

# Theoretical investigation of the relations between quantum decoherence and weak-to-strong measurement transition

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This paper delves into the crucial aspects of pointer-induced quantum decoherence and the transition between von Neumann's projective strong measurement and Aharonov's weak measurement. Both phenomena significantly impact the dynamical understanding of quantum measurement processes. Specifically, we focus on the interplay between quantum decoherence and the transition from weak to strong measurement by deducing and comparing the quantum decoherence and weak-to-strong measurement transition factors within a general model and using the well-known Stern-Gerlach experiment as an illustrative example. Our findings reveal that both phenomena can be effectively characterized by a universal transition factor intricately linked to the coupling between the system and the measurement apparatus. The analysis presented can clarify the mechanism behind the relations of quantum decoherence to the weak measurement and weak-to-strong measurement transition.

## I. INTRODUCTION

Measurements in quantum mechanics have posed a longstanding and formidable challenge, playing a fundamental role in exploring the properties of quantum systems [1, 2]. It is widely recognized that von Neumann developed the first model to describe strong quantum measurements by treating both the system under test and the quantum measuring instrument, with strong interactions between them [3]. The well-known Stern-Gerlach (SG) experiment is a typical model of quantum strong measurements, which can be interpreted as a quantum measurement process that measures the spin of the particles through their spatial distribution [4]. During the strong measurement process, the system's state collapses to one of its eigenstates due to the significant interaction between the system and the measuring instrument. This process is advantageous because it allows the acquisition of the desired system information through a single measurement. However, the wave packet collapse induced by strong measurement is irreversible, suggesting that the measured quantum state is unlikely to revert to its original state.

The issues associated with strong quantum measurement have been extensively elucidated within the framework of quantum mechanics, focusing on the interaction between the measuring device and the system [5]. One of these important issues is quantum decoherence, whereby the measurement devices disrupt over time the quantum coherence of superpositions [6]. In 1970, Zeh [7] authored the first paper on decoherence, highlighting that realistic macroscopic quantum systems are inherently open, undergoing strong interactions with their environments. However, a crucial advancement in decoherence occurred in the 1980s with the introduction of the term decoherence. This milestone was notably propelled by the seminal contributions of Zurek [8, 9], who underscored the

paramount significance of preserving quantum correlations, establishing it as a pivotal criterion for discerning preferred states within the decoherence framework [10, 11]. The reader is referred to recent reviews in the field for further details of quantum decoherence [12–15].

As previously discussed, when the interaction between the measured system and the measuring apparatus is intense, decoherence takes place. This results in the collapse of the measured system into its corresponding eigenstate of the measured observable, imposing information loss. Moreover, the irreversible nature of strong measurements implies that the measured quantum state is unlikely to return to its original state. Thus, the weak measurement term was introduced to address the challenges of strong quantum measurement. In weak measurement, the coupling between the system and the measuring apparatus is minimal, thereby avoiding wave function collapse [16]. The measurement value is obtained by incorporating a suitable post-selection step in the weak measurement process, often termed the *weak value*, which may fall beyond the observable's eigenvalue spectrum. This leads to a phenomenon known as *weak value amplification* (WVA), which has proven beneficial for detecting and examining minute effects within linear optical systems [17–20]. It has also assisted in exploring quantum mechanics and its applications [21–24].

Despite the successful explanations of various quantum measurement phenomena achieved by both the theory of strong and weak measurements, an unavoidable question naturally arises regarding the feasibility of transitioning from weak to strong measurement by modifying the interaction between the system and the apparatus [25]. Transitioning from weak to strong measurement can be traced back to the investigation of Zhu et al. [26]. Their study examined quantum measurements involving pre-selection and post-selection and studied the pointer position and momentum shifts without relying on approximations, thus broadening the scope to encompass strong interactions. Ban's research [27] focused on exploring whether another form of observable average ex-

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isted distinct from weak and strong values in a post-selected quantum system. The author's findings reveal that if eigen-projectors of a measured observable solely represent the measurement's impact on the system, the conditional average is a combination of strong and weak observable values. Later, several studies explored the transition from weak to strong measurement by selecting different pointers. For instance, Pan et al. [28] designed a system to experimentally observe the transition from weak to strong measurement in a Gaussian state by modulating a global transition factor. Orszag et al. [29] investigated the measurement transition for a coherent squeezed pointer state. Their study demonstrates a pathway from weak to strong measurements while preserving the constancy of the global transition factor. This methodology offers an alternative pathway for exploring the measurement transition. Furthermore, a general approach addressing the transition from weak to strong measurement employed Fock state-based states as the pointer state. This approach is based on the principle that the Hermitian nature of the photon number operator allows for any state to be expanded based on  $|n\rangle$  [30].

Although quantum decoherence and weak-to-strong measurement transitions have been extensively studied, their potential relationship remains unexplored. Hence, this paper investigates the connection between these two phenomena to uncover the underlying physical implications by employing a generic computational model for each phenomenon. Surprisingly, our analytic results reveal that the quantum decoherence and transition factors share a common mathematical form. Besides, we validate our findings further by applying this model to the renowned SG experiment and obtain consistent outcomes for both factors. Moreover, upon separately analyzing the two asymptotic states with factor values approaching 0 and 1, we observe that the measured system demonstrates consistent trends under identical factors, whether within the context of decoherence or the transition from weak to strong measurements.

The remainder of this paper is organized as follows. Section. II briefly introduces the quantum decoherence and weak-to-strong measurement transition models. Section. III discusses the dynamical measurement process of the SG experiment as a typical example of our proposal. Section. IV discusses our findings and concludes this work.

