LÉVY PROCESSES RESURRECTED IN THE POSITIVE HALF-LINE

MARÍA EMILIA CABALLERO, LOÏC CHAUMONT, AND VÍCTOR RIVERO

Abstract. A Lévy processes resurrected in the positive half-line is a Markov process obtained by removing successively all jumps that make it negative. A natural question, given this construction, is whether the resulting process is absorbed at 0 or not. We first describe the law of the resurrected process in terms of that of the initial Lévy process. Then in many important classes of Lévy processes, we give conditions for absorption and conditions for non absorption bearing on the characteristics of the initial Lévy process.

1. Introduction

Let $X^{(1)}$ be a real Lévy process starting from a nonnegative level. If $X^{(1)}$ becomes negative by a jump, then remove this jump and if it reaches 0 from above, then let the process be absorbed at 0. Then call $X^{(2)}$ the process thus obtained and apply to $X^{(2)}$ the same transformation as for $X^{(1)}$. Call $X^{(3)}$ the new process, and so on. The process Z obtained by repeating this procedure as long as $X^{(n)}$ crosses 0 by a jump is called the Lévy process $X^{(1)}$ resurrected in the positive half-line. The level 0 is clearly absorbing for the resurrected process Z and a natural question that will occupy much of this article is whether or not Z hits 0 in a finite time.

The resurrected process Z is actually a special case of Markov process constructed by piecing out as first introduced by Ikeda, Nagasawa and Watanabe [9] and studied in more detail by Meyer [13]. The problem of the finiteness of the lifetime (that is non conservativeness) of resurrected processes was first pointed out and studied in [9], Proposition 4.3, see also Sato [15], Theorem 4.5. These results are obtained in a very general setting and only allow us to solve the case where the negative half line is not regular for the Lévy process $X^{(1)}$, see Corollary 2 below. Later, Bogdan, Burdzy and Chen [3] considered a similar question for multidimensional symmetric stable processes resurrected in open sets with finite Lebesgue measure. This work was then extended by Wagner [16] to any symmetric Lévy process, see Theorem 2.6 therein. The papers [3] and [16] strongly bear on the symmetry of the process and the powerful tools provided by Dirichlet forms that can be used in this case. Then recently, Kim, Song and Vondraček [10] tackled the problem of conservativeness for positive self-similar Markov processes resurrected in the positive half line. They provide a complete solution of the problem in this case, which includes stable Lévy processes, as a direct application of the Lamperti transformation, see Section 6 below.

The main objective of the present paper is to give conditions for conservativeness of a real Lévy process resurrected in the positive half line. In the next section, we give a detailed definition of this process whose law is described in terms of that of the process killed when it reaches the negative half line. In particular, we specify the explicit form of the resurrection kernel. Then in Section 3 we prove that, when the initial Lévy process $X^{(1)}$ creeps downward and satisfy certain additional condition, the resurrected process is

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absorbed at 0 with probability one, independently of its starting point. Some criteria for conservativeness (non absorption) are given in Section 4 and some criteria for absorption are given in Section 5. This section actually contains the most delicate case, that is when $X^{(1)}$ enters immediately in the negative half line and drifts to $-\infty$. We give a sufficient condition for absorption in Theorem 5.2 but up to now, even when $X^{(1)}$ is the negative of a subordinator, we don't know whether this condition can be dropped or not. The stable case already mentioned above is treated in Section 6 and we address some perspectives in the last section.

We close this introduction by pointing out that even though we provide a rather large set of criteria to determine whether a resurrected process is conservative or not, there remain various open questions related to this. Thus, we see this paper as an invitation for a broader audience to work on the subject.

2. The resurrected process

2.1. **Basic definition.** Let $X=(X_t)_{t\geq 0}$ be any real Lévy process starting from 0. The *resurrected Lévy process* takes its name from the following recursive pathwise construction. Let $x\geq 0$ and let $X^{(n)}$, $n\geq 1$ be the sequence of stochastic processes defined by $X^{(1)}=X+x$ and for $n\geq 2$, if $\tau_{n-1}:=\inf\{t\geq 0: X_t^{(n-1)}\leq 0\}<\infty$, then

$$X_t^{(n)}=\begin{cases} X_t^{(n-1)}, & \text{if } t<\tau_{n-1},\\ X_t^{(n-1)}-(X_{\tau_{n-1}}^{(n-1)}-X_{\tau_{n-1}-}^{(n-1)}), & \text{if } t\geq \tau_{n-1},\end{cases} \tag{2.1}$$

where $X_{0-}=0$ and $X^{(n)}=X^{(n-1)}$, if $\tau_{n-1}=\infty$. The process $X^{(n)}$ is obtained by removing from $X^{(n-1)}$ its first jump through 0. Note that if for some $n\geq 1$, $X^{(n)}$ hits 0 continuously, that is $X_{\tau_n}=X_{\tau_n-}$, then $X^{(k)}=X^{(n)}$, for all $k\geq n$. Note also that $(\tau_n)_{n\geq 1}$ is a non decreasing sequence of random times and that $X_t^{(k)}=X_t^{(n)}$, for all $k\geq n$, whenever $t\leq\tau_n$. This allows us to define for each $t\geq 0$ the random variable $Z_t:=\lim_{n\to\infty}X_t^{(n)}\mathbb{I}_{\{t<\tau_n\}}$. Then $Z=(Z_t)_{t\geq 0}$ defines a càdlàg stochastic process which is nonnegative, absorbed at 0 and satisfies $Z_0=x$, a.s. The first hitting time of 0 by Z is obtained as the limit of the sequence $(\tau_n, n\geq 0)$, that is,

$$\zeta:=\inf\{t: Z_t=0\}=\lim_{n\to\infty}\tau_n.$$

We will also use the following more synthetic expression of Z,

$$Z_t=\sum_{n\geq 1}X_t^{(n)}\mathbb{I}_{\{\tau_{n-1}\leq t<\tau_n\}}\,,\quad t\geq 0\,, \tag{2.2}$$

where we set $\tau_0=0$. Then the process Z is called the resurrected Lévy process X starting from x.

