# Quantitative Results on Symplectic Barriers 

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#### Abstract

In this paper we present some quantitative results concerning symplectic barriers. In particular, we answer a question raised by Sackel, Song, Varolgunes, and Zhu regarding the symplectic size of the $2 n$-dimensional Euclidean ball with a codimension-two linear subspace removed.


## 1 Introduction and Results

In a recent work 6, we established a new type of rigidity for symplectic embeddings that originates from obligatory intersections with symplectic submanifolds. Inspired by the terminology introduced by Biran [2] for analogous results regarding Lagrangian submanifolds, we refer to such symplectic obstructions as symplectic barriers. More precisely, if $(M, \omega)$ is a $2 n$-dimensional symplectic manifold, a symplectic submanifold $S \subset M$ is said to be a symplectic barrier if

$$
c(M \backslash S, \omega)<c(M, \omega)
$$

where $c$ is some (normalized) symplectic capacity. Recall that symplectic capacities are numerical invariants that, roughly speaking, measure the size of a symplectic manifold (see, e.g., Chapter 2 of 9$]$ ). More precisely,

Definition 1.1. A symplectic capacity is a map which associates to every symplectic manifold $(M, \omega)$ an element of $[0, \infty]$ with the following properties:

- $c(M, \omega) \leq c(N, \tau)$ if $(M, \omega) \stackrel{\mathrm{s}}{\hookrightarrow}(N, \tau)$ (Monotonicity)
- $c(M, \alpha \omega)=|\alpha| c(M, \omega)$ for all $\alpha \in \mathbb{R}, \alpha \neq 0$ (Conformality)
- $c\left(B^{2 n}(1)\right)=1=c\left(Z^{2 n}(1)\right)$ (Nontriviality and Normalization)

Here $B^{2 n}(r)$ denotes the $2 n$-dimensional Euclidean ball with radius $\sqrt{r / \pi}$, $Z^{2 n}(r)$ denotes the cylinder $B^{2}(r) \times \mathbb{R}^{2 n-2}$, and $\stackrel{\mathrm{s}}{\hookrightarrow}$ stands for a symplectic embedding. Note that both the ball $B^{2 n}(r)$ and the cylinder $Z^{2 n}(r)$ are equipped with the standard symplectic form $\omega=d x \wedge d y$ on $\mathbb{R}^{2 n}$. Two examples of symplectic capacities, which naturally arise from Gromov's celebrated non-squeezing theorem [5], are the Gromov width and the cylindrical capacity:

$$
\underline{c}(M)=\sup \left\{r: B^{2 n}(r) \stackrel{\mathrm{s}}{\hookrightarrow} M\right\}, \quad \bar{c}(M)=\inf \left\{r: M \stackrel{s}{\hookrightarrow} Z^{2 n}(r)\right\} .
$$

Another important example is the Hofer-Zhender capacity $c_{\mathrm{HZ}}$, which is closely related with Hamiltonian dynamics (see, e.g., 8]). It follows immediately from

Definition 1.1 that $\underline{c}(M, \omega) \leq c(M, \omega) \leq \bar{c}(M, \omega)$ for any symplectic capacity $c$. For more information on symplectic capacities, see e.g., the survey [4].

In this paper we present some quantitative results concerning symplectic barriers. Our first result answers a question raised in Section 6.2 of [11] regarding the capacity of the $2 n$-dimensional Euclidean ball in $\mathbb{R}^{2 n}$ with a codimensiontwo linear subspace removed, where the classical Kähler angle is used to measure the "defect" of the subspace from being complex. More precisely, let $E_{t} \subset \mathbb{R}^{2 n}$ be a codimension-two linear subspace with Kähler angle $t$, i.e, the unit outer normals $n_{1} \perp n_{2}$ to $E_{t}$ satisfy $\left|\omega\left(n_{1}, n_{2}\right)\right|=t$. The following result shows that the $(2 n-2)$-ball $E_{t} \cap B^{2 n}(1)$ is a symplectic barrier in $B^{2 n}(1)$ when $0 \leq t<1$.

Theorem 1.2. Let $n>1$ and $0 \leq t \leq 1$. For any symplectic capacity $c$ one has

$$
c\left(B^{2 n}(1) \backslash E_{t}\right)=\frac{1+t}{2} .
$$

We remark that the complex case $t=1$ follows from Proposition 1.6 in [6] (cf. Theorem 3.1.A in [10]). For the Gromov width, the Lagrangian case $t=0$ follows from [2], where it is proved that $\underline{c}\left(\mathbb{C P}^{n} \backslash \mathbb{R} \mathbb{P}^{n}\right)=\frac{1}{2}$. Moreover, one can check that Theorem 1.3 in [11 implies that $\underline{c}\left(B^{2 n}(1) \backslash E_{0}\right)=\bar{c}\left(B^{2 n}(1) \backslash E_{0}\right)=\frac{1}{2}$.

Our next result concerns the symplectic barriers introduced in 6. For $\varepsilon>0$ denote by $\Sigma_{\varepsilon}$ the following union of symplectic codimension-two subspaces in $\mathbb{R}^{2 n} \simeq \mathbb{C}^{n}$ :

$$
\Sigma_{\varepsilon}:=\bigcup\left\{\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in \mathbb{C}^{n}: z_{n} \in \varepsilon \mathbb{Z}^{2}\right\}
$$

Moreover, define $\Sigma_{\varepsilon}^{t}$ to be a linear image of $\Sigma_{\varepsilon}$ such that the Kähler angle of the corresponding planes is $t$, i.e., $\left|\omega\left(n_{1}, n_{2}\right)\right|=t$, where $n_{1} \perp n_{2}$ are the unit outer normals to the subspaces in $\Sigma_{\varepsilon}^{t}$. Note that any two such configurations of subspaces with the same Kähler angle are unitarily equivalent. We also note that when $\varepsilon$ is sufficiently large, the intersection $B^{2 n}(1) \cap \Sigma_{\varepsilon}^{t}$ becomes $B^{2 n}(1) \cap E_{t}$, where $E_{t}$ is a single codimension-two subspace of Kähler angle $t$ as above. Thus we are especially interested in the case when $\varepsilon$ is small. In [6] it was proved that for small $\varepsilon$, the configurations $\Sigma_{\varepsilon}^{t}$ are symplectic barriers of the ball $B^{2 n}(1)$ with respect to any (normalized) symplectic capacity. Here we provide more precise bounds for the symplectic size of the complement of $\Sigma_{\varepsilon}^{t}$ in $B^{2 n}(1)$ when $\varepsilon \rightarrow 0$.

Theorem 1.3. For any $t \in(0,1)$ and $n>1$

$$
\lim _{\varepsilon \rightarrow 0} c_{\mathrm{HZ}}\left(B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}\right)=\lim _{\varepsilon \rightarrow 0} \bar{c}\left(B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}\right)=t
$$

We suspect that Theorem 1.3 also holds for the Gromov width. This is supported by the following claim that provides an almost exact lower bound:

Theorem 1.4. For any $t \in(0,1)$, any $\varepsilon>0$, and $n>1$

$$
f(t) \leq \underline{c}\left(B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}\right)
$$

where $f(t)$ is an explicit function given by (5) below and satisfies $f(t) \geq t-0.07$.
While Theorem 1.2 shows that the complement of a single codimension-two linear subspace with Kähler angle $t$ has capacity $\frac{1+t}{2}$, the two theorems above
show that the symplectic size of the complement of a large number of such spaces is strictly smaller, and takes a value around the Kähler angle $t$.

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## 2 The Complement of a Single Subspace

In this section we prove Theorem 1.2 . We first introduce the following notations. Equip $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$ with coordinates $\left(z_{1}, \ldots, z_{n}\right)$, where $z_{j}=x_{j}+i y_{j}$, and with the standard symplectic form $\omega=d x \wedge d y$. Let $P_{t}$ be the real two-dimensional plane in $\mathbb{C}^{2}$ spanned by the two vectors $(s, t)$ and $(0, i)$, where $t, s$ are positive real numbers satisfying $t^{2}+s^{2}=1$. It is not hard to check that using a linear unitary transformation in $\mathbb{C}^{n}$, we can assume without loss of generality that any codimension-two linear space $E_{t} \subset \mathbb{R}^{2 n}$ with Kähler angle $t$ is of the form $P_{t} \times \mathbb{C}^{n-2}$. This implies that the proof of Theorem 1.2 is, roughly speaking, four-dimensional.

The strategy for the proof of Theorem 1.2 is as follows: first we prove the required lower bound for the Gromov width by an explicit embedding of a 4 dimensional ball in $B^{4}(1) \backslash P_{t}$ (Proposition 2.2), and then extend the argument to any dimension in Proposition 2.3. Next, in Proposition 2.4 we develop the main ingredient needed for the required upper bound for the cylindrical capacity, which in turn is proved in Proposition 2.12.

We start with some preparation. First, note that the plane $P_{t}$ lies in the hyperplane $\Sigma:=\left\{y_{1}=0\right\}$. Moreover, let $\operatorname{proj}_{z_{k}}: \mathbb{C}^{2} \rightarrow \mathbb{C}$ be the projection onto the $z_{k}$-plane, for $k=1,2$. Then, one has

$$
\operatorname{proj}_{z_{1}}\left(P_{t} \cap B^{4}(1)\right)=\left\{x_{1} \in[-s, s], y_{1}=0\right\} \text { and } \operatorname{proj}_{z_{2}}\left(P_{t} \cap B^{4}(1)\right)=E
$$

where $E$ is the ellipse with axes $[-t / \sqrt{\pi}, t / \sqrt{\pi}] \times\{0\},\{0\} \times[-1 / \sqrt{\pi}, 1 / \sqrt{\pi}]$, and area $t$ (see Figure 1).