## II. BRIEF DESCRIPTION OF QUANTUM DECOHERENCE AND WEAK-TO-STRONG MEASUREMENT TRANSITION

### A. Quantum decoherence and decoherence function

Any standard quantum measurement model has two components: system (measured system) and pointer (measurement device/measuring system/measurement

apparatus). Its Hamiltonian is formulated as follows,

$$H = H_s + H_p + H_I = H_0 + H_I. \quad (1)$$

In the equation above,  $H_s$  and  $H_p$  represent the Hamiltonian of the system and measurement pointer, respectively, and  $H_I$  describes the interaction between the system and the pointer. It is important to note that this work solely focuses on the ideal measurement model without considering the noise caused by a reservoir. In general, the interaction Hamiltonian  $H_I$  takes the von Neumann measurement form as

$$H_I = gA \otimes Q, \quad (2)$$

where  $A$  is the system observable we want to obtain information from the measurement model and  $Q = Q^\dagger$  is an arbitrary pointer operator. The  $Q$  is usually the position ( $X$ ) or momentum ( $P$ ) operator, easing the system information acquisition in the lab. The interaction strength  $g$  between the system and the pointer is usually an impulsive function, which is only effective over a very short time interval to guarantee the precision of the measurement result. We assume that the system Hamiltonian commutes with the observable  $A$ , which yields,

$$[H, A] = [H_I, A] = [H_s, A] = 0, \quad (3)$$

such that the system observable  $A$  involves conserved quantities. As a consequence, the mean energy of the system is constant in time, i.e.,

$$\frac{d}{dt} \langle H_s(t) \rangle = 0. \quad (4)$$

From the above assumption, the system and observable  $A$  have the following eigenvalue function, i.e.,

$$H_s |a_i\rangle = E_i |a_i\rangle, \quad A |a_i\rangle = a_i |a_i\rangle. \quad (5)$$

Initially, the system and the pointer are relatively independent and the initial state of the composite system is written as

$$|\Psi(0)\rangle = |\psi_i\rangle \otimes |\phi\rangle. \quad (6)$$

Here  $|\psi_i\rangle = \sum_i \alpha_i |a_i\rangle$  with  $\alpha_i = \langle a_i | \psi_i \rangle$  and  $\phi$  representing the initial states of the system and the pointer, respectively. The initial system state  $|\psi_i\rangle$  can be expressed in terms of density matrix as

$$\begin{aligned} \rho_s(0) &= |\psi_i\rangle \langle \psi_i| \\ &= \sum_i |\alpha_i|^2 |a_i\rangle \langle a_i| + \sum_{j,i} \alpha_i \alpha_j^* |a_i\rangle \langle a_j|, \end{aligned} \quad (7)$$

where  $|\alpha_i|^2$  is the measuring probability of eigenvalue  $a_i$  of the observable  $A$  corresponding to the eigenstate  $|a_i\rangle$  if the system is prepared in  $|\psi_i\rangle$ . In order to obtain a determined outcome, the second term (off-diagonal, the

part that represented the coherence) of  $\rho$  has to vanish after the measurement. This means that, after the measurement, the system is a mixture of the eigenstates of the measured observable. Next, we describe this dynamic transition process.

After completing measurements, the time evolution of the total system is characterized by

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle, \quad (8)$$

where  $U(t)$  is a unitary time evolution operator defined as

$$\begin{aligned} U(t) &= \exp[-iHt] = \exp[-i(H_s + H_p + H_I)t] \\ &= \sum_i e^{-iE_i t} e^{-i(H_p + g_0 a_i Q)t} |a_i\rangle\langle a_i|. \end{aligned} \quad (9)$$

By substituting Eq. (9) into Eq. (8) we obtain

$$|\Psi(t)\rangle = \sum_i \alpha_i e^{-iE_i t} |a_i\rangle |\phi_i(t)\rangle. \quad (10)$$

Here  $|\phi_i(t)\rangle = e^{-i(H_p + g_0 a_i Q)t} |\phi\rangle$  represents the final states of the pointer and contains the information of the system observable  $A$ .  $|\Psi(t)\rangle$  indicates that the system and pointer become entangled after time evolution and cannot be separated. However, we can determine the system state by tracing out the degrees of freedom of the pointer as

$$\begin{aligned} \rho_s(t) &= \text{Tr}_p(|\Psi(t)\rangle\langle\Psi(t)|) \\ &= \sum_i |\alpha_i|^2 |a_i\rangle\langle a_i| \\ &\quad + \sum_{i \neq j} \alpha_j^* \alpha_i e^{-i(E_i - E_j)t} |a_i\rangle\langle a_j| \langle\phi_j(t)|\phi_i(t)\rangle. \end{aligned} \quad (11)$$

The above expression reveals that the diagonal terms of  $\rho_s(t)$  remain unchanged with time, while the off-diagonal terms vary over time. The dependence of matrix element  $\langle a_i | \rho_s(t) | a_j \rangle$  on time is given in the form of overlapping integrals of  $|\phi_i(t)\rangle$  and  $|\phi_j(t)\rangle$ , and the impact exerted by the pointer (measuring apparatus) on the statistical measurement outcomes is effectively subsumed in the overlap. The amount of overlap is a quantitative measure delineating the degree of interference. Generally,

$$F(t) = |\langle\phi_i(t)|\phi_j(t)\rangle| = \exp[-\Gamma_{ij}(t)], \Gamma_{ij}(t) \geq 0. \quad (12)$$