This process is actually a special case of constructing a Markov process by piecing out, as described in [9] and [13]. More specifically, let $\mathbb{P}_x$, $x\in\mathbb{R}$ be a family of probability measures under which X is a Lévy process such that $\mathbb{P}_x(X_0=x)=1$ and define,

$$\tau=\inf\{t\geq 0: X_t\leq 0\}<\infty.$$

Then the law of the process Z is obtained by resurrecting under $\mathbb{P}_x$, $x\geq 0$, the killed Lévy process

$$Y_t=X_t\mathbb{I}_{\{t<\tau\}}\,,\quad t\geq 0\,, \tag{2.3}$$

according to the resurrection kernel,

$$K(\omega, dy) = \delta_{X_{\tau-}(\omega)}(dy). \tag{2.4}$$

(See Subsection 2.3 for more details.) From Theorem 1.1 in [9] and Théorème 1 in [13], Z is a strong Markov process with state space $[0, \infty)$, in a filtration where $(\tau_n,\, n \geq 1)$ are stopping times. Note that $Y = (Y_t)_{t\geq 0}$ is also a strong Markov process with state space $[0, \infty)$ and 0 as an absorbing state. We will keep the notation $(\mathbb{P}_x)_{x\in[0,\infty)}$ for the family of probability measures associated to the process Y, and $(P_x)_{x\in[0,\infty)}$ will denote the family of probability measures associated to Z. From our construction, when ζ is finite, Z reaches 0 continuously, that is, for all $x \geq 0$,

$$Z_{\zeta-} = 0\,, \quad P_x\text{-a.s. on the set } \{\zeta < \infty\}.$$

So, either Z reaches 0 continuously at a finite time or it never reaches 0. In all the cases treated here, these events have probability 0 or 1, independently of x. The major part of this paper is devoted to determine conditions on the characteristics of the Lévy process X for ζ to be finite.

2.2. **The distribution of the resurrected process.** Let us now describe the distribution of the process Z in more detail. Note that $\tau_1 = \tau$, $\mathbb{P}_x$-a.s., for all $x \geq 0$. Then it follows from (2.2) and (2.3) that for all nonnegative measurable function f such that $f(0) = 0$, $f(Z_t) = f(Y_t) + \sum_{n\geq 2} f(X_t^{(n)})\mathbb{1}_{\{\tau_{n-1}\leq t<\tau_n\}}$, $\mathbb{P}_x$-a.s., for all $t \geq 0$, so that Y is a subprocess of Z in the sense of part III.3 in [4], that is $\mathbb{E}_x(f(Y_t)) \leq E_x(f(Z_t))$, for all $t \geq 0$ and $x \geq 0$. This implies the existence of a multiplicative functional $M = (M_t)_{t\geq 0}$ of Z such that for all t, x and f as above,

$$\mathbb{E}_x(f(Y_t)) = E_x(f(Z_t)M_t)\,, \tag{2.5}$$

see Theorem 2.3, p.101 in [4]. It also suggests that the distribution of Z_t can be expressed from the process Y, at least in a non formal way, as follows

$$E_x(f(Z_t)) = \mathbb{E}_x(f(Y_t)M_t^{-1}(Y))\,. \tag{2.6}$$

The aim of Theorem 2.11 below is to make the functional $M_t^{-1}(Y)$ explicit and to give a direct proof of identity (2.6).

For that end, we next quote a result describing the joint distribution of $(\tau, X_{\tau-})$ under $\mathbb{P}_x$, for $x > 0$. This is more general than needed right now, but it will be handy all over our work. We denote by π the Lévy measure of X and we set $\bar{\pi}^-(x) = \pi((-\infty, x])$, $x \in \mathbb{R}$. We will also denote by

$$\mathbb{E}_x\left(\mathbb{1}_{\{X_s\in dy,s<\tau\}}\right) ds = U^0(x; ds, dy), \qquad s, x, y \geq 0, \tag{2.7}$$

the potential measure, in time and space, of X killed at its first passage time below 0 (that is the process Y). By U and U^*, we denote the renewal measure of the bivariate ascending, respectively descending, ladder time and height process associated to X, see Chap. VI in [1]. The Wiener-Hopf factorization in time and space ensures that for any non negative and measurable function h,

$$\begin{aligned}
&\iint_{[0,\infty)\times(0,\infty)} U^0(x; ds, dy)h(s, y) \\
&= \int_{[0,\infty)\times[0,x)} U^*(ds, d\ell) \int_{(0,\infty)\times[0,\infty)} U(du, dv)h(s + u, x - \ell + v)\mathbb{1}_{\{x-\ell+v>0\}},
\end{aligned}$$

where $U^*(ds,dy)=\delta_{(0,0)}(ds,dy)$ (resp. $U(ds,dy)=\delta_{(0,0)}(ds,dy)$) if X (resp. $-X$) is a subordinator.

Lemma 1. *The joint distribution of $(\tau, X_{\tau-})$ is characterized by the following identity, which holds for any Borel function $h:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$,*

$$\begin{aligned}&\iint_{[0,\infty)\times[0,\infty)}\mathbb{P}_x\left(\tau\in dt, X_{\tau-}\in dy,\tau<\infty\right)h(t,y)\\ &=\int_{[0,\infty)}a^*u^*(ds,x)h(s,0)\\ &\quad+\iint_{[0,\infty)\times(0,\infty)}U^0(x;ds,dy)h(s,y)\overline{\pi}^-(-y),\end{aligned}\tag{2.8}$$

where a^* is the drift coefficient of the descending ladder height process of X and $u^*(ds,x)$ denotes the density in the spatial coordinate of $U^*(ds,d\ell)$, which exists when $a^*>0$.

For all $t>0$, $x>0$ and for all positive and measurable function $f:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$,

$$\mathbb{E}_x(f(X_{\tau-},\tau)\mathbb{I}_{\{\tau\le t,\,X_\tau<X_{\tau-}\}}) = \int_0^t\mathbb{E}_x(f(X_s,s)\bar{\pi}^-(-X_s)\mathbb{I}_{\{s\le\tau\}})\,ds\,,\tag{2.9}$$

and

$$\mathbb{P}_x(\tau\le t,\,X_{\tau-}=0)=\int_{(0,t]}a^*u^*(ds,x).\tag{2.10}$$

Moreover, if X drifts to ∞, then

$$\mathbb{P}_x(\tau=\infty)=\kappa^*U([0,\infty)\times[0,x]),\qquad x>0;$$

if X drifts to $-\infty$, then

$$\mathbb{E}_x(\tau)=\kappa U^*([0,\infty)\times[0,x]),\qquad x>0;$$

while, if X oscillates, then

$$\mathbb{P}_x(\tau=\infty)=0\quad\textit{and}\quad\mathbb{E}_x(\tau)=\infty,\quad\textit{for all } x>0.$$

Here κ and κ^* are the killing rates of the upward and downward ladder height processes, respectively.

