# Well-posedness of McKean-Vlasov SDEs with density-dependent drift* 

Anh-Dung Le<br>Toulouse School of Economics, 1 Esplanade de l'Université, 31000 Toulouse, France<br>leanhdung1994@gmail.com

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#### Abstract

In this paper, we study the well-posedness of McKean-Vlasov stochastic differential equations (SDE) whose drift depends pointwisely on marginal density and satisfies a condition about local integrability in time-space variables. The drift is assumed to be Lipschitz continuous in distribution variable with respect to Wasserstein metric $W_{p}$. Our approach is by approximation with mollified SDEs. We establish a new estimate about Hölder continuity in time of marginal density. Then we deduce that the marginal distributions (resp. marginal densities) of the mollified SDEs converge in $W_{p}$ (resp. topology of compact convergence) to the solution of the Fokker-Planck equation associated with the density-dependent SDE. We prove strong existence of a solution. Weak and strong uniqueness are obtained when $p=1$, the drift coefficient is bounded, and the diffusion coefficient is distribution free.


AMS subject classification: $60 \mathrm{~B} 10,60 \mathrm{H} 10$.
Keywords: McKean-Vlasov SDEs, density-dependent SDEs, well-posedness, local integrability.

## 1 Introduction

The study of distribution-dependent SDEs started with McKean's seminal work [1] about Vlasov equation of plasma which had been proposed in [2]. Classical results about the solvability of McKeanVlasov equations include [3, 4, 5]. Since then, the literature on the well-posedness of McKean-Vlasov SDEs has been extended significantly. For recent results, we refer to [6, 7, 8, 9, 10, 11], the survey [12], and references therein. For applications on mean-field games, see the two-volume monograph [13].

In this paper, we consider the density-dependent McKean-Vlasov SDEs studied in [14]. More precisely, let $p \in[1, \infty)$ and $\mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ be the space of Borel probability measures on $\mathbb{R}^{d}$ with finite $p$-th moment. We endow $\mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ with the Wasserstein metric $W_{p}$. Let $T>0$ and $\mathbb{T}$ be the interval $[0, T]$. We consider measurable functions

$$
\begin{aligned}
& b: \mathbb{T} \times \mathbb{R}^{d} \times \mathbb{R}_{+} \times \mathcal{P}_{p}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d}, \\
& \sigma: \mathbb{T} \times \mathbb{R}^{d} \times \mathcal{P}_{p}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{m}
\end{aligned}
$$

[^0]Let $\left(B_{t}, t \geq 0\right)$ be a given $m$-dimensional Brownian motion and $\mathbb{F}:=\left(\mathcal{F}_{t}, t \geq 0\right)$ a given admissible filtration on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We consider the $\operatorname{SDE}$

$$
\begin{equation*}
\mathrm{d} X_{t}=b\left(t, X_{t}, \ell_{t}\left(X_{t}\right), \mu_{t}\right) \mathrm{d} t+\sigma\left(t, X_{t}, \mu_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{1.1}
\end{equation*}
$$

where the distribution of $X_{0}$ is $\nu$, that of $X_{t}$ is $\mu_{t}$, and the probability density function (p.d.f.) of $\mu_{t}$ is $\ell_{t}$.

The papers $[15,16,17]$ study (1.1) in case $\sigma$ is constant and $b$ belongs to a Besov space. On the other hand, $[18,19,20,21]$ study the generalized porous media equation where $b=0$ and $\sigma$ is density-dependent but time-independent. The papers [22, 23, 24, 25, 14, 26, 27, 28, 29, 30, 31] study the case where $b, \sigma$ are density-dependent but time-independent. More generally, [32, 33] allow $b, \sigma$ to be time-dependent. However, $b$ and $\sigma$ do not depend on probability distribution in those previously mentioned papers. Our paper is closely related to [34, 35] in which $b$ is both distribution-dependent and density-dependent. In particular, [34] utilizes the total variation norm $\|\cdot\|_{\text {var }}$ on $\mathcal{P}\left(\mathbb{R}^{d}\right)$ while $[35]$ utilizes the $L^{\infty}$-norm $\|\cdot\|_{\infty}$ on the space of bounded p.d.f.'s. For a recent survey on existence result of density-dependent SDEs, we refer to [36].

The current paper contains the following contributions:
(a) Our approach is based on a mollifying argument whereas [34, 35] employ a Picard iteration argument.
(b) We obtain Hölder continuity of marginal density in both space and time whereas [34, 35] obtain only Hölder continuity in space.
(c) To obtain uniqueness of a solution, $[34,35]$ estimate $L^{\infty}$-norm $\|\cdot\|_{\infty}$ between marginal densities of two weak solutions. In our case, the presence of Wasserstein metric $W_{p}$ renders this approach inapplicable. To overcome this difficulty, we use another metric between marginal densities.

We recall two different notions of a solution of (1.1):

## Definition 1.1.

1. A strong solution to (1.1) is a continuous $\mathbb{R}^{d}$-valued $\mathbb{F}$-adapted process $\left(X_{t}, t \in \mathbb{T}\right)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $t \in \mathbb{T}$ :

$$
\begin{equation*}
X_{t}=X_{0}+\int_{0}^{t} b\left(s, X_{s}, \ell_{s}\left(X_{s}\right), \mu_{s}\right) \mathrm{d} s+\int_{0}^{t} \sigma\left(s, X_{s}, \mu_{s}\right) \mathrm{d} B_{s} \quad \mathbb{P} \text {-a.s. } \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{t} \mathbb{E}\left[\left|b\left(s, X_{s}, \ell_{s}\left(X_{s}\right), \mu_{s}\right)\right|+\left|\sigma\left(s, X_{s}, \mu_{s}\right)\right|^{2}\right] \mathrm{d} s<\infty . \tag{1.3}
\end{equation*}
$$

2. A weak solution to (1.1) is a continuous $\mathbb{R}^{d}$-valued process $\left(X_{t}, t \in \mathbb{T}\right)$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where there exist some $m$-dimensional Brownian motion ( $B_{t}, t \geq 0$ ) and some admissible filtration $\mathbb{F}:=\left(\mathcal{F}_{t}, t \geq 0\right)$ such that $\left(X_{t}, t \in \mathbb{T}\right)$ is $\mathbb{F}$-adapted and that (1.2) and (1.3) hold.

Correspondingly, we have notions of uniqueness of (1.1):

## Definition 1.2.

1. Equation (1.1) has strong uniqueness if any two strong solutions on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for a given Brownian motion and a given admissible filtration coincide $\mathbb{P}$-a.s. on the path space $C\left(\mathbb{T} ; \mathbb{R}^{d}\right)$. Equation (1.1) has weak uniqueness if any weak solution induces the same distribution on $C\left(\mathbb{T} ; \mathbb{R}^{d}\right)$.
2. Equation (1.1) is strongly well-posed if it has strong solution and strong uniqueness. Equation (1.1) is weakly well-posed if it has weak solution and weak uniqueness. Equation (1.1) is well-posed if it is both strongly and weakly well-posed.

In Section 2, we state our main results about existence, Hölder regularity, and uniqueness of a solution of (1.1). In Section 3, we remind some facts about optimal transport. Also, we recall and establish estimates of marginal density/distribution of classical SDEs. We explicitly mention the dependence of an estimate on parameters of a given assumption. This is crucial in ensuring that those estimates are stable in our mollifying argument. We prove our theorems in Section 4 and Section 5 respectively. To facilitate the navigation of the readers, we summarize the results in our papers as follows:

| Result | Content | Assumption |
| :--- | :--- | :--- |
| Theorem 2.2 <br> Theorem 2.3 | Main results | Assumption 2.1 |
| Lemma 3.1 <br> Lemma 3.2 | Properties of $W_{p}$ |  |
| Theorem 3.4 | Operator norms of semigroups | Assumption 3.3 |
| Lemma 3.5 | Heat kernel estimates |  |
| Proposition 3.8 | Krylov's and Khasminskii's estimates | Assumption 3.6 |
| Theorem 3.9 | Moment estimates |  |
| Theorem 3.10 | $L^{p}$ estimates | Assumption 3.11 |
| Theorem 3.12 | Hölder continuity in space |  |
| Corollary 3.13 | Duhamel representation |  |

Above, Assumption 3.11 implies both Assumption 3.6 and Assumption 3.3. On the other hand, Assumption 3.6 does not imply Assumption 3.3, and vice versa. Throughout this paper, we use the following conventions:
(a) The set $\mathbb{R}^{m} \otimes \mathbb{R}^{n}$ is the space of matrices of size $m \times n$ with real entries. For $x \in \mathbb{R}^{m} \otimes \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n} \otimes \mathbb{R}^{k}$, let $x y$ be their matrix product. For $x, y \in \mathbb{R}^{m} \otimes \mathbb{R}^{n}$, let $\langle x, y\rangle$ be their Frobenius inner product and $|x|$ the induced Frobenius norm of $x$.
(b) Let $\mathbb{R}_{+}:=\{x \in \mathbb{R}: x \geq 0\}$. We denote by $\mathcal{B}\left(\mathbb{R}^{d}\right)$ the Borel $\sigma$-algebra on $\mathbb{R}^{d}$. For brevity, we write $\infty$ for $+\infty$. We denote $x \vee y:=\max \{x, y\}$ and $x \wedge y:=\min \{x, y\}$ for $x, y \in \mathbb{R}$.
(c) We denote by $\nabla, \nabla^{2}$ the gradient and the Hessian with respect to (w.r.t.) the spatial variable. We denote by $\partial_{t}$ the derivative w.r.t. time. For $\alpha \in(0,1]$ and a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$, we denote by $[f]_{\alpha}$ the best $\alpha$-Hölder constant of $f$, i.e.,

$$
[f]_{\alpha}:=\sup _{\substack{x, y \in \mathbb{R}^{d} \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} .
$$

(d) Let $L^{0}\left(\mathbb{R}^{d}\right)$ be the space of real-valued measurable functions on $\mathbb{R}^{d}$. Let $L_{+}^{0}\left(\mathbb{R}^{d}\right)$ be the subset of $L^{0}\left(\mathbb{R}^{d}\right)$ that consists of non-negative functions. Let $L_{b}^{0}\left(\mathbb{R}^{d}\right)$ be the subset of $L^{0}\left(\mathbb{R}^{d}\right)$ that consists of bounded functions.
(e) Let $\mathcal{P}\left(\mathbb{R}^{d}\right)$ be the space of Borel probability measures on $\mathbb{R}^{d}$. The weak topology (and thus weak convergence $\rightharpoonup$ ) of $\mathcal{P}\left(\mathbb{R}^{d}\right)$ is the topology induced by $C_{b}\left(\mathbb{R}^{d}\right)$. The weak* topology (and thus weak convergence $\stackrel{*}{\rightharpoonup})$ of $\mathcal{P}\left(\mathbb{R}^{d}\right)$ is the topology induced by $C_{c}\left(\mathbb{R}^{d}\right)$.

## 2 Main results

Let $p, q \in[1, \infty]$. Let $L^{p}\left(\mathbb{R}^{d}\right) \subset L^{0}\left(\mathbb{R}^{d}\right)$ be the usual Lebesgue space. The localized version $\tilde{L}^{p}\left(\mathbb{R}^{d}\right)$ of $L^{p}\left(\mathbb{R}^{d}\right)$ is defined by the norm

$$
\|f\|_{\tilde{L}^{p}}:=\sup _{x \in \mathbb{R}^{d}}\left\|1_{B(x, 1)} f\right\|_{L^{p}}
$$

where $B(x, r)$ is the open ball centered at $x$ with radius $r$. We define the Bochner space

$$
L_{q}^{p}\left(t_{0}, t_{1}\right):=L^{q}\left(\left[t_{0}, t_{1}\right] ; L^{p}\left(\mathbb{R}^{d}\right)\right), \quad 0 \leq t_{0}<t_{1} \leq T
$$

The localized version $\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)$ of $L_{q}^{p}\left(t_{0}, t_{1}\right)$ is defined by the norm

$$
\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}:=\sup _{x \in \mathbb{R}^{d}}\left\|1_{B(x, 1)} g\right\|_{L_{q}^{p}\left(t_{0}, t_{1}\right)} .
$$

In particular, $\|g\|_{\tilde{L}_{\infty}^{\infty}\left(t_{0}, t_{1}\right)}=\|g\|_{L_{\infty}^{\infty}\left(t_{0}, t_{1}\right)}$ and we have for $p, q \in[1, \infty)$ :

$$
\begin{aligned}
& \|g\|_{L_{q}^{p}\left(t_{0}, t_{1}\right)}=\left(\int_{t_{0}}^{t_{1}}\left(\int_{\mathbb{R}^{d}}|g(s, y)|^{p} \mathrm{~d} y\right)^{\frac{q}{p}} \mathrm{~d} s\right)^{\frac{1}{q}}, \\
& \|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}=\sup _{x \in \mathbb{R}^{d}}\left(\int_{t_{0}}^{t_{1}}\left(\int_{B(x, 1)}|g(s, y)|^{p} \mathrm{~d} y\right)^{\frac{q}{p}} \mathrm{~d} s\right)^{\frac{1}{q}} .
\end{aligned}
$$

For brevity, we denote

$$
L_{q}^{p}(t):=L_{q}^{p}(0, t), \quad \tilde{L}_{q}^{p}(t):=\tilde{L}_{q}^{p}(0, t), \quad L_{q}^{p}:=L_{q}^{p}(0, T), \quad \tilde{L}_{q}^{p}:=\tilde{L}_{q}^{p}(0, T)
$$

The class $\mathcal{K}$ of exponent parameter is defined by

$$
\mathcal{K}:=\left\{(p, q) \in(2, \infty]^{2}: \frac{d}{p}+\frac{2}{q}<1\right\} .
$$

For $\alpha \in(0,1)$, let $C_{b}^{\alpha}\left(\mathbb{R}^{d}\right)$ be the Hölder space of functions $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ with the norm

$$
\|f\|_{C_{b}^{\alpha}}:=\|f\|_{\infty}+[f]_{\alpha} .
$$

We denote by $M_{p}(\rho)$ the $p$-th moment of $\rho \in \mathcal{P}\left(\mathbb{R}^{d}\right)$, i.e.,

$$
M_{p}(\rho):=\int_{\mathbb{R}^{d}}|x|^{p} \mathrm{~d} \rho(x) .
$$

Below, we introduce the main assumption about the initial distribution and the coefficients of (1.1). Let $a:=\sigma \sigma^{\top}$. We denote $b_{t}(x, r, \rho):=b(t, x, r, \rho), \sigma_{t}(x, \rho):=\sigma(t, x, \rho)$ and $a_{t}(x, \rho):=$ $a(t, x, \rho)$. We consider the set of assumptions:

Assumption 2.1. There exist constants $\alpha \in(0,1), \beta \in(0,1), p \in[1, \infty), C>0$ such that for $t \in \mathbb{T} ; x, y \in \mathbb{R}^{d} ; r, \tilde{r} \in \mathbb{R}_{+}$and $\rho, \tilde{\rho} \in \mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ :
(A1) $a_{t}$ is invertible and $\sigma_{t}$ is differentiable w.r.t. the spatial variable.
(A2) There exists $1 \leq f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$ with $\left(p_{0}, q_{0}\right) \in \mathcal{K}$ such that $\left|b_{t}(x, r, \rho)\right| \leq f_{0}(t, x)$.
(A3) $\alpha \in\left(0,1-\frac{d}{p_{0}}-\frac{2}{q_{0}}\right)$ and $\nu \in \mathcal{P}_{p+\alpha}\left(\mathbb{R}^{d}\right)$ has a density $\ell_{\nu} \in C_{b}^{\alpha}\left(\mathbb{R}^{d}\right)$.
(A4) The following conditions hold:

$$
\begin{aligned}
\left|b_{t}(x, r, \rho)-b_{t}(x, \tilde{r}, \tilde{\rho})\right| & \leq C\left\{|r-\tilde{r}|+W_{p}(\rho, \tilde{\rho})\right\}, \\
\left\|\nabla \sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}+\left\|a_{t}^{-1}\right\|_{\infty} & \leq C, \\
\left|\sigma_{t}(x, \rho)-\sigma_{t}(x, \tilde{\rho})\right| & \leq C W_{p}(\rho, \tilde{\rho}), \\
\left|\nabla \sigma_{t}(\cdot, \rho)(x)-\nabla \sigma_{t}(\cdot, \rho)(y)\right| & \leq C\left(|x-y|+|x-y|^{\beta}\right) .
\end{aligned}
$$

We gather parameters in Assumption 2.1:

$$
\Theta_{1}:=\left(d, T, \alpha, \beta, C, p_{0}, q_{0}, f_{0}, p\right) .
$$

There is no continuity condition on the spatial variable of the drift. If $b$ is bounded, then it satisfies Assumption 2.1[A2]. Assumption 2.1[A4] implies for $t \in \mathbb{T}, \gamma \in(0,1) ; x, y \in \mathbb{R}^{d}$ and $\rho \in$ $\mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ that $\left|\sigma_{t}(x, \rho)-\sigma_{t}(y, \rho)\right| \leq 2 C|x-y|^{\gamma}$. Assumption 2.1[A2] means that the marginal density and marginal distribution do not affect the local integrability of the drift. In Assumption 2.1[A3], we requires $\nu \in \mathcal{P}_{p+\alpha}\left(\mathbb{R}^{d}\right)$ instead of just $\nu \in \mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$. This gives us a uniform control on the tail behavior of the marginal distribution in (3.17) of Theorem 3.9.