Figure 1: The projections of $P_{t} \cap B^{4}(1)$.
Set $E^{+}=E \cap\left\{x_{2} \geq 0\right\}, E^{-}=E \cap\left\{x_{2} \leq 0\right\}, \partial E^{+}=\partial E \cap\left\{x_{2} \geq 0\right\}$, and $\partial E^{-}=\partial E \cap\left\{x_{2} \leq 0\right\}$. Note that any subset $U$ of the intersection $\Sigma \cap B^{4}(1)$
satisfying $\operatorname{proj}_{z_{2}}(U) \cap \partial E^{+}=\emptyset$ can be displaced from $P_{t}$ using a Hamiltonian diffeomorphism of $\Sigma \cap B^{4}(1)$ that sends points of the form $\left(x_{1}, 0, x_{2}, y_{2}\right)$ to $\left(f\left(x_{1}, x_{2}, y_{2}\right), 0, x_{2}, y_{2}\right)$, with $f\left(x_{1}, x_{2}, y_{2}\right) \geq x_{1}$. This can be done, e.g., via the following simple lemma.

Lemma 2.1. Let $N \subset(M, \omega)$ be a submanifold and let $X_{N}$ be a vector field on $N$ tangent to $\operatorname{ker}\left(\left.\omega\right|_{N}\right)$ whose time-1 flow defines a diffeomorphism $\psi_{f}$ of $N$. Then there exists a Hamiltonian diffeomorphism of $M$ which preserves $N$ and restricts to $\psi_{f}$ on $N$.

Proof. As the 1-form $\left.\eta=X_{N}\right\rfloor \omega$ vanishes on $T N$, one can find a function $H: M \rightarrow \mathbb{R}$ which vanishes on $N$, and satisfies $d H=\eta$. Clearly the function $H$ generates the required Hamiltonian diffeomorphism.

With these preliminaries in place, we turn now to prove the required lower bound for the Gromov width in dimension four.

Proposition 2.2. For any $t \in(0,1)$ one has,

$$
\underline{c}\left(B^{4}(1) \backslash P_{t}\right) \geq \frac{1+t}{2}
$$

Proof. By the remark proceeding Lemma 2.1. for the proof of the proposition it suffices to find a symplectic embedding $\phi: B^{4}\left(\frac{1+t}{2}\right) \stackrel{\mathrm{s}}{\hookrightarrow} B^{4}(1)$ with image $\mathbf{B}$ satisfying proj${\underset{z}{2}}(\mathbf{B} \cap \Sigma) \cap \partial E^{+}=\emptyset$. Denote by $D(r)$ the disk of area $r$ centered at the origin. Note that $\partial E^{+}$divides the disc $D(1)$ into two regions, and set $R$ to be the one with area $\frac{1+t}{2}$. Next, let $S_{h}:=R \cap D(1-h)$ (see Figure 2). Note that $S_{h}$ is the set of all points in $R$ such that the fibers of the map $\operatorname{proj}_{z_{2}}$ in $B^{4}(1)$ have area at least $h$. It is not hard to check that the area of $S_{h}$ is at least $\frac{1-h}{2}+\frac{t}{2} \sqrt{1-h}$. Indeed, the area of $S_{h} \cap\left\{x_{2} \leq 0\right\}$ is $\frac{1-h}{2}$, while the area of $S_{h} \cap\left\{x_{2} \geq 0\right\}$ can be bounded from below by half the area of an ellipse centered at the origin with radii $r_{1}$ and $r_{2}$, where $\pi r_{1}^{2}=1-h$ and $\pi r_{2}^{2}=t^{2}$.

We shall construct an area preserving map $\varphi$ from $D\left(\frac{1+t}{2}\right)$ to $R$, such that the product $\mathrm{Id} \times \varphi$ is the required symplectic embedding $\phi$. Note that the image $\mathbf{B} \subset B^{4}(1)$ if $\varphi\left(D\left(\frac{1+t}{2}-h\right)\right) \subset S_{h}$ for all $0 \leq h \leq \frac{1+t}{2}$. Indeed, let $\left(z_{1}^{\prime}, z_{2}^{\prime}\right) \in B^{4}\left(\frac{1+t}{2}\right)$ such that $z_{1}^{\prime} \in D\left(h^{\prime}\right)$ and $z_{2}^{\prime} \in D\left(\frac{1+t}{2}-h^{\prime}\right)$ for some $h^{\prime}$. If $\varphi\left(z_{2}^{\prime}\right) \in S_{h^{\prime}}$, then, as $S_{h^{\prime}} \subset D\left(1-h^{\prime}\right)$, one as $\left(z_{1}^{\prime}, \varphi\left(z_{2}^{\prime}\right)\right) \subset B^{4}(1)$. Using Lemma 3.1.5 in [12] (see Figure 3), one can construct such a map $\varphi$ since

$$
\text { Area }\left(D\left(\frac{1+t}{2}-h\right)\right)=\frac{1+t}{2}-h \leq \frac{1-h}{2}+\frac{t}{2} \sqrt{1-h} \leq \text { Area }\left(S_{h}\right)
$$

for all $0 \leq h \leq \frac{1+t}{2}$. Finally, as the $z_{2}$-component of $\phi$ lies in $R$, such a map automatically satisfies $\operatorname{proj}_{z_{2}}(\mathbf{B} \cap \Sigma) \cap \partial E^{+}=\emptyset$, as required. This completes the proof of the proposition.

We can extend Proposition 2.2 to higher dimensions as follows.
Proposition 2.3. For any $t \in(0,1)$ and $n \geq 2$

$$
\underline{c}\left(B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right)\right) \geq \frac{1+t}{2}
$$



Figure 2: The domain $S_{h}$ in blue, and the domain $R$ in purple. The domain in green is used to bound the area of $S_{h}$ from below.

Proof. We would like to construct a Hamiltonian function $G\left(z_{1}, \ldots, z_{n}, t\right)$ whose corresponding time-1 flow maps the ball $B^{2 n}\left(\frac{1+t}{2}\right)$ into $\left.B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right)\right)$. From the proof of Proposition 2.2 it follows that there exists a Hamiltonian function $H\left(z_{1}, z_{2}, t\right)$ whose time-1 flow maps the ball $B^{4}\left(\frac{1+t}{2}\right)$ into $B^{4}(1) \backslash P_{t}$. It follows that for every $r>0$ the time-1 map of the Hamiltonian function $H_{r}\left(z_{1}, z_{2}, t\right):=r H\left(z_{1} / \sqrt{r}, z_{2} / \sqrt{r}, t\right)$ maps $B^{4}\left(\frac{r(1+t)}{2}\right)$ into $B^{4}(r) \backslash P_{t}$. Next, for a point $z=\left(z_{1}, \ldots, z_{n}\right) \in B^{2 n}\left(\frac{1+t}{2}\right)$, set $\xi(z)^{2}=\pi \sum_{k=3}^{n}\left|z_{k}\right|^{2}$. Note that $\left(z_{1}, z_{2}\right) \in B^{4}\left(\frac{1+t}{2}-\xi(z)\right) \subset B^{4}\left((1-\xi(z)) \frac{1+t}{2}\right)$. Hence, since $\phi_{H_{1-\xi(z)}}\left(z_{1}, z_{2}\right) \in$ $B^{4}(1-\xi(z)) \backslash P_{t}$, the time-1 map of the Hamiltonian function

$$
G\left(z_{1}, \ldots, z_{n}, t\right):=H_{1-\xi(z)}\left(z_{1}, z_{2}, t\right),
$$

satisfies

$$
\left.\phi_{G}\left(z_{1}, \ldots z_{n}\right) \in B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right)\right)
$$

as required (cf. [3], Section 2.1). This completes the proof of the proposition.


Figure 3: The map $\varphi$ from $D\left(\frac{1+t}{2}\right)$ into $R$.
Next we turn to establish the upper bound in Theorem 1.2. For this, let $L$ be the Lagrangian plane spanned by $(0, i)$ and $(1,0)$. Note that both $L$ and $P_{t}$ lie in the hyperplane $\Sigma:=\left\{y_{1}=0\right\}$, and one has $\operatorname{proj}_{z_{2}}(L)=\left\{x_{2}=0\right\}$. The main ingredient we need is the following:

Theorem 2.4. Let $\mathbf{K} \subset B^{4}(1) \backslash P_{t}$ be a compact subset. Then there exists $a$ symplectomorphism $\phi$ of $\mathbb{C}^{2}$ with $\phi(\mathbf{K}) \subset B^{4}(1+t) \backslash L$.

Proof. Set $I:=\mathbf{K} \cap \Sigma=A \sqcup B$ with $A, B$ defined as follows:

$$
\begin{aligned}
& A:=\left\{p \in I \mid p+(\lambda, 0) \in P_{t} \text { for some } \lambda<0\right\} \\
& B:=\left\{p \in I \mid p+(\lambda, 0) \in P_{t} \text { for some } \lambda>0\right\}
\end{aligned}
$$

Since $P_{t}$ is a graph over the $z_{2}$-plane, $I$ is the disjoint union of $A$ and $B$. Further, since $s, t$ are positive we have that $\left\{\left(z_{1}, z_{2}\right) \in P_{t} \mid z_{2} \in E^{+}\right\} \subset\left\{x_{1} \geq 0\right\}$ and $\left\{\left(z_{1}, z_{2}\right) \in P_{t} \mid z_{2} \in E^{-}\right\} \subset\left\{x_{1} \leq 0\right\}$. This implies in particular that $\operatorname{proj}_{z_{2}}(A) \subset\left\{x_{2} \leq 0\right\} \cup E^{+}$and $\operatorname{proj}_{z_{2}}(B) \subset\left\{x_{2} \geq 0\right\} \cup E^{-}$. Indeed, for $p \in A$, if $x_{2}(p) \geq 0$, then $x_{1}(p) \geq 0$, and then $p+(\lambda, 0) \in B^{4}(1) \cap P_{t}$. Hence $\operatorname{proj}_{z_{2}}(p)=\operatorname{proj}_{z_{2}}(p+(\lambda, 0)) \in E^{+}$. A similar argument holds for points in $B$.

Our proof has two steps. In Step 1 we apply a symplectic diffeomorphism to $\mathbf{K}$, with support in $B^{4}(1+t)$, moving first the subsets $A$ and $B$ away from $x_{1}=0$, and then moving $\mathbf{K}$ sufficiently away from the $z_{2}$-axis. In Step 2 we describe a Hamiltonian diffeomorphism of $B^{4}(1+t)$ displacing the re-positioned K obtained in Step 1 from $L$ as required.