Proof. The proof of identity (2.8) can be found in Theorem 3.1 of [8], for the creeping part and in Lemma 11 of [7], for the jump part. The proof of (2.9) follows that of Lemma 11 of [7] up to a slight extension from the case $f\equiv 1$ to the general case. The rest of the identities comes from Proposition 17 in Chapter VI of [1]. □

Theorem 1. *For all $x\ge 0$, $t\ge 0$ and for all positive measurable function f,*

$$E_x(f(Z_t)\mathbb{I}_{\{t<\zeta\}})=\mathbb{E}_x\left(f(X_t)\exp\left(\int_0^t\bar{\pi}^-(-X_s)\,ds\right)\mathbb{I}_{\{t<\tau\}}\right).\tag{2.11}$$

Proof. First note that identity (2.11) is trivial for $x=0$. Moreover, since $(\tau_n,n\ge 1)$ is a non decreasing sequence which satisfies $\zeta=\lim_{n\to\infty}\tau_n$, it suffices to show that for all $x>0$, $t\ge 0$ and $n\ge 1$,

$$(n-1)!\,E_x\left(f(Z_t)\mathbb{I}_{\{\tau_{n-1}\le t<\tau_n\}}\right)=\mathbb{E}_x\left(f(X_t)\left(\int_0^t\bar{\pi}^-(-X_s)\,ds\right)^{n-1}\mathbb{I}_{\{t<\tau\}}\right).\tag{2.12}$$

For $n=1$, the equality is trivial for all $x>0$ and $t\ge 0$ (recall that $\tau_0=0$ and $\tau_1=\tau$, $\mathbb{P}_x$-a.s.). Then let us prove (2.12) by induction. Recall that $(\tau_n,n\ge 1)$ are stopping times in a filtration making Z a strong Markov process. Moreover, they satisfy $\tau_n = \tau_1 + \tau_{n-1} \circ \theta_{\tau_1}$.

Then let us fix $x > 0$ and $n \geq 2$. Assume that (2.12) holds for $n-1$ and for all $t > 0$, and apply the strong Markov property at time τ_1 in order to obtain,

$$
\begin{aligned}
& E_x\left(f(Z_t)\mathbb{1}_{\{\tau_{n-1}\le t<\tau_n\}}\right) = E_x\left(f(Z_t)\mathbb{1}_{\{\tau_{n-1}\le t<\tau_n,\, Z(\tau_{n-1})>0\}}\right) \\
= \;& E_x\left(\mathbb{1}_{\{\tau_1\le t,\, Z(\tau_1)>0\}}\left(\mathbb{1}_{\{\tau_{n-2}\le t-\tau_1<\tau_{n-1},\, Z(\tau_{n-2})>0\}} f(Z_{t-\tau_1})\right)\circ\theta_{\tau_1}\right) \\
= \;& E_x\left(\mathbb{1}_{\{\tilde\tau_1\le t,\, \tilde Z(\tilde\tau_1)>0\}} E_{\tilde Z(\tilde\tau_1)}\left(\mathbb{1}_{\{\tau_{n-2}\le t-s<\tau_{n-1},\, Z(\tau_{n-2})>0\}} f(Z_{t-s})\right)_{|s=\tilde\tau_1}\right),
\end{aligned}
$$

where in the last equality $(\tilde\tau_1, \tilde Z(\tilde\tau_1))$ is integrated under P_x and has the same law as $(\tau_1, Z(\tau_1))$. Then by applying successively (2.9) in the second equality below and our induction hypothesis in the third one, we obtain,

$$
\begin{aligned}
& E_x\left(\mathbb{1}_{\{\tilde\tau_1\le t,\, \tilde Z(\tilde\tau_1)>0\}} E_{\tilde Z(\tilde\tau_1)}\left(\mathbb{1}_{\{\tau_{n-2}\le t-s<\tau_{n-1},\, Z(\tau_{n-2})>0\}} f(Z_{t-s})\right)_{|s=\tilde\tau_1}\right) \\
= \;& \mathbb{E}_x\left(\mathbb{1}_{\{\tilde\tau\le t,\, \tilde X(\tilde\tau-)>0\}} E_{\tilde X(\tilde\tau-)}\left(\mathbb{1}_{\{\tau_{n-2}\le t-s<\tau_{n-1},\, Z(\tau_{n-2})>0\}} f(Z_{t-s})\right)_{|s=\tilde\tau}\right) \\
= \;& \int_0^t \mathbb{E}_x\left(\overline{\pi}^-(-X_s)\mathbb{1}_{\{s\le\tau\}} E_{X_s}\left(\mathbb{1}_{\{\tau_{n-2}\le t-s<\tau_{n-1},\, Z(\tau_{n-2})>0\}} f(Z_{t-s})\right)\right) ds \\
= \;& \frac{1}{(n-2)!}\int_0^t \mathbb{E}_x\Big(\overline{\pi}^-(-X_s)\mathbb{1}_{\{s\le\tau\}} \\
& \qquad\qquad \times \mathbb{E}_{X_s}\left(f(X_{t-s})\left(\int_0^{t-s}\overline{\pi}^-(-X_u)\,du\right)^{n-2}\mathbb{1}_{\{t-s<\tau\}}\right)\Big)\, ds \\
= \;& \frac{1}{(n-2)!}\mathbb{E}_x\left(\int_0^t \overline{\pi}^-(-X_s)\left(\int_s^t \overline{\pi}^-(-X_u)\,du\right)^{n-2} ds\, f(X_t)\mathbb{1}_{\{t<\tau\}}\right) \\
= \;& \frac{1}{(n-1)!}\mathbb{E}_x\left(\left(\int_0^t \overline{\pi}^-(-X_s)\,du\right)^{n-1} f(X_t)\mathbb{1}_{\{t<\tau\}}\right),
\end{aligned}
$$

where in the first equality, $(\tilde\tau, \tilde X(\tilde\tau-))$ is integrated under $\mathbb{P}_x$ and has the same law as $(\tau, X(\tau-))$. This shows (2.12) and ends the proof of (2.11). $\square$

Theorem 1, up to a few additional justifications, shows that the multiplicative functional of Z involved in (2.5) has the following expression,

$$
\mathbb{E}_x(f(Y_t)) = E_x\left(f(Z_t)\exp\left(-\int_0^t \overline{\pi}^-(-Z_s)\,ds\right)\mathbb{1}_{\{t<\zeta\}}\right).
$$

However, the interest of our work lies mainly in the identity (2.11) which describes the law of Z in terms of that of X.