The above formula describes the behavior of the non-diagonal elements of the reduced density matrix  $\rho_s(t)$  when  $i \neq j$ . Its time dependence is related to many elements, such as the specific form or the system-pointer coupling, on the underlying model's various parameters and the initial state's properties. Therefore,  $F(t)$  is called the *decoherence function*. For many physical systems, the irreversible dynamics induced by the system-pointer (system-reservoir) interaction rapidly decreases the overlap  $\langle\phi_j(t)|\phi_i(t)\rangle$  when  $i \neq j$ . Thus, to quantitatively

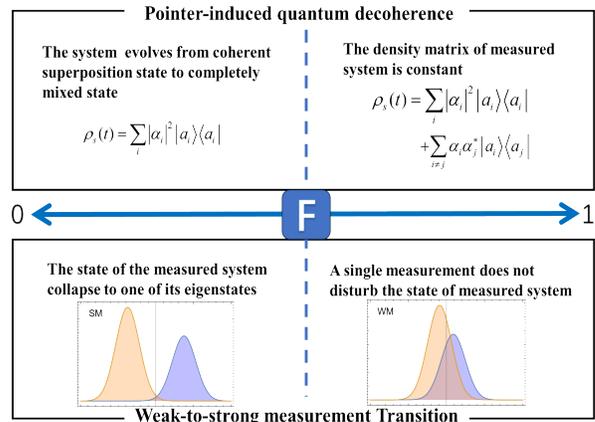


Figure 1. Schematic relations between decoherence and weak-to-strong measurement transition model.

describe the decreasing  $F(t)$ , we introduce the *decoherence time*  $\tau_D$  term. If for  $i \neq j$  the overlap of the states  $|\phi_i(t)\rangle$  and  $|\phi_j(t)\rangle$  approaches to zero after large times compared to  $\tau_D$  such that

$$\langle\phi_j(t)|\phi_i(t)\rangle \rightarrow \delta_{ij}, \text{ for } t \gg \tau_D, \quad (13)$$

then, the reduced density matrix of the system becomes as

$$\rho_s(t) \rightarrow \sum_i |\alpha_i|^2 |a_i\rangle\langle a_i|. \quad (14)$$

This result shows that the coherence of our system's density matrix of our system vanishes after a long time ( $t \gg \tau_D$ ) interaction with the pointer. In the measurement problem, after  $t \gg \tau_D$  the state  $\rho_s(t)$  of the system behaves as an incoherent mixture of the state  $|a_i\rangle$ , so that the interference terms of the form  $\langle a_i | A | a_j \rangle$  ( $i \neq j$ ) of any system observable  $A$  no longer occurs in the expectation value. In other words, after a longer time of interaction between the system and the pointer or environment, the superpositions of the states  $|a_i\rangle$  are destroyed locally, meaning they are unobservable for all measurements executed exclusively on the system. The dynamic transition process expressed by Eq. (14) is called *decoherence*. The main idea of our proposal to reveal the relations between quantum decoherence and weak-to-strong measurement transition is illustrated in Fig. (1).

## B. A model of pointer-induced decoherence

This subsection introduces a pointer-induced decoherence model. As mentioned previously, in the ideal measurement, the Hamiltonian part only adds a phase factor to the total system state after time evolution and does not affect measurement results. Thus, we assume the Hamiltonian of one measurement as

$$H = \frac{p^2}{2m} - g(t)x \otimes A. \quad (15)$$

Here,  $A = \sum_i a_i |a_i\rangle\langle a_i|$  as defined is a system observable and  $m$  is the mass of a quantum object. The initial state of the total system is  $|\psi_i\rangle \otimes |\phi\rangle$ , where  $|\psi_i\rangle$  as defined in Sec. (II A) and the pointer is assumed to be a Gaussian profile as

$$|\phi\rangle = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}} \int \exp\left(-\frac{x^2}{4\sigma^2}\right) dx |x\rangle, \quad (16)$$

where  $\sigma$  is the width of the Gaussian wave packet. The evolution of the total system can be written as

$$|\Psi(t)\rangle = U(t)|\psi_i\rangle \otimes |\phi\rangle \quad (17)$$

with  $U(t) = \exp(-iHt)$ . The explicit form of  $|\Psi(t)\rangle$  is obtained by factorizing the unitary operator  $U(t)$ , accomplished by adopting the Wei-Norman technique [31] as

$$U(t) = \sum_i e^{g_1} e^{g_2 p^2} e^{g_3 p} e^{g_4 x} |a_i\rangle\langle a_i| \quad (18)$$

with

$$g_1 = -\frac{i(ga_i)^2}{6m} t^3, \quad (19)$$

$$g_2 = -\frac{it}{2m}, \quad (20)$$

$$g_3 = \frac{it^2 ga_i}{2m}, \quad (21)$$

$$g_4 = i g a_i t. \quad (22)$$

By substituting Eq. (18) into Eq. (17), we obtain

$$|\Psi(t)\rangle = \sum_i \alpha_i |a_i\rangle |\phi_i(t)\rangle. \quad (23)$$

Here,  $|\phi_i(t)\rangle$  are final states of the pointer, and reads as

$$|\phi_i(t)\rangle = \frac{(\sigma^2/2\pi)^{1/4}}{\sqrt{\sigma^2 + \frac{it}{2m}}} e^{-i\theta(t)} e^{-i g a_i t x} \times \exp\left[-\frac{\left(x - \frac{g a_i t^2}{2m}\right)^2}{4\left(\sigma^2 + \frac{it}{2m}\right)}\right] \quad (24)$$

with  $\theta(t) = \frac{(g a_i)^2}{6m} t^3$ . The expression above represents a Gaussian wavepacket with width  $\sigma(t) = \sigma \left(1 + \frac{t^2}{4m^2\sigma^4}\right)^{1/2}$  and a central position at  $x_i = \frac{g a_i t^2}{2m}$ .