Step 1. The repositioning of $\mathbf{K}$ is achieved via the following two lemmas.
Lemma 2.5. For every $\delta>0$ sufficiently small, there exists a Hamiltonian diffeomorphism $\psi_{1}$ with compact support in $B^{4}(1) \backslash P_{t}$ such that the sets $A$ and $B$ defined for $\psi_{1}(\mathbf{K})$ satisfy $A \subset\left\{x_{1}>\delta\right\}$ and $B \subset\left\{x_{1}<-\delta\right\}$.

Proof. We can find such a diffeomorphism $\psi_{1}$ which preserves $\Sigma$ by applying Lemma 2.1, since moving points of $A$ in the positive $x_{1}$-direction, and points of $B$ in the negative $x_{1}$-direction does not introduce intersections with $P_{t}$.

To simplify notations, in what follows we denote the image $\psi_{1}(\mathbf{K})$ provided by Lemma 2.5 also by $\mathbf{K}$.
Lemma 2.6. For every $\delta>0$ sufficiently small there is a symplectic diffeomorphism $\psi_{2}: \mathbb{R}^{4} \rightarrow \mathbb{R}^{4}$ which is the identity on $\Sigma$, and satisfies

$$
\psi_{2}(\mathbf{K}) \subset\left(\left\{\pi\left|z_{1}\right|^{2}>t\right\} \cup\left\{\left|x_{1}\right|>\delta \text { and }\left|y_{1}\right|<\delta\right\}\right) \cap B^{4}(1+t) .
$$

Proof. We use symplectic polar coordinates on the $z_{1}$-plane, with $R=\pi\left|z_{1}\right|^{2}$ and $\theta \in S^{1}=\mathbb{R} / \mathbb{Z}$. Let $f: S^{1} \rightarrow[0, t]$ satisfy $f(\theta)=t$ when both $|\theta|>\delta$ and $|\theta-1 / 2|>\delta$, and $f=0$ when $\theta=0$ or $\theta=1 / 2$. Then, consider the closed 1-form $\eta=f(\theta) d \theta$. As $\mathbf{K}$ is disjoint from $\left\{z_{1}=0\right\}$, the form $\eta$ defines a symplectic isotopy of $\mathbf{K}$ increasing the $R$ coordinate by $f(\theta)$ and preserving $z_{2}$. As $0 \leq f(\theta)<t$, the ball remains disjoint from $\left\{z_{1}=0\right\}$ but stays within $B^{4}(1+t)$, as required (see Figure 4). This completes the proof of the lemma.

Step 2. Displacing the repositioned $\mathbf{K}$ from $L$.
To further simplify notations, using Step 1 , in what follows we assume that $\mathbf{K} \subset\left(\left\{\pi\left|z_{1}\right|^{2}>t\right\} \cup\left\{\left|x_{1}\right|>\delta\right.\right.$ and $\left.\left.\left|y_{1}\right|<\delta\right\}\right) \cap B^{4}(1+t)$ for some sufficiently small $\delta>0$. In addition, note that $\mathbf{K} \cap \Sigma=A \sqcup B \subset B^{4}(t)$, where $A \subset\left\{x_{1}>\delta\right\}$ and $B \subset\left\{x_{1}<-\delta\right\}$. Our goal is to find a Hamiltonian diffeomorphism $\phi$ of $B^{4}(1+t)$ which displaces $\mathbf{K}$ from $L$. We must consider both $I=\mathbf{K} \cap \Sigma$ and


Figure 4: An illustration of the area preserving map $\psi_{2}$ which "pushes" $\mathbf{K}$ into the set $\left\{\pi\left|z_{1}\right|^{2}>t\right\} \cup\left\{\left|x_{1}\right|>\delta\right.$ and $\left.\left|y_{1}\right|<\delta\right\}$.
$J=\mathbf{K} \cap \Sigma^{c}$, and give sufficient conditions for a Hamiltonian function to generate a diffeomorphism displacing $I$ from $L$, while leaving $J$ disjoint from $L \subset \Sigma$. The proof will conclude by showing that Hamiltonian functions satisfying the sufficient conditions exist.

## Displacing I from L.

We can find two Hamiltonian functions $a\left(z_{2}\right)$ and $b\left(z_{2}\right)=a\left(-z_{2}\right)$, with compact support in $D(1+t)$ whose corresponding time- 1 flows, denoted by $\phi_{a}$ and $\phi_{b}$ respectively, satisfy $\phi_{a}\left(\operatorname{proj}_{z_{2}}(A)\right) \cap\left\{x_{2}=0\right\}=\emptyset$ and $\phi_{b}\left(\operatorname{proj}_{z_{2}}(B)\right) \cap$ $\left\{x_{2}=0\right\}=\emptyset$. This is because $\operatorname{proj}_{z_{2}}(A)$ and $\operatorname{proj}_{z_{2}}(B)$ are compact subsets of $\left\{x_{2} \leq 0\right\} \cup E^{+}$and $-\left(\left\{x_{2} \leq 0\right\} \cup E^{+}\right)$, respectively, and both sets have area $(1+t) / 2$. In fact, as $A$ and $B$ are compact, choosing $\delta$ smaller if necessary we may assume $a$ and $b$ have compact support in $D(1+t-\delta)$. Note that one can choose $a$ such that $0 \leq a \leq t / 2-\delta$. In addition, we can let $a=0$ on $\left\{x_{2}=0\right\} \backslash D(1)$, and make sure the flow $\phi_{a}$ of $\left\{x_{2}=0\right\} \cap D(1)$ remains in $D(1)$.

Next, let $\chi(x)$ be such that $\chi(x)=0$ if $x<0, \chi(x)=1$ if $x>\delta$, and $0 \leq \chi^{\prime}(x)<1 / \delta$. We consider the Hamiltonian function $F: \mathbb{C}^{2} \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
F\left(x_{1}, y_{1}, x_{2}, y_{2}\right)=\chi\left(x_{1}\right) a\left(x_{2}, y_{2}\right)+\chi\left(-x_{1}\right) b\left(x_{2}, y_{2}\right) . \tag{1}
\end{equation*}
$$

Note that the Hamiltonian flow satisfies $\phi_{F}(I) \cap L=\phi_{F}(A \sqcup B) \cap L=\emptyset$. More generally we have the following.
Lemma 2.7. Suppose $H: \mathbb{C}^{2} \rightarrow \mathbb{R}$ satisfies $H=F$ in $\Sigma$ and in a neighborhood of $\Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$. Then the time-1 flow satisfies $\phi_{H}(A \sqcup B) \cap L=\emptyset$.

Proof. Suppose $p=\left(z_{1}, z_{2}\right) \in A$ (the argument for points in $B$ is identical), then $x_{1}>\delta$. The component of the Hamiltonian vector field $X_{H}$ in the direction $\frac{\partial}{\partial x_{1}}$ is given by $\frac{\partial H}{\partial y_{1}}$, which vanishes when $x_{1}=\delta$ as $H$ agrees with $F$. Also, as $p \in \Sigma$, when $x_{1}>\delta$ we have $\frac{\partial H}{\partial x_{1}}=\frac{\partial F}{\partial x_{1}}=0$. Therefore the flow $\phi_{H}$ of $p$ remains in $\Sigma$, and as the $z_{2}$-component of $X_{F}$ on $\Sigma \cap\left\{x_{1}>\delta\right\}$ is independent of $x_{1}$, we see that $\phi_{H}$ also displaces $A$ from $L=\Sigma \cap\left\{x_{2}=0\right\}$.

## Controlling the flow of $J$.

Note that showing that $J=\mathbf{K} \cap \Sigma^{c}$ remains disjoint from $L$ under a Hamiltonian flow is equivalent to showing that the inverse flow applied to $L$ remains disjoint from $J$. Consider the Hamiltonian flow $\phi_{-F}^{s}$ generated by $-F$, so $\phi_{-F}^{1}=\left(\phi_{F}\right)^{-1}$. This flow has the following property:

Lemma 2.8. Let $0 \leq s \leq 1$. With the above notations one has

$$
\phi_{-F}^{s}(L) \subset \Sigma \cup\left\{-\delta<x_{1}<\delta, 0 \leq \operatorname{sign}\left(x_{1}\right) y_{1}<\frac{t}{2 \delta}-1, z_{2} \in D(1)\right\}
$$

Proof. Suppose $x_{1} \geq 0$. Then we have

$$
X_{-F}=\chi^{\prime}\left(x_{1}\right) a\left(z_{2}\right) \frac{\partial}{\partial y_{1}}+\chi\left(x_{1}\right) X_{a}
$$

For a point $\left(x_{1}, 0,0, y_{2}\right) \in L$, set $p:=\left(x_{1}^{\prime}, y_{1}^{\prime}, x_{2}^{\prime}, y_{2}^{\prime}\right)$ to be its image under the flow of $X_{-F}$. Note that if $x_{1}>\delta$, then $\chi^{\prime}=0$ and in particular $p \in \Sigma$. Assume $0<x_{1}<\delta$. Note that if the projection $\operatorname{proj}_{z_{2}}\left(x_{1}, 0,0, y_{2}\right)=\left(0, y_{2}\right) \notin D(1)$, then by our assumption on the Hamlitonian function $a$ one has $a\left(0, y_{2}\right)=0$, and thus $y_{1}^{\prime}=0$, i.e., $p \in \Sigma$. If $\left(0, y_{2}\right) \in D(1)$, then, by the assumptions on $\chi$ and $a$, one has that $0 \leq y_{1}^{\prime} \leq \frac{t}{2 \delta}-1$, and that $\left(x_{2}^{\prime}, y_{2}^{\prime}\right) \in D(1)$. This completes the proof of the lemma for $x_{1} \geq 0$. A similar argument works for the case $x_{1}<0$.