2.3. **The resurrection kernel.** Let us define the kernel,

$$
\mathcal{K}_\lambda(x, dy) = E_x\left(e^{-\lambda\tau_1}\mathbb{1}_{\{Z_{\tau_1}\in dy,\, \tau_1<\infty\}}\right), \quad x, y, \lambda \geq 0, \tag{2.13}
$$

and the function $f_\zeta^{(\lambda)}(x) = E_x\left(e^{-\lambda\zeta}\right)$, for $x \geq 0$ and $\lambda \geq 0$. Then we first note that $f_\zeta^{(\lambda)}$ is invariant for $\mathcal{K}_\lambda$. We will set $\mathcal{K} := \mathcal{K}_0$ and $f_\zeta := f_\zeta^{(0)}$.

Proposition 1. *For all $x \geq 0$ and $\lambda \geq 0$,*

$$
\mathcal{K}_\lambda f_\zeta^{(\lambda)}(x) = f_\zeta^{(\lambda)}(x). \tag{2.14}
$$

In particular, the function $f_\zeta(x) = P_x(\zeta < \infty)$, $x \geq 0$, *satisfies,*

$$\mathcal{K} f_\zeta(x) = f_\zeta(x)\,. \tag{2.15}$$

Proof. From the strong Markov property applied for Z at time τ_1 and the identity $\zeta = \tau_1 + \zeta \circ \theta_{\tau_1}$, we obtain

$$\begin{aligned} f_\zeta^{(\lambda)}(x) &= E_x(\mathbb{I}_{\{\tau_1<\infty\}} e^{-\lambda\tau_1} \mathbb{E}_{Z_{\tau_1}}(e^{-\lambda\zeta})) \\ &= \int_{y\in[0,\infty)} \mathcal{K}_\lambda(x,dy) f_\zeta^{(\lambda)}(y)\,, \end{aligned}$$

which proves (2.14). Then (2.15) is obtained by taking $\lambda = 0$. □

From (2.8) in Lemma 1, the kernel $\mathcal{K}_\lambda$ can be made explicit as follows,

$$\begin{aligned} \mathcal{K}_\lambda(x,dy) &= \mathbb{E}_x\left(e^{-\lambda\tau} \mathbb{I}_{\{X_{\tau-}\in dy,\, \tau<\infty\}}\right) \\ &= U_\lambda^0(x,dy)\overline{\pi}^-(-y)\mathbb{I}_{\{y>0\}} + a^* u_\lambda^*(x)\delta_0(dy), \end{aligned} \tag{2.16}$$

where $U_\lambda^0(x,dy) = \int_0^\infty e^{-\lambda s}\mathbb{P}_x(X_s \in dy,\, s < \tau)\, ds$ is the λ-potential measure of the killed process Y defined in the previous subsection, $u_\lambda^*(x)$ is the density of the λ-potential of the downward ladder height process of X and a^* is its drift coefficient.

Following our objective, we wish to obtain more information on the function $f_\zeta(x) = P_x(\zeta < \infty)$. Note that when the process X drifts toward ∞, $\mathbb{P}_x(\tau < \infty) = P_x(\tau_1 < \infty) < 1$ and since $\zeta \geq \tau_1$, P_x-a.s., we have $f_\zeta(x) < 1$. On the other hand, when the process X does not drift toward ∞, we have $\mathbb{P}_x(\tau < \infty) = P_x(\tau_1 < \infty) = 1$ so that the kernel

$$\mathcal{K}(x,dy) = U^0(x,dy)\overline{\pi}^-(-y)\mathbb{I}_{\{y>0\}} + a^* u^*(x)\delta_0(dy)$$

is Markovian. It completes the description of the resurrection kernel given in (2.4). It is actually the transition kernel of the Markov chain $(Z_{\tau_n})_{n\geq 0}$, that is for all bounded Borel function f,

$$\mathcal{K}^{(n)} f(x) = E_x(f(Z_{\tau_n})),$$

where $\mathcal{K}^{(n)}$ denotes the n-th composition of $\mathcal{K}$ with itself. Equation (2.15) tells us that the function f_ζ is a bounded invariant function for $\mathcal{K}$. In our forthcoming analysis of cases, we will encounter that a zero-one law arises, either $f_\zeta \equiv 1$ or $f_\zeta \equiv 0$. From there, we conjecture that

- *If X does not drift towards ∞, then either $f_\zeta(x) = 1$, for all $x \in (0,\infty)$, or $f_\zeta(x) = 0$, for all $x \in (0,\infty)$.*

A possible approach to prove this conjecture would require studying the totality of invariant functions for the Markovian kernel $\mathcal{K}$. In particular, if one is able to prove that the totality of bounded invariant functions for $\mathcal{K}$ are the constant functions, then necessarily f_ζ would be a constant function, and hence equal to 0 or 1. We invite the interested reader to prove or disprove this conjecture.

3. The creeping case

Let us start with the case where the Lévy process X creeps downward. Recall that by definition, this means that for all $x > 0$,

$$\mathbb{P}_x(X_{\tau-} = 0,\, \tau < \infty) = a^* u^*(x) > 0$$

and that X creeps downward if and only if the drift a^* is positive. Moreover, in this case, u^* is continuous on $[0,\infty)$ and satisfies $\lim_{x\to 0+} a^* u^*(x) = 1$.

Theorem 2. *Assume that X creeps downward and that it does not drift toward ∞. If either $-X$ is a subordinator or if the downward ladder height process of X has finite mean, then $f_\zeta(x) = P_x(\zeta < \infty) = 1$, for all $x \geq 0$.*

Proof. First observe that since X does not drift toward ∞, $P_x(\tau_n < \infty) = 1$, for all $n \geq 1$. In particular $\mathbb{P}_x(\tau < \infty) = 1$ and $\mathbb{P}_x(X_{\tau-} = 0) = P_x(Z_{\tau_1} = 0) = a^* u^*(x)$. Then from the Markov property and the identity $\tau_n = \tau_1 + \tau_{n-1} \circ \theta_{\tau_1}$,

$$
\begin{aligned}
P_x(Z_{\tau_n} > 0) &= E_x(\mathbb{1}_{\{Z_{\tau_{n-1}}>0\}} P_{Z_{\tau_{n-1}}}(Z_{\tau_1} > 0)) \\
&= E_x(\mathbb{1}_{\{Z_{\tau_{n-1}}>0\}}[1 - a^* u^*(Z_{\tau_{n-1}})]). \qquad (3.1)
\end{aligned}
$$

If $-X$ is a subordinator, then $Z_{\tau_n} \leq x$, P_x-a.s., for all n, so that

$$1 - a^* u^*(Z_{\tau_{n-1}}) \leq 1 - \inf_{y \in [0,x]} a^* u^*(y) := k < 1, \quad P_x - a.s.,$$

and hence, from equality (3.1), $P_x(Z_{\tau_n} > 0) \leq k P_x(Z_{\tau_{n-1}} > 0)$, which implies $P_x(Z_{\tau_n} > 0) < k^n$. Then we derive from Borel-Cantelli lemma that P_x-a.s., $Z_{\tau_n} > 0$ holds only a finite number of times and therefore Z is absorbed at a finite time.