Our main results are the following:
Theorem 2.2 (Existence and Regularity). Let Assumption 2.1 hold.

1. Equation (1.1) has a strong solution whose marginal distribution is denoted by $\left(\mu_{t}, t \in \mathbb{T}\right)$ and marginal density is denoted by $\left(\ell_{t}, t \in \mathbb{T}\right)$.
2. There exist constants $c>0$ (depending only on $\Theta_{1}$ ), $c_{1}, c_{2}>0$ (depending only on $\Theta_{1}, \nu$ ), $\delta_{1} \in\left(0, \frac{1}{2}\right)$ (depending only on $\left.d, p_{0}, q_{0}, \alpha\right)$, and $\delta_{2} \in\left(0, \frac{1}{2}\right)$ (depending only on $q_{0}$ ) such that

$$
\begin{aligned}
& \sup _{t \in \mathbb{T}}\left\|\ell_{t}\right\|_{C_{b}^{\alpha}} \leq c\left\|\ell_{\nu}\right\|_{C_{b}^{\alpha}}, \\
& \left\|\ell_{t}-\ell_{s}\right\|_{\infty} \leq c_{1}|t-s|^{\delta_{1}}, \\
& W_{p}\left(\mu_{s}, \mu_{t}\right) \leq c_{2}|t-s|^{\delta_{2}}, \quad s, t \in \mathbb{T} .
\end{aligned}
$$

Theorem 2.3 (Uniqueness). Let Assumption 2.1 hold. Assume in addition that $p=1,\|b\|_{\infty} \leq C$ and that $\sigma$ does not depend on marginal distribution. Let $X^{1}, X^{2}$ be two weak solutions of (1.1). Let $\nu_{1}, \nu_{2}$ be their initial distributions and $\ell^{1}, \ell^{2}$ their marginal densities. We assume that $\nu_{1}, \nu_{2}$ satisfy Assumption 2.1[A3] for the same parameters.

1. There exists an increasing function $\Lambda: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$(depending only on $\Theta_{1}$ ) such that

$$
\begin{aligned}
& \sup _{t \in \mathbb{T}} \int_{\mathbb{R}^{d}}(|x|+1)\left|\ell_{t}^{1}(x)-\ell_{t}^{2}(x)\right| \mathrm{d} x \\
\leq & \Lambda\left(\left\|\ell_{\nu_{1}}\right\|_{\infty}+M_{1}\left(\nu_{1}\right)\right) \int_{\mathbb{R}^{d}}(|x|+1)\left|\ell_{\nu_{1}}(x)-\ell_{\nu_{2}}(x)\right| \mathrm{d} x .
\end{aligned}
$$

2. Equation (1.1) has both weak and strong uniqueness.

## 3 Preliminaries

### 3.1 Some facts from optimal transport

Let $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$. We denote by $|\mu-\nu|$ the variation of the signed measure $\mu-\nu$ as in $[37$, Section 6.1]. By [37, Theorems 6.2 and 6.4], $|\mu-\nu|$ is a non-negative finite measure. The set of transport plans (or couplings) between $\mu$ and $\nu$ is defined by

$$
\Gamma(\mu, \nu):=\left\{\xi \in \mathcal{P}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right): \pi_{\sharp}^{1} \xi=\mu \text { and } \pi_{\sharp}^{2} \xi=\nu\right\},
$$

where $\pi^{i}$ is the projection of $\mathbb{R}^{d} \times \mathbb{R}^{d}$ onto its $i$-th coordinate and $\pi_{\sharp}^{i} \xi$ is the push-forward of $\xi$ by $\pi^{i}$. Their extended Wasserstein metric is defined by

$$
W_{p}(\mu, \nu):=\inf _{\xi \in \Gamma(\mu, \nu)}\left(\int_{\mathbb{R}^{d}}|x-y|^{p} \mathrm{~d} \xi(x, y)\right)^{1 / p}
$$

By [38, Theorem 6.18], $\left(\mathcal{P}_{p}\left(\mathbb{R}^{d}\right), W_{p}\right)$ is a Polish space. By [38, Theorem 6.9], we have for $\mu_{n}, \mu \in \mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ that $W_{p}\left(\mu_{n}, \mu\right) \rightarrow 0$ if and only if $\mu_{n} \rightharpoonup \mu$ and $M_{p}\left(\mu_{n}\right) \rightarrow M_{p}(\mu)$. For more information about optimal transport, we refer to [39, 38, 40, 41, 42, 43, 44].

Let $\Phi_{p}$ be the collection of all $(\varphi, \psi) \in C_{b}\left(\mathbb{R}^{d}\right) \times C_{b}\left(\mathbb{R}^{d}\right)$ such that $\varphi(x)+\psi(y) \leq|x-y|^{p}$ for $x, y \in \mathbb{R}^{d}$. We recall results needed for the proofs of our theorems:

## Lemma 3.1.

1. [39, Theorem 1.3] We have for $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ and $p \in[1, \infty)$ that

$$
\left(W_{p}(\mu, \nu)\right)^{p}=\sup \left\{\int_{\mathbb{R}^{d}} \varphi \mathrm{~d} \mu+\int_{\mathbb{R}^{d}} \psi \mathrm{~d} \nu:(\varphi, \psi) \in \Phi_{p}\right\} .
$$

2. [39, Theorem 1.14] We have for $\mu, \nu \in \mathcal{P}_{1}\left(\mathbb{R}^{d}\right)$ that

$$
W_{1}(\mu, \nu)=\sup \left\{\int_{\mathbb{R}^{d}} f \mathrm{~d}(\mu-\nu): f \in L^{1}(|\mu-\nu|) \text { with }[f]_{1} \leq 1\right\} .
$$

3. [39, Remark 7.1.2] We have for $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ and $1 \leq p \leq q<\infty$ that $W_{p}(\mu, \nu) \leq W_{q}(\mu, \nu)$.

Above, the first claim is called Kantorovich duality while the second one is called KantorovichRubinstein theorem. The next result states that the extended Wasserstein metric is controlled by the weighted $L^{1}$-metric between the corresponding p.d.f.'s.

Lemma 3.2. Let $p \in[1, \infty)$ and $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ be absolutely continuous with $\ell_{\mu}, \ell_{\nu}$ being their p.d.f.'s respectively. Then

$$
\left(W_{p}(\mu, \nu)\right)^{p} \leq\left(1 \vee 2^{p-1}\right) \int_{\mathbb{R}^{d}}|x|^{p}\left|\ell_{\mu}-\ell_{\nu}\right|(x) \mathrm{d} x
$$

Proof. For $B \in \mathcal{B}\left(\mathbb{R}^{d}\right)$, we denote by $\Pi(B)$ the collection of all finite measurable partitions of $B$. This means $\left(B_{1}, \ldots, B_{n}\right) \in \Pi(B)$ if and only if $\left\{B_{1}, \ldots, B_{n}\right\} \subset \mathcal{B}\left(\mathbb{R}^{d}\right)$ are pairwise disjoint and $B=\bigcup_{k=1}^{n} B_{k}$. We have

$$
\begin{aligned}
|\mu-\nu|(B) & =\sup \left\{\sum_{k=1}^{n}\left|(\mu-\nu)\left(B_{k}\right)\right|:\left(B_{1}, \ldots, B_{n}\right) \in \Pi(B)\right\} \\
& =\sup \left\{\sum_{k=1}^{n}\left|\int_{B_{k}}\left(\ell_{\mu}-\ell_{\nu}\right)(x) \mathrm{d} x\right|:\left(B_{1}, \ldots, B_{n}\right) \in \Pi(B)\right\} \\
& \leq \sup \left\{\sum_{k=1}^{n} \int_{B_{k}}\left|\ell_{\mu}-\ell_{\nu}\right|(x) \mathrm{d} x:\left(B_{1}, \ldots, B_{n}\right) \in \Pi(B)\right\} \\
& =\int_{B}\left|\ell_{\mu}-\ell_{\nu}\right|(x) \mathrm{d} x .
\end{aligned}
$$

On the other hand, we have from [39, Proposition 7.10] that

$$
\left(W_{p}(\mu, \nu)\right)^{p} \leq\left(1 \vee 2^{p-1}\right) \int_{\mathbb{R}^{d}}|x|^{p} \mathrm{~d}|\mu-\nu|(x) .
$$

The claim then follows.

### 3.2 Heat kernel estimates of transition density

In the rest of Section 3, we consider measurable functions

$$
\begin{aligned}
& b: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \\
& \sigma: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{m} .
\end{aligned}
$$

Let $a:=\sigma \sigma^{\top}$. We denote $b_{t}:=b(t, \cdot), \sigma_{t}:=\sigma(t, \cdot)$ and $a_{t}:=a(t, \cdot)$. We consider the following set of assumptions:

Assumption 3.3. There exist constants $\beta \in(0,1)$ and $C>0$ such that for $t \in \mathbb{T}$ and $x, y \in \mathbb{R}^{d}$ :
(A1) $a_{t}$ is invertible.
(A2) The following conditions hold:

$$
\begin{aligned}
\left\|b_{t}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}+\left\|a_{t}^{-1}\right\|_{\infty} & \leq C \\
\left|b_{t}(x)-b_{t}(y)\right| & \leq C\left(|x-y|+|x-y|^{\beta}\right), \\
\left|\sigma_{t}(x)-\sigma_{t}(y)\right| & \leq C|x-y|^{\beta} .
\end{aligned}
$$

We gather parameters in Assumption 3.3:

$$
\Theta_{2}:=(d, T, \beta, C) .
$$

Throughout the remaining of Section 3.2, we assume that $(b, \sigma)$ satisfies Assumption 3.3.
For any $(s, x) \in[0, T) \times \mathbb{R}^{d}$, the $\operatorname{SDE}$

$$
\begin{equation*}
\mathrm{d} X_{s, t}^{x}=b\left(t, X_{s, t}^{x}\right) \mathrm{d} t+\sigma\left(t, X_{s, t}^{x}\right) \mathrm{d} B_{t}, \quad t \in[s, T], X_{s, s}^{x}=x, \tag{3.1}
\end{equation*}
$$

is weakly well-posed by [45, Theorem 1.2] and has transition density denoted by $\left(p_{s, t}^{b, \sigma}\right)_{0 \leq s<t \leq T}$, i.e., $p_{s, t}^{b, \sigma}(x, \cdot)$ is the density of $X_{s, t}^{x}$. The associated semigroup $\left(P_{s, t}^{b, \sigma}\right)_{0 \leq s<t \leq T}$ is defined for $x \in \mathbb{R}^{d}$ and $f \in L_{+}^{0}\left(\mathbb{R}^{d}\right) \cup L_{b}^{0}\left(\mathbb{R}^{d}\right)$ by

$$
\begin{equation*}
P_{s, t}^{b, \sigma} f(x):=\mathbb{E}\left[f\left(X_{s, t}^{x}\right)\right]=\int_{\mathbb{R}^{d}} p_{s, t}^{b, \sigma}(x, y) f(y) \mathrm{d} y \tag{3.2}
\end{equation*}
$$

As in [45, Section 1.2], we construct a family $\left(\psi_{s, t}\right)_{s, t \in \mathbb{T}}$ of $C^{\infty}$-diffeomorphisms on $\mathbb{R}^{d}$. Let $\rho: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a smooth symmetric p.d.f. whose support is contained in the unit ball of $\mathbb{R}^{d}$. We define $\underline{b}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ by $\underline{b}(t, \cdot):=b(t, \cdot) * \rho$ where $*$ is the convolution operator. By [45, Inequalities 1.9 and 1.10],

$$
\sup _{t \in \mathbb{T}}\left\{\left\|\nabla^{n} \underline{b}(t, \cdot)\right\|_{\infty}+\|b(t, \cdot)-\underline{b}(t, \cdot)\|_{\infty}\right\}<\infty, \quad n \in \mathbb{N} .
$$

For a fixed $(s, x) \in \mathbb{T} \times \mathbb{R}^{d}$, we consider the ODE

$$
\begin{cases}\frac{\mathrm{d}}{\mathrm{~d} t} \psi_{s, t}(x) & =\underline{b}\left(t, \psi_{s, t}(x)\right), \quad t \in \mathbb{T}, \\ \psi_{s, s}(x) & =x\end{cases}
$$

For $\kappa>0$, we consider the Gaussian heat kernel

$$
p_{t}^{\kappa}(x):=\frac{1}{(\kappa \pi t)^{\frac{d}{2}}} \exp \left(-\frac{|x|^{2}}{\kappa t}\right), \quad t>0, x \in \mathbb{R}^{d} .
$$

The following results give density and gradient estimates for (3.1):
Theorem 3.4. Let Assumption 3.3 hold.