Let $\xi$ be a symplectomorphism of the $z_{1}$-plane mapping the region

$$
\left\{-\delta<x_{1}<\delta, 0 \leq \operatorname{sign}\left(x_{1}\right) y_{1}<\frac{t}{2 \delta}-1\right\}
$$

into the disk $D(t) \backslash\left\{\left|x_{1}\right|>\delta\right.$ and $\left.\left|y_{1}\right|<\delta\right\}$. Furthermore, suppose that $\xi$ is the identity near $\left\{y_{1}=0\right\}$. Define $G: \mathbb{C}^{2} \rightarrow \mathbb{R}$ by $G=F \circ(\xi \times \mathrm{Id})^{-1}$. The following is a corollary of Lemma 2.8 .
Corollary 2.9. Let $0 \leq s \leq 1$. Then

$$
\phi_{-G}^{s}(L) \subset \Sigma \cup\left(\left(\left\{\pi\left|z_{1}\right|^{2}<t\right\} \backslash\left\{\left|x_{1}\right|>\delta \text { and }\left|y_{1}\right|<\delta\right\}\right) \cap B^{4}(1+t)\right) .
$$

Proof. The proof follows immediately from Lemma 2.8 and the fact that the flow of $G$ is given by $\phi_{G}^{s}=(\xi \times \mathrm{Id}) \circ \phi_{G}^{s} \circ(\xi \times \mathrm{Id})^{-1}$.

Our repositioning Lemma 2.6 now implies that $\phi_{G}(J) \cap L=\emptyset$. Slightly more generally, let $U$ be a neighborhood of the union of the sets $\phi_{-G}^{s}(L)$ for $0 \leq s \leq 1$. Then we have the following.

Corollary 2.10. Let $H: \mathbb{C}^{2} \rightarrow \mathbb{R}$ so that $H=G$ on $U$. Then $\phi_{H}(J) \cap L=\emptyset$.

## Displacing K.

Recall that the flow generated by a Hamiltonian function $H: \mathbb{C}^{2} \rightarrow \mathbb{R}$ preserves the ball $B^{4}(1+t)$ provided it is constant on the characteristic circles in $\partial B^{4}(1+t)$. Summarizing our discussion above, combining Lemma 2.7 . Corollary 2.10, and the symplectomorphism from Step 1, give the following.

Corollary 2.11. Suppose $H: \mathbb{C}^{2} \rightarrow \mathbb{R}$ is such that

1. $H$ is constant on characteristics of $\partial B^{4}(1+t)$,
2. $H=F$ in $\Sigma$ and in a neighborhood of $\Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$,
3. $H=G$ on $U$.

Then, the time-1 map $\phi_{H}$ restricts to give a diffeomorphism of $B^{4}(1+t)$ which displaces $\mathbf{K}=I \sqcup J$ from $L$.

It remains to show that such functions $H$ exist. We will show that the three conditions in Corollary 2.11 are consistent.

We start by showing the compatibility of the first two conditions. For this we need to show that for each Hopf circle on $\partial B^{4}(1+t)$, the restriction of $F$ to the intersection of the Hopf circle with $\Sigma$, and to a neighborhood of $\Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$, is constant.

Note that there is a single characteristic, $\left\{z_{1}=0\right\} \cap \partial B^{4}(1+t)$, lying entirely in $\Sigma$. As our functions $a$ and $b$ have compact support in $D(1+t)$, the function $F$ is identically 0 on this circle. The remaining characteristics intersect $\Sigma$ in exactly two points, say $\left(x_{1}, 0, x_{2}, y_{2}\right)$ and $\left(-x_{1},, 0,-x_{2},-y_{2}\right)$. Assume $x_{1} \geq 0$ (a similar argument holds for $x_{1}<0$ ), note that

$$
F\left(x_{1}, 0, x_{2}, y_{2}\right)=\chi\left(x_{1}\right) a\left(x_{2}, y_{2}\right)=\chi\left(-x_{1}\right) b\left(-x_{2},-y_{2}\right)=F\left(-x_{1}, 0,-x_{2},-y_{2}\right),
$$

and so for each Hopf circle $F$ is constant on its intersection with $\Sigma$. Thus $F$ can be extended over $\partial B^{4}(1+t)$ to a function $H$ constant on the characteristics.

Now we consider characteristics intersecting the neighborhood $\left\{\left|z_{1}\right|<\delta\right\}$ of $\partial B^{4}(1+t) \cap \Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$. These characteristics intersect $\Sigma$ inside the region

$$
\partial B^{4}(1+t) \cap \Sigma \cap\left\{\left|x_{1}\right|<\delta\right\} \subset \partial B^{4}(1+t) \cap \Sigma \cap\left\{\pi\left|z_{2}\right|^{2}>1+t-\delta\right\}
$$

where $F$ is identically 0 . Hence a function $H$ on $\partial B^{4}(1+t)$ which agrees with $F$ on $\Sigma$ and is constant on characteristics will be identically 0 on $\left\{\left|z_{1}\right|<\delta\right\}$. In particular $H$ agrees with $F$ on a neighborhood of $\Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$ and we have shown the compatibility of conditions 1 and 2 .

Regarding condition 3, as $U$ is relatively compact in the ball, its only intersection with the boundary is on $\Sigma$. Also $F=G$ in a neighborhood of $\Sigma$. Thus we can define a smooth function $H$ simultaneously equal to $G$ on $U \backslash \Sigma$, equal to $F$ on $\Sigma$ and in a neighborhood of $\Sigma \cap\left\{\left|x_{1}\right|<\delta\right\}$, and equal to our extension of $F$ over $\partial B^{4}(1+t)$ which is constant on the characteristics. In other words, one can find a smooth function $H$ as required. This completes the proof of Theorem 2.4

Equipped with Theorem 2.4, we turn now to the final ingredient needed for the proof of Theorem 1.2 .

Proposition 2.12. For any $t \in(0,1)$ and $n \geq 2$

$$
\bar{c}\left(B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right)\right) \leq \frac{1+t}{2}
$$

Proof. We need to produce a symplectic embedding

$$
B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right) \stackrel{\mathrm{s}}{\hookrightarrow} Z^{2 n}\left(\frac{1+t}{2}+\delta\right)
$$

for all $\delta>0$. Given a compact $\mathbf{K} \subset B^{4}(1) \backslash P_{t}$, Theorem 2.4 gives a symplectic embedding $\mathbf{K} \stackrel{\mathrm{s}}{\hookrightarrow} B^{4}(1+t) \backslash L$. Theorem 1.3 from [11] says that there exists a symplectic embedding $B^{4}(1+t) \backslash L \stackrel{\mathrm{~s}}{\hookrightarrow} Z^{4}\left(\frac{1+t}{2}\right)$, and so composing gives an embedding $\mathbf{K} \stackrel{\mathrm{s}}{\hookrightarrow} Z^{4}\left(\frac{1+t}{2}\right)$. Next we observe that $B^{4}(1) \backslash P_{t}$ embeds into a compact subset of $B^{4}(1+\delta) \backslash P_{t}$ for all $\delta>0$. Indeed, a suitable embedding is
the restriction of a symplectic embedding $\Xi$ of $\mathbb{C}^{2}$ into itself defined as follows. Write $\mathbb{C}^{2}=P_{t} \oplus Q_{t}$, where $Q_{t}$ is the symplectic complement of $P_{t}$. Let $\zeta$ be a $C^{0}$-small symplectic embedding $Q_{t} \backslash\{0\} \stackrel{\mathrm{s}}{\hookrightarrow} Q_{t} \backslash B_{\delta^{\prime}}$, where $B_{\delta^{\prime}}$ is small neighborhood of the origin. Thus, the required embedding is

$$
\Xi=\mathrm{Id} \times \zeta: P_{t} \oplus Q_{t} \rightarrow P_{t} \oplus Q_{t}
$$

Compose the above embedding $\mathbf{K} \stackrel{\mathrm{s}}{\hookrightarrow} Z^{4}\left(\frac{1+t}{2}\right)$ with $\Xi$, we obtain an embedding

$$
\phi: B^{4}(1) \backslash P_{t} \stackrel{\mathrm{~s}}{\hookrightarrow} Z^{4}\left(\frac{1+t}{2}+\delta\right)
$$

To conclude, we observe that

$$
\phi \times \operatorname{Id}:\left(z_{1}, \ldots, z_{n}\right) \mapsto\left(\phi\left(z_{1}, z_{2}\right), z_{3}, \ldots, z_{n}\right)
$$

satisfies

$$
\phi \times \operatorname{Id}\left(\left(B^{2 n}(1) \backslash\left(P_{t} \times \mathbb{C}^{n-2}\right)\right)\right) \subset \phi\left(B^{4}(1) \backslash P_{t}\right) \times \mathbb{C}^{n-2} \subset Z^{2 n}\left(\frac{1+t}{2}+\delta\right)
$$

This completes the proof of the proposition.
Proof of Theorem 1.2. The proof follows immediately by combining Proposition 2.3. Proposition 2.12, and the unitary equivalence between the codimensiontwo linear space $E_{t}$ and $P_{t} \times \mathbb{C}^{n-2}$.

## 3 The Complement of Parallel Subspaces

In this section we prove Theorem 1.3 and Theorem 1.4. As before, assume that $n>1$ and equip $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$ with coordinates $\left(z_{1}, \ldots, z_{n}\right)=\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right)$, and with the standard symplectic form $\omega=d x \wedge d y$.

The proof of Theorem 1.3 is broken into two parts: obtaining an upper bound for the cylindrical capacity, and a lower bound for the Hofer-Zehnder capacity. For the upper bound we need the following observation. Let $K \subset \mathbb{R}^{2 n} \simeq \mathbb{C}^{n}$ be a convex body, and denote by $c_{\text {Енz }}(K)$ the Ekeland-Hofer-Zehder capacity associated with $K$, i.e., the minimal action among the closed characteristics on the boundary $\partial K$ (see, e.g., Section 1.5 in [8). Moreover, for $L>0$, let $A^{L}: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ be the linear map that takes $z_{n}$ to $L z_{n}$, and leaves $z_{i}$ fixed for $1 \leq i \leq n-1$.