Let us now consider the case where X does not drift toward ∞ and creeps downward, and assume that the downward ladder height process of X has finite mean. Then recall from [2] that $\lim_{y\to\infty} u^*(y) = a^*/m > 0$, where m is the mean of the downward ladder height process. This yields,

$$1 - a^* u^*(Z_{\tau_{n-1}}) \leq 1 - \inf_{y \geq 0} a^* u^*(y) < 1, \quad P_x - a.s.,$$

and the same argument as above leads to the same conclusion. □

In view of Theorem 3 below, it seems that integrability of the downward ladder height process when $-X$ is not a subordinator is not a necessary condition in Theorem 2. However, although it is a little counterintuitive at first glance, it is possible that 'big' negative jumps of X at its infimum play an important role for the conservativeness property of the resurrected process.

4. Some criteria for non absorption

Let us denote by τ_z^-, the first passage time below z by X, that is

$$\tau_z^- = \inf\{t \geq 0 : X_t \leq z\}, \qquad z \in \mathbb{R}.$$

By our construction in Subsection 2.1 and from the strong Markov property, conditionally on the $n-1$ first positions where the process is resurrected, say $(Z(\tau_0) = x_0,\, Z(\tau_1) = x_1, \ldots, Z(\tau_{n-1}) = x_{n-1})$, the n-th resurrection time under P_{x_0} has the same distribution as

$$\tau_n = \sum_{i=0}^{n-1} \tau_{-x_i}^{-,i}, \quad n \geq 1, \qquad (4.1)$$

where $(\tau_{-x_0}^{-,0}, \tau_{-x_1}^{-,1}, \ldots, \tau_{-x_n}^{-,n}, \ldots)$ are independent random variables, and the law of $\tau_{-z}^{-,i}$ is the same as that of τ_{-z}^- under $\mathbb{P}$. We deduce therefrom that, conditionally on the resurrection positions $(Z(\tau_0) = x_0,\, Z(\tau_1) = x_1, \ldots, Z(\tau_n) = x_n, \ldots)$, the resurrected process Z will be absorbed at 0 in a finite time if and only if

$$\sum_{n \geq 0} \tau_{-x_n}^{-,n} < \infty.$$

This argument yields the following identity in law,

$$\zeta \stackrel{\text{(Law)}}{=} \sum_{n\geq 0} \tau^{-,n}_{-Z(\tau_n)}, \tag{4.2}$$

where the family of processes $\{(\tau^{-,i}_{-x_i}),\ x_i \geq 0, i \geq 0\}$ is independent of the sequence $(Z(\tau_i))_{i\geq 0}$. As one can easily guess, there is no standard technique for determining necessary and sufficient conditions for the convergence of such a series. Hence, according to the case, we will develop a technique that will lead us to sufficient conditions for their convergence, respectively, divergence.

As a first instance, in the following Lemma we will establish a curious identity in law, which in turn will prove handy in establishing a stochastic domination for the resurrection times. See the forthcoming inequality (4.3), in order to study the limit in (4.2). A version of this Lemma for general Markov processes has been obtained in [15].

Lemma 2. *Assume that the Lévy process X does not creep downward and does not drift to ∞. Then the random variable*

$$\int_0^\tau \overline{\pi}^-(-X_t)dt$$

is exponentially distributed with parameter 1.

Proof. It follows from Lemma 1 that in the present setting the first passage time below 0, in the event where the process does not creep below 0 has a density given by

$$\frac{\mathbb{P}_x(\tau \in dt, X_{\tau-} > 0)}{dt} = \mathbb{E}_x(\overline{\pi}^-(-X_t)\mathbb{I}_{\{t<\tau\}}), \qquad t > 0, \qquad x > 0.$$

We hence have

$$\mathbb{P}_x(X_{\tau-} > 0, \tau < \infty) = \mathbb{E}_x\left(\int_0^\tau \overline{\pi}^-(-X_t)dt\right).$$

On the other hand, since we assumed that X does not creep downward and does not drift to ∞ we have that

$$\mathbb{P}_x(X_{\tau-} > 0, \tau < \infty) = 1, \qquad x > 0.$$

The n-th moment of the r.v. of interest can be calculated using Kac's moment formula to get the identity

$$\begin{aligned}&\mathbb{E}_x\left[\left(\int_0^\tau \overline{\pi}(-X_t)dt\right)^n\right]\\ &= n!\mathbb{E}_x\left[\int_{0<s_1<s_2<\cdots<s_n<\tau} \prod_{i=1}^n \overline{\pi}^-(-X_{s_i})ds_n ds_{n-1}\cdots ds_1\right].\end{aligned}$$

Then we can apply the Markov property to obtain that the latter expression equals

$$\begin{aligned}&n!\mathbb{E}_x\left[\int_{0<s_1<s_2<\cdots<s_{n-1}<\tau} \prod_{i=1}^{n-1} \overline{\pi}^-(-X_{s_i})\left(\mathbb{E}_{X_{s_{n-1}}}\left[\int_0^\tau \overline{\pi}^-(-X_{s_n})ds_n\right]\right) ds_{n-1}\cdots ds_1\right]\\ &= n!\mathbb{E}_x\left[\int_{0<s_1<s_2<\cdots<s_{n-1}<\tau} \prod_{i=1}^{n-1} \overline{\pi}^-(-X_{s_i})\mathbb{P}_{X_{s_{n-1}}}(X_{\tau-} > 0, \tau < \infty)ds_{n-1}\cdots ds_1\right]\\ &= n!\mathbb{E}_x\left[\int_{0<s_1<s_2<\cdots<s_{n-1}<\tau} \prod_{i=1}^{n-1} \overline{\pi}^-(-X_{s_i})ds_{n-1}\cdots ds_1\right] = n!\mathbb{P}_x(X_{\tau-} > 0, \tau < \infty) = n!,\end{aligned}$$

and the above holds for any $n \geq 1$. We deduce that the moments of $\int_0^\tau \overline{\pi}^-(-X_t)dt$ coincide with those of a standard exponential r.v. Since the latter is moment determinate we get our claim. $\square$

We derive from the above lemma the two following corollaries.

