text{fission},1a}(E) = \mathcal{T}_{1,a} Y \quad (24)$$

where

$$Y = (1/\lambda^2) \sum_{\mu\nu} \left| \sum_{mn} (V_1)_{\mu m} (G_{\text{tr}})_{mn} (V_2)_{\nu n} \right|^2 \quad (25)$$

and

$$G_{\text{tr}} = (\mathbf{E} - H_{\text{tr}} + iV_1^T V_1/\lambda + iV_2^T V_2/\lambda)^{-1} = (\mathbf{E} - H_{\text{eff}})^{-1}. \quad (26)$$

Here \mathbf{E} is the product of the energy E of the reaction and of the unit matrix in transition space. The operator G_{tr} is the effective propagator in transition space, and H_{eff} is the effective

Hamiltonian in that space. It consists of the real symmetric Hamiltonian H_{tr} in Eq. (2) and an imaginary symmetric matrix that is due to the coupling of transition space with space 1 and space 2 via the matrices V_1 and V_2 in Eq. (2). For a physical interpretation of Y we introduce the average GOE level spacing $d = 2\pi\lambda/N$ and obtain

$$Y = \frac{1}{N^2} \sum_{\mu\nu} \left| \frac{2\pi}{d} \sum_{mn} (V_1)_{\mu m} (G_{\text{tr}})_{mn} (V_2)_{\nu n} \right|^2. \quad (27)$$

The expression within absolute signs is the dimensionless amplitude for passing via the transition region from state μ in system 1 to state ν in system 2, and Y is the average (taken over states in space 1 and space 2) of the square of that amplitude. In applications, d^2 must be replaced by the product $d_1 d_2$ of the mean level spacings of system 1 and system 2.

The propagator $G_{\text{tr}}(E)$ in Eq. (26) can be written in terms of the complex eigenvalues \mathcal{E}_l with $\Im(\mathcal{E}_l) < 0$ for all $l = 1, \dots, k$ and the complex orthonormal eigenfunctions ψ_l , $l = 1, \dots, k$, of H_{eff} . The factor Y in Eq. (25) takes the form [12]

$$Y = \frac{1}{N^2} \sum_{mn} \left| \sum_l \zeta_{1,ml} \frac{1}{E - \mathcal{E}_l} \zeta_{2,nl} \right|^2. \quad (28)$$

Except for a factor $(2\pi/d)^{1/2}$, the parameters $\zeta_{1,ml}$ and $\zeta_{2,nl}$ are projections of the eigenfunctions ψ_l onto the matrices V_1 and V_2 , respectively. The parameters $\zeta_{1,ml}$ and $\zeta_{2,nl}$ have dimension (energy) $^{1/2}$. The amplitude within absolute square signs in Eq. (28) is the sum of k overlapping Breit-Wigner resonances. A similar but less general result was obtained in Ref. [14] under restrictive assumptions on the matrices V_1 and V_2 .

D. Discussion

In deriving expressions (21) to (27) we have used that $P_{1a,2b}(E)$ in Eq. (17) factorizes, one factor being given by $\mathcal{T}_{2,b}$. Such factorization is the result of the statistical assumption (11) for H_2 and of the inequality $\sum_b T_{2,b} \gg 1$. Factorization makes it possible to sum over exit channels explicitly and to obtain closed expressions for $P_{\text{fission},1a}(E)$. Eqs. (22, 23) and (24) show that factorization likewise holds with respect to the entrance-channel dependence of $P_{\text{fission},1a}(E)$, owing to the statistical assumption (11) on H_1 and the inequality $\sum_a T_{1,a} \gg 1$. In all these cases, the orthogonal invariance of the GOE removes all reference to intermediate states in system 2 and system 1 and leaves us for $P_{\text{fission},1a}(E)$ with a first factor that depends only upon transmission coefficients, and a second factor that depends only upon the dynamics of the transition through or over the barrier. Factorization in Eq. (21) is the result of first-order perturbation theory and the statistical assumption (11) on H_1 without any additional conditions on the resonances in system 1.