The final state of the system after time evolution is

$$\begin{aligned} \rho'_s(t) &= \text{Tr}_p(|\Psi(t)\rangle\langle\Psi(t)|) \\ &= \sum_i |\alpha_i|^2 |a_i\rangle\langle a_i| + \sum_{i \neq j} \alpha_i \alpha_j^* |a_i\rangle\langle a_j| F_{ij}. \end{aligned} \quad (25)$$

Here, the decoherence factor is given by

$$\begin{aligned} F &= |F_{ij}| = |\langle\phi_i(t)|\phi_j(t)\rangle| \\ &= \exp\left[-\frac{5}{8} \frac{(\Delta x)^2}{\sigma^2(t)} - \frac{t^2}{32\sigma^4 m^2} \frac{(\Delta x)^2}{\sigma^2(t)} - \frac{2\sigma^4 m^2 (\Delta x)^2}{t^2 \sigma^2(t)}\right], \end{aligned} \quad (26)$$

where  $\Delta x = g t^2 (a_i - a_j) / 2m$ . This function can quantify the degree of decoherence as a function of time  $t$ . It should be noted that the distinguishability condition of the wavepackets at time  $t$  is the distance of the center of two near wavepackets larger than its width, i.e.,  $\Delta x \gg \sigma(t)$ . This condition can easily be satisfied if time  $t$  is long enough.

### C. Weak-to-strong measurement transition model

Let the weak-to-strong measurement transition model have the same Hamiltonian as in the above subsection. Using the pointer shift, we read the system's observable values in all measurement schemes. Therefore, three value types of the observable  $A$  correspond to different measurement circumstances. To clearly understand the mechanism of measurement transition, let's first briefly introduce each value of  $A$ :

1. *Expectation value.* Suppose a system state  $|\psi_i\rangle = \sum_j \alpha_j |a_j\rangle$  with  $\sum_j |\alpha_j|^2 = 1$ , then the expectation value of the observable under the state  $|\psi_i\rangle$  is given by

$$\langle A \rangle = \sum_j a_j |\alpha_j|^2. \quad (27)$$

This expectation value can be obtained by reading the position shift  $\delta x$  under the state given in Eq. (23), i.e.,

$$\delta x = \langle \Psi(t) | X | \Psi(t) \rangle - \langle \phi | X | \phi \rangle = \frac{gt^2}{2m} \langle A \rangle. \quad (28)$$

2. *Conditional Expectation value.* In time-symmetric quantum mechanics [32–34], if we take a post-selection by using the state  $|\psi_f\rangle = \sum_j \beta_j |a_j\rangle$  with  $\sum_j |\beta_j|^2 = 1$  after some evolution, the conditional expectation value of the observable  $A$  is determined by [32]

$$\begin{aligned} \langle A \rangle_c &= \frac{\sum_j a_j |\langle \psi_f | a_j \rangle \langle a_j | \psi_i \rangle|^2}{\sum_j |\langle \psi_f | a_j \rangle \langle a_j | \psi_i \rangle|^2} \\ &= \frac{\sum_j a_j |\alpha_j \beta_j^*|^2}{\sum_j |\alpha_j \beta_j^*|^2}. \end{aligned} \quad (29)$$

This value is also called the post-selected strong value of  $A$ . The above processes we assume that the coupling between the measured system and the pointer is strong enough so that the spatial sub-wave-packets of the pointer corresponding to the different eigenvalues of the observable are distinguishable, i.e.,  $g\Delta a \gg \sigma$ . Here,  $\Delta a = a_i - a_{i-1}$  and  $\sigma$  represent the differences between

neighboring eigenvalues and the width of the sub-wave packets, respectively.

3. *Weak value.* The weak value of observable  $A$  with pre- and post-selected state reads as

$$\begin{aligned} \langle A \rangle_w &= \frac{\langle \psi_i | A | \psi_f \rangle}{\langle \psi_i | \psi_f \rangle} = \frac{\sum_j \sum_k \alpha_j \beta_k^* \langle a_k | A | a_j \rangle}{\sum_j \sum_k \alpha_j \beta_k^* \langle a_k | a_j \rangle} \\ &= \frac{\sum_j a_j \alpha_j \beta_j^*}{\sum_j \alpha_j \beta_j^*}. \end{aligned} \quad (30)$$

It can be seen that, in general, the conditional expectation value  $\langle A \rangle_c$  and the weak value  $\langle A \rangle_w$  are different [please see the Eq. (29) and Eq. (30)], and correspond to different measurement strengths. Hence, the (conditional) expectation value of the system observable is related to the (post-selected) strong measurement, while the post-selected weak measurement causes the weak value. If  $\beta_i = \alpha_i$ , the above-introduced conditional expectation value and weak value are reduced to the typical expectation value of the observable  $A$  as given in Eq. (27). Actually, the  $\langle A \rangle_w$  and  $\langle A \rangle_c$  are the two extreme values of the transition value of observable  $A$  defined below

$$\begin{aligned} A_T &= \frac{\langle \psi_f | A \rho'_s(t) | \psi_f \rangle}{\langle \psi_f | \rho'_s(t) | \psi_f \rangle} \\ &= \frac{\sum_i a_i |\alpha_i \beta_i|^2 + \sum_{i \neq j} a_i \beta_j \beta_i^* \alpha_i \alpha_j^* \langle \phi_i(t) | \phi_j(t) \rangle}{\sum_i |\alpha_i \beta_i|^2 + \sum_{i \neq j} \beta_j \beta_i^* \alpha_i \alpha_j^* \langle \phi_i(t) | \phi_j(t) \rangle}. \end{aligned} \quad (31)$$