Corollary 1. *Assume that X does not creep downward and that $0 < \pi(-\infty, 0) < \infty$. Then the resurrected process has an infinite lifetime, that is $P_x(\zeta = \infty) = 1$, for all $x > 0$.*

Proof. We assume first that X does not drift towards ∞. Recall that for $x > 0$, under $\mathbb{P}_x$ the random variable $\int_0^\tau \overline{\pi}^-(-X_t)dt$ follows an exponential distribution with parameter 1. Now, remark the following inequality,

$$\mathbf{e} := \int_0^\tau \overline{\pi}^-(-X_t)dt \leq \overline{\pi}^-(0)\tau.$$

Recall also that, in the notation of the beginning of this section, under $\mathbb{P}_x$, τ has the same law as τ_{-x}^- under $\mathbb{P}$. Then from the above observation and the representation (4.1) of τ_n, we obtain the stochastic domination

$$\tau_n \overset{\text{Law}}{\geq} \frac{1}{\overline{\pi}^-(0)} \sum_{i=0}^{n-1} \mathbf{e}_i, \tag{4.3}$$

where $\mathbf{e}_i$, $i \geq 0$ are i.i.d. standard exponential r.v.'s. It follows that $\tau_n \to \infty$, a.s. when $n \to \infty$.

Assume now that X drifts towards ∞. In this case, we have that

$$P_x(\zeta = \infty) \geq \mathbb{P}_x(\tau = \infty) > 0, \qquad x > 0.$$

To prove that the latter probability equals one, independently of the starting point, we will prove that the event $\{\zeta < \infty\}$ has zero probability. Since the process does not creep downward, the only way in which the resurrected process gets absorbed at 0, viz. $\zeta < \infty$, is by infinite resurrections whose sum of lengths is finite. But this is impossible because, in this case, each resurrection time is stochastically bounded by below by an exponential random variable of parameter $\pi^-(0)$. Indeed, to have a resurrection, at least there should be a negative jump, which happens at an exponential time with parameter $\pi^-(0)$. This concludes the proof. $\square$

The following corollary is a consequence of Corollary 1. It is obtained by a domination argument.

Corollary 2. *If 0 is not regular for $(-\infty, 0)$, then the resurrected process has infinite lifetime, that is $P_x(\zeta = \infty) = 1$, for all $x > 0$.*

Proof. If 0 is not regular for the half-line $(-\infty, 0)$, the downward ladder height process $\widehat{H}$ has a finite Lévy measure and zero drift, so the process X can not creep downwards. We assume for a moment that X does not drift towards ∞, and hence $\widehat{H}$ has also an infinite lifetime. Then Corollary 1 ensures that the process obtained by resurrection of $\widehat{H}$ is never absorbed at zero. Since in the local time scale this process bounds by below the process Z, we infer that the latter is never absorbed at 0.

Then we deal with the case where X drifts towards ∞. As in Corollary 1,we see that the event $\{\zeta < \infty\}$, has zero probability. Indeed, in the local time scale, the downward ladder height process is never absorbed at zero, which bounds the resurrected process from below, and then an infinite excursion from the infimum, inside the latest resurrection, arises, and from there onwards there is no need to resurrect the process again as it never goes below the reached infimum. $\square$

Let us point out that the result of Corollary 2 can also be derived from Proposition 4.3 in [9] or Theorem 4.5 in [15], but we have chosen to give a proof here for sake of completeness and to explain the applicability of our results.

5. When X drifts towards $-\infty$

Let us recall the notations introduced before Lemma 1 in Section 2 and denote the renewal functions of the downward and upward ladder height processes respectively by

$$U^*([0,\infty)\times[0,x]) := U^*(x)\,, \qquad U([0,\infty)\times[0,x]) := U(x), \qquad x>0.$$

Recall that $\kappa\in[0,\infty)$ is the killing rate of the upward ladder process, that is $U([0,\infty)\times[0,\infty))=\kappa^{-1}$. Moreover, from Lemma 1, when X drifts towards $-\infty$, $\kappa>0$ and

$$\mathbb{E}_x(\tau) = \kappa U^*(x), \qquad x\ge 0. \tag{5.1}$$

In this section we give sufficient conditions for the lifetime of the resurrected process to be finite.

Theorem 3. *Assume that,*
(i) 0 is regular for $(-\infty,0)$,
(ii) X drifts towards $-\infty$,
(iii) the following condition is satisfied,

$$\sup_{y>0} U^*(y)\overline{\pi}^-(-y) < \kappa. \tag{5.2}$$

Then $E_x(\zeta)<\infty$, and in particular $P_x(\zeta<\infty)=1$, for all $x\ge 0$.

Remark 1. It is worth pointing out that the constant κ depends on the chosen normalization of the local time at the supremum, which is actually related to that chosen at the infimum, thus, changing it, would lead to a change the representation of U^*.

Proof. We derive from (4.2) that

$$E_x(\zeta) = \sum_{n\ge 0} E_x\left(\tau^{-,n}_{-Z(\tau_n)}\right), \tag{5.3}$$

where we recall that, under P_x, the family of processes $\{(\tau^{-,n}_{-x_n}),\, x_n\ge 0, n\ge 0\}$ is independent of the sequence $(Z(\tau_n))_{n\ge 0}$. Moreover, under P_x, the variables $\tau^{-,n}_{-x_n}$, $x_n\ge 0$, $n\ge 0$ are independent and $\tau^{-,n}_{-x_n}$ has the same law as $\inf\{t\ge 0 : X_t\le -x_n\}$ under $\mathbb{P}$. Note that from assumption (ii) of the statement, $\tau_n<\infty$, P_x-a.s. for all $x\ge 0$. Then from (5.1), the relation $\tau_n=\tau_{n-1}+\tau_1\circ\theta_{\tau_{n-1}}$ and the strong Markov property applied at time τ_{n-1}, we obtain that for all $n\ge 1$,

$$\begin{aligned} E_x\left(\tau^{-,n}_{-Z(\tau_n)}\right) &= \kappa E_x\left(U^*(Z(\tau_n))\right) \\ &= \kappa E_x\left(E_{Z(\tau_{n-1})}\left(U^*(Z(\tau_1))\right)\right). \end{aligned}$$