Proposition 3.1. For convex $K \subset \mathbb{R}^{2 n} \simeq \mathbb{C}^{n}$ such that $K=-K$ one has

$$
\lim _{L \rightarrow \infty} c_{\mathrm{EHZ}}\left(A^{L} K\right) \leq c_{\mathrm{EHZ}}\left(K \cap\left\{z_{n}=0\right\}\right) .
$$

Note that $K \cap\left\{z_{n}=0\right\} \subset \mathbb{C}^{n-1}$, and hence its capacity is taken with respect to the standard symplectic form restricted to $\mathbb{C}^{n-1}$.

Proof. Assume without loss of generality that $K$ is also strictly convex and smooth. Recall that, by Clarke's dual action principle (see, e.g., Section 1.5 of [9]), one has that for every convex body $T \subset \mathbb{R}^{2 n}$

$$
\begin{equation*}
c_{\mathrm{EHZ}}(T)=\min _{\eta \in \mathcal{E}_{n}, \mathcal{A}(\eta)=1} \frac{\pi}{2} \int_{0}^{2 \pi} h_{T}^{2}(\dot{\eta}(t)) d t, \tag{2}
\end{equation*}
$$

where $\mathcal{E}_{n}=\left\{\eta \in W^{1,2}\left(S^{1}, \mathbb{R}^{2 n}\right) \mid \int_{0}^{2 \pi} \eta(t) d t=0\right\}$, the function $h_{T}$ is the support function of $T$ defined by $h_{T}(u):=\sup \{\langle x, u\rangle \mid x \in T\}$, and $\mathcal{A}(\eta)$ is the symplectic action of $\eta$. Moreover, one has that for $\eta$, a minimizer of (2), and some $\lambda \in \mathbb{R}^{+}$ and $b \in \mathbb{R}^{2 n}$, the orbit $\gamma(t):=\lambda J \eta(t)+b$ is a closed characteristic on the boundary of $T$ with minimal action. Denote by $\eta_{1}$ a minimizer of (2) for the body $K \cap\left\{z_{n}=0\right\}$, and by $\gamma_{1}$ the corresponding closed characteristic. In order to bound the capacity of $A^{L} K$, we consider the loop $\eta_{2}$ defined by

$$
\dot{\eta}_{2}(t)=\left\|\dot{\eta}_{1}(t)\right\| \frac{h_{K \cap\left\{z_{n}=0\right\}}\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right)\right)}{h_{A^{L} K}\left(n_{A^{L} K}\left(\gamma_{1}(t)\right)\right)} n_{A^{L} K}\left(\gamma_{1}(t)\right),
$$

where $n_{K}(\cdot)$ is the unit outer normal to $K$ at a point. Recall that as $K=-K$ one has that $\gamma_{1}$ can be assumed to be centrally symmetric (see Corollary 2.2. in [1]), and hence $\eta_{2}$ is a closed loop. Define $\alpha_{1}(t)$ and $\alpha_{2}(t)$ so that

$$
n_{K}\left(\gamma_{1}(t)\right)=\frac{\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right), \alpha_{1}(t), \alpha_{2}(t)\right)}{\sqrt{1+\alpha_{1}(t)^{2}+\alpha_{2}(t)^{2}}} .
$$

Note that since $\gamma_{1} \in\left\{z_{n}=0\right\}$,

$$
\begin{aligned}
n_{A^{L} K}\left(\gamma_{1}(t)\right) & =\frac{A^{1 / L} n_{K}\left(A^{1 / L} \gamma_{1}(t)\right)}{\left\|A^{1 / L} n_{K}\left(A^{1 / L} \gamma_{1}(t)\right)\right\|} \\
& =\frac{\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right), \frac{\alpha_{1}(t)}{L}, \frac{\alpha_{2}(t)}{L}\right)}{\sqrt{1+\alpha_{1}(t)^{2}+\alpha_{2}(t)^{2}}\left\|A^{1 / L} n_{K}\left(A^{1 / L} \gamma_{1}(t)\right)\right\|} .
\end{aligned}
$$

Moreover, since for every $p \in \partial K$ one has that $h_{K}\left(n_{K}(p)\right)=\left\langle n_{K}(p), p\right\rangle$,

$$
\begin{aligned}
\frac{n_{A^{L} K}\left(\gamma_{1}(t)\right)}{h_{A^{L} K}\left(n_{A^{L} K}\left(\gamma_{1}(t)\right)\right)} & =\frac{\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right), \frac{\alpha_{1}(t)}{L}, \frac{\alpha_{2}(t)}{L}\right)}{\left\langle\left(\gamma_{1}(t), 0,0\right),\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right), \frac{\alpha_{1}(t)}{L}, \frac{\alpha_{2}(t)}{L}\right)\right\rangle} \\
& =\frac{\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right), \frac{\alpha_{1}(t)}{L}, \frac{\alpha_{2}(t)}{L}\right)}{h_{K \cap\left\{z_{n}=0\right\}}\left(n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right)\right)} .
\end{aligned}
$$

Next, since $\dot{\eta}_{1}(t)$ is parallel to $n_{K \cap\left\{z_{n}=0\right\}}\left(\gamma_{1}(t)\right)$, one has

$$
\dot{\eta}_{2}(t)=\left(\dot{\eta}_{1}(t), \frac{\left\|\dot{\eta}_{1}\right\| \alpha_{1}(t)}{L}, \frac{\left\|\dot{\eta}_{1}\right\| \alpha_{2}(t)}{L}\right) .
$$

This together with the definition of the support function implies that

$$
h_{K \cap\left\{z_{n}=0\right\}}\left(\dot{\eta}_{1}(t)\right)=\left\langle\dot{\eta}_{1}(t), \gamma_{1}(t)\right\rangle=\left\langle\dot{\eta}_{2}(t), \gamma_{1}(t)\right\rangle=h_{A^{L} K}\left(\dot{\eta}_{2}(t)\right) .
$$

Note that one has $\mathcal{A}\left(\eta_{2}\right)=\mathcal{A}\left(\eta_{1}\right)+\frac{\left\|\dot{\eta}_{1}\right\|^{2}}{L^{2}} \mathcal{A}(\alpha)$, where $\dot{\alpha}(t)=\left(\alpha_{1}(t), \alpha_{2}(t)\right)$. As $\mathcal{A}(\alpha)$ is bounded, one gets that $\mathcal{A}\left(\eta_{2}\right) \rightarrow \mathcal{A}\left(\eta_{1}\right)=1$ as $L \rightarrow \infty$. Using (2) and normalizing $\eta_{2}$ appropriately, complete the proof.

Recall that $\Sigma_{\varepsilon}^{t}$ is a linear image of the set of codimension-two subspaces

$$
\begin{equation*}
\Sigma_{\varepsilon}:=\bigcup\left\{\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in \mathbb{C}^{n}: z_{n} \in \varepsilon \mathbb{Z}^{2}\right\} \tag{3}
\end{equation*}
$$

with Kähler angle $t$, i.e., one has $\left|\omega\left(n_{1}, n_{2}\right)\right|=t$, where $n_{1}, n_{2}$ are the unit outer normals to the subspaces in $\Sigma_{\varepsilon}^{t}$.

Proposition 3.2. For any $\delta>0$ and $t \in(0,1)$, there exists $\varepsilon>0$ such that

$$
\bar{c}\left(B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}\right) \leq t+\delta
$$

Proof. Let $0<t<1$, and consider the symplectic matrix (cf. Example 2.2. in [6)

$$
M_{t}=\left(\right)
$$

It follows from the proof of Theorem 1.3 in [6] that for every $L>1$

$$
\bar{c}\left(M_{t} B^{2 n}(1) \backslash \Sigma_{\varepsilon}\right) \leq\left(1+\frac{\sqrt{2} \varepsilon L}{\lambda\left(A^{L} K\right)}\right)^{2} \bar{c}\left(A^{L} M_{t} B^{2 n}(1)\right)
$$

where $\lambda(K):=\inf _{u \in \mathbb{S}^{2 n-1}} h_{K}(u)$, and $\Sigma_{\varepsilon}$ is the family of complex planes (3). Since $A^{L} M_{t} B^{2 n}(1)$ is a centrally symmetric ellipsoid, Proposition 3.1 implies that for every (normalized) symplectic capacity $c$ one has

$$
\lim _{L \rightarrow \infty} c\left(A^{L} M_{t} B^{2 n}(1)\right) \leq c\left(M_{t} B^{2 n}(1) \cap\left\{z_{n}=0\right\}\right) .
$$

A direct computation shows that $M_{t} B^{2 n}(1) \cap\left\{z_{n}=0\right\}$ is linearly symplectomorphic to the symplectic ellipsoid $E(1, \ldots, 1, t)=\left\{\pi\left(\sum_{i=1}^{n-2}\left|z_{i}\right|^{2}+\frac{\left|z_{n-1}\right|^{2}}{t}\right) \leq 1\right\}$ which has capacity $t$. Note that the normals to $M_{t}^{-1} \Sigma_{\varepsilon}$ (in the $(\bar{x}, \bar{y})$-coordinate system) are

$$
n_{1}=\left(0, \ldots, 0,0,-\sqrt{1-t^{2}}, t, 0\right), \quad n_{2}=(0, \ldots, 0,0,0,0,1)
$$

and hence $\omega\left(n_{1}, n_{2}\right)=t$. Overall, for any $\delta>0$ and $0<t<1$, there exist $L \gg 1$ and $\varepsilon>0$ such that

$$
\bar{c}\left(B^{2 n}(1) \backslash M_{t}^{-1} \Sigma_{\varepsilon}\right) \leq t+\delta
$$

and the proof of the proposition is thus complete.
We turn now to obtain the required lower bound for the Hofer-Zehnder capacity. To this end we shall need the following

Proposition 3.3. Let $t \in(0,1)$ and $\varepsilon>0$, and let $A$ be a symplecitc matrix such that $A \Sigma_{\varepsilon}^{t}=\Sigma_{\varepsilon}$. Then for every $\delta>0$ there is a symplectic embedding

$$
(1-\delta)\left(A B^{2 n}(1) \cap W^{2 n}\right) \stackrel{\mathrm{s}}{\hookrightarrow} B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}
$$

where

$$
W^{2 n}:=\left(\left\{x_{n}=0, y_{n}=0\right\} \cap A B^{2 n}(1)\right) \times \operatorname{span}\left\{x_{n}, y_{n}\right\}
$$