The expression above reveals that the transition value depends on the overlap  $\langle \phi_i(t) | \phi_j(t) \rangle$  of the states  $|\phi_i(t)\rangle$  and  $|\phi_j(t)\rangle$ , and its module  $F$  is decoherence function of the system [see Eq. (26)].  $F$  is an exponentially decreasing function of time  $t$  and coupling strength  $g$ . If  $gt^2|a_i - a_j|/2m \gg \sigma(t)$ , then  $F$  approaches zero and  $A_T$  becomes as

$$(A_T)_{F \rightarrow 0} = \frac{\sum_j a_j |\alpha_j \beta_j^*|^2}{\sum_j |\alpha_j \beta_j^*|^2} = \langle A \rangle_c. \quad (32)$$

On the contrary, using  $gt^2|a_i - a_j|/2m \ll \sigma(t)$ , the overlap  $\langle \phi_i(t) | \phi_j(t) \rangle$  approximately equals one, and  $A_T$  is reduced to

$$\begin{aligned} (A_T)_{F \rightarrow 1} &= \frac{\sum_i a_i |\alpha_i \beta_i|^2 + \sum_{i \neq j} a_i \beta_j \beta_i^* \alpha_i \alpha_j^*}{\sum_i |\alpha_i \beta_i|^2 + \sum_{i \neq j} \beta_j \beta_i^* \alpha_i \alpha_j^*} \\ &= \frac{\sum_j a_j \alpha_j \beta_j^*}{\sum_j \beta_j^* \alpha_j} = \langle A \rangle_w. \end{aligned} \quad (33)$$

From an experimental point of view, we obtain the above values of the system observable  $A$  by reading the position and momentum shifts of the pointer. As given in the above subsection, the  $|\Psi(t)\rangle$  [see Eq. (23)] is the total system state of the our system described by the Hamiltonian in Eq. (15) after the time evolution. If we

take a post-selection on it using the post-selected state  $|\psi_f\rangle$ , then the unnormalized final state of the pointer is given as

$$|\Xi(t)\rangle = \sum_i \beta_i^* \alpha_i |\phi_i(t)\rangle. \quad (34)$$

Using this final state provides the position and momentum shifts of the pointer, and their expression are expressed as

$$\begin{aligned} \delta x &= \frac{\langle \Xi(t) | x | \Xi(t) \rangle}{\langle \Xi(t) | \Xi(t) \rangle} - \langle \phi | x | \phi \rangle \\ &= \frac{1}{\sum_{i,j} \alpha_i \alpha_j^* \beta_i^* \beta_j F_{ij}} \left\{ \sum_{i,j} \alpha_i \alpha_j^* \beta_i^* \beta_j \frac{gt^2(a_i + a_j)}{4m} F_{ij} \right. \\ &\quad \left. + i g t \sum_{i,j} \alpha_i \alpha_j^* \beta_i^* \beta_j \left( \frac{t^2(a_i - a_j)}{8\sigma^2 m^2} - (a_i - a_j) \sigma^2(t) \right) F_{ij} \right\} \\ &= \frac{gt^2}{2m} \text{Re}(A_T) + \frac{gt^3}{4\sigma^2 m^2} \text{Im}(A_T) - 2gt\sigma^2(t) \text{Im}(A_T), \end{aligned} \quad (35)$$

and

$$\begin{aligned} \delta p &= \frac{\langle \Xi(t) | p | \Xi(t) \rangle}{\langle \Xi(t) | \Xi(t) \rangle} - \langle \phi | p | \phi \rangle \\ &= \frac{\sum_{i,j} \alpha_i \alpha_j^* \beta_i^* \beta_j \left[ \frac{gt(a_i + a_j)}{2} - i \frac{gt^2(a_i - a_j)}{8m\sigma^2} \right] F_{ij}}{\sum_{i,j} \alpha_i \alpha_j^* \beta_i^* \beta_j F_{ij}} \\ &= gt \text{Re}(A_T) - \frac{gt^2}{4\sigma^2 m} \text{Im}(A_T), \end{aligned} \quad (36)$$

respectively. If the transition factor  $F$  approaches one, the shift in the pointer's position and momentum behave as

$$(\delta x)_{F \rightarrow 1} = \frac{gt^2}{2m} \text{Re}[\langle A \rangle_w] - \frac{gt^3 + 8gt\sigma^4 m^2}{4\sigma^2 m^2} \text{Im}[\langle A \rangle_w], \quad (37)$$

$$(\delta p)_{F \rightarrow 1} = gt \text{Re}[\langle A \rangle_w] - \frac{gt^2}{4\sigma^2 m} \text{Im}[\langle A \rangle_w]. \quad (38)$$

On the other hand, if  $F = 0$  then Eqs.(35) and (36) reduce to

$$(\delta x)_{F \rightarrow 0} = \frac{gt^2}{2m} \langle A \rangle_c, \quad (39)$$

and

$$(\delta p)_{F \rightarrow 0} = gt \langle A \rangle_c. \quad (40)$$

Since the decoherence factor  $F$  is a continuous function, its two extreme cases of can establish a relationship between weak and strong measurements. As noticed in this work, both displacements in the weak and strong measurement regimes do not coincide with the results obtained by Josza [35] and Turek [30], owing to the Hamiltonian of our scheme. Since we aim to investigate the mechanism behind the weak-to-strong measurement transition and

its relations with pointer-induced decoherence, we consider the pointer's kinetic energy. However, in previous works, we only considered the interaction of the Hamiltonian between the measured system and the pointer. However, one interesting point of our scheme is that if particle mass is assumed to be too heavy, the above displacements reproduce the previous results, i.e.,