1. 146, Inequalities 2.7 and 2.8] There exist constants $c, \kappa>0$ (depending only on $\Theta_{2}$ ) such that for $i \in\{0,1\}, 0 \leq s<t \leq T$ and $x, y \in \mathbb{R}^{d}$ :

$$
\left|\nabla_{x}^{i} p_{s, t}^{b, \sigma}(x, y)\right| \leq c(t-s)^{-\frac{i}{2}} p_{t-s}^{\kappa}(x-y) .
$$

2. [45, Lemma A.1] For $\alpha \in(0, \beta)$, there exist constants $c, \kappa>0$ (depending only on $\Theta_{2}, \alpha$ ) such that for $0 \leq s<t \leq T$ and $x, y, y^{\prime} \in \mathbb{R}^{d}$ :

$$
\begin{aligned}
& \left|\nabla_{x} p_{s, t}^{b, \sigma}(x, y)-\nabla_{x} p_{s, t}^{b, \sigma}\left(x, y^{\prime}\right)\right| \\
\leq & c\left|y-y^{\prime}\right|^{\alpha}(t-s)^{-\frac{1+\alpha}{2}}\left\{p_{t-s}^{\kappa}\left(\psi_{s, t}(x)-y\right)+p_{t-s}^{\kappa}\left(\psi_{s, t}(x)-y^{\prime}\right)\right\} .
\end{aligned}
$$

We define for $\kappa>0, f \in L_{+}^{0}\left(\mathbb{R}^{d}\right) \cup L_{b}^{0}\left(\mathbb{R}^{d}\right), x \in \mathbb{R}^{d}$ and $0 \leq s<t \leq T$ :

$$
\begin{align*}
& P_{t}^{\kappa} f(x):=\int_{\mathbb{R}^{d}} p_{t}^{\kappa}(x-y) f(y) \mathrm{d} y,  \tag{3.3}\\
& \hat{P}_{s, t}^{\kappa} f(x):=\int_{\mathbb{R}^{d}} p_{t-s}^{\kappa}\left(\psi_{s, t}(x)-y\right) f(y) \mathrm{d} y, \\
& \tilde{P}_{s, t}^{\kappa} f(x):=\int_{\mathbb{R}^{d}} p_{t-s}^{\kappa}\left(\psi_{s, t}(y)-x\right) f(y) \mathrm{d} y . \tag{3.4}
\end{align*}
$$

For brevity, we denote $\frac{q-p}{p q}:=\frac{1}{p}-\frac{1}{q}$ for $p, q \in[1, \infty]$. By Young's inequality for convolution, there exists a constant $c>0$ (depending only on $d, \kappa$ ) such that for $t>0$ and $1 \leq p \leq \bar{p} \leq \infty$ :

$$
\left\|P_{t}^{\kappa}\right\|_{L^{p} \rightarrow L^{\bar{p}}}:=\sup _{\|f\|_{L^{p} \leq 1}}\left\|P_{t}^{\kappa} f\right\|_{L^{\bar{p}}} \leq c t^{-\frac{d(\overline{\bar{p}}-p)}{2 p \bar{p}}}
$$

We recall an essential generalization for dealing with unbounded drift:
Lemma 3.5. [35] Let Assumption 3.3 hold. There exists a constant $c>0$ (depending only on $\Theta_{2}$ ) such that for $0 \leq s<t \leq T$ and $1 \leq p \leq \bar{p} \leq \infty$ :

$$
\left\|P_{t-s}^{\kappa}\right\|_{\tilde{L}^{p} \rightarrow \tilde{L}^{\bar{p}}}+\left\|\hat{P}_{s, t}^{\kappa}\right\|_{\tilde{L}^{p} \rightarrow \tilde{L}^{\bar{p}}}+\left\|\tilde{P}_{s, t}^{\kappa}\right\|_{\tilde{L}^{p} \rightarrow \tilde{L}^{\bar{p}}} \leq c(t-s)^{-\frac{d(\overline{\bar{p}}-\bar{p})}{2 \bar{p}}} .
$$

For the sake of being self-contained, we also include the proof.
Proof. Let $z \in \mathbb{R}^{d}, f \in L_{+}^{0}\left(\mathbb{R}^{d}\right)$ and $\mathbf{B}_{n}:=\left\{v=\left(v_{1}, \ldots, v_{d}\right) \in \mathbb{Z}^{d}: \sum_{i=1}^{d}\left|v_{i}\right|=n\right\}$ for $n \in \mathbb{N}$. We write $M_{1} \lesssim M_{2}$ if there exists a constant $c>0$ (depending only on $\Theta_{2}$ ) such that $M_{1} \leq c M_{2}$. By [35, Inequality 3.11], there exists a constant $c_{1}>0$ (depending only on $\Theta_{2}$ ) such that

$$
\left\|1_{B(z, 1)} \hat{P}_{s, t}^{\kappa} f\right\|_{L^{\bar{p}}} \lesssim(t-s)^{-\frac{d(\overline{\bar{p}}-p)}{2 p \bar{p}}} \sum_{n=0}^{\infty} \sum_{v \in \mathbf{B}_{n}} e^{-\frac{n^{2}}{c_{1}(t-s)}}\left\|1_{B(z+v, d)} f\right\|_{L^{p}}
$$

We have $\operatorname{card}\left(\mathbf{B}_{n}\right) \leq(2 n+1)^{d}$, so

$$
\sum_{n=0}^{\infty} \sum_{v \in \mathbf{B}_{n}} e^{-\frac{n^{2}}{c_{1}(t-s)}} \leq \sum_{n=0}^{\infty}(2 n+1)^{d} e^{-\frac{n^{2}}{c_{1} T}} \lesssim 1 .
$$

On the other hand, $\left\|1_{B(z+v, d)} f\right\|_{L^{p}} \lesssim\|f\|_{\tilde{L}^{p}}$. Then

$$
\left\|\hat{P}_{s, t}^{\kappa} f\right\|_{\tilde{L}^{\bar{p}}} \lesssim(t-s)^{-\frac{d(\overline{\bar{p}}-p)}{2 p \bar{p}}}\|f\|_{\tilde{L}^{p}}
$$

Clearly, $P_{t-s}^{\kappa}$ is a special case of $\hat{P}_{s, t}^{\kappa}$ in which $\psi_{s, t}$ is the identity map on $\mathbb{R}^{d}$. Then

$$
\left\|P_{t-s}^{\kappa} f\right\|_{\tilde{L}^{\bar{p}}} \lesssim(t-s)^{-\frac{d(\overline{\bar{p}}-\bar{p})}{2 p \bar{p}}}\|f\|_{\tilde{L}^{p}}
$$

It remains to prove for $\tilde{P}_{s, t}^{\kappa}$. By [35, Inequality 3.2], there exists a constant $c_{2} \geq 1$ (depending only on $\Theta_{2}$ ) such that

$$
\begin{equation*}
\sup _{0 \leq s \leq t \leq T}\left\{\left\|\nabla \psi_{s, t}\right\|_{\infty}+\left\|\nabla \psi_{s, t}^{-1}\right\|_{\infty}\right\} \leq c_{2} \tag{3.5}
\end{equation*}
$$

We have

$$
\begin{align*}
\tilde{P}_{s, t}^{\kappa} f(x) & =\int_{\mathbb{R}^{d}} p_{t-s}^{\kappa}\left(\psi_{s, t}(y)-x\right) f(y) \mathrm{d} y \quad \text { by }(3.4) \\
& \leq \int_{\mathbb{R}^{d}} p_{t-s}^{\kappa}\left(\frac{\left.\psi_{s, t}^{-1}(x)-y\right)}{c_{2}}\right) f(y) \mathrm{d} y \quad \text { by }(3.5) \\
& \lesssim \int_{\mathbb{R}^{d}} p_{t-s}^{\bar{\kappa}}\left(\psi_{s, t}^{-1}(x)-y\right) f(y) \mathrm{d} y \quad \text { where } \bar{\kappa}:=\kappa\left|c_{2}\right|^{2} \\
& =\left(P_{t-s}^{\kappa} f\right) \circ \psi_{s, t}^{-1}(x) . \tag{3.6}
\end{align*}
$$

It suffices to consider $\bar{p}<\infty$. We have

$$
\begin{align*}
\left\|1_{B(z, 1)} \tilde{P}_{s, t}^{\kappa} f\right\|_{L^{\bar{p}}}^{\bar{p}} & \lesssim \int_{B(z, 1)}\left|\left(P_{t-s}^{\bar{\kappa}} f\right) \circ \psi_{s, t}^{-1}(x)\right|^{\bar{p}} \mathrm{~d} x \quad \text { by }(3.6) \\
& =\int_{\psi_{s, t}^{-1}(B(z, 1))}\left|\left(P_{t-s}^{\bar{\kappa}} f\right)(x)\right|^{\bar{p}}\left|\operatorname{det} \nabla \psi_{s, t}(x)\right| \mathrm{d} x  \tag{3.7}\\
& \lesssim \int_{\psi_{s, t}^{-1}(B(z, 1))}\left|\left(P_{t-s}^{\bar{\kappa}} f\right)(x)\right|^{\bar{p}} \mathrm{~d} x  \tag{3.8}\\
& =\left\|1_{\psi_{s, t}^{-1}(B(z, 1))} P_{t-s}^{\bar{\kappa}} f\right\|_{L^{\bar{p}}}^{\bar{p}} \\
& \lesssim\left\|P_{t-s} f\right\|_{\tilde{L}^{\bar{p}}}^{\bar{p}} \quad \text { by }(3.5) .
\end{align*}
$$

Above, (3.7) is due to change of variables formula and (3.8) due to Hadamard's inequality for determinants. It follows that

$$
\left\|1_{B(z, 1)} \tilde{P}_{s, t}^{\kappa} f\right\|_{L^{\bar{p}}} \lesssim\left\|P_{t-s}^{\bar{\kappa}} f\right\|_{\tilde{L}^{\bar{p}}} .
$$

This completes the proof.

### 3.3 Moment and $L^{p}$ estimates of marginal density

We consider another set of assumption:
Assumption 3.6. There exist constants $\beta \in(0,1), C>0, l \in \mathbb{N}^{*}$ such that for $t \in \mathbb{T}$ and $x, y \in \mathbb{R}^{d}$ :
(A1) $\nu \in \mathcal{P}_{1}\left(\mathbb{R}^{d}\right)$ has a density $\ell_{\nu}$.
(A2) $a_{t}$ is invertible and $\sigma_{t}$ is weakly differentiable.
(A3) There exist measurable maps $b^{(0)}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $b^{(1)}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ such that $b_{t}(x)=$ $b_{t}^{(0)}(x)+b_{t}^{(1)}(x)$.
(A4) There exists $1 \leq f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$ with $\left(p_{0}, q_{0}\right) \in \mathcal{K}$ such that $\left|b_{t}^{(0)}(x)\right| \leq f_{0}(t, x)$.
(A5) For $i \in\{1,2, \ldots, l\}$, there exists $1 \leq f_{i} \in \tilde{L}_{q_{i}}^{p_{i}}$ with $\left(p_{i}, q_{i}\right) \in \mathcal{K}$ such that $\left|\nabla \sigma_{t}(x)\right| \leq$ $\sum_{i=1}^{l} f_{i}(t, x)$.
(A6) The following conditions hold:

$$
\begin{aligned}
\left|b_{t}^{(1)}(x)-b_{t}^{(1)}(y)\right| & \leq C|x-y| \\
\left|b_{t}^{(1)}(0)\right|+\left\|\sigma_{t}\right\|_{\infty}+\left\|a_{t}^{-1}\right\|_{\infty} & \leq C, \\
\left|\sigma_{t}(x)-\sigma_{t}(y)\right| & \leq C|x-y|^{\beta} .
\end{aligned}
$$

Assumption 3.6 is appealing because it is a general but sufficient condition to obtain Krylov's and Khasminskii's estimates, which are a key ingredient for establishing the other estimates in the remaining of Section 3. We gather parameters in Assumption 3.6:

$$
\Theta_{3}:=\left(d, T, \beta, C, l,\left(p_{i}, q_{i}, f_{i}\right)_{i=0}^{l}\right) .
$$

The class $\overline{\mathcal{K}}$ of exponent parameter is defined by

$$
\overline{\mathcal{K}}:=\left\{(p, q) \in(1, \infty)^{2}: \frac{d}{p}+\frac{2}{q}<2\right\} .
$$

Remark 3.7. If $f \in \tilde{L}_{q}^{p}$ for some $(p, q) \in \mathcal{K}$ then there exists $(\bar{p}, \bar{q}) \in \overline{\mathcal{K}}$ such that $|f|^{2} \in \tilde{L}_{\bar{q}}^{\bar{p}}$.
Let $\nu \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ and $\ell_{\nu}$ be its density. We consider the SDE

$$
\begin{equation*}
\mathrm{d} X_{t}=b\left(t, X_{t}\right) \mathrm{d} t+\sigma\left(t, X_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{3.9}
\end{equation*}
$$

where the distribution of $X_{0}$ is $\nu$. By [10, Theorem 1.1(1)], (3.9) is well-posed under Assumption 3.6. First, we establish an essential result for proving Theorem 3.9:

Proposition 3.8. Let $(b, \sigma)$ satisfy Assumption 3.6 and $\left(X_{t}, t \in \mathbb{T}\right)$ be the solution of (3.9).

1. (Khasminskii's estimate) For each $(p, q) \in \bar{K}$, there exist constants $c>0, k>1$ (depending only on $\left.\Theta_{3}, p, q\right)$ such that for $0 \leq t_{0}<t_{1} \leq T$ and $g \in \tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)$ :

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right) \mid \mathcal{F}_{t_{0}}\right] \leq \exp \left(c\left(1+\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{k}\right)\right) . \tag{3.10}
\end{equation*}
$$

2. (Krylov's estimate) For each $(p, q) \in \bar{K}$ and $j \geq 1$, there exists a constant $c>0$ (depending only on $\left.\Theta_{3}, p, q, j\right)$ such that for $0 \leq t_{0}<t_{1} \leq T$ and $g \in \tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)$ :

$$
\begin{equation*}
\mathbb{E}\left[\left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right)^{j} \mid \mathcal{F}_{t_{0}}\right] \leq c\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{j} . \tag{3.11}
\end{equation*}
$$

Proof. We fix $(p, q) \in \bar{K}$.

1. There exists $\bar{q} \in(1, q)$ such that $(p, \bar{q}) \in \bar{K}$. By [47, Theorem 3.1], there exists a constant $c_{1}>0$ (depending only on $\left.\Theta_{3}, p, \bar{q}\right)$ such that for $0 \leq t_{0}<t_{1} \leq T$, a stopping time $\tau$ and $g \in \tilde{L}_{\tilde{q}}^{p}\left(t_{0}, t_{1}\right):$

$$
\mathbb{E}\left[\int_{t_{0} \wedge \tau}^{t_{1} \wedge \tau}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s \mid \mathcal{F}_{t_{0}}\right] \leq c_{1}\|g\|_{\tilde{L}_{\tilde{q}}^{p}\left(t_{0}, t_{1}\right)} .
$$

Let $\delta:=\frac{1}{\bar{q}}-\frac{1}{q} \in(0,1)$. By Hölder's inequality, we have for $0 \leq t_{0}<t_{1} \leq T$ and $g \in \tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)$ :

$$
\begin{equation*}
\|g\|_{\tilde{L}_{\tilde{q}}^{p}\left(t_{0}, t_{1}\right)} \leq\left(t_{1}-t_{0}\right)^{\delta}\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)} . \tag{3.12}
\end{equation*}
$$

We denote by $I_{j}^{n}$ the open interval $\left(\frac{(j-1)\left(t_{1}-t_{0}\right)}{n}, \frac{j\left(t_{1}-t_{0}\right)}{n}\right)$ for $j=1, \ldots, n$. We fix $g \in$ $\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right) \subset \tilde{L}_{\tilde{q}}^{p}\left(t_{0}, t_{1}\right)$. Let $n \geq 2$ be the smallest integer such that

$$
\begin{equation*}
\|g\|_{\tilde{L}_{\bar{q}}^{p}\left(I_{j}^{n}\right)} \leq \frac{1}{2 c_{1}}, \quad j=1, \ldots, n . \tag{3.13}
\end{equation*}
$$

By [48, Lemma 3.5],

$$
\mathbb{E}\left[\exp \left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right) \mid \mathcal{F}_{t_{0}}\right] \leq 2^{n}
$$

By (3.13), there exists $\bar{j} \in\{1, \ldots, n-1\}$ such that

$$
\begin{equation*}
\|g\|_{\tilde{L}_{\tilde{q}}^{p}\left(I_{\bar{j}}^{n-1}\right)}>\frac{1}{2 c_{1}} . \tag{3.14}
\end{equation*}
$$

It follows from (3.12) and (3.14) that

$$
\left(\frac{t_{1}-t_{0}}{n-1}\right)^{\delta}\|g\|_{\tilde{L}_{q}^{p}\left(I_{\bar{j}}^{n-1}\right)}>\frac{1}{2 c_{1}}
$$

which implies

$$
n<1+T\left(2 c_{1}\right)^{-\frac{1}{\delta}}\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{1 / \delta} .
$$

The estimate (3.10) then follows with $k:=\frac{1}{\delta}$.
2. We fix $j \geq 1$. We adapt an elegant idea from [10, Lemma 2.3] into our simpler setting. Let $C_{j}:=e^{j-1}$. Then $h(r):=\left|\log \left(C_{j}+r\right)\right|^{j}$ is concave for $r \in \mathbb{R}_{+}$. We have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right)^{j} \mid \mathcal{F}_{t_{0}}\right] \\
\leq & \mathbb{E}\left[\left\{\log \left(C_{j}+\exp \left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right)\right)\right\}^{j} \mid \mathcal{F}_{t_{0}}\right] \\
\leq & \left\{\log \left(C_{j}+\mathbb{E}\left[\exp \left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right) \mid \mathcal{F}_{t_{0}}\right]\right)\right\}^{j} \quad \text { by Jensen's inequality } \\
\leq & \left\{\log \left[C_{j}+\exp \left(c\left(1+\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{k}\right)\right]\right\}^{j},\right.
\end{aligned}
$$

where the constants $c, k>0$ are given by (3.10). As a result, there exists a constant $\bar{C}_{j}>0$ (depending only on $c, j$ ) such that

$$
\mathbb{E}\left[\left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right)^{j} \mid \mathcal{F}_{t_{0}}\right] \leq \bar{C}_{j}\left(1+\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{k}\right)^{j} .
$$

Replacing $g$ with $\frac{g}{\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}}$ in the above inequality, we obtain

$$
\mathbb{E}\left[\left(\int_{t_{0}}^{t_{1}}\left|g\left(s, X_{s}\right)\right| \mathrm{d} s\right)^{j} \mid \mathcal{F}_{t_{0}}\right] \leq \bar{C}_{j} 2^{j}\|g\|_{\tilde{L}_{q}^{p}\left(t_{0}, t_{1}\right)}^{j}
$$

The estimate (3.11) then follows. This completes the proof.