Proof. The idea of the proof is to "push" $\Sigma_{\varepsilon}$ to a small neighborhood of the boundary of $A B^{2 n}(1) \cap W^{2 n}$ via a symplectic isotopy $\phi_{t}$ (see Figure 5). Then, $\phi^{-1}$ will give the required embedding. More precisely, let $\left\{u_{i}\right\}$ be the set of all the directions connecting points in $\varepsilon \mathbb{Z}^{2} \backslash\{0\} \cap \operatorname{proj}_{z_{n}} A B^{2 n}(1)$ to the origin. For every $v \in\left\{u_{i}\right\}$, set $l_{v}:=\left\{\lambda v: \lambda>\delta_{v}\right\}$ such that $\cup_{i}\left\{l_{u_{i}}\right\}$ cover all the points in $\varepsilon \mathbb{Z}^{2} \backslash\{0\}$ (see Figure 6 ). Next, consider

$$
W_{v}^{2 n}:=\left(\left\{x_{n}=0, y_{n}=0\right\} \cap A B^{2 n}(1)\right) \times l_{v}
$$

and define a function $\alpha_{v}: A B^{2 n}(1) \cap\{x:\langle x, J v\rangle=0\} \rightarrow \mathbb{R}$ such that it vanishes outside $W_{v}^{2 n}$, is equal to 1 in the interior of $W_{v}^{2 n}$, and with some smooth cutoff in between (see Figure 7). Define a smooth function $H_{v}(x)$ such that it equals $-\alpha_{v}(x)\langle x, J v\rangle$ in a small neighborhood of $W_{v}^{2 n} \subset\{x:\langle x, J v\rangle=0\}$, and vanishes outside a slightly larger neighborhood (by choosing the former neighborhood small enough, one can make sure that the supports of $H_{v_{i}}$ are disjoint). Note that the Hamiltonian vector field is equal to $X_{H_{v}}(x)=\alpha_{v}(x) v$ for $x$ such that $\langle x, J v\rangle=0$ and $\langle x, v\rangle>\delta_{v}$. Hence, the Hamiltonian function $\sum_{i} H_{v_{i}}$ generates the required flow. Note that in a similar way one can also push the codimension-two hyperplane passing through the origin in $\Sigma_{\varepsilon}$ to the boundary of $A B^{2 n}(1) \cap W^{2 n}$. This completes the proof of the lemma.


Figure 5: Pushing the hyperplanes arbitrarily close to the boundary.


Figure 6: The line $l_{v}$ together with the support of the function $H_{v}$.


Figure 7: The domain $W_{v}^{2 n}$ in the $(2 n-1)$-space $\{\langle x, J v\rangle=0\}$. The flow of $H_{v}$ pushes $\left\{z_{n}=z_{0}\right\}$ close to the boundary.

From Proposition 3.3 it follows that in order to obtain a lower bound for the capacity of $B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}$ it is enough to find a lower bound for the capacity of the intersection $A B^{2 n}(1) \cap W^{2 n}$, which is a convex domain in $\mathbb{R}^{2 n}$. We will show below that the minimal action capacity of this domain equals $t$, which is suffices for the proof of Theorem 1.3

Remark 3.4. In light of the well-known conjecture that all symplectic capacities coincide on convex domains in the classical phase space, one expects that the Gromov width of the intersection $A B^{2 n}(1) \cap W^{2 n}$ also equals $t$. This, in turn, would imply that Theorem 1.3 holds for the Gromov width as well. However, we are only able to show that the Gromov width of the above intersection is bounded below by $t-0.07$ as stated in Theorem 1.4

Proposition 3.5. With the above notations one has

$$
c_{\mathrm{EHZ}}\left(A B^{2 n}(1) \cap W^{2 n}\right)=t .
$$

Proof. We start with the 4-dimensional case. Based on the definition of the Ekeland-Hofer-Zehnder capacity and the fact that $A$ is a symplectic matrix, it suffices to show that any closed characteristic with minimal action on the boundary of $B^{4}(1) \cap A^{-1} W^{4}$ has action $t$. For this end, without loss of generality, one can choose $A$ to be the matrix

$$
A^{-1}=\frac{1}{\sqrt{2}}\left(\begin{array}{cccc}
\frac{\sqrt{1+t}}{\sqrt{t}} & 0 & \frac{\sqrt{1-t}}{\sqrt{t}} & 0  \tag{4}\\
0 & \frac{\sqrt{1+t}}{\sqrt{t}} & 0 & -\frac{\sqrt{1-t}}{\sqrt{t}} \\
\frac{\sqrt{1-t}}{\sqrt{t}} & 0 & \frac{\sqrt{1+t}}{\sqrt{t}} & 0 \\
0 & -\frac{\sqrt{1-t}}{\sqrt{t}} & 0 & \frac{\sqrt{1+t}}{\sqrt{t}}
\end{array}\right)
$$

In this case the base of the cylinder $A^{-1} W^{4}$ is spanned by

$$
v_{1}:=\frac{1}{\sqrt{2}}(\sqrt{1+t}, 0, \sqrt{1-t}, 0) \text { and } v_{2}:=\frac{1}{\sqrt{2}}(0, \sqrt{1+t}, 0,-\sqrt{1-t})
$$

Complete $v_{1}, v_{2}$ into an orthonormal basis with

$$
n_{1}:=\frac{1}{\sqrt{2}}(0, \sqrt{1-t}, 0, \sqrt{1+t}) \text { and } n_{2}:=\frac{1}{\sqrt{2}}(-\sqrt{1-t}, 0, \sqrt{1+t}, 0)
$$

Denote $S_{1}:=\partial B^{4}(1) \cap A^{-1} W^{4}$ and $S_{2}:=B^{4}(1) \cap \partial A^{-1} W^{4}$, and note that $\partial\left(B^{4}(1) \cap A^{-1} W^{4}\right)=S_{1} \cup S_{2}$. We classify the closed characteristics on the boundary $\partial\left(B^{4}(1) \cap A^{-1} W^{4}\right)$ by how many time they "visit" the sets $S_{1}$ and $S_{2}$. More precisely, note that a closed characteristic which lies entirely in $S_{1}$ has action 1, and a closed characteristic which is entirely in $S_{2}$ has action $t$. Moreover, such a characteristic in $S_{2}$ exists in the subspace spanned by $v_{1}, v_{2}$. The other options include closed characteristics that pass between $S_{1}$ and $S_{2}$ and vice versa (maybe several times), and closed characteristics that stay in the intersection $S_{1} \cap S_{2}$ for all time. We analyse the latter two options below. We remark that from the proof below it follows that any closed characteristic spending a non-discrete time in $S_{1} \cap S_{2}$ must remain within this intersection indefinitely.

We start with some general observations regarding the Reeb dynamics that will be useful later on. Observe that the characteristics in $S_{1}$ are moving along
two centred circles in the $z_{1}$ and $z_{2}$-coordinate planes respectively, in the same direction and the same angular speed. The sum of the areas enclosed by the two circles is 1 . On the other hand, a direct computation shows that the characteristics in $S_{2}$ are moving along two non-centred circles in the $z_{1}, z_{2}$-coordinate planes respectively, in opposite directions, and with the same angular speed. The areas enclosed by these circles are $\frac{1+t}{2}$ and $\frac{1-t}{2}$, respectively.



Figure 8: The projections to the $z_{1}, z_{2}$ planes of the closed characteristic of $B^{4}(1)$ in black, and the closed characteristic of $A^{-1} W^{4}$ in red. The bold part is the section of the characteristic contained in $B^{4}(1) \cap A^{-1} W^{4}$.

Using the fact that $p \in A^{-1} W^{4}$ if and only if

$$
\left\langle p, J v_{1}\right\rangle^{2}+\left\langle p, J v_{2}\right\rangle^{2} \leq \frac{t^{2}}{\pi}
$$

one can write

$$
\begin{aligned}
& S_{1}=\left\{x_{1} J n_{1}+x_{2} J n_{2}+x_{3} J v_{1}+x_{4} J v_{2} \left\lvert\, \sum_{i=1}^{4} x_{i}^{2}=\frac{1}{\pi}\right., x_{3}^{2}+x_{4}^{2} \leq \frac{t^{2}}{\pi}\right\} \\
& S_{2}=\left\{x_{1} J n_{1}+x_{2} J n_{2}+x_{3} J v_{1}+x_{4} J v_{2} \left\lvert\, \sum_{i=1}^{4} x_{i}^{2} \leq \frac{1}{\pi}\right., x_{3}^{2}+x_{4}^{2}=\frac{t^{2}}{\pi}\right\}
\end{aligned}
$$

Note that for a point $p \in S_{1}$, the (not normalized) outer normal equals $p$, and the characteristic direction is $J p$. On the other hand, for $p \in S_{2}$, the characteristic direction is $J n_{A^{-1} W^{4}}(p)$, where

$$
n_{A^{-1} W^{4}}(p)=\left\langle p, J v_{1}\right\rangle J v_{1}+\left\langle p, J v_{2}\right\rangle J v_{2}
$$

is the outer normal to $A^{-1} W^{4}$ at $p$. From now on we set $x_{1}, x_{2}, x_{3}, x_{4}$ for a point $p$ so that

$$
p=x_{1} J n_{1}+x_{2} J n_{2}+x_{3} J v_{1}+x_{4} J v_{2}
$$

Moreover, a point $p$ in $A^{-1} W^{4}$ can be written as $p=\alpha_{1} v_{1}+\alpha_{2} v_{2}+\alpha_{3} J n_{1}+$ $\alpha_{4} J n_{2}$, where $\alpha_{1}^{2}+\alpha_{2}^{2} \leq \frac{1}{\pi}$. One can check that

$$
\alpha_{3}=\frac{\left\langle p, n_{2}\right\rangle}{t}=-\frac{\sqrt{1-t^{2}}}{t} x_{4}+x_{1}, \quad \alpha_{4}=-\frac{\left\langle p, n_{1}\right\rangle}{t}=\frac{\sqrt{1-t^{2}}}{t} x_{3}+x_{2}
$$

A direct computation shows that the projection of a Hopf circle passing through a point $\left(z_{1}, z_{2}\right)$ on the plane spanned by $J v_{1}, J v_{2}$ is an ellipse with area given
by $\left|\pi\left\|z_{1}\right\|^{2}-\frac{1-t}{2}\right|$. Denote the projection of the Hopf circle on $z_{i}$ by $\gamma_{i}$, and the projection to the $J v_{1}, J v_{2}$ plane by $\widetilde{\gamma}$. Denote the projection of the characteristics of $A^{-1} W^{4}$ on $z_{i}$ by $\eta_{i}$ (non-centred circles) and the projection on the plane spanned by $J v_{1}, J v_{2}$ by $\widetilde{\eta}$ (centred circle of radius $\left.\frac{t}{\sqrt{\pi}}\right)$.