$$(\delta x)_{F \rightarrow 1, m \rightarrow \infty} = -2gt\sigma^2 \text{Im}[\langle A \rangle_w], \quad (41)$$

$$(\delta p)_{F \rightarrow 1, m \rightarrow \infty} = gt \text{Re}[\langle A \rangle_w], \quad (42)$$

and

$$(\delta x)_{F \rightarrow 0, m \rightarrow \infty} = 0, \quad (43)$$

$$(\delta p)_{F \rightarrow 0, m \rightarrow \infty} = gt \langle A \rangle_c, \quad (44)$$

respectively. Most existing studies consider the interaction between Hamiltonian  $H_I$  to be in  $g\hat{A} \otimes \hat{P}$  form, whereas in this work,  $H_I = g\hat{A} \otimes \hat{X}$ . Thus, contrary to the previous results, in a weak measurement regime, the position shift of the pointer is proportional to the imaginary part of the weak value, and the momentum shift gives the real part of the weak value. It is worth noting that the original paper of Aharonov [36] used the same interaction Hamiltonian as in this paper. Fig.2 highlights the above relations. The next section provides a feasible example of the proposed scheme.

### III. A TYPICAL EXAMPLE—THE STERN-GERLACH (SG) EXPERIMENT

The Stern-Gerlach (SG) experiment is a very important quantum measurement model, which reflects the relationship between spin and spatial degrees of freedom in atoms, and makes it possible to distinguishing different spin states from spatial distributions. In the SG experiment, a silver atom in the ground state with orbital angular momentum  $L = 0$  moves along the  $x$  direction and enters the non-uniform magnetic field directed on the  $z$ -axis. This process is described by the Hamiltonian, which can be written as

$$H = \frac{p^2}{2m} - \mu B(x)\sigma_z. \quad (45)$$

Here,  $m$  is the mass of the atom. If we take a linear approximation  $B(x) \approx \frac{\partial B}{\partial x}|_{x=0}x$ , then the above Hamiltonian becomes as

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (46)$$

where  $H_{\pm} = \frac{p^2}{2m} \mp fx$  and  $f = \mu \frac{\partial B}{\partial x}|_{x=0}$ . If we assume that the initial system and pointer prepared to  $|\psi_i\rangle = \cos\theta_1|\uparrow\rangle + e^{i\delta_1}\sin\theta_1|\downarrow\rangle$  and  $|\phi\rangle$  [see Eq. (16)], the time evolution of the total system is given by

$$|\Phi(t)\rangle = \cos\theta_1|\uparrow\rangle|\phi_+(t)\rangle + e^{i\delta}\sin\theta_1|\downarrow\rangle|\phi_-(t)\rangle. \quad (47)$$

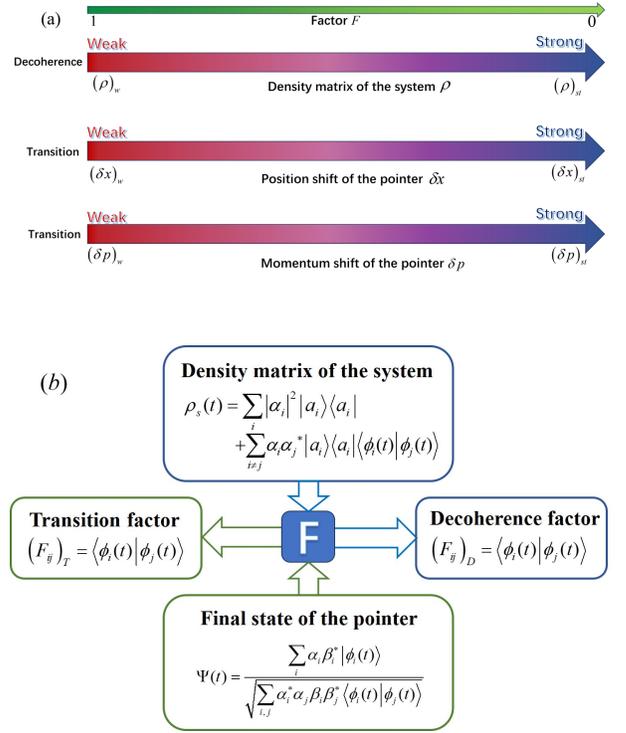


Figure 2. Weak-to-strong measurement model and its relation with the quantum decoherence factor. (a) measurement readout by the displacements of position and momentum observables. (b) decoherence and transition factors describing different physical processes but share the same expression.

Here,  $|\phi_{\pm}(t)\rangle = e^{-iH_{\pm}t}|\phi\rangle$  and their explicit expression in position representation is obtained as

$$\phi_{\pm}(x, t) = \frac{(\sigma^2/2\pi)^{1/4}}{\sqrt{\sigma^2 + \frac{it}{2m}}} e^{-i\theta'(t)} e^{\mp iftx} \times \exp \left[ -\frac{\left(x \pm \frac{ft^2}{2m}\right)^2 \left(\sigma^2 - \frac{it}{2m}\right)}{4\sigma^4 + \frac{t^2}{m^2}} \right], \quad (48)$$

with  $\theta'(t) = \frac{f^2 t^3}{6m}$ . The width of this Gaussian wavepacket is  $\sigma(t)$  with a central position at  $x_{\pm} = \pm \frac{ft^2}{2m}$ . Every wavepacket has the same group velocity but propagates in opposite directions, i.e.,  $v_{\pm} = \pm ft/m$ . The expression presented above infers that the central motion of wavepackets obeys the rule of classical dynamics, i.e., an object with mass  $m$  has acceleration  $f/m$  under the external force  $f$ .