Since X drifts towards $-\infty$, we kave $\kappa > 0$. Then it follows from (2.8) in Lemma 1 that

$$\begin{aligned}
&\kappa E_x\left(E_{Z(\tau_{n-1})}\left(U^*(Z(\tau_1))\right)\right) = \kappa E_x\left(\mathbb{E}_{Z(\tau_{n-1})}\left(U^*(X(\tau-))\right)\right)\\
= \;& \kappa E_x\left(\int_0^{Z(\tau_{n-1})} U^*(dy)\int_0^\infty U(dz)U^*(Z(\tau_{n-1})-y+z)\overline{\pi}^-(y-Z(\tau_{n-1})-z)\right)\\
&+a^*U^*(0)E_x(u^*(Z(\tau_{n-1}))\\
\le \;& \kappa U([0,\infty)\times[0,\infty))\sup_{y>0}U^*(y)\overline{\pi}^-(-y)E_x\left(U^*(Z(\tau_{n-1}))\right)+a^*U^*(0)E_x(u^*(Z(\tau_{n-1}))\\
= \;& \sup_{y>0}U^*(y)\overline{\pi}^-(-y)E_x\left(U^*(Z(\tau_{n-1}))\right) = \kappa^{-1}\sup_{y>0}U^*(y)\overline{\pi}^-(-y)E_x\left(\tau^{-,n-1}_{-Z(\tau_{n-1})}\right).
\end{aligned}$$

Note that $U^*(0)=0$ since 0 is regular for $(-\infty,0)$. It follows from the above inequalities that for all $n\ge 1$, $E_x\left(\tau^{-,n}_{-Z(\tau_n)}\right)\le cE_x\left(\tau^{-,n-1}_{-Z(\tau_{n-1})}\right)$, where $c:=\kappa^{-1}\sup_{y>0}U^*(y)\overline{\pi}^-(-y)$ and hence $E_x\left(\tau^{-,n}_{-Z(\tau_n)}\right)\le c^nU^*(x)$. Together with (5.2) and (5.3), this implies that for any $x>0$, $\mathbb{E}_x\left(\zeta\right)\le \frac{1}{1-c}U^*(x)<\infty$. $\square$

We emphasize that creeping Lévy processes always satisfy condition (i) of Theorem 3 and when they also satisfy conditions of Theorem 2, then conditions (ii) and (iii) of Theorem 3 are not needed for the associated resurrected process to be absorbed at 0 in a finite time.

For the remainder of this section, we will focus on the special case where X is a non increasing Lévy process, that is the negative of a subordinator. Condition (5.2) can be very useful in this particular case. Indeed, when X is decreasing, U^* is the renewal function of the process X itself. In this case, U^* will be written as,

$$U^*(x)=\int_0^\infty \mathbb{P}(0\le -X_t\le x)\,dt\,.$$

Moreover, recall that the renewal measure $U(dt,dx)$ has the simple form $U(dt,dx)=\delta_{(0,0)}(dt,dx)$, so that $\kappa=1$. Then conditions of Theorem 3 are satisfied whenever X has no negative drift, $\pi(-\epsilon,0)=\infty$, for all $\epsilon>0$ and (5.2) holds and we obtain the following corollary.

Corollary 3. *Assume that X is the negative of a subordinator with no (negative) drift and such that $\pi(-\epsilon,0)=\infty$, for all $\epsilon>0$. If*

$$\sup_{y>0}U^*(y)\overline{\pi}^-(-y)<1, \tag{5.4}$$

then $P_x(\zeta<\infty)=1$, for all $x\ge 0$.

This result leads to the following corollary that covers many commonly found examples of subordinators.

Corollary 4. *Assume that X is the negative of a subordinator whose tail Lévy measure satisfies that there are $0<\alpha,\beta<1$, such that $\overline{\pi}^-$ is regularly varying at 0 with index α and at infinity with index β, viz. for all $c>0$,*

$$\lim_{x\to 0+}\frac{\overline{\pi}^-(-xc)}{\overline{\pi}^-(-x)}=c^{-\alpha},\qquad \lim_{x\to\infty}\frac{\overline{\pi}^-(-xc)}{\overline{\pi}^-(-x)}=c^{-\beta}.$$

In this case, we have that $P_x(\zeta<\infty)=1$, for all $x\ge 0$.

Proof. From the estimates in page 75 in [1] and the reflection formula for the Gamma function, we know that under the assumptions of the Corollary we have

$$\lim_{y\to 0} U^*(y)\overline{\pi}^-(-y) = \frac{\sin \pi\alpha}{\pi\alpha}, \qquad \lim_{y\to 0} U^*(y)\overline{\pi}^-(-y) = \frac{\sin \pi\beta}{\pi\beta}.$$

The latter are both valued in $(0,1)$. It follows from Theorem 3 in [6] that for all $x>0$, $\mathbb{P}(X_{\tau^-_{-x}-} - X_{\tau^-_{-x}} > x) = U^*(x)\overline{\pi}^-(-x)$. The assumptions imply that $\overline{\pi}^-(-x) > 0$, $\forall x > 0$, and hence the latter probability is necessarily in $(0,1)$. We can conclude from here that the condition (5.2) is satisfied. □

Remark 2. The previous corollary suggest a method to build examples of Lévy process for which

$$\sup_{x>0} U^*(x)\overline{\pi}^-(-x) = 1. \tag{5.5}$$

For instance, this is the case when $-X$ is a Gamma subordinator, that is

$$\mathbb{E}(\exp(\lambda X_1)) = (1+\lambda/b)^{-a} = \exp\left(-\int_0^\infty (1-e^{-\lambda x})ax^{-1}e^{-bx}\,dx\right), \quad a,b,\lambda > 0,$$

see [1] p.75. In this interesting setting, Theorem 3 is not conclusive and other techniques seem to be necessary.

6. The stable case

The case where X is a stable Lévy process is very particular as our problem can be tackled and entirely solved by using the Lamperti transformation. The method we will develop in this section has recently been extended to all positive self-similar Markov processes in [10].

We consider X a stable Lévy process with index $\alpha \in (0,2]$ and recall from (2.3) the definition of the killed process Y. In this setting, both Markov processes Y and Z clearly inherit from X the scaling property of index $1/\alpha$. As positive self-similar Markov processes, Y and Z can each be represented as the exponential of some possibly killed Lévy process, time changed by the inverse of its exponential functional, see [12]. Let ξ^Y (resp. ξ^Z) be the underlying Lévy process in the Lamperti representation of Y, (resp. Z). Then our construction of Z from Y and the Lamperti representation of both processes show that ξ^Y is obtained from (the non killed Lévy process) ξ^Z by killing it at an independent exponential time of some parameter, say $\beta \ge 0$. Note that $\beta = 0$ if and only if X has no negative jumps. In this case, $Z = Y$ and the latter process clearly hits 0 in a finite time almost surely. Therefore, we can assume that $\beta > 0$.