For another proof of Proposition 3.8, see [49, Lemma 4.1]. Second, we establish moment estimates:

Theorem 3.9. Let $p \in[1, \infty)$ and $(b, \sigma)$ satisfy Assumption 3.6. Let $\mu_{t}$ be the distribution of $X_{t}$ in (3.9).

1. $\mu_{t}$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^{d}$.
2. There exist constants $c>0$ (depending only on $\left.\Theta_{3}, p\right)$ and $\delta \in\left(0, \frac{1}{2}\right)$ (depending only on $q_{0}$ ) such that for $\alpha>0$ and $0 \leq u \leq t \leq T$ :

$$
\begin{align*}
\mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{s}\right|^{p}\right] & \leq c\left(1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]\right),  \tag{3.15}\\
\mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{s}-X_{u}\right|^{p}\right] & \leq c|t-u|^{\delta p}\left(1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]\right),  \tag{3.16}\\
\sup _{s \in[u, t]} \int_{B_{R}^{c}}|x|^{p} \mathrm{~d} \mu_{s}(x) & \leq \frac{c\left(1+\mathbb{E}\left[\left|X_{u}\right|^{p+\alpha}\right]\right)}{R}, \quad R>0, \tag{3.17}
\end{align*}
$$

where $B_{R}^{c}:=\mathbb{R}^{d} \backslash B(0, R)$. Estimate (3.17) also holds for $p=0$.
In particular, (3.17) is essential in proving Lemma 4.3 in Section 4.2.
Proof. By Assumption 3.6[A3], there exist measurable maps $b^{(0)}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $b^{(1)}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow$ $\mathbb{R}^{d}$ such that $b_{t}(x)=b_{t}^{(0)}(x)+b_{t}^{(1)}(x)$. By Assumption 3.6[A4], there exists $f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$ with $\left(p_{0}, q_{0}\right) \in \mathcal{K}$ such that $\left|b_{t}^{(0)}(x)\right| \leq f_{0}(t, x)$. We consider the SDE

$$
\begin{equation*}
\mathrm{d} \bar{X}_{t}=b^{(1)}\left(t, \bar{X}_{t}\right) \mathrm{d} t+\sigma\left(t, \bar{X}_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{3.18}
\end{equation*}
$$

where the distribution of $\bar{X}_{0}$ is $\nu$. Clearly, $\left(b^{(1)}, \sigma\right)$ satisfies Assumption 3.6, so (3.18) is well-posed. We define

$$
\begin{aligned}
\xi_{t} & :=\left\{\sigma_{t}^{\top} a_{t}^{-1} b_{t}^{(0)}\right\}\left(\bar{X}_{t}\right), \\
\bar{B}_{t} & :=B_{t}-\int_{0}^{t} \xi_{s} \mathrm{~d} s, \\
R_{t} & :=\exp \left(\int_{0}^{t} \xi_{t}^{\top} \mathrm{d} B_{t}-\frac{1}{2} \int_{0}^{t}\left|\xi_{t}\right|^{2} \mathrm{~d} t\right), \\
I_{t} & :=\mathbb{E}\left[\exp \left(\frac{1}{2} \int_{0}^{t}\left|\xi_{t}\right|^{2} \mathrm{~d} t\right)\right], \quad t \in \mathbb{T} .
\end{aligned}
$$

By the uniform boundedness of $\sigma_{t}^{\top} a_{t}^{-1}$, Remark 3.7 and Proposition 3.8[1], we get $I_{T}<\infty$. So $R_{T}$ is an exponential martingale with $\mathbb{E}\left[R_{T}\right]=1$. By Girsanov's theorem, the process ( $\bar{B}_{t}, t \in \mathbb{T}$ ) is an $m$-dimensional Brownian motion under the probability measure $\overline{\mathbb{P}}:=R_{T} \mathbb{P}$. We denote by $\overline{\mathbb{E}}$ the expectation w.r.t. $\overline{\mathbb{P}}$. Clearly, (3.18) can be written under $\overline{\mathbb{P}}$ as

$$
\begin{equation*}
\mathrm{d} \bar{X}_{t}=b\left(t, \bar{X}_{t}\right) \mathrm{d} t+\sigma\left(t, \bar{X}_{t}\right) \mathrm{d} \bar{B}_{t}, \quad t \in \mathbb{T} . \tag{3.19}
\end{equation*}
$$

1. By Assumption 3.6[A6] and [45, Theorem 1.2], the distribution of $\bar{X}_{t}$ under $\mathbb{P}$ admits a density. Notice that $\overline{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, so the distribution of $\bar{X}_{t}$ under $\overline{\mathbb{P}}$ also admits a density. Because $\bar{X}_{0}$ is $\mathcal{F}_{0}$-measurable, we have for each $\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ that

$$
\overline{\mathbb{E}}\left[\varphi\left(\bar{X}_{0}\right)\right]=\mathbb{E}\left[\varphi\left(\bar{X}_{0}\right) R_{0}\right]=\mathbb{E}\left[\varphi\left(\bar{X}_{0}\right)\right] .
$$

It follows that the distribution of $\bar{X}_{0}$ under $\overline{\mathbb{P}}$ is also $\nu$. By weak uniqueness of (3.9) and (3.19), the distribution of $X_{t}$ under $\mathbb{P}$ is the same as that of $\bar{X}_{t}$ under $\overline{\mathbb{P}}$. As a result, the distribution of $X_{t}$ under $\mathbb{P}$ admits a density.
2. We combine the localization argument (see, for instance, [50, Theorem 9.1]) with Krylov's estimate.
(a) We fix $p \in[1, \infty)$ and $u \in[0, T)$ such that $\mathbb{E}\left[\left|X_{u}\right|^{p}\right]<\infty$. For $R>0$, let $\tau_{R}:=\inf \{t \in$ $\left.[u, T]:\left|X_{t}\right| \geq R\right\}$ be the exit time of $X_{t}$ from the open ball $B(0, R)$. We adopt the convention that $\tau_{R}=T$ if $\left|X_{t}\right|<R$ for all $t \in[u, T]$. We denote $X_{R}(t):=X_{t \wedge \tau_{R}}$. We have for $t \in[u, T]$ :

$$
\begin{aligned}
X_{R}(t)= & X_{u}+\int_{u}^{t \wedge \tau_{R}} b\left(r, X_{r}\right) \mathrm{d} r+\int_{u}^{t \wedge \tau_{R}} \sigma\left(r, X_{r}\right) \mathrm{d} B_{r} \\
= & X_{u}+\int_{u}^{t} b\left(r, X_{r}\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} r+\int_{u}^{t} \sigma\left(r, X_{r}\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} B_{r} \\
= & X_{u}+\int_{u}^{t} b^{(1)}\left(r, X_{R}(r)\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} r+\int_{u}^{t} b^{(0)}\left(r, X_{r}\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} r \\
& +\int_{u}^{t} \sigma\left(r, X_{R}(r)\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} B_{r} .
\end{aligned}
$$

By Hardy's inequality, we have for $n \in \mathbb{N}^{*}, p \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathbb{R}^{d}$ :

$$
\left|x_{1}+\cdots+x_{n}\right|^{p} \leq n^{p}\left(\left|x_{1}\right|^{p}+\cdots+\left|x_{n}\right|^{p}\right) .
$$

We write $M_{1} \lesssim M_{2}$ if there exists a constant $c>0$ (depending only on $\Theta_{3}, p$ ) such that $M_{1} \leq c M_{2}$. Then

$$
\begin{aligned}
& \mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{R}(s)\right|^{p}\right] \lesssim \mathbb{E}\left[\left|X_{u}\right|^{p}\right]+\mathbb{E}\left[\left(\int_{u}^{t} f_{0}\left(r, X_{r}\right) \mathrm{d} r\right)^{p}\right] \\
&+\mathbb{E}\left[\left(\int_{u}^{t}\left|b^{(1)}\left(r, X_{R}(r)\right)\right| \mathrm{d} r\right)^{p}\right] \\
&+\mathbb{E}\left[\sup _{s \in[u, t]}\left|\int_{u}^{s} \sigma\left(r, X_{R}(r)\right) 1_{\left\{r<\tau_{R}\right\}} \mathrm{d} B_{r}\right|^{p}\right] \\
&=: \mathbb{E}\left[\left|X_{u}\right|^{p}\right]+I_{1}+I_{2}+I_{3} .
\end{aligned}
$$

By Proposition 3.8[2], $I_{1} \lesssim 1$. We have $\left|b^{(1)}(r, x)\right| \lesssim 1+|x|$, so

$$
\begin{aligned}
I_{2} & \lesssim|t-u|^{p-1} \mathbb{E}\left[\int_{u}^{t}\left|b^{(1)}\left(r, X_{R}(r)\right)\right|^{p} \mathrm{~d} r\right] \quad \text { by Hölder's inequality } \\
& \lesssim|t-u|^{p-1} \mathbb{E}\left[\int_{u}^{t}\left(1+\left|X_{R}(r)\right|^{p}\right) \mathrm{d} r\right] \\
& \lesssim 1+|t-u|^{p-1} \mathbb{E}\left[\int_{u}^{t}\left|X_{R}(r)\right|^{p} \mathrm{~d} r\right]
\end{aligned}
$$

By Burkholder-Davis-Gundy inequality (see, for instance, [51, Theorem 19.20]) and the boundedness of $\sigma$ (from Assumption 3.6[A6]),

$$
I_{3} \lesssim \mathbb{E}\left[\left(\int_{u}^{t}\left|\sigma\left(r, X_{R}(r)\right)\right|^{2} \mathrm{~d} r\right)^{p / 2}\right] \lesssim|t-u|^{\frac{p}{2}}
$$

As a result,

$$
\begin{aligned}
\eta_{R}(t) & :=\mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{R}(s)\right|^{p}\right] \\
& \lesssim 1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]+\int_{u}^{t} \mathbb{E}\left[\left|X_{R}(r)\right|^{p}\right] \mathrm{d} r, \quad t \in[u, T] .
\end{aligned}
$$

By construction, $\left|X_{R}(s)\right| \leq\left|X_{u}\right| \vee R$ for $s \in[u, t]$. This implies $\eta_{R}(t) \leq \mathbb{E}\left[\left|X_{u}\right|^{p} \vee R^{p}\right]<\infty$ for $t \in[u, T]$. We have

$$
\eta_{R}(t) \lesssim 1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]+\int_{u}^{t} \eta_{R}(r) \mathrm{d} r, \quad t \in[u, T] .
$$

By Grönwall's lemma,

$$
\eta_{R}(t) \lesssim 1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right], \quad t \in[u, T], R>0 .
$$

Because $X$ has continuous sample paths, $\tau_{R} \uparrow T$ a.s. as $R \rightarrow \infty$. Hence

$$
\sup _{s \in[u, t]}\left|X_{R}(s)\right|^{p} \uparrow \sup _{s \in[u, t]}\left|X_{s}\right|^{p} \quad \text { a.s. as } R \rightarrow \infty
$$

The estimate (3.15) then follows from an application of monotone convergence theorem.
(b) We have

$$
\begin{aligned}
\mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{s}-X_{u}\right|^{p}\right] \lesssim \mathbb{E}[ & \left.\left(\int_{u}^{t} f_{0}\left(r, X_{r}\right) \mathrm{d} r\right)^{p}\right] \\
& +\mathbb{E}\left[\left(\int_{u}^{t}\left|b^{(1)}\left(r, X_{r}\right)\right| \mathrm{d} r\right)^{p}\right] \\
& +\mathbb{E}\left[\sup _{s \in[u, t]}\left|\int_{u}^{s} \sigma\left(r, X_{r}\right) \mathrm{d} B_{r}\right|^{p}\right] \\
=: & J_{1}+J_{2}+J_{3} .
\end{aligned}
$$

There exists $\bar{q}_{0} \in\left(2, q_{0}\right)$ such that $\left(p_{0}, \bar{q}_{0}\right) \in \bar{K}$. Let $\delta:=\frac{1}{\bar{q}_{0}}-\frac{1}{q_{0}} \in\left(0, \frac{1}{2}\right)$. By Hölder's inequality,

$$
\left\|f_{0}\right\|_{\tilde{L}_{\tilde{q}_{0}}^{p_{0}}(u, t)} \leq(t-u)^{\delta}\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}(u, t)} .
$$

By Proposition 3.8[2],

$$
J_{1} \lesssim\left\|f_{0}\right\|_{\tilde{\tilde{q}}_{\tilde{q}_{0}}^{p_{0}}(u, t)}^{p} \leq(t-u)^{\delta p}\left\|f_{0}\right\|_{\tilde{L}_{\tilde{q}_{0}}^{p_{0}}(u, t)}^{p} \lesssim(t-u)^{\delta p}
$$

As for $I_{2}$ and $I_{3}$, we get

$$
\begin{aligned}
J_{2} & \lesssim|t-u|^{p-1} \mathbb{E}\left[\int_{u}^{t}\left(1+\left|X_{r}\right|^{p}\right) \mathrm{d} r\right] \\
& \lesssim|t-u|^{p}\left(1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]\right) \quad \text { by }(3.15), \\
J_{3} & \lesssim|t-u|^{\frac{p}{2}} .
\end{aligned}
$$

It follows that

$$
\mathbb{E}\left[\sup _{s \in[u, t]}\left|X_{s}-X_{u}\right|^{p}\right] \lesssim|t-u|^{\delta p}\left(1+\mathbb{E}\left[\left|X_{u}\right|^{p}\right]\right)
$$

The estimate (3.16) then follows.
(c) We have

$$
\begin{aligned}
\sup _{s \in[u, t]} \int_{B_{R}^{c}}|x|^{p} \mathrm{~d} \mu_{s}(x) & \leq \frac{1}{R} \sup _{s \in[u, t]} \int_{\mathbb{R}^{d}}|x|^{p+\alpha} \mathrm{d} \mu_{s}(x) \quad \text { by Markov's inequality } \\
& \lesssim \frac{1+\mathbb{E}\left[\left|X_{u}\right|^{p+\alpha}\right]}{R} \text { by (3.15). }
\end{aligned}
$$

The estimate (3.17) then follows. This completes the proof.

Third, we recall a Duhamel presentation and an $L^{p}$ estimate of marginal density:
Theorem 3.10. [35, Lemma 4.1] Let $(b, \sigma)$ satisfy Assumption 3.6. Let $\ell_{t}$ be the p.d.f. of $X_{t}$ in (3.9). Then

1. Let $v: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ be another drift such that $(v, \sigma)$ satisfies Assumption 3.3. We assume there exists $1 \leq g \in \tilde{L}_{\bar{q}}^{\bar{p}}$ with $(\bar{p}, \bar{q}) \in \overline{\mathcal{K}}$ such that $\left|b_{t}(x)-v_{t}(x)\right| \leq g(t, x)$ for $t \in \mathbb{T}$ and $x \in \mathbb{R}^{d}$. Then we have for $t \in \mathbb{T}$ and $x \in \mathbb{R}^{d}$ :

$$
\begin{aligned}
\ell_{t}(x)= & \int_{\mathbb{R}^{d}} p_{0, t}^{v, \sigma}(y, x) \ell_{\nu}(y) \mathrm{d} y \\
& +\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{s}(y)\left\langle b_{s}(y)-v_{s}(y), \nabla_{y} p_{s, t}^{v, \sigma}(y, x)\right\rangle \mathrm{d} y \mathrm{~d} s .
\end{aligned}
$$

2. There exists a constant $c>0$ (depending only on $\Theta_{3}$ ) such that

$$
\sup _{t \in \mathbb{T}}\left\|\ell_{t}\right\|_{\tilde{L}^{k}} \leq c\left\|\ell_{\nu}\right\|_{\tilde{L}^{k}}, \quad k \in\left[p_{0}^{*}, \infty\right] .
$$

We remind that $p_{0}$ is a parameter in $\Theta_{3}$. Also, $p_{0}^{*}$ is the Hölder conjugates of $p_{0}$, i.e., $1=\frac{1}{p_{0}}+\frac{1}{p_{0}^{*}}$. Although the statement of Theorem 3.10[1] does not appear in [35], its proof is contained in part (c) of the proof of [35, Lemma 4.1].