Figure 9: The projection of the characteristics to the $\left(J v_{1}, J v_{2}\right)$-plane.
We analyse first closed characteristics entirely contained in the intersection of $S_{1}$ and $S_{2}$. Note that for $p \in S_{1} \cap S_{2}$ the characteristic direction is a non-negative linear combination of the two outer normals $J p$ and $J n_{A^{-1} W^{4}}(p)$, denoted by $s:=\alpha J p+\beta J n_{A^{-1} W^{4}}(p)$ for some $\alpha, \beta \geq 0$. In order for the characteristic to stay in the intersection, $s$ needs to be tangent to $S_{1} \cap S_{2}$, meaning $\langle s, p\rangle=$ $\left\langle s, n_{A^{-1} W^{4}}(p)\right\rangle=0$, which is equivalent to $\left\langle p, J n_{A^{-1} W^{4}}(p)\right\rangle=0$. This condition, together with the fact that $x_{1}^{2}+x_{2}^{2}=\frac{1-t^{2}}{\pi}, x_{3}^{2}+x_{4}^{2}=\frac{t^{2}}{\pi}$, implies that

$$
x_{1}=\lambda x_{4}, \quad x_{2}=-\lambda x_{3}, \quad \text { where }, \quad \lambda= \pm \frac{\sqrt{1-t^{2}}}{t}
$$

We start with the case $\lambda=\frac{\sqrt{1-t^{2}}}{t}$. In order for the characteristic to remain in the intersection one also needs to require that $\left\langle s, J n_{1}\right\rangle=\lambda\left\langle s, J v_{2}\right\rangle$, and $\left\langle s, J n_{2}\right\rangle=-\lambda\left\langle s, J v_{1}\right\rangle$. This condition implies that $\alpha=0$, i.e. the velocity is only "coming" from $S_{2}$. The point $p$ in Euclidean coordinates is of the form

$$
\left(\frac{\sqrt{1+t}}{\sqrt{2} t} x_{4},-\frac{\sqrt{1+t}}{\sqrt{2} t} x_{3}, \frac{\sqrt{1-t}}{\sqrt{2} t} x_{4}, \frac{\sqrt{1-t}}{\sqrt{2} t} x_{3}\right)
$$

and in this case $\alpha_{3}=\alpha_{4}=0$. This is the same closed characteristic as in the case when the characteristic is contained in $S_{2}$, which has action $t$. In addition, we claim that moving in the direction of the Hopf circle (and possibly leaving the intersection) is not possible. Indeed, recall that the projection of the Hopf circle $\widetilde{\gamma}$ passing through $p$ is an ellipse, which is tangent to $\widetilde{\eta}$, a circle of radius $t / \pi$ (and area $t^{2}$ ). As the area of the ellipse is $t$, it must contain the circle, and hence the Hopf circle intersects $A^{-1} W^{4}$ only in the tangency points. This observation also implies that a characteristic cannot get to $p$ from $S_{1}$.

We turn to case that $\lambda=-\frac{\sqrt{1-t^{2}}}{t}$. The conditions $\left\langle s, J n_{1}\right\rangle=\lambda\left\langle s, J v_{2}\right\rangle$ and $\left\langle s, J n_{2}\right\rangle=-\lambda\left\langle s, J v_{1}\right\rangle$ imply that $\beta=\left(\frac{1}{2 t^{2}}-2\right) \alpha$. As $\alpha, \beta>0$, we get that this case holds only for $t<\frac{1}{2}$. In this case,

$$
s=\frac{\alpha}{2 t}\left(-x_{2} J n_{1}+x_{1} J n_{2}-x_{4} J v_{1}+x_{3} J v_{2}\right) .
$$

This creates a simultaneous circular movement in the $J n_{1}, J n_{2}$ and $J v_{1}, J v_{2}$ planes. One can check that the projection of this orbit to the $z_{1}, z_{2}$ coordinates are also centred circles (rotating in opposite directions). The point $p$ in Euclidean coordinates is of the form

$$
\left(\frac{\sqrt{1+t}}{\sqrt{2}}\left(2-\frac{1}{t}\right) x_{4},-\frac{\sqrt{1+t}}{\sqrt{2}}\left(2-\frac{1}{t}\right) x_{3},-\frac{\sqrt{1-t}}{\sqrt{2}}\left(2+\frac{1}{t}\right) x_{4},-\frac{\sqrt{1-t}}{\sqrt{2}}\left(2+\frac{1}{t}\right) x_{3}\right) .
$$

The symplectic action is thus

$$
\pi\left\|z_{2}\right\|^{2}-\pi\left\|z_{1}\right\|^{2}=t\left(3-4 t^{2}\right) .
$$

Since $t<\frac{1}{2}$, the action is larger then $t$. Similarly to the case of $\lambda=\frac{\sqrt{1-t^{2}}}{t}$, the projection of the Hopf circle, $\widetilde{\gamma}$, is an ellipse of area $t\left(1-2 t^{2}\right)$, which is larger than $t^{2}$ (as $t<\frac{1}{2}$ ). Hence the ellipse is tangent to the circle of radius $t$ from the outside, which means that the direction of the Hopf circle does not interact with this characteristic.

It remains to consider the case of a characteristic which alternates between $S_{1}$ and $S_{2}$. Assume without loss of generality that the starting point of the characteristic is in $S_{1} \cap S_{2}$ and it moves along $S_{2}$ until it hits again the intersection. We claim that the norms of the $z_{1}$ and $z_{2}$ coordinates are the same at these two points (before and after moving in $S_{2}$ ). To show this, we calculate the intersection of a characteristic of $S_{2}$ with the boundary of the unit ball. Recall that a possible representation of this characteristic is

$$
\alpha_{1} v_{1}+\alpha_{2} v_{2}+\alpha_{3} J n_{1}+\alpha_{4} J n_{2}
$$

for fixed $\alpha_{3}, \alpha_{4}$ and $\alpha_{1}^{2}+\alpha_{2}^{2}=\frac{1}{\pi}$. In Euclidean coordinates this becomes

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}\left(\alpha_{1} \sqrt{1+t}+\alpha_{3} \sqrt{1-t}, \alpha_{2} \sqrt{1+t}+\alpha_{4} \sqrt{1-t}\right. \\
& \left.\quad \alpha_{1} \sqrt{1-t}+\alpha_{3} \sqrt{1+t},-\alpha_{2} \sqrt{1-t}-\alpha_{4} \sqrt{1+t}\right)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& \left\|z_{1}\right\|^{2}=\frac{1}{2}\left(\frac{1+t}{\pi}+(1-t)\left(\alpha_{3}^{2}+\alpha_{4}^{2}\right)+2 \sqrt{1-t^{2}}\left(\alpha_{1} \alpha_{3}+\alpha_{2} \alpha_{4}\right)\right) \\
& \left\|z_{2}\right\|^{2}=\frac{1}{2}\left(\frac{1-t}{\pi}+(1+t)\left(\alpha_{3}^{2}+\alpha_{4}^{2}\right)+2 \sqrt{1-t^{2}}\left(\alpha_{1} \alpha_{3}+\alpha_{2} \alpha_{4}\right)\right)
\end{aligned}
$$

In the intersection $\left\|z_{1}\right\|^{2}+\left\|z_{2}\right\|^{2}=\frac{1}{\pi}$, and hence

$$
\left\|z_{1}\right\|^{2}=\frac{1+t}{2 \pi}-\frac{t\left(\alpha_{3}^{2}+\alpha_{4}^{2}\right)}{2}, \quad\left\|z_{2}\right\|^{2}=\frac{1-t}{2 \pi}+\frac{t\left(\alpha_{3}^{2}+\alpha_{4}^{2}\right)}{2}
$$

Since these expressions are independent of $\alpha_{1}, \alpha_{2}$, we get that $\left\|z_{i}\right\|$, for $i=1,2$, is the same before and after the movement in $S_{2}$. In addition, we note that as one varies $\alpha_{1}$ and $\alpha_{2}$, the change in $\left\|z_{1}\right\|$ and $\left\|z_{2}\right\|$ is the same. Since $\left\|z_{1}\right\|^{2}+\left\|z_{2}\right\|^{2} \leq$ $\frac{1}{\pi}$, this means that the movement along $\eta_{i}$ lies inside the disc enclosed by $\gamma_{i}$. We continue along movement on $S_{1}$, which has the same projection to $z_{2}$ as $\gamma_{2}$. Consider a closed characteristic which starts with movement in $S_{2}$ with angular
change $\theta_{1}$ along $\eta_{2}$, then movement on $S_{1}$ with angular change $\tau_{1}$ (see Figure 100, continuing with movements that alternate between $S_{2}$ and $S_{1}$ with angular movements $\theta_{2}, \tau_{2}, \ldots, \theta_{k}, \tau_{k}$. Denote by $\widetilde{\theta}_{i}$ the angular change along $\gamma_{2}$ which corresponds to $\theta_{i}$.


Figure 10: The projection of a closed orbit alternating between $S_{1}$ and $S_{2}$ to the $z_{2}$ plane.