When written in the form

$$\langle |(S_1)_{1a,1a'}(E)|^2 \rangle = \mathcal{T}_{1,a} T_{1,a'}, \quad (29)$$

Eq. (15) for the average compound-nucleus reaction probability displays a striking similarity to $P_{\text{fission},1a}(E)$ in Eqs. (22, 23) and (24). The factor $\mathcal{T}_{1,a}$ appears in all these expressions and accounts for compound-nucleus formation. The factor $T_{1,a'}$ in Eq. (29) (which accounts for the decay of the compound nucleus) is in Eqs. (22, 23) and (24) replaced by a factor that is given in terms of $P_{\text{tun}}(E)$ or of Y and accounts for transmission through or over the barrier. The difference between Eq. (29) and the fission probability is that in Eq. (29) the sum in the denominator of $\mathcal{T}_{1,a}$ extends over all channels. In $P_{\text{fission},1a}(E)$ that same sum does not comprise the fission channel. With that slight proviso our results show that neutron-induced fission may be seen as a compound-nucleus reaction that feeds the fission channel.

For practical applications it is important that the factor $\mathcal{T}_{1,a}$ in Eqs. (22, 23), and (24) occurs likewise in the parametrization of the compound-nucleus cross section for strongly overlapping resonances in Eqs. (15, 16). Calculations of compound-nucleus reaction cross sections are completely standard. The factor $\mathcal{T}_{1,a}$ is, therefore, known very precisely and easily available. The factor in the second line of Eq. (19) involves the threefold integral of compound-nucleus reaction theory. It can be obtained by a slight modification of the standard numerical program for that integral. We conclude that the channel-dependent factors in Eqs. (19, 22, 23) and (24) can all be calculated reliably within standard compound-nucleus reaction theory.

In principle, the probability $P_{\text{tun}}(E)$ in Eqs. (21) and (22) can be computed semiclassically with the help of the WKB approximation and the use of collective variables. For spontaneous fission, that is the standard procedure, see Ref. [16] and references therein. However, as stated in the Introduction, the present theory deals with states of fixed spin J and parity π of the neutron-induced fission reaction and $P_{\text{tun}}(E)$, written explicitly as $P_{\text{tun}}(E; J, \pi)$, must, in principle, be determined separately for each pair of values (J, π) . We expect that collective motion and, therefore, $P_{\text{tun}}(E; J, \pi)$ is fairly independent of (J, π) . We, thus, expect that $P_{\text{fission},1a}(E)$ in Eqs. (21, 22) and (23) can be estimated fairly reliably. For the case of passage over the barrier, the parameters \mathcal{E}_l and $\zeta_{1,ml}$, $\zeta_{2,ml}$ in Eq. (28) are also determined by collective features of the system. Therefore, we expect that here, too, Y is nearly independent of spin and parity of the system. A prediction of value and energy dependence of Y has to be based upon a theoretical model for the transition Hamiltonian H_{tr} , i.e., a collective dynamical model for the transition states and their coupling to the states in system 1 and system 2. That seems realistic only for $k = 1$ and $k = 2$ because the number of parameters in Eq. (28) increases strongly with increasing k .

The total cross section $\sigma_{\text{fission}}(E)$ for neutron-induced fission at energy E is obtained by multiplying the fission probability $P_{\text{fission},1a}(E; J, \pi)$ (now written with its full dependence on J and π) with an entrance-channel dependent factor $C_{1,a}(E; J, \pi)$ that depends upon geometrical and kinematical factors, and summing over both values of π and all values of J participating in the reaction,

$$\sigma_{\text{fission}}(E) = \sum_{J,\pi} C_{1,a}(E; J, \pi) P_{\text{fission},1a}(E; J, \pi). \quad (30)$$

We do not write $C_{1,a}(E; J, \pi)$ in full because that expression is well known, is lengthy, and only deflects attention from the essential elements of the theory. The explicit expression may be found, for instance, in Ref. [15].

For tunneling through the barrier we consider as an example Eq. (23). For brevity we define

$$\mathcal{P}_{\text{tun}}(E; J, \pi) = \frac{P_{\text{tun}}(E; J, \pi)}{(1 + P_{\text{tun}}(E; J, \pi))^2}. \quad (31)$$

Use of Eq. (31) in Eq. (23) and insertion of the latter into Eq. (30) gives

$$\sigma_{\text{fission}}(E) = \sum_{J,\pi} C_{1,a}(E; J, \pi) \mathcal{T}_{1,a}(J, \pi) \mathcal{P}_{\text{tun}}(E; J, \pi). \quad (32)$$

If $P_{\text{tun}}(E; J, \pi)$ depends upon (J, π) only weakly,

$$P_{\text{tun}}(E; J, \pi) \approx P_{\text{tun}}(E), \quad (33)$$

Eq. (32) takes the form

$$\begin{aligned} \sigma_{\text{fission}}(E) &\approx \left(\sum_{J,\pi} C_{1,a}(E; J, \pi) \mathcal{T}_{1,a}(J, \pi) \right) \mathcal{P}_{\text{tun}}(E) \\ &= \sigma_{\text{compound}}(E) \mathcal{P}_{\text{tun}}(E). \end{aligned} \quad (34)$$