It can be observed that the spatial degree of freedom and internal degree (spin) of freedom of the silver atoms are entangled. Since the atoms in different spin states  $|\uparrow\rangle$  and  $|\downarrow\rangle$  experience opposite forces, the atom with the initial superposition of the two spin states will eventually form two macroscopic distinguishable spots on the detection screen. Once a particle is found at the position associated

with the  $|\uparrow\rangle$  state, the state of the system's state is said to collapse to the  $|\uparrow\rangle$  state, and vice versa.

The reduced density operator corresponding to the spin degrees of freedom of the atom is given as

$$\begin{aligned}\rho_s(t) &= Tr_p(|\Phi(t)\rangle\langle\Phi(t)|) \\ &= \cos^2\theta_1|\uparrow\rangle\langle\uparrow| + \sin^2\theta_1|\downarrow\rangle\langle\downarrow| \\ &\quad + \frac{1}{2}e^{-i\delta_1}\sin 2\theta_1|\uparrow\rangle\langle\downarrow|F'(t) + h.c.\end{aligned}\quad (49)$$

with  $F'(t) = \langle\phi_+(t)|\phi_-(t)\rangle$  and its explicit expression reads as

$$F'(t) = \exp\left[-\frac{\gamma^2}{8} - \frac{1}{32}\frac{t^2}{m^2\sigma^4}\gamma^2 - 2\sigma^2ft^2\right].\quad (50)$$

where  $\gamma' = ft^2/m\sigma(t)$ . This expression can be obtained directly from Eq. (26) by substituting the  $a_{i,j} = \pm 1$  eigenvalues. Furthermore, if the post-selection with the system state  $|\psi_f\rangle = \cos\theta_2|\uparrow\rangle + e^{i\delta_2}\sin\theta_2|\downarrow\rangle$  is placed onto  $|\Phi(t)\rangle$ , the normalized final state of the pointer in the position representation is obtained as

$$\Omega(x, t) = \frac{\cos\theta_1\cos\theta_2\phi_+(x, t) + \sin\theta_1\sin\theta_2e^{i(\delta_1-\delta_2)}\phi_-(x, t)}{\beta},\quad (51)$$

Here,  $\beta$  is the normalized coefficient, which is given by

$$\begin{aligned}\beta^2 &= \cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2 \\ &\quad + \frac{1}{2}\sin 2\theta_1\sin 2\theta_2\cos(\delta_1 - \delta_2)F'(t).\end{aligned}\quad (52)$$

We obtain the explicit expressions of position and momentum shift by using  $\Omega(x, t)$ , and the results expressed as

$$\begin{aligned}\delta x &= \frac{1}{\beta^2}\left\{\frac{ft^2}{2m}(\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2) \right. \\ &\quad \left. + \frac{ft^3 + 8ft\sigma^4m^2}{8\sigma^2m^2}\sin 2\theta_1\sin 2\theta_2\sin(\delta_1 - \delta_2)F'(t)\right\},\end{aligned}\quad (53)$$

and

$$\begin{aligned}\delta p &= \frac{1}{\beta^2}\left\{ft(\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2) \right. \\ &\quad \left. + \frac{ft^2}{8\sigma^2m}\sin 2\theta_1\sin 2\theta_2\sin(\delta_1 - \delta_2)F'(t)\right\},\end{aligned}\quad (54)$$

respectively.

Then, if  $F' \rightarrow 0$ , we consider the larger value of the coupling strength parameter to know the position and momentum shifts of the pointer in the post-selected strong measurement regime, which can be written as

$$(\delta x)_{F'\rightarrow 0} = \frac{ft^2}{2m}\frac{\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2}{\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2} = \frac{ft^2}{2m}\langle\sigma_z\rangle_c.\quad (55)$$

and

$$(\delta p)_{F'\rightarrow 0} = ft\frac{\cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2}{\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2} = ft\langle\sigma_z\rangle_c.\quad (56)$$

Here,  $\langle\sigma_z\rangle_c$  is the conditional expectation value of observable  $\sigma_z$ , which is obtained in a conditional strong measurement.

Furthermore, if one wants to know the position shift formula for the post-selected weak measurement regime, a limit  $F' \rightarrow 1$  should be taken for this extreme case. Then, the position and momentum shifts become as

$$(\delta x)_{F'\rightarrow 1} = \frac{ft^2}{2m}Re[\langle\sigma_z\rangle_w] - \frac{ft^3 + 8ft\sigma^4m^2}{4\sigma^2m^2}Im[\langle\sigma_z\rangle_w],\quad (57)$$

and

$$(\delta p)_{F'\rightarrow 1} = ftRe[\langle\sigma_z\rangle_w] - \frac{ft^2}{4\sigma^2m}Im[\langle\sigma_z\rangle_w].\quad (58)$$

These are the general results of the SG experiment in the post-selected weak measurement. However, in the dynamical evolution of the measurement process, we usually assume the mass of the pointer to be too large and do not consider the effects of the pointer caused by itself. In this case, we omit the terms in the above results associated with mass  $m$ , thereby recovering the typical displacements presented in previous studies.

#### IV. DISCUSSION AND CONCLUSION

The experimental results reveal that whether we propose a general model of ideal measurement or explain it through the specific SG experiment, the decoherence factor obtained from the decoherence process and the transition factor in the weak-to-strong measurement transition exhibit the same mathematical form. This similarity raises the question of whether the decoherence process of the measured system caused the weak-to-strong measurement transition and weak measurement procedure as well.