### 3.4 Hölder estimates of marginal density

For Hölder continuity of $\ell_{t}$, we need another set of stronger assumptions:
Assumption 3.11. There exist constants $\beta \in(0,1)$ and $C>0$ such that for $t \in \mathbb{T}$ and $x \in \mathbb{R}^{d}$ :
(A1) $\nu \in \mathcal{P}_{1}\left(\mathbb{R}^{d}\right)$ has a density $\ell_{\nu}$.
(A2) $a_{t}$ is invertible and $\sigma_{t}$ is weakly differentiable.
(A3) There exists $1 \leq f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$ with $\left(p_{0}, q_{0}\right) \in \mathcal{K}$ such that $\left|b_{t}(x)\right| \leq f_{0}(t, x)$.
(A4) The following conditions hold:

$$
\begin{aligned}
& \left\|\nabla \sigma_{t}\right\|_{\infty}+\left\|\sigma_{t}\right\|_{\infty}+\left\|a_{t}^{-1}\right\|_{\infty} \leq C, \\
& \left|\nabla \sigma_{t}(x)-\nabla \sigma_{t}(y)\right| \leq C\left(|x-y|+|x-y|^{\beta}\right) .
\end{aligned}
$$

We gather parameters in Assumption 3.11:

$$
\Theta_{4}:=\left(d, T, \beta, C, p_{0}, q_{0}, f_{0}\right)
$$

Assumption 3.11 is stronger than Assumption 3.6 in two aspects: $b$ is not allowed to have linear growth and $\nabla \sigma_{t}$ is uniformly bounded and uniformly continuous (in space). Assumption 3.11[A4] implies for $t \in \mathbb{T}, \gamma \in(0,1)$ and $x, y \in \mathbb{R}^{d}$ that $\left|\sigma_{t}(x)-\sigma_{t}(y)\right| \leq 2 C|x-y|^{\gamma}$.

First, we recall an estimate about Hölder continuity in space:
Theorem 3.12. [35, Lemma 5.1] Let $(b, \sigma)$ satisfy Assumption 3.11 and $\alpha \in\left(0,1-\frac{d}{p_{0}}-\frac{2}{q_{0}}\right)$. Let $\ell_{t}$ be the p.d.f. of $X_{t}$ in (3.9). There exists a constant $c>0$ (depending only on $\Theta_{4}, \alpha$ ) such that

$$
\sup _{t \in \mathbb{T}}\left\|\ell_{t}\right\|_{C_{b}^{\alpha}} \leq c\left\|\ell_{\nu}\right\|_{C_{b}^{\alpha}} .
$$

Second, we define $\bar{b}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ by

$$
\begin{equation*}
\left(\bar{b}_{t}\right)^{i}:=\frac{1}{2} \sum_{j=1}^{d} \frac{\partial\left(a_{t}\right)^{i, j}}{\partial x_{j}}, \quad i \in\{1,2, \ldots, d\} \tag{3.20}
\end{equation*}
$$

where

1. $\left(\bar{b}_{t}\right)^{i}$ is the entry in the $i$-th row of $\bar{b}_{t}$.
2. $\left(a_{t}\right)^{i, j}$ is the entry in the $i$-th row and $j$-th column of $a_{t}$.

By construction, $\bar{b}$ depends only on $\sigma$. By [52, Equation 1.6], $p_{s, t}^{\bar{b}, \sigma}$ is symmetric, i.e., $p_{s, t}^{\bar{b}, \sigma}(x, y)=$ $p_{s, t}^{\bar{b}, \sigma}(y, x)$. Clearly, if $(b, \sigma)$ satisfies Assumption 3.11, then $(\bar{b}, \sigma)$ satisfies both Assumption 3.3 and Assumption 3.11. We apply Theorem $3.10[1]$ to the pair $(b, \sigma)$ and $v=\bar{b}$. As a consequence, we get

Corollary 3.13. Let $(b, \sigma)$ satisfy Assumption 3.11. Let $\ell_{t}$ be the p.d.f. of $X_{t}$ in (3.9). Let $\bar{b}$ be defined by (3.20). We have for $t \in \mathbb{T}$ and $x \in \mathbb{R}^{d}$ :

$$
\ell_{t}(x)=P_{0, t}^{\bar{b}, \sigma} \ell_{\nu}(x)+\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{s}(y)\left\langle b_{s}(y)-\bar{b}_{s}(y), \nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right\rangle \mathrm{d} y \mathrm{~d} s,
$$

where $P_{0, t}^{\bar{b}, \sigma} \ell_{\nu}$ is defined by (3.2).

Third, we establish an essential estimate about Hölder continuity in time:
Theorem 3.14. Let $(b, \sigma)$ satisfy Assumption 3.11 and $\ell_{\nu} \in C_{b}^{\alpha}\left(\mathbb{R}^{d}\right)$ for some $\alpha \in\left(0,1-\frac{d}{p_{0}}-\frac{2}{q_{0}}\right)$. Let $\ell_{t}$ be the p.d.f. of $X_{t}$ in (3.9). Then there exist a constant $\delta \in\left(0, \frac{1}{2}\right)$ (depending only on $\left.d, p_{0}, q_{0}, \alpha\right)$ and a constant $c>0$ (depending only on $\Theta_{4}, \alpha, \nu$ ) such that

$$
\left\|\ell_{t}-\ell_{s}\right\|_{\infty} \leq c|t-s|^{\delta}, \quad s, t \in \mathbb{T}
$$

We remind that the constants $C, p_{0}, q_{0}$ are parameters in $\Theta_{4}$. Theorem 3.12 and Theorem 3.14 are prerequisite for applying Arzelà-Ascoli theorem in Section 4.2. The proof of Theorem 3.14 is by extending an idea from [53, Corollary 2.5 (iii)].

Proof. We write $M_{1} \lesssim M_{2}$ if there exists a constant $c>0$ (depending only on $\left.\Theta_{4}, \alpha, \nu\right)$ such that $M_{1} \leq c M_{2}$. Let $\bar{b}$ be defined by (3.20). Recall that $(\bar{b}, \sigma)$ satisfies Assumption 3.3. By Theorem 3.4,
(a) There exists a constant $\bar{\kappa}>0$ (depending only on $\Theta_{4}$ ) such that for $i \in\{0,1\}, 0 \leq s<t \leq T$ and $x, y \in \mathbb{R}^{d}$ :

$$
\begin{equation*}
\left|\nabla_{y}^{i} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \lesssim(t-s)^{-\frac{i}{2}} p_{t-s}^{\bar{\kappa}}(y-x) \tag{3.21}
\end{equation*}
$$

(b) There exist a constant $\kappa>0$ (depending only on $\left.\Theta_{4}, \alpha\right)$ and a family $\left(\psi_{s, t}\right)_{0 \leq s<t \leq T}$ of $C^{\infty}$ diffeomorphisms on $\mathbb{R}^{d}$ such that for $0 \leq s<t \leq T$ and $x, y, z \in \mathbb{R}^{d}$ :

$$
\begin{align*}
&\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, z)-\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \lesssim|z-x|^{\alpha}(t-s)^{-\frac{1+\alpha}{2}}\left\{p_{t-s}^{\kappa}\left(\psi_{s, t}(y)-z\right)\right.  \tag{3.22}\\
&\left.+p_{t-s}^{\kappa}\left(\psi_{s, t}(y)-x\right)\right\}
\end{align*}
$$

Recall that $p_{r, t}^{\bar{b}, \sigma}$ is symmetric, so

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z=\int_{\mathbb{R}^{d}} p_{s, t}^{\bar{b}, \sigma}(x, z) \mathrm{d} z=1 \tag{3.23}
\end{equation*}
$$

WLOG, we assume $s<t$. We have for $i \in\{0,1\}$ that

$$
\begin{align*}
& \nabla_{y}^{i} p_{r, t}^{\bar{b}, \sigma}(y, x)-\nabla_{y}^{i} p_{r, s}^{\bar{b}, \sigma}(y, x) \\
= & \nabla_{y}^{i} \int_{\mathbb{R}^{d}} p_{r, s}^{\bar{b}, \sigma}(y, z) p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z-\nabla_{y}^{i} p_{r, s}^{\bar{b}, \sigma}(y, x)  \tag{3.24}\\
= & \nabla_{y}^{i} \int_{\mathbb{R}^{d}} p_{r, s}^{\bar{b}, \sigma}(y, z) p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z-\nabla_{y}^{i} \int_{\mathbb{R}^{d}} p_{r, s}^{\bar{b}, \sigma}(y, x) p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z  \tag{3.25}\\
= & \int_{\mathbb{R}^{d}}\left\{\nabla_{y}^{i} p_{r, s}^{\bar{b}, \sigma}(y, z)-\nabla_{y}^{i} p_{r, s}^{\bar{b}, \sigma}(y, x)\right\} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \tag{3.26}
\end{align*}
$$

Above, (3.24) is due to Chapman-Kolmogorov equation, (3.25) due to (3.23), and (3.26) due to Leibniz integral rule. Recall that $|\bar{b}| \lesssim 1 \leq f_{0}$ and $|b| \leq f_{0}$, so

$$
\begin{equation*}
|b-\bar{b}| \lesssim f_{0} \tag{3.27}
\end{equation*}
$$

By Corollary 3.13,

$$
\begin{align*}
& \ell_{t}(x)=P_{0, t}^{\bar{b}, \sigma} \ell_{\nu}(x)+\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{r}(y)\left\langle b_{r}(y)-\bar{b}_{r}(y), \nabla_{y} p_{r, t}^{\bar{b}, \sigma}(y, x)\right\rangle \mathrm{d} y \mathrm{~d} r \\
& \ell_{s}(x)=P_{0, s}^{\bar{b}, \sigma} \ell_{\nu}(x)+\int_{0}^{s} \int_{\mathbb{R}^{d}} \ell_{r}(y)\left\langle b_{r}(y)-\bar{b}_{r}(y), \nabla_{y} p_{r, s}^{\bar{b}, \sigma}(y, x)\right\rangle \mathrm{d} y \mathrm{~d} r \tag{3.28}
\end{align*}
$$

By (3.27) and (3.28),

$$
\begin{aligned}
\left|\ell_{t}(x)-\ell_{s}(x)\right| \lesssim & \left|\int_{\mathbb{R}^{d}} \ell_{\nu}(y)\left\{p_{0, t}^{\bar{b}, \sigma}(y, x)-p_{0, s}^{\bar{b}, \sigma}(y, x)\right\} \mathrm{d} y\right| \\
& +\int_{0}^{s} \int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y)\left|\nabla_{y} p_{r, t}^{\bar{b}, \sigma}(y, x)-\nabla_{y} p_{r, s}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} r \\
& \quad+\int_{s}^{t} \int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y)\left|\nabla_{y} p_{r, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} r \\
= & I_{1}(x)+I_{2}(x)+I_{3}(x)
\end{aligned}
$$

We are going to upper bound $I_{1}(x), I_{2}(x)$ and $I_{3}(x)$ separately.

1. We consider the SDE

$$
\mathrm{d} Y_{t}=\bar{b}\left(t, Y_{t}\right) \mathrm{d} t+\sigma\left(t, Y_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}
$$

where the distribution of $Y_{0}$ is $\nu$. Recall that $(\bar{b}, \sigma)$ satisfies Assumption 3.11. Let $\bar{\ell}_{t}$ be the p.d.f. of $Y_{t}$. Then

$$
\begin{align*}
\left|\int_{\mathbb{R}^{d}} \ell_{\nu}(y)\left\{p_{0, s}^{\bar{b}, \sigma}(y, z)-p_{0, s}^{\bar{b}, \sigma}(y, x)\right\} \mathrm{d} y\right| & =\left|\bar{\ell}_{s}(z)-\bar{\ell}_{s}(x)\right| \\
& \lesssim|z-x|^{\alpha} \quad \text { by Theorem 3.12. } \tag{3.29}
\end{align*}
$$

By (3.26) with $i=r=0$,

$$
\begin{aligned}
I_{1}(x) & \leq \int_{\mathbb{R}^{d}}\left|\int_{\mathbb{R}^{d}} \ell_{\nu}(y)\left\{p_{0, s}^{\bar{b}, \sigma}(y, z)-p_{0, s}^{\bar{b}, \sigma}(y, x)\right\} \mathrm{d} y\right| p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \\
& \lesssim \int_{\mathbb{R}^{d}}|z-x|^{\alpha} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \quad \text { by }(3.29) \\
& =: J(x) \\
& \lesssim \int_{\mathbb{R}^{d}}|z-x|^{\alpha} p_{t-s}^{\bar{\kappa}}(z-x) \mathrm{d} z \quad \text { by }(3.21) \\
& =\int_{\mathbb{R}^{d}}|z|^{\alpha} p_{t-s}^{\bar{\kappa}}(z) \mathrm{d} z \\
& \lesssim|t-s|^{\frac{\alpha}{2}}
\end{aligned}
$$

2. By (3.26) with $i=1$ and (3.22),

$$
\begin{align*}
& I_{2}(x) \lesssim \int_{0}^{s}(s-r)^{-\frac{1+\alpha}{2}} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y)|z-x|^{\alpha}\left\{p_{s-r}^{\kappa}\left(\psi_{r, s}(y)-z\right)\right.  \tag{3.30}\\
&\left.+p_{s-r}^{\kappa}\left(\psi_{r, s}(y)-x\right)\right\} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \mathrm{~d} y \mathrm{~d} r
\end{align*}
$$

By Theorem 3.10[2],

$$
\begin{equation*}
\sup _{r \in \mathbb{T}}\left\|\ell_{r}\right\|_{\infty} \lesssim\left\|\ell_{\nu}\right\|_{\infty} \tag{3.31}
\end{equation*}
$$

By Lemma 3.5 with $(p, \bar{p})=\left(p_{0}, \infty\right)$,

$$
\begin{equation*}
\left\|\tilde{P}_{r, s}^{\kappa}\left\{(s-r)^{\frac{d}{2 p_{0}}} f_{0}(r, \cdot)\right\}\right\|_{\infty} \lesssim\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} . \tag{3.32}
\end{equation*}
$$

Then

$$
\begin{align*}
& \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y)|z-x|^{\alpha} p_{s-r}^{\kappa}\left(\psi_{r, s}(y)-z\right) p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \mathrm{~d} y \\
\lesssim & \int_{\mathbb{R}^{d}}\left(\int_{\mathbb{R}^{d}} f_{0}(r, y) p_{s-r}^{\kappa}\left(\psi_{r, s}(y)-z\right) \mathrm{d} y\right)|z-x|^{\alpha} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \quad \text { by } \quad(3.31) \\
= & (s-r)^{-\frac{d}{2 p_{0}}} \int_{\mathbb{R}^{d}}|z-x|^{\alpha} p_{s, t}^{\bar{b}, \sigma}(z, x) \tilde{P}_{r, s}^{\kappa}\left\{(s-r)^{\frac{d}{2 p_{0}}} f_{0}(r, \cdot)\right\}(z) \mathrm{d} z \quad \text { by }(3.4) \\
\lesssim & (s-r)^{-\frac{d}{2 p_{0}}}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} \int_{\mathbb{R}^{d}}|z-x|^{\alpha} p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \quad \text { by }(3.32) \\
= & (s-r)^{-\frac{d}{2 p_{0}}}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} J(x) . \tag{3.33}
\end{align*}
$$