Using these arguments and standard facts from the theory of self-similar Markov processes, we obtain that Z hits 0 in a finite time if and only if ξ^Z drifts towards $-\infty$. On the other hand, we know from [5] and [11] that the process ξ^Y is a process of the hypergeometric type, with characteristic exponent,

$$\mathbb{E}\left(e^{i\lambda\xi_1^Y}, 1 < \mathrm{e}\right) = \exp\{-\Psi(\lambda)\}, \qquad \Psi(\lambda) = \frac{\Gamma(\alpha - i\lambda)}{\Gamma(\alpha\overline{\rho} - i\lambda)}\frac{\Gamma(1+i\lambda)}{\Gamma(1-\alpha\overline{\rho}+i\lambda)},$$

where e denotes the lifetime of ξ^Y and $\overline{\rho} = \mathbb{P}(X_1 < 0)$. It is then easily verified that there is a constant $C_\alpha > 0$ such that,

$$\mathbb{E}\left(\xi_1^Z\right) = \mathbb{E}\left(\xi_1^Y \mid 1 < \mathrm{e}\right) = C_\alpha\left((\psi(1-\alpha\overline{\rho}) - \psi(1)) - (\psi(\alpha\overline{\rho}) - \psi(\alpha))\right), \tag{6.1}$$

where $\psi(\beta)$ denotes the digamma function $\psi(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$. We are now able to solve the problem of the finiteness of the lifetime of Z in the stable case.

Theorem 4. *Assume that X is a stable Lévy process with index $\alpha \in (0, 2]$.*
Then $P_x(\zeta < \infty) = 1$, for all $x \geq 0$ if and only if α and $\overline{\rho}$ satisfy,

$$\cot(\pi\alpha\overline{\rho}) < \int_0^\infty \frac{dt}{1 - e^{-t}} \left(e^{-\alpha t} - e^{-t}\right). \tag{6.2}$$

Proof. As argued before the statement of the theorem, $P_x(\zeta < \infty) = 1$, for all $x \geq 0$ if and only if ξ^Z drifts towards $-\infty$, which is equivalent to $\mathbb{E}\left(\xi_1^Z\right) < 0$. From (6.1) we are then left to find the values α and $\overline{\rho}$ such that $\psi(1 - \alpha\overline{\rho}) - \psi(1) - \psi(\alpha\overline{\rho}) + \psi(\alpha) < 0$.

Then note that by the reflection formula for the digamma function,

$$\psi(1 - \alpha\overline{\rho}) - \psi(\alpha\overline{\rho}) = \cot(\pi\alpha\overline{\rho}).$$

On the other hand, the following identity for the digamma function is well known

$$\psi(\delta) - \psi(\gamma) = \int_0^\infty \frac{dt}{1 - e^{-t}} \left(e^{-\gamma t} - e^{-\delta t}\right), \qquad \delta, \gamma \geq 0,$$

and this allows us to conclude. □

Since $\cot(\pi\alpha\overline{\rho}) \geq 0$ if $\alpha\overline{\rho} \in [0, 1/2]$ and $\cot(\pi\alpha\overline{\rho}) < 0$ if $\alpha\overline{\rho} \in (1/2, 1]$, it follows from (6.2), that if $\alpha < 1$ and $\alpha\overline{\rho} > 1/2$ then $P_x(\zeta < \infty) = 1$, for all $x \geq 0$, whereas if $\alpha\overline{\rho} \leq 1/2$ and $\alpha \geq 1$ then $P_x(\zeta < \infty) = 0$, for all $x > 0$. Note also that when $-X$ is a stable subordinator, Corollary 4 cannot be recovered from (6.2) without any further development. However, it can be derived directly from Lamperti's transformation. Indeed, in this case, Z is a decreasing self-similar Markov process whose only alternative is to hit 0 through an accumulation of jumps in a finite time.

7. Open question and perspectives

The various open questions raised throughout this note actually boil down to the fact that some class of Lévy processes resists our investigations. This is the class of Lévy processes which do not drift towards infinity, for which 0 is regular for $(-\infty, 0)$ and which fall outside the cases covered by Theorems 2, 3 and 4. We would like to know which among the latter processes satisfy the property,

$$(P) \qquad P_x(\zeta < \infty) = 1 \text{ for all } x \in [0, \infty).$$

When the initial Lévy process X either drifts toward infinity or when 0 is not regular for $(-\infty, 0)$, the property (P) fails, as shown in the comment at the end of Subsection 2.3 and in Corollary 2. On the other hand, the stable case in Theorem 4 shows that it is not enough that X does not drift towards infinity and that 0 is regular for $(-\infty, 0)$ for the property (P) to be satisfied.

Let us finally emphasize that some of our results can be extended to general $\mathbb{R}^d$-valued Markov processes. We consider the resurrection of such a process when leaving an open subset $D \subset \mathbb{R}^d$. The rate function $x \mapsto \bar{\pi}^-(-x)$ involved in the killing of the process is then given by $x \mapsto N(x, \mathbb{1}_{D^c})$, where $N(x, dy)$ is the kernel of the Lévy system of the process and Theorem 1 remains valid where the multiplicative functional $\exp\left(\int_0^t N(X_s, \mathbb{1}_{D^c})\, ds\right)$ now defines the resurrected process. We can also claim, as an extension of Theorem 3, that provided the condition $\sup_{x \in D} \mathbb{E}_x(\tau_{D^c}) N(x, \mathbb{1}_{D^c}) < 1$ is satisfied, where τ_{D^c} is the first exist time from D, the lifetime of the resurrected process has finite mean. These few extensions allow us to believe that other more refined results can be obtained in fairly general frameworks thus offering some perspectives in this direction.

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María Emilia Caballero, Instituto de Matemáticas, Universidad Nacional Autónoma de México. Distrito federal CP 04510, México.
Email address: mariaemica@gmail.com

Loïc Chaumont, Univ Angers, CNRS, LAREMA, SFR MATHSTIC, F-49000 Angers, France
Email address: loic.chaumont@univ-angers.fr

Víctor Rivero, Centro de Investigación en Matemáticas (CIMAT A.C.). Calle Jalisco s/n, 36240 Guanajuato, Guanajuato México.
Email address: rivero@cimat.mx