The decoherence factor given in Eq.(26) can be rewritten as

$$F = \exp\left[-\frac{1}{8}\frac{(\Delta x)^2}{\sigma^2(t)} - \frac{t^2}{32\sigma^4m^2}\frac{(\Delta x)^2}{\sigma^2(t)} - \frac{2m^2\sigma^2(\Delta x)^2}{t^2}\right].\quad (59)$$

This factor also occurred in the weak-to-strong measurement transition process. The value of  $F$  depends on some parameters, including time  $t$ , mass of the atom  $m$ , coupling strength  $g$ , and atomic beam width  $\sigma$ . Among these parameters, we can easily control the time  $t$  and coupling strength  $g$ . After the dynamical evolution, the atomic beam width changed from  $\sigma$  to  $\sigma(t) = \sigma\left(1 + \frac{t^2}{4m^2\sigma^4}\right)^{1/2}$ . Thus, during dynamic evolution, the wavepacket of the

atomic beam spreads in space. Decoherence arises from the interaction between the measured system and the measuring apparatus during the quantum measurement. Under the evolution of time, the system will change from a superposition state that embodies quantum coherence to a mixed state. Hence, utilizing a density matrix becomes essential to describe the local system with greater relevance. However, providing the exact decoherence time of our decoherence factor  $F$  is impossible, but we can discuss the two extreme cases with the time scale given in  $\sigma(t)$ . If the dynamical evolution time  $t$  is too long so that  $t \gg m\sigma^2$ , then

$$F \approx \exp \left[ -\frac{g^2(a_i - a_j)^2}{32\sigma^2 m^2} t^4 \right]. \quad (60)$$

If  $t$  or  $g$  or both are large, this factor tends to zero, and it can characterize the complete decoherence. In the context of decoherence, the density matrix of the system's final state approaches this limit, causing the off-diagonal elements that characterize coherence to become zero. Taking the SG experiment as an example, when  $F = 0$ , the system's final state density matrix transforms into a completely mixed state  $\hat{\rho}_s(t) = \cos^2\theta_1 |\uparrow\rangle\langle\uparrow| + \sin^2\theta_1 |\downarrow\rangle\langle\downarrow|$ . Moreover, since the overlap is zero, indicating orthogonality between  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , and the information about the apparatus's final state can be effectively distinguished. Additionally, the decoherence discussed here arises from the strong interaction between the system and the apparatus induced during the measurement process. On the other hand, when  $F$  approaches zero, the measurement is considered strong in the weak-to-strong measurement of the transition process. In this case, the displacement of the apparatus's position is proportional to the conditional expectation value. When the factor approaches zero, the prepared quantum system will collapse, allowing us to differentiate the information about the apparatus's final state effectively within a single measurement.

If we consider the very short time case, i.e.,  $t \ll m\sigma^2$ , then the decoherence factor Eq. (59) is reduced to

$$F \approx \exp \left[ -\frac{\sigma^2 g^2 (a_i - a_j)^2 t^2}{2} \right] \approx 1 - \tau^2 t^2, \quad (61)$$

where  $\tau = g\sigma(a_i - a_j)/\sqrt{2}$ . This kind of Gauss attenuation can be considered a quantum Zero effect [37]. In this process, transitions between quantum states are inhibited by frequent state measurements. The inhibition phenomena arises because the measurement causes wave function collapse. If the time between measurements is short enough, the wave function usually collapses back to the initial state, and the main point of the post-selected

weak measurement could occur. In this case, since the factors on the off-diagonal elements representing coherence tend to be one, the system's density matrix is constant. As given in the SG experiment, the amount of overlap quantifies the degree of interference based on the system. When  $F = 1$ , the reduced density matrix of the measured system after dynamical evolution becomes  $\rho_s = \cos^2\theta_1 |\uparrow\rangle\langle\uparrow| + \sin^2\theta_1 |\downarrow\rangle\langle\downarrow| + e^{-i\delta} \cos\theta_1 \sin\theta_1 |\uparrow\rangle\langle\downarrow| + e^{i\delta} \cos\theta_1 \sin\theta_1 |\downarrow\rangle\langle\uparrow|$ , as the initial state never touched.

Similarly, in the weak-to-strong measurement transition, when the  $F \rightarrow 1$ , the measurement is considered weak. In this process, the position displacements and momentum of the pointer are proportional to the imaginary and real parts of weak value. Since the interaction strength between the system and the apparatus is weak in the weak-measurement process, the shifts displayed on the dial of the pointer during a single measurement are insufficient to achieve the desired measurement outcome. However, a zero-effective process in enough short time or weak coupling cases allows us to take multiple consecutive measurements to obtain statistical results. Thus, we can confirm that controlling the quantum decoherence is essential to performing the weak measurement.

This paper studied the relations between quantum decoherence and weak-to-strong measurement transition. We observed that there exists a certain connection between decoherence and weak-to-strong transitions, as they share common features and exhibit analogous behaviors regarding these factors. We presented the general expression of the decoherence factor in weak-to-strong measurement transition by taking the pointer's Hamiltonian into account. We found that in the pointer-based decoherence process, the mass  $m$  of the pointer is highly important. Additionally, we noticed that, in general, quantum measurement's dynamical evolution, whether weak or strong, displacements of position and momentum of the pointer are not zero. Specifically, in a weak measurement regime, both displacements of position and momentum observables are proportional to the real and imaginary parts of the weak value, respectively. In contrast, in a strong measurement regime, they are both associated with conditional expectation value. This result is not very common for the measurement community. However, assuming the mass of this pointer is too heavy to disturb the measurement result, the results presented in this paper recovered the previous results. Overall, the proposed scheme can help deepen the understanding of weak-to-strong measurement and clarify the hiding mechanism behind the weak measurement theory.

## ACKNOWLEDGMENTS

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