Similarly,

$$
\begin{align*}
& \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y)|z-x|^{\alpha} p_{s-r}^{\kappa}\left(\psi_{r, s}(y)-x\right) p_{s, t}^{\bar{b}, \sigma}(z, x) \mathrm{d} z \mathrm{~d} y  \tag{3.34}\\
\lesssim & (s-r)^{-\frac{d}{2 p_{0}}}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} J(x) .
\end{align*}
$$

It follows from (3.30), (3.33) and (3.34) that

$$
I_{2}(x) \lesssim J(x) \int_{0}^{s}(s-r)^{-\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} \mathrm{~d} r .
$$

By Hölder's inequality,

$$
\begin{aligned}
& \int_{0}^{s}(s-r)^{-\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} \mathrm{~d} r \\
\leq & \left(\int_{0}^{s}(s-r)^{-q_{0}^{*}\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)} \mathrm{d} r\right)^{\frac{1}{q_{0}^{*}}}\left(\int_{0}^{s}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}}^{q_{0}} \mathrm{~d} r\right)^{\frac{1}{q_{0}}} \\
= & \left(\int_{0}^{s}(s-r)^{-q_{0}^{*}\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)} \mathrm{d} r\right)^{\frac{1}{q_{0}^{*}}}\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}(s)} .
\end{aligned}
$$

Above, $q_{0}^{*} \in[1,2)$ is the Hölder conjugate of $q_{0}$, i.e., $\frac{1}{q_{0}}+\frac{1}{q_{0}^{*}}=1$. Notice that $\alpha \in\left(0,1-\frac{d}{p_{0}}-\frac{2}{q_{0}}\right)$ implies $q_{0}^{*}\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)<1$ and thus

$$
\sup _{s \in \mathbb{T}} \int_{0}^{s}(s-r)^{-q_{0}^{*}\left(\frac{1+\alpha}{2}+\frac{d}{2 p_{0}}\right)} \mathrm{d} r<\infty .
$$

Recall from (1) that $J(x) \lesssim|t-s|^{\frac{\alpha}{2}}$, so $I_{2}(x) \lesssim|t-s|^{\frac{\alpha}{2}}$.
3. By Lemma 3.5 with $(p, \bar{p})=\left(p_{0}, \infty\right)$,

$$
\begin{equation*}
\left\|P_{t-r}^{\bar{\kappa}}\left\{(t-r)^{\frac{d}{p_{0}}} f_{0}(r, \cdot)\right\}\right\|_{\infty} \lesssim\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} \tag{3.35}
\end{equation*}
$$

We have

$$
\begin{aligned}
I_{3}(x) & \lesssim \int_{s}^{t}(t-r)^{-\frac{1}{2}}\left(\int_{\mathbb{R}^{d}} \ell_{r}(y) f_{0}(r, y) p_{t-r}^{\bar{\kappa}}(y-x) \mathrm{d} y\right) \mathrm{d} r \quad \text { by }(3.21) \\
& \lesssim \int_{s}^{t}(t-r)^{-\frac{1}{2}}\left(\int_{\mathbb{R}^{d}} f_{0}(r, y) p_{t-r}^{\bar{\kappa}}(y-x) \mathrm{d} y\right) \mathrm{d} r \quad \text { by }(3.31) \\
& =\int_{s}^{t}(t-r)^{-\left(\frac{1}{2}+\frac{d}{2 p_{0}}\right)} P_{t-r}^{\bar{\kappa}}\left\{(t-r)^{\frac{d}{2 p_{0}}} f_{0}(r, \cdot)\right\}(x) \mathrm{d} r \quad \text { by }(3.3) \\
& \lesssim \int_{s}^{t}(t-r)^{-\left(\frac{1}{2}+\frac{d}{2 p_{0}}\right)}\left\|f_{0}(r, \cdot)\right\|_{\tilde{L}^{p_{0}}} \mathrm{~d} r \quad \text { by }(3.35) \\
& \leq\left(\int_{s}^{t}(t-r)^{-q_{0}^{*}\left(\frac{1}{2}+\frac{d}{2 p_{0}}\right)} \mathrm{d} r\right)^{\frac{1}{q_{0}^{*}}}\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}(s, t)} \quad \text { by Hölder's inequality. }
\end{aligned}
$$

Notice that $\left(p_{0}, q_{0}\right) \in \mathcal{K}$ implies $\varepsilon:=q_{0}^{*}\left(\frac{1}{2}+\frac{d}{2 p_{0}}\right) \in(0,1)$. Then

$$
\int_{s}^{t}(t-r)^{-\varepsilon} \mathrm{d} r=\frac{(t-s)^{1-\varepsilon}}{1-\varepsilon} .
$$

It follows that

$$
I_{3}(x) \lesssim|t-s|^{\frac{1-\varepsilon}{q_{0}^{*}}} .
$$

The claim then follows by picking

$$
\delta:=\frac{\alpha}{2} \wedge \frac{1-\varepsilon}{q_{0}^{*}} .
$$

## 4 Proof of Theorem 2.2

We recall that $\Theta_{1}=\left(d, T, \alpha, \beta, C, p_{0}, q_{0}, f_{0}, p\right)$ contains the parameters in Assumption 2.1. We write $M_{1} \lesssim M_{2}$ if there exists a constant $c>0$ (depending only on $\left.\Theta_{1}, \nu\right)$ such that $M_{1} \leq c M_{2}$. We construct a sequence ( $\rho^{n}$ ) of mollifiers as follows. We fix a smooth p.d.f. $\rho: \mathbb{R}^{d} \rightarrow \mathbb{R}$ whose support is contained in the unit ball of $\mathbb{R}^{d}$. For each $n \in \mathbb{N}^{*}$, we define $\rho^{n}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by $\rho^{n}(x):=n^{d} \rho(n x)$ and consider the SDE

$$
\begin{equation*}
\mathrm{d} X_{t}^{n}=b\left(t, X_{t}^{n},\left(\rho^{n} * \mu_{t}^{n}\right)\left(X_{t}^{n}\right), \mu_{t}^{n}\right) \mathrm{d} t+\sigma\left(t, X_{t}^{n}, \mu_{t}^{n}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{4.1}
\end{equation*}
$$

where the distribution of $X_{0}^{n}$ is $\nu$ and that of $X_{t}^{n}$ is $\mu_{t}^{n}$. Above, $*$ is the convolution operator, i.e.,

$$
\left(\rho^{n} * \mu_{t}^{n}\right)(x):=\int_{\mathbb{R}^{d}} \rho^{n}(x-y) \mathrm{d} \mu_{t}^{n}(y) .
$$

Clearly, (4.1) is the mollified version of (1.1).

### 4.1 Stability estimates for mollified SDEs

We define the map $b^{n}: \mathbb{T} \times \mathbb{R}^{d} \times \mathcal{P}_{p}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d}$ by $b^{n}(t, x, \mu):=b\left(t, x,\left(\rho^{n} * \mu\right)(x), \mu\right)$. Then

$$
\begin{align*}
\left|b^{n}(t, x, \mu)-b^{n}(t, x, \tilde{\mu})\right| & \lesssim\left|\int_{\mathbb{R}^{d}} \rho^{n}(x-y) \mathrm{d}(\mu-\tilde{\mu})(y)\right|+W_{p}(\mu, \tilde{\mu})  \tag{4.2}\\
& \leq\left\|\nabla \rho^{n}\right\|_{\infty} W_{1}(\mu, \tilde{\mu})+W_{p}(\mu, \tilde{\mu})  \tag{4.3}\\
& \leq\left(1+\left\|\nabla \rho^{n}\right\|_{\infty}\right) W_{p}(\mu, \tilde{\mu}) . \tag{4.4}
\end{align*}
$$

Above, (4.2) is due to Assumption 2.1[A4], (4.3) due to Lemma 3.1[2], and (4.4) due to Lemma 3.1[3]. It follows that $b^{n}$ is Lipschitz in distribution. We consider the McKean-Vlasov SDE

$$
\begin{equation*}
\mathrm{d} Y_{t}=b^{n}\left(t, Y_{t}, \xi_{t}\right) \mathrm{d} t+\sigma\left(t, Y_{t}, \xi_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{4.5}
\end{equation*}
$$

where the distribution of $Y_{0}$ is $\nu$ and that of $Y_{t}$ is $\xi_{t}$. It follows from [10, Theorem 1.1(1)] that (4.5) is well-posed. Clearly, $\left(X_{t}^{n}, t \in \mathbb{T}\right)$ satisfies (4.5). As a consequence, (4.1) is well-posed. We define the maps $\bar{b}^{n}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\bar{\sigma}^{n}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{m}$ by

$$
\begin{aligned}
& \bar{b}^{n}(t, x):=b^{n}\left(t, x, \mu_{t}^{n}\right), \\
& \bar{\sigma}^{n}(t, x):=\sigma\left(t, x, \mu_{t}^{n}\right) .
\end{aligned}
$$

Clearly,

$$
\begin{equation*}
\mathrm{d} X_{t}^{n}=\bar{b}^{n}\left(t, X_{t}^{n}\right) \mathrm{d} t+\bar{\sigma}^{n}\left(t, X_{t}^{n}\right) \mathrm{d} B_{t} . \tag{4.6}
\end{equation*}
$$

Let $\bar{a}^{n}:=\bar{\sigma}^{n}\left(\bar{\sigma}^{n}\right)^{\top}$. We denote $\bar{b}_{t}^{n}:=\bar{b}^{n}(t, \cdot), \bar{\sigma}_{t}^{n}:=\bar{\sigma}^{n}(t, \cdot)$ and $\bar{a}_{t}^{n}:=\bar{a}^{n}(t, \cdot)$.
Remark 4.1. We emphasize that every pair $\left(\bar{b}^{n}, \bar{\sigma}^{n}\right)$ satisfies Assumption 3.11 for the same set of parameters. More precisely, we have for $n \in \mathbb{N}^{*}, t \in \mathbb{T}$ and $x, y \in \mathbb{R}^{d}$ :

$$
\begin{gathered}
\left|\bar{b}_{t}^{n}(x)\right| \leq f_{0}(t, x) \\
\left\|\nabla \bar{\sigma}_{t}^{n}\right\|_{\infty}+\left\|\bar{\sigma}_{t}^{n}\right\|_{\infty}+\left\|\left(\bar{a}_{t}^{n}\right)^{-1}\right\|_{\infty} \leq C \\
\left|\nabla \bar{\sigma}_{t}^{n}(x)-\nabla \bar{\sigma}_{t}^{n}(y)\right| \leq C\left(|x-y|+|x-y|^{\beta}\right) .
\end{gathered}
$$

By Theorem 3.9[1], each $X_{t}^{n}$ admits a p.d.f. denoted by $\ell_{t}^{n}$. By Theorem 3.12,

$$
\begin{equation*}
\sup _{n \in \mathbb{N}^{*}} \sup _{t \in \mathbb{T}}\left\|\ell_{t}^{n}\right\|_{C_{b}^{\alpha}} \lesssim 1 \tag{4.7}
\end{equation*}
$$

By Theorem 3.14, there exists a constant $\delta_{1} \in\left(0, \frac{1}{2}\right)$ (depending only on $\left.d, p_{0}, q_{0}, \alpha\right)$ such that

$$
\begin{equation*}
\sup _{n \in \mathbb{N}^{*}}\left\|\ell_{t}^{n}-\ell_{s}^{n}\right\|_{\infty} \lesssim|t-s|^{\delta_{1}}, \quad s, t \in \mathbb{T} \tag{4.8}
\end{equation*}
$$

By Theorem 3.9[2], there exists a constant $\delta_{2} \in\left(0, \frac{1}{2}\right)$ (depending only on $q_{0}$ ) such that

$$
\begin{align*}
\sup _{n \in \mathbb{N}^{*}} \sup _{t \in \mathbb{T}} M_{p}\left(\mu_{t}^{n}\right) & \lesssim 1,  \tag{4.9}\\
\sup _{n \in \mathbb{N}^{*}} W_{p}\left(\mu_{s}^{n}, \mu_{t}^{n}\right) & \lesssim|t-s|^{\delta_{2}}, \quad s, t \in \mathbb{T},  \tag{4.10}\\
\sup _{n \in \mathbb{N}^{*}} \sup _{t \in \mathbb{T}} \int_{B_{R}^{c}}|x|^{p} \mathrm{~d} \mu_{t}^{n}(x) & \lesssim \frac{1}{R},  \tag{4.11}\\
\sup _{n \in \mathbb{N}^{*}} \sup _{t \in \mathbb{T}} \mu_{t}^{n}\left(B_{R}^{c}\right) & \lesssim \frac{1}{R}, \quad R>0 . \tag{4.12}
\end{align*}
$$

By (4.10), the map $\mathbb{T} \rightarrow \mathcal{P}_{p}\left(\mathbb{R}^{d}\right), t \mapsto \mu_{t}^{n}$ is continuous.

### 4.2 Convergence of marginal densities of mollified SDEs

By (4.7), (4.8) and Arzelà-Ascoli theorem, there exist a sub-sequence (also denoted by ( $\ell^{n}$ ) for simplicity) and a continuous function $\ell: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}_{+}$such that

$$
\begin{equation*}
\lim _{n} \sup _{t \in \mathbb{T}} \sup _{x \in B(0, R)}\left|\ell_{t}^{n}(x)-\ell_{t}(x)\right|=0, \quad R>0 \tag{4.13}
\end{equation*}
$$

where $\ell_{t}:=\ell(t, \cdot)$. Clearly, $\ell_{0}=\ell_{\nu}$. Taking the limit $n \rightarrow \infty$ in (4.7) and (4.8), we get

$$
\begin{align*}
& \sup _{t \in \mathbb{T}}\left\|\ell_{t}\right\|_{C_{b}^{\alpha}} \lesssim\left\|\ell_{\nu}\right\|_{C_{b}^{\alpha}},  \tag{4.14}\\
& \left\|\ell_{t}-\ell_{s}\right\|_{\infty} \lesssim|t-s|^{\delta_{1}}, \quad s, t \in \mathbb{T} .
\end{align*}
$$

We remark that the constant in (4.14) depends only on $\Theta_{1}$. Next we verify that $\ell_{t}$ is indeed a p.d.f. for $t \in \mathbb{T}$. We have

$$
\begin{aligned}
\int_{B(0, R)} \ell_{t}^{n}(x) \mathrm{d} x & =1-\int_{B_{R}^{c}} \ell_{t}^{n}(x) \mathrm{d} x \\
& \gtrsim 1-\frac{1}{R} \quad \text { by }(4.12) .
\end{aligned}
$$

By (4.13), (4.14) and dominated convergence theorem (DCT),

$$
\int_{B(0, R)} \ell_{t}(x) \mathrm{d} x=\lim _{n \rightarrow \infty} \int_{B(0, R)} \ell_{t}^{n}(x) \mathrm{d} x .
$$

It follows that

$$
1-\frac{1}{R} \lesssim \int_{B(0, R)} \ell_{t}(x) \mathrm{d} x \leq 1,
$$

which implies

$$
\int_{\mathbb{R}^{d}} \ell_{t}(x) \mathrm{d} x=\lim _{R \rightarrow \infty} \int_{B(0, R)} \ell_{t}(x) \mathrm{d} x=1
$$

Let $\mu_{t} \in \mathcal{P}\left(\mathbb{R}^{d}\right)$ be the probability measure induced by $\ell_{t}$, i.e.,

$$
\mu_{t}(B):=\int_{B} \ell_{t}(x) \mathrm{d} x, \quad B \in \mathcal{B}\left(\mathbb{R}^{d}\right)
$$

Lemma 4.2. We have for each $t \in \mathbb{T}$ that $\mu_{t}^{n} \rightharpoonup \mu_{t}$ as $n \rightarrow \infty$.
Proof. By (4.13),

$$
\begin{equation*}
\mu_{t}^{n} \stackrel{*}{\rightharpoonup} \mu_{t} \quad \text { as } \quad n \rightarrow \infty . \tag{4.15}
\end{equation*}
$$