The last line defines the total cross section $\sigma_{\text{compound}}(E)$ for compound-nucleus formation. The value of $\sigma_{\text{compound}}(E)$ is completely determined by nuclear reaction data and can be calculated reliably. The ratio $\sigma_{\text{fission}}(E)/\sigma_{\text{compound}}(E)$ yields $\mathcal{P}_{\text{tun}}(E)$. If the approximation (33) does not apply, the ratio

$$\sigma_{\text{fission}}(E)/\sigma_{\text{compound}}(E) = \langle \mathcal{P}_{\text{tun}}(E; J, \pi) \rangle \quad (35)$$

gives the weighted average of $\mathcal{P}_{\text{tun}}(E; J, \pi)$ taken over the values of (J, π) that participate in the reaction. Analogous conclusions hold for the results in Eqs. (21), (23), and (24).

The values of $\mathcal{P}_{\text{tun}}(E)$, of $\langle \mathcal{P}_{\text{tun}}(E; J, \pi) \rangle$, of Y , and of $\langle Y \rangle$ obtained from data on total fission cross sections may serve as tests for collective models of barrier penetration. That is true, in particular, of the energy dependence of $P_{\text{tun}}(E)$ which relates directly to the shape of the barrier. For small neutron energies, $P_{\text{tun}}(E)$ should be close to the barrier penetration factor for spontaneous fission in neighboring nuclei.

VI. HEAVY-ION FUSION

As an example we consider the case where the excitation energies both of system 1 and of system 2 are in the regime of overlapping resonances. Other cases may be discussed in analogy to Section V D. We use the second of Eqs. (18). For the case of barrier penetration, summation of the result in Eq. (32) of Ref. [8] over channels (1a), the first of Eqs. (31) of Ref. [8], and Eq. (31) give

$$P_{\text{fusion},2b}(E) = \mathcal{T}_{2,b} \mathcal{P}_{\text{tun}}(E) \quad (36)$$

while for barrier transition via a set of k transition states Eq. (25) of Ref. [12] gives

$$P_{\text{fusion},2b}(E) = \mathcal{T}_{2,b}Y . \quad (37)$$

Here $P_{\text{tun}}(E)$ and Y are defined in Eqs. (20) and (25), respectively.

For practical applications, numerator and denominator of the factor $\mathcal{T}_{2,b}$ in Eqs. (36) and (37) are given for each value of J and π in terms of the transmission coefficients $T_{2,b'}$ for channels (2 b') containing two heavy ions each in the ground or in an excited state, see Eqs. (16). These transmission coefficients can be calculated from the optical model for elastic scattering. The optical-model potential for scattering of two heavy nuclei seems less well investigated both phenomenologically and theoretically than that for nucleon-nucleus scattering, see, for instance, Refs. [17, 18]. Evaluation of the denominator in $\mathcal{T}_{2,b}$ requires, in addition, the average level densities of the fission products. These are, in general, known as well as that of the target nucleus in a neutron-induced reaction. We conclude that the factor $\mathcal{T}_{2,b}$ in Eqs. (36, 37) can perhaps not be calculated as precisely as $\mathcal{T}_{1,a}$ for a neutron-induced fission reaction but can be predicted at least semiquantitatively from known nuclear data.

As for the factors that determine penetration through or transition over the barrier, much of the discussion in Section V D applies in the present case as well and is not repeated here.

The total average fusion probability in Eqs. (36, 37) is similar in form to the probability of a compound-nucleus reaction that starts in channel (2 b) and feeds transmission through or over the barrier. The probability for that to happen is determined by the factors $P_{\text{tun}}(E)$ and Y . In full analogy to Section V D, these may be determined theoretically with the help of collective variables.