As the radius of $\eta_{2}$ (i.e., $\left.\sqrt{(1-t) /(2 \pi)}\right)$ is always smaller then the radius of $\gamma_{2}$ (i.e., $\left\|z_{2}\right\|$ ), we get that $\widetilde{\theta}_{i}<\theta_{i}$. Hence the action of the loop is

$$
\mathcal{A}=\sum_{i=1}^{k} \frac{\theta_{i}}{2 \pi} t+\frac{\tau_{i}}{2 \pi}>\sum_{i=1}^{k} \frac{\tilde{\theta}_{i}}{2 \pi} t+\frac{\tau_{i}}{2 \pi}>\left(\sum_{i=1}^{k} \frac{\tilde{\theta}_{i}}{2 \pi}+\frac{\tau_{i}}{2 \pi}\right) t \geq t
$$

where the last inequality is due to the fact that the loop is closed and the orientation of the loops does not change. This completes the proof of the proposition in the 4 -dimensional case. For a general dimension $n>2$, note that one can assume that the symplectic matrix, which we now denote by $A_{2 n}$ to distinguish it from the 4 -dimensional case above, is of the form

$$
A_{2 n}=\left(\begin{array}{c|c}
\mathbb{1}_{\mathbb{R}^{2 n-4}} & \mathbf{0} \\
\hline \mathbf{0} & A
\end{array}\right)
$$

where $A$ is the matrix (4). In this case one has

$$
A_{2 n} B^{2 n}(1) \cap W^{2 n}=B^{2 n-4}(1) \times_{2}\left(A B^{4}(1) \cap W^{4}\right)
$$

where $\times_{2}$ stands for the symplectic 2-product defined more generally for two convex domains $K_{1} \in \mathbb{R}^{2 n}$ and $K_{2} \in \mathbb{R}^{2 m}$ by

$$
K_{1} \times_{2} K_{2}:=\bigcup_{0 \leq s \leq 1}\left((1-s)^{1 / 2} K_{1} \times s^{1 / 2} K_{2}\right) .
$$

From Proposition 1.5 in [7] one has
$c_{\mathrm{EHZ}}\left(B^{2 n-4}(1) \times_{2}\left(A B^{4}(1) \cap W^{4}\right)\right)=\min \left\{c_{\mathrm{EHZ}}\left(B^{2 n-4}(1)\right), c_{\mathrm{EHZ}}\left(A B^{4}(1) \cap W^{4}\right)\right\}=t$,
which completes the proof of the proposition.
Proof of Theorem 1.3. The upper bound follows immediately from Proposition 3.2, and the lower bound follows from the combination of Proposition 3.3, Proposition 3.5, and the well known fact that the Hofer-Zehnder capacity coincide with the minimal action capacity for convex domains.

We turn now to the proof of Theorem 1.4 , which shows, roughly speaking that the Gromov width of $B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}$ is bounded below by an almost linear function.

Proof of Theorem 1.4. For $0<t<1$, consider the function $f(t)$ defined by

$$
\begin{equation*}
f(t):=\sqrt{2\left(\frac{1}{t^{2}}-1\right)\left(\sqrt{1-t^{2}}-1\right)+1} \tag{5}
\end{equation*}
$$

A direct computation shows that $f(t) \geq t-0.07$ (see Figure 11).


Figure 11: The different lower bounds together with the upper bound $t$.

From Proposition 3.3 it follows that it is enough to show that

$$
\underline{c}\left(A B^{2 n}(1) \cap W^{2 n}\right) \geq f(t)
$$

where $A$ is a symplectic matrix such that $A \Sigma_{\varepsilon}^{t}=\Sigma_{\varepsilon}$. We may assume without loss of generality that the outer normals to the hyperplanes in $\Sigma_{\varepsilon}^{t}$ are given by $n_{1}=\left(0,0, \ldots, 0, \sqrt{1-t^{2}}, 0,-t, 0\right)$ and $n_{2}=(0,0, \ldots, 0,0,0,0,1)$, written here in the $(\bar{x}, \bar{y})$-coordinate system. Note that as symplectic matrix $A$ that maps
$\Sigma_{\varepsilon}^{t}$ to $\Sigma_{\varepsilon}$ one can now take

$$
A=\left(\right) .
$$

We first describe two immediate bounds for $\underline{c}\left(A B^{2 n}(1) \cap W^{2 n}\right)$. The first comes from the largest Euclidean ball contained in the domain $B^{2 n}(1) \cap A^{-1} W^{2 n}$. More precisely, let $E$ be the linear subspace such that $E \perp n_{1}$ and $E \perp n_{2}$. Note that

$$
E=\mathbb{C}^{n-2} \times \operatorname{span}\left\{\left(t, 0, \sqrt{1-t^{2}}, 0\right),(0,1,0,0)\right\}
$$

and the corresponding symplectic orthogonal subspace is

$$
E^{\omega}=0 \times \operatorname{span}\left\{\left(0,-\sqrt{1-t^{2}}, 0, t\right),(0,0,1,0)\right\}
$$

In addition, the orthogonal complement of $E^{\omega}$ is

$$
\left(E^{\omega}\right)^{\perp}=\mathbb{C}^{n-2} \times \operatorname{span}\left\{(1,0,0,0),\left(0, t, 0, \sqrt{1-t^{2}}\right)\right\}
$$

and since the orthogonal projection of $E \cap B^{2 n}(1)$ to $\left(E^{\omega}\right)^{\perp}$ is

$$
\left\{\left(\lambda_{1}, \ldots, \lambda_{2 n-4}, \lambda_{2 n-3} t, \lambda_{2 n-2} t^{2}, 0, \lambda_{2 n-2} t \sqrt{1-t^{2}}\right): \pi \sum_{i=1}^{2 n-2} \lambda_{i}^{2} \leq 1\right\}
$$

one has $B^{2 n}\left(t^{2}\right) \subset B^{2 n}(1) \cap A^{-1} W^{2 n}$, which gives $\underline{c}\left(A B^{2 n}(1) \cap W^{2 n}\right) \geq t^{2}$.
The second lower bound is provided by the largest Euclidean ball inside $A B^{2 n}(1) \cap W$. More precisely, note that $\left\{x_{n}=0, y_{n}=0\right\} \cap A B^{2 n}(1)$ is of the form

$$
\begin{aligned}
& \left\{\left(x_{1}, y_{1}, \ldots, x_{n-2}, y_{n-2}, \frac{x_{n-1}}{\sqrt{t}}, y_{n-1} \sqrt{t}, 0,0\right) \left\lvert\, \pi\left(\sum_{i=1}^{n-2} x_{i}^{2}+y_{i}^{2}+\frac{x_{n-1}^{2}}{t^{2}}+y_{n-1}^{2}\right) \leq 1\right.\right\} \\
& =\left\{\left(x_{1}, y_{1}, \ldots, x_{n-2}, y_{n-2}, x_{n-1}, y_{n-1}, 0,0\right) \left\lvert\, \pi\left(\sum_{i=1}^{n-2} x_{i}^{2}+y_{i}^{2}+\frac{x_{n-1}^{2}}{t}+\frac{y_{n-1}^{2}}{t}\right) \leq 1\right.\right\}
\end{aligned}
$$

which is a $(2 n-2)$-dimensional symplectic ellipsoid containing the ball $B^{2 n-2}(t)$ of capacity $t$. On the other hand, for the largest ball inside $A B^{2 n}(1)$ one has

$$
\operatorname{inrad}\left(A B^{2 n}(1)\right)=\min _{\|v\|=1 / \sqrt{\pi}}\|A v\|=\frac{1}{\sqrt{\pi}\left\|A^{-1}\right\|_{\mathrm{op}}}=\sqrt{\frac{t}{\pi\left(\sqrt{1-t^{2}}+1\right)}} .
$$

Thus, $B^{2 n}\left(\frac{t}{\sqrt{1-t^{2}}+1}\right) \subseteq A B^{2 n}(1) \cap W$, which gives $\underline{c}\left(B^{2 n}(1) \backslash \Sigma_{\varepsilon}^{t}\right) \geq \frac{t}{\sqrt{1-t^{2}+1}}$.

Note that none of the above bounds dominates the other (see Figure 11). Moreover, in the first case the constraint for embedding is coming from the intersection of the ball with the cylinder $A^{-1} W$, while in the second case, the constraint is due to the intersection of a ball with $A B^{2 n}(1)$. A way to improve the two bounds above is to consider a symplectic linear image of the ball $S B^{2 n}(r)$ such that the largest $r$ for which it fits in the ball $B^{2 n}(1)$ is equal to the largest $r$ for which the image fits in the cylinder $A^{-1} W$. For this, consider the following symplectic matrix for some parameters $d_{1}, d_{2}>0$,

$$
S=\left(\right) .
$$

Note that when $d_{1}=d_{2}=1$ this corresponds to the first embedding described above, and up to a unitary transformation (which does not change the embedding of the ball), there exist $d_{1}$ and $d_{2}$ which correspond to the second embedding, i.e., $A^{-1}$ has this form after multiplying with a unitary matrix. Now we choose the parameters $d_{1}$ and $d_{2}$ such that the projection of the image of the ball to $\left(E^{\omega}\right)^{\perp}$ is a symplectic ellipsoid, or, in other words, that the base of the relevant symplectic image of the cylinder is always a disc. Moreover, as is often the case with similar optimization problems, we require that:

$$
\sup _{S B^{2 n}(r) \subset B^{2 n}(1)} r=\sup _{S B^{2 n}(r) \subset A^{-1} W} r .
$$

These two assumptions determine $d_{1}$ and $d_{2}$, and when plugging these solutions into $S$ we conclude that one can fit into $B^{2 n}(1) \cap A^{-1} W$ a ball of capacity

$$
\sqrt{2\left(\frac{1}{t^{2}}-1\right)\left(\sqrt{1-t^{2}}-1\right)+1}
$$

Remark 3.6. Numerical tests suggest that the embedding above of a ball with capacity given by (5) is the best embedding one can find using only linear symplectic maps.

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