Let $f \in C_{b}\left(\mathbb{R}^{d}\right)$ and $g \in C_{c}\left(\mathbb{R}^{d}\right)$ such that $0 \leq g \leq 1$. Then $g f \in C_{c}\left(\mathbb{R}^{d}\right)$ and $f=(1-g) f+g f$. We have

$$
\left|\int_{\mathbb{R}^{d}} f \mathrm{~d}\left(\mu_{t}^{n}-\mu_{t}\right)\right| \leq\|f\|_{\infty} \int_{\mathbb{R}^{d}}(1-g) \mathrm{d}\left(\mu_{t}^{n}+\mu_{t}\right)+\left|\int_{\mathbb{R}^{d}} g f \mathrm{~d}\left(\mu_{t}^{n}-\mu_{t}\right)\right| .
$$

By (4.15),

$$
\limsup \int_{\mathbb{R}^{d}} g f \mathrm{~d}\left(\mu_{t}^{n}-\mu_{t}\right)=0,
$$

which implies

$$
\limsup _{n}\left|\int_{\mathbb{R}^{d}} f \mathrm{~d}\left(\mu_{t}^{n}-\mu_{t}\right)\right| \leq\|f\|_{\infty} \limsup _{n} \int_{\mathbb{R}^{d}}(1-g) \mathrm{d}\left(\mu_{t}^{n}+\mu_{t}\right) .
$$

Notice that

$$
\begin{aligned}
\underset{n}{\limsup } \int_{\mathbb{R}^{d}}(1-g) \mathrm{d} \mu_{t}^{n} & =1-\liminf _{n} \int_{\mathbb{R}^{d}} g \mathrm{~d} \mu_{t}^{n} \\
& =1-\int_{\mathbb{R}^{d}} g \mathrm{~d} \mu_{t} \quad \text { by }(4.15) \\
& =\int_{\mathbb{R}^{d}}(1-g) \mathrm{d} \mu_{t}
\end{aligned}
$$

which implies

$$
\begin{equation*}
\underset{n}{\limsup }\left|\int_{\mathbb{R}^{d}} f \mathrm{~d}\left(\mu_{t}^{n}-\mu_{t}\right)\right| \leq 2\|f\|_{\infty} \int_{\mathbb{R}^{d}}(1-g) \mathrm{d} \mu_{t} . \tag{4.16}
\end{equation*}
$$

Because $\mu_{t}$ is a probability measure,

$$
\begin{equation*}
\sup \left\{\int_{\mathbb{R}^{d}} g \mathrm{~d} \mu_{t}: g \in C_{c}\left(\mathbb{R}^{d}\right) \text { and } 0 \leq g \leq 1\right\}=1 . \tag{4.17}
\end{equation*}
$$

The claim then follows from (4.16) and (4.17).
By monotone convergence theorem (MCT),

$$
\begin{aligned}
\int_{B_{R}^{c}}|x|^{p} \mathrm{~d} \mu_{t}(x) & =\lim _{K \rightarrow \infty} \int_{B_{R}^{c} \cap B(0, K)}|x|^{p} \mathrm{~d} \mu_{t}(x) \\
& =\lim _{K \rightarrow \infty} \lim _{n \rightarrow \infty} \int_{B_{R}^{c} \cap B(0, K)}|x|^{p} \mathrm{~d} \mu_{t}^{n}(x) \quad \text { by (4.13) } \\
& \lesssim \frac{1}{R} \quad \text { by }(4.11),
\end{aligned}
$$

which implies

$$
\begin{equation*}
\sup _{t \in \mathbb{T}} \int_{B_{R}^{c}}|x|^{p} \mathrm{~d} \mu_{t}(x) \lesssim \frac{1}{R} . \tag{4.18}
\end{equation*}
$$

Clearly, $|\cdot|^{p}$ is continuous and bounded from below. By Lemma 4.2 and Portmanteau's theorem,

$$
\begin{aligned}
\sup _{t \in \mathbb{T}} M_{p}\left(\mu_{t}\right) & \leq \sup _{t \in \mathbb{T}} \lim _{n} \inf M_{p}\left(\mu_{t}^{n}\right) \\
& \lesssim 1 \text { by (4.9), }
\end{aligned}
$$

which implies $\mu_{t} \in \mathcal{P}_{p}\left(\mathbb{R}^{d}\right)$ for $t \in \mathbb{T}$. We have

$$
\begin{align*}
\left(W_{p}\left(\mu_{s}, \mu_{t}\right)\right)^{p} & =\sup \left\{\int_{\mathbb{R}^{d}} \varphi \mathrm{~d} \mu_{s}+\int_{\mathbb{R}^{d}} \psi \mathrm{~d} \mu_{t}:(\varphi, \psi) \in \Phi_{p}\right\}  \tag{4.19}\\
& =\sup \left\{\lim _{n}\left(\int_{\mathbb{R}^{d}} \varphi \mathrm{~d} \mu_{s}^{n}+\int_{\mathbb{R}^{d}} \psi \mathrm{~d} \mu_{t}^{n}\right):(\varphi, \psi) \in \Phi_{p}\right\}  \tag{4.20}\\
& \leq \limsup _{n} \sup \left\{\int_{\mathbb{R}^{d}} \varphi \mathrm{~d} \mu_{s}^{n}+\int_{\mathbb{R}^{d}} \psi \mathrm{~d} \mu_{t}^{n}:(\varphi, \psi) \in \Phi_{p}\right\} \\
& =\limsup _{n}\left(W_{p}\left(\mu_{s}^{n}, \mu_{t}^{n}\right)\right)^{p}  \tag{4.21}\\
& \lesssim|t-s|^{\delta_{2} p} . \tag{4.22}
\end{align*}
$$

Above, (4.19) and (4.21) are due to Lemma 3.1[1], (4.20) due to Lemma 4.2, and (4.22) due to (4.10). It follows that

$$
W_{p}\left(\mu_{s}, \mu_{t}\right) \lesssim|t-s|^{\delta_{2}}, \quad s, t \in \mathbb{T}
$$

Next we establish an essential result about convergence:
Lemma 4.3. We have

$$
\limsup _{n} \sup _{t \in \mathbb{T}} W_{p}\left(\mu_{t}^{n}, \mu_{t}\right)=0
$$

Proof. Let $t \in \mathbb{T}, n \in \mathbb{N}^{*}$ and $R>0$. We have

$$
\begin{aligned}
\left(W_{p}\left(\mu_{t}^{n}, \mu_{t}\right)\right)^{p} & \lesssim \int_{\mathbb{R}^{d}}|x|^{p}\left|\ell_{t}^{n}(x)-\ell_{t}(x)\right| \mathrm{d} x \quad \text { by Lemma } 3.2 \\
& \leq \int_{B(0, R)}|x|^{p}\left|\ell_{t}^{n}(x)-\ell_{t}(x)\right| \mathrm{d} x+\int_{B_{R}^{c}}|x|^{p}\left(\ell_{t}^{n}(x)+\ell_{t}(x)\right) \mathrm{d} x \\
& =: J(t, n, R)+K(t, n, R) .
\end{aligned}
$$

By (4.13),

$$
\lim _{n} \sup _{t \in \mathbb{T}} J(t, n, R)=0 .
$$

By (4.11) and (4.18),

$$
\sup _{n \in \mathbb{N}^{*}} \sup _{t \in \mathbb{T}} K(t, n, R) \lesssim \frac{1}{R}
$$

As such,

$$
\begin{aligned}
& \limsup _{n} \sup _{t \in \mathbb{T}}\left(W_{p}\left(\mu_{t}^{n}, \mu_{t}\right)\right)^{p} \\
\lesssim & \lim _{n} \sup _{\sup _{t \in \mathbb{T}}} J(t, n, R)+\lim \sup _{n} \sup _{t \in \mathbb{T}} K(t, n, R) \\
\lesssim & \frac{1}{R} .
\end{aligned}
$$

The claim then follows by taking the limit $R \rightarrow \infty$.

### 4.3 Existence of a weak solution

For convenience, we write $\ell^{n}(t, x):=\ell_{t}^{n}(x)$. Notice that $\rho^{n} * \mu_{t}^{n}=\rho^{n} * \ell_{t}^{n}$. The Fokker-Planck equation (in distributional sense) associated with (4.1) is

$$
\begin{aligned}
\partial_{t} \ell^{n}(t, x)=-\sum_{i=1}^{d} & \frac{\partial}{\partial x_{i}}\left\{b\left(t, x,\left(\rho^{n} * \ell_{t}^{n}\right)(x), \mu_{t}^{n}\right) \ell^{n}(t, x)\right\} \\
& +\frac{1}{2} \sum_{i, j=1}^{d} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left\{a^{i, j}\left(t, x, \mu_{t}^{n}\right) \ell^{n}(t, x)\right\}
\end{aligned}
$$

which means for each $(\varphi, \psi) \in C_{c}^{\infty}(\mathbb{T}) \times C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ that

$$
\begin{align*}
&-\int_{\mathbb{T}} \int_{\mathbb{R}^{d}} \varphi^{\prime}(t) \psi(x) \mathrm{d} \mu_{t}^{n}(x) \mathrm{d} t \\
&=\sum_{i=1}^{d} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} b\left(t, x,\left(\rho^{n} * \ell_{t}^{n}\right)(x), \mu_{t}^{n}\right) \varphi(t) \frac{\partial \psi}{\partial x_{i}}(x) \mathrm{d} \mu_{t}^{n}(x) \mathrm{d} t  \tag{4.23}\\
&+\frac{1}{2} \sum_{i, j=1}^{d} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} a^{i, j}\left(t, x, \mu_{t}^{n}\right) \varphi(t) \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}(x) \mathrm{d} \mu_{t}^{n}(x) \mathrm{d} t .
\end{align*}
$$

Above, $a^{i, j}$ is the entry in the $i$-th row and $j$-th column of $a$. We recall from Lemma 4.3 and (4.13) that

$$
\begin{align*}
\sup _{t \in \mathbb{T}} W_{p}\left(\mu_{t}^{n}, \mu_{t}\right) \xrightarrow{n \rightarrow \infty} 0,  \tag{4.24}\\
\sup _{t \in \mathbb{T}} \sup _{x \in B(0, R)}\left|\ell_{t}^{n}(x)-\ell_{t}(x)\right| \xrightarrow{n \rightarrow \infty} 0, \quad R>0 . \tag{4.25}
\end{align*}
$$

We fix $(\varphi, \psi) \in C_{c}^{\infty}(\mathbb{T}) \times C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. By (4.24), the boundedness of $a$, and the continuity of $a$ w.r.t. the distribution variable, we get for $t \in \mathbb{T}$ :

$$
\begin{gathered}
\int_{\mathbb{R}^{d}} \psi(x) \mathrm{d} \mu_{t}^{n}(x) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^{d}} \psi(x) \mathrm{d} \mu_{t}(x), \\
\int_{\mathbb{R}^{d}} a^{i, j}\left(t, x, \mu_{t}^{n}\right) \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}(x) \mathrm{d} \mu_{t}^{n}(x) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^{d}} a^{i, j}\left(t, x, \mu_{t}\right) \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}(x) \mathrm{d} \mu_{t}(x) .
\end{gathered}
$$

Let $S:=B(0,1)+\operatorname{supp} \psi$. By triangle inequality,

$$
\begin{aligned}
\left\|1_{S}\left\{\left(\rho^{n} * \ell_{t}^{n}\right)-\ell_{t}\right\}\right\|_{\infty} & \leq\left\|1_{S}\left\{\rho^{n} *\left(\ell_{t}^{n}-\ell_{t}\right)\right\}\right\|_{\infty}+\left\|1_{S}\left(\rho^{n} * \ell_{t}-\ell_{t}\right)\right\|_{\infty} \\
& \leq\left\|\rho^{n} *\left\{1_{S}\left(\ell_{t}^{n}-\ell_{t}\right)\right\}\right\|_{\infty}+\left\|1_{S}\left(\rho^{n} * \ell_{t}-\ell_{t}\right)\right\|_{\infty} \\
& \leq\left\|1_{S}\left(\ell_{t}^{n}-\ell_{t}\right)\right\|_{\infty}+\left\|1_{S}\left(\rho^{n} * \ell_{t}-\ell_{t}\right)\right\|_{\infty} .
\end{aligned}
$$

By (4.25), $\left\|1_{S}\left(\ell_{t}^{n}-\ell_{t}\right)\right\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. By [54, Proposition 4.21], $\left\|1_{S}\left(\rho^{n} * \ell_{t}-\ell_{t}\right)\right\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\left\|1_{S}\left\{\left(\rho^{n} * \ell_{t}^{n}\right)-\ell_{t}\right\}\right\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. Hence we have for $t \in \mathbb{T}$ :

$$
\begin{equation*}
\left\|1_{S}\left\{b\left(t, \cdot,\left(\rho^{n} * \ell_{t}^{n}\right)(\cdot), \mu_{t}^{n}\right)-b\left(t, \cdot, \ell_{t}(\cdot), \mu_{t}\right)\right\}\right\|_{\infty} \xrightarrow{n \rightarrow \infty} 0 . \tag{4.26}
\end{equation*}
$$

Recall that $|b| \leq f_{0}$ and $f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$. It follows from (4.24), (4.26) and DCT that

$$
\int_{\mathbb{R}^{d}} b\left(t, x,\left(\rho^{n} * \ell_{t}^{n}\right)(x), \mu_{t}^{n}\right) \frac{\partial \psi}{\partial x_{i}}(x) \mathrm{d} \mu_{t}^{n}(x) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^{d}} b\left(t, x, \ell_{t}(x), \mu_{t}\right) \frac{\partial \psi}{\partial x_{i}}(x) \mathrm{d} \mu_{t}(x) .
$$

Taking the limit $n \rightarrow \infty$ in (4.23), we get

$$
\begin{aligned}
& -\int_{\mathbb{T}} \int_{\mathbb{R}^{d}} \varphi^{\prime}(t) \psi(x) \mathrm{d} \mu_{t}(x) \mathrm{d} t \\
= & \sum_{i=1}^{d} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} b\left(t, x, \ell_{t}(x), \mu_{t}\right) \varphi(t) \frac{\partial \psi}{\partial x_{i}}(x) \mathrm{d} \mu_{t}(x) \mathrm{d} t \\
& +\frac{1}{2} \sum_{i, j=1}^{d} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} a^{i, j}\left(t, x, \mu_{t}\right) \varphi(t) \frac{\partial^{2} \psi}{\partial x_{i} \partial x_{j}}(x) \mathrm{d} \mu_{t}(x) \mathrm{d} t, \quad(\varphi, \psi) \in C_{c}^{\infty}(\mathbb{T}) \times C_{c}^{\infty}\left(\mathbb{R}^{d}\right) .
\end{aligned}
$$

So $\ell$ satisfies the Fokker-Planck equation

$$
\begin{aligned}
\partial_{t} \ell(t, x)= & -\sum_{i=1}^{d} \frac{\partial}{\partial x_{i}}\left\{b\left(t, x, \ell_{t}(x), \mu_{t}\right) \ell(t, x)\right\} \\
& +\frac{1}{2} \sum_{i, j=1}^{d} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left\{a^{i, j}\left(t, x, \mu_{t}\right) \ell(t, x)\right\} .
\end{aligned}
$$

Moreover, $\ell$ satisfies the following integrability estimate:
Lemma 4.4. There exists a constant $c>0$ (depending only on $\Theta_{1}$ ) such that

$$
\int_{\mathbb{T}} \int_{\mathbb{R}^{d}}\left\{\left|b\left(t, x, \ell_{t}(x), \mu_{t}\right)\right|+\left|a\left(t, x, \mu_{t}\right)\right|\right\} \mathrm{d} \mu_{t}(x) \mathrm{d} t \leq c\left(1+\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}}\right) .
$$

Proof. By (4.6), $\mathrm{d} X_{t}^{n}=\bar{b}^{n}\left(t, X_{t}^{n}\right) \mathrm{d} t+\bar{\sigma}^{n}\left(t, X_{t}^{n}\right) \mathrm{d} B_{t}$. By Remark 4.1, every pair $\left(\bar{b}^{n}, \bar{\sigma}^{n}\right)$ satisfies Assumption 3.6 for the same set of parameters. Then

$$
\begin{align*}
\int_{\mathbb{T}} \int_{\mathbb{R}^{d}} f_{0}(t, x) \mathrm{d} \mu_{t}^{n}(x) \mathrm{d} t & =\mathbb{E}\left[\int_{0}^{T} f_{0}\left(t, X_{t}^{n}\right) \mathrm{d} t\right] \quad \text { by Tonelli's theorem } \\
& \lesssim 1+\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}} \quad \text { by Proposition } 3.8[2] . \tag{4.27}
\end{align*}
$$

We have

$$
\begin{align*}
\int_{\mathbb{T}} \int_{\mathbb{R}^{d}} f_{0}(t, x) \mathrm{d} \mu_{t}(x) \mathrm{d} t & =\int_{\mathbb{T}} \lim _{k} \int_{\mathbb{R}^{d}} 1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}(x) \mathrm{d} x \mathrm{~d} t  \tag{4.28}\\
& \leq \liminf _{k} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} 1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}(x) \mathrm{d} x \mathrm{~d} t  \tag{4.29}\\
& =\liminf _{k} \int_{\mathbb{T}} \lim _{n} \int_{\mathbb{R}^{d}} 1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}^{n}(x) \mathrm{d} x \mathrm{~d} t  \tag{4.30}\\
& \leq \liminf _{k} \liminf _{n} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} 1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}^{n}(x) \mathrm{d} x \mathrm{~d} t  \tag{4.31}\\
& \leq \liminf _{n} \int_{\mathbb{T}} \int_{\mathbb{R}^{d}} f_{0}(t, x) \ell_{t}^{n}(x) \mathrm{d} x \mathrm{~d} t \\
& \lesssim 1+\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}} \text { by }(4.27) . \tag{4.32}
\end{align*}
$$

Above, (4.28) is due to MCT; (4.29) and (4.31) are due to Fatou's lemma. We will justify how (4.30) follows from $f_{0} \in \tilde{L}_{q_{0}}^{p_{0}}$ and DCT:

1. From (4.7), we get $1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}^{n}(x) \lesssim 1_{B(0, k)}(x) f_{0}(t, x)$.
2. From (4.25), we get $1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}^{n}(x) \rightarrow 1_{B(0, k)}(x) f_{0}(t, x) \ell_{t}(x)($ as $n \rightarrow \infty)$ for all $x \in \mathbb{R}^{d}$.