In analogy to Eq. (30), the total fusion cross section is written as

$$\sigma_{\text{fusion}}(E) = \sum_{J\pi} C_{2,b}(E; J, \pi) P_{\text{fusion},2b}(E; J, \pi) . \quad (38)$$

With the help of Eq. (36) that gives

$$\sigma_{\text{fusion}}(E) = \sum_{J\pi} C_{2,b}(E; J, \pi) \mathcal{T}_{2,b}(J, \pi) \mathcal{P}_{\text{tun}}(E; J, \pi) , \quad (39)$$

and correspondingly for Eq. (37). Defining the cross section for merger as

$$\sigma_{\text{merger}}(E) = \sum_{J\pi} C_{2,b}(E; J, \pi) \mathcal{T}_{2,b}(J, \pi) , \quad (40)$$

we note that $\sigma_{\text{merger}}(E)$ is available from nuclear data and conclude that $\mathcal{P}_{\text{tun}}(E)$ or $\langle \mathcal{P}_{\text{tun}}(E; J, \pi) \rangle$ may be determined from the ratio of the measured total fusion cross section and the cross section for merger, and correspondingly for Y and $\langle Y \rangle$.

VII. CONCLUSIONS

For neutron-induced fission, we have modeled both the compound nucleus generated by absorption of the incident neutron and the nuclear system near the scission point as two statistically independent dynamical systems that can be described in terms of random-matrix theory. For heavy-ion fusion, we have used the same model for the compound system reached after merger of the two ions, and for the compound nucleus at the other side of the barrier. In both cases we have assumed that, seen from the entrance channel, the system on the other side of the barrier is in the regime of strongly overlapping resonances. That makes it possible to derive, for fixed quantum numbers (J, π) , explicit expressions for the average probability for neutron-induced fission and for heavy-ion fusion leading to specific final channels. These expressions factorize. The first factor describes the reaction up to the point where the system has passed the barrier. The second factor describes how the resulting highly excited system de-excites and/or decays into fragments. Because of our statistical assumptions these two factors are completely independent of each other, in the same sense in which the decay of the compound nucleus in the regime of strongly overlapping resonances is independent of its mode of formation.

Summation over all final channels yields the total average probabilities in Eqs. (18) for neutron-induced fission or heavy-ion fusion. These expressions are independent of what happens after the system has passed the barrier. The expressions (18) relate the probability for neutron-induced fission and for heavy-ion fusion to elements of the statistical theory of nuclear reactions. Parts of these expressions can, therefore, be calculated accurately within existing compound-nucleus theory. The parts of the theory that go beyond the standard statistical theory are the probabilities for penetration through or passage over the fission barrier. These must be calculated within a model for collective motion or, alternatively, may be determined experimentally from total fission or fusion cross sections. In that way our approach promises direct insight into the dynamics of barrier penetration and of transmission over the barrier.

The present theory does not specify what happens after the system has passed the barrier. In principle, that question must be addressed separately and independently for every value of (J, π) . It is conceivable, however, that collective dynamics beyond the barrier provides a description independent of (J, π) . In that case, the fission (fusion) reaction would be described by two factors: The total cross section for fission (fusion) would be multiplied by the probability (given by the collective dynamics) for the system on the other side of the barrier to decay into a series of reaction products.

-
- [1] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
- [2] G. F. Bertsch and K. Hagino, Phys. Rev. C **105**, 034618 (2022).
- [3] M. B. Chadwick *et al.*, Nuclear Data Sheets **112**, 2887 (2011).
- [4] C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions*, North-Holland Publishing Company, Amsterdam (1969).
- [5] M. L. Mehta, *Random Matrices*, Third Edition, Elsevier, Amsterdam (2006).
- [6] G. E. Mitchell, A. Richter, and H. A. Weidenmüller, Rev. Mod. Phys. **82**, 2645 (2010).
- [7] T. Kawano, P. Talou, and H. A. Weidenmüller, Phys. Rev. C **92**, 044617 (2015).
- [8] H. A. Weidenmüller, Phys. Rev. E **105**, 044143 (2022).
- [9] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 368 (1985).
- [10] K. Hagino and G. F. Bertsch, Phys. Rev. E **104**, L052104 (2021).
- [11] G. F. Bertsch and K. Hagino, J. Phys. Soc. Jpn. **90**, 114005 (2021).
- [12] H. A. Weidenmüller, Phys. Rev. E **109**, 034117 (2024).
- [13] J. J. M. Verbaarschot, Ann. Phys. (N.Y.) **168**, 368 (1986).
- [14] K. Hagino and G. F. Bertsch, arXiv:2310.09537.
- [15] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).
- [16] N. Schunck and D. Regnier, Prog. Part. Nucl. Phys. **125**, 103963 (2022).
- [17] Dao T. Khoa, Nguyen Hoang Phuc, Doan Thi Loan, and Bui Minh Loc, Phys. Rev. C **94**, 034612 (2016).
- [18] T. Furumoto, K. Tsubakihara, S. Ebata, and W. Horiuchi, Phys. Rev. C **99**, 034605 (2019).