We denote by $I$ the LHS of the inequality in the statement of Lemma 4.4. Then

$$
\begin{aligned}
I & \lesssim 1+\int_{\mathbb{T}} \int_{\mathbb{R}^{d}} f_{0}(t, x) \mathrm{d} \mu_{t}(x) \mathrm{d} t \\
& \lesssim 1+\left\|f_{0}\right\|_{\tilde{L}_{q_{0}}^{p_{0}}} \text { by }(4.32) .
\end{aligned}
$$

This completes the proof.
Clearly,

1. The maps $(t, x) \mapsto b\left(t, x, \ell_{t}(x), \mu_{t}\right)$ and $(t, x) \mapsto a\left(t, x, \mu_{t}\right)$ are measurable.
2. By Lemma 4.4,

$$
\int_{\mathbb{T}} \int_{\mathbb{R}^{d}}\left\{\left|b\left(t, x, \ell_{t}(x), \mu_{t}\right)\right|+\left|a\left(t, x, \mu_{t}\right)\right|\right\} \mathrm{d} \mu_{t}(x) \mathrm{d} t<\infty .
$$

3. The map $\mathbb{T} \rightarrow \mathcal{P}_{p}\left(\mathbb{R}^{d}\right), t \mapsto \mu_{t}$ is continuous by (4.22).

By the same application of the superposition principle [55, 56, 57] as in [14, Section 2], (1.1) has a weak solution whose marginal distribution is exactly ( $\mu_{t}, t \in \mathbb{T}$ ).

### 4.4 Existence of a strong solution

By the previous sub-section, there exists a probability space $(\bar{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}})$ on which there exist an $m$ dimensional Brownian motion ( $\bar{B}_{t}, t \geq 0$ ), an admissible filtration $\overline{\mathbb{F}}:=\left(\overline{\mathcal{F}}_{t}\right)_{t \geq 0}$, and a continuous $\overline{\mathbb{F}}$-adapted process $\left(\bar{X}_{t}, t \in \mathbb{T}\right)$ such that

$$
\mathrm{d} \bar{X}_{t}=b\left(t, \bar{X}_{t}, \ell_{t}\left(\bar{X}_{t}\right), \bar{\mu}_{t}\right) \mathrm{d} t+\sigma\left(t, \bar{X}_{t}, \bar{\mu}_{t}\right) \mathrm{d} \bar{B}_{t}, \quad t \in \mathbb{T},
$$

where the distribution of $\bar{X}_{0}$ is $\nu$, that of $\bar{X}_{t}$ is $\bar{\mu}_{t}$, and the p.d.f. of $\bar{X}_{t}$ is $\ell_{t}$. We define the map $\bar{b}: \mathbb{T} \times \mathbb{R}^{d} \times \mathcal{P}_{p}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d}$ by $\bar{b}(t, x, \tilde{\rho}):=b\left(t, x, \ell_{t}(x), \tilde{\rho}\right)$. We consider the SDE

$$
\begin{equation*}
\mathrm{d} Y_{t}=\bar{b}\left(t, Y_{t}, \mu_{t}\right) \mathrm{d} t+\sigma\left(t, Y_{t}, \mu_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{4.33}
\end{equation*}
$$

where the distribution of $Y_{0}$ is $\nu$ and that of $Y_{t}$ is $\mu_{t}$. We recall that $\left(B_{t}, t \geq 0\right)$ is the fixed $m$ dimensional Brownian motion on the fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$ introduced in Section 1. By [10, Theorem 1.1(1)], (4.33) is well-posed. On the other hand,

$$
\mathrm{d} \bar{X}_{t}=\bar{b}\left(t, \bar{X}_{t}, \bar{\mu}_{t}\right) \mathrm{d} t+\sigma\left(t, \bar{X}_{t}, \bar{\mu}_{t}\right) \mathrm{d} \bar{B}_{t} .
$$

It follows that $\mu_{t}=\bar{\mu}_{t}$ and thus the p.d.f. of $Y_{t}$ is $\ell_{t}$. In particular,

$$
\mathrm{d} Y_{t}=b\left(t, Y_{t}, \ell_{t}\left(Y_{t}\right), \mu_{t}\right) \mathrm{d} t+\sigma\left(t, Y_{t}, \mu_{t}\right) \mathrm{d} B_{t} .
$$

This completes the proof.

## 5 Proof of Theorem 2.3

For $k \in\{1,2\}$, we consider

$$
\begin{equation*}
\mathrm{d} X_{t}^{k}=b\left(t, X_{t}^{k}, \ell_{t}^{k}\left(X_{t}^{k}\right), \mu_{t}^{k}\right) \mathrm{d} t+\sigma\left(t, X_{t}^{k}\right) \mathrm{d} B_{t}^{k}, \quad t \in \mathbb{T}, \tag{5.1}
\end{equation*}
$$

where the distribution of $X_{0}^{k}$ is $\nu_{k}$, that of $X_{t}^{k}$ is $\mu_{t}^{k}$, the p.d.f. of $X_{t}^{k}$ is $\ell_{t}^{k}$, and $\left(B_{t}^{k}, t \geq 0\right)$ is a $d$-dimensional Brownian motion on a probability space $\left(\Omega^{k}, \mathcal{A}^{k}, \mathbb{P}^{k}\right)$ with an admissible filtration $\mathbb{F}^{k}:=\left(\mathcal{F}_{t}^{k}\right)_{t \geq 0}$. We define measurable maps $b^{k}: \mathbb{T} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ by $b^{k}(t, x):=b\left(t, x, \ell_{t}^{k}(x), \mu_{t}^{k}\right)$.

### 5.1 Uniqueness of marginal density

Clearly, $\left(b^{k}, \sigma\right)$ satisfies Assumption 3.11. Let $\bar{b}$ be defined by (3.20). We denote $b_{t}^{k}(x):=b^{k}(t, x)$. By Corollary 3.13,

$$
\ell_{t}^{k}(x)=P_{0, t}^{\bar{b}, \sigma} \ell_{\nu_{k}}(x)+\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{s}^{k}(y)\left\langle b_{s}^{k}(y)-\bar{b}_{s}(y), \nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right\rangle \mathrm{d} y \mathrm{~d} s
$$

which implies

$$
\begin{align*}
\left|\ell_{t}^{2}(x)-\ell_{t}^{1}(x)\right| \leq & P_{0, t}^{\bar{b}, \sigma}\left\{\left|\ell_{\nu_{1}}(x)-\ell_{\nu_{2}}\right|\right\}(x)  \tag{5.2}\\
& \quad+\int_{0}^{t} \int_{\mathbb{R}^{d}}\left|b_{s}^{2}(y)-\bar{b}_{s}(y)\right|\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right|\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s \\
& \quad+\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{s}^{1}(y)\left|b_{s}^{2}(y)-b_{s}^{1}(y)\right|\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s .
\end{align*}
$$

We write $M_{1} \preccurlyeq M_{2}$ if there exists a constant $c>0$ (depending only on $\Theta_{1}$ ) such that $M_{1} \preccurlyeq c M_{2}$. We have $\left\|b^{k}\right\|_{\infty}+\|\bar{b}\|_{\infty}<\infty$, so

$$
\begin{equation*}
\left|b^{k}-\bar{b}\right| \preccurlyeq 1 \text {. } \tag{5.3}
\end{equation*}
$$

By (5.2) and (5.3),

$$
\begin{align*}
&\left|\ell_{t}^{2}(x)-\ell_{t}^{1}(x)\right| \preccurlyeq P_{0, t}^{\bar{b}, \sigma}\left\{\left|\ell_{\nu_{1}}(x)-\ell_{\nu_{2}}\right|\right\}(x)  \tag{5.4}\\
&+\int_{0}^{t} \int_{\mathbb{R}^{d}}\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right|\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s \\
&+\int_{0}^{t} \int_{\mathbb{R}^{d}} \ell_{s}^{1}(y)\left|b_{s}^{2}(y)-b_{s}^{1}(y)\right|\left|\nabla_{y}{ }_{s}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s
\end{align*}
$$

By Theorem 3.10[2],

$$
\begin{equation*}
\sup _{t \in \mathbb{T}}\left\|\ell_{t}^{1}\right\|_{\infty} \preccurlyeq\left\|\ell_{\nu_{1}}\right\|_{\infty} \tag{5.5}
\end{equation*}
$$

By Assumption 2.1[A4],

$$
\begin{equation*}
\left|b_{s}^{2}(y)-b_{s}^{1}(y)\right| \preccurlyeq\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right|+W_{p}\left(\mu_{s}^{2}, \mu_{s}^{1}\right) . \tag{5.6}
\end{equation*}
$$

By (5.4), (5.5) and (5.6),

$$
\begin{aligned}
\left|\ell_{t}^{2}(x)-\ell_{t}^{1}(x)\right| \preccurlyeq & P_{0, t}^{\bar{b}, \sigma}\left\{\left|\ell_{\nu_{1}}(x)-\ell_{\nu_{2}}\right|\right\}(x) \\
& +\left(1+\left\|\ell_{\nu_{1}}\right\|_{\infty}\right) \int_{0}^{t} \int_{\mathbb{R}^{d}}\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right|\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s \\
& \quad+\int_{0}^{t} W_{p}\left(\mu_{s}^{2}, \mu_{s}^{1}\right) \int_{\mathbb{R}^{d}} \ell_{s}^{1}(y)\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} y \mathrm{~d} s \\
= & : I_{1}(t, x)+\left(1+\left\|\ell_{\nu_{1}}\right\|_{\infty}\right) I_{2}(t, x)+I_{3}(t, x)
\end{aligned}
$$

The pair $(\bar{b}, \sigma)$ satisfies Assumption 3.3. By Theorem 3.4[1], there exists a constant $\kappa>0$ (depending only on $\Theta_{1}$ ) such that for $i \in\{0,1\}, 0 \leq s<t \leq T$ and $x, y \in \mathbb{R}^{d}$ :

$$
\begin{equation*}
\left|\nabla_{y}^{i} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \preccurlyeq(t-s)^{-\frac{i}{2}} p_{t-s}^{\kappa}(y-x) \tag{5.7}
\end{equation*}
$$

Then

$$
\begin{align*}
& \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right)\left|\nabla_{y}^{i} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} x \\
\preccurlyeq & (t-s)^{-\frac{i}{2}} \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right) p_{t-s}^{\kappa}(y-x) \mathrm{d} x \quad \text { by }(5.7) \\
\preccurlyeq & (t-s)^{-\frac{i}{2}}\left(|y|^{p}+1\right) . \tag{5.8}
\end{align*}
$$

We define a measurable map $f: \mathbb{T} \rightarrow \mathbb{R}_{+}$by

$$
f(s):=\int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right)\left|\ell_{s}^{2}(x)-\ell_{s}^{1}(x)\right| \mathrm{d} x, \quad s \in \mathbb{T}
$$

By (3.15), $f$ is bounded. First,

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right) I_{1}(t, x) \mathrm{d} x \\
= & \int_{0}^{t} \int_{\mathbb{R}^{d}}\left|\ell_{\nu_{1}}(y)-\ell_{\nu_{2}}(y)\right| \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right)\left|p_{0, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} x \mathrm{~d} y \mathrm{~d} s \\
\preccurlyeq & \int_{0}^{t} \int_{\mathbb{R}^{d}}\left|\ell_{\nu_{1}}(y)-\ell_{\nu_{2}}(y)\right|\left(|y|^{p}+1\right) \mathrm{d} y \mathrm{~d} s \quad \text { by }(5.8) \\
\preccurlyeq & \int_{\mathbb{R}^{d}}\left(|y|^{p}+1\right)\left|\ell_{\nu_{1}}(y)-\ell_{\nu_{2}}(y)\right| \mathrm{d} y=f(0)
\end{aligned}
$$

Second,

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right) I_{2}(t, x) \mathrm{d} x \\
= & \int_{0}^{t} \int_{\mathbb{R}^{d}}\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right| \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right)\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} x \mathrm{~d} y \mathrm{~d} s \\
\preccurlyeq & \int_{0}^{t}(t-s)^{-\frac{1}{2}} \int_{\mathbb{R}^{d}}\left|\ell_{s}^{2}(y)-\ell_{s}^{1}(y)\right|\left(|y|^{p}+1\right) \mathrm{d} y \mathrm{~d} s \quad \text { by }(5.8) \\
= & \int_{0}^{t}(t-s)^{-\frac{1}{2}} f(s) \mathrm{d} s .
\end{aligned}
$$

Third,

$$
\begin{aligned}
& \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right) I_{3}(t, x) \mathrm{d} x \\
= & \int_{0}^{t} W_{p}\left(\mu_{s}^{2}, \mu_{s}^{1}\right) \int_{\mathbb{R}^{d}} \ell_{s}^{1}(y) \int_{\mathbb{R}^{d}}\left(|x|^{p}+1\right)\left|\nabla_{y} p_{s, t}^{\bar{b}, \sigma}(y, x)\right| \mathrm{d} x \mathrm{~d} y \mathrm{~d} s \\
\preccurlyeq & \int_{0}^{t}(t-s)^{-\frac{1}{2}} W_{p}\left(\mu_{s}^{2}, \mu_{s}^{1}\right) \int_{\mathbb{R}^{d}} \ell_{s}^{1}(y)\left(|y|^{p}+1\right) \mathrm{d} y \mathrm{~d} s \quad \text { by }(5.8) \\
\preccurlyeq & \left(1+M_{p}\left(\nu_{1}\right)\right) \int_{0}^{t}(t-s)^{-\frac{1}{2}} W_{p}\left(\mu_{s}^{2}, \mu_{s}^{1}\right) \mathrm{d} s \quad \text { by }(3.15) \\
\preccurlyeq & \left(1+M_{p}\left(\nu_{1}\right)\right) \int_{0}^{t}(t-s)^{-\frac{1}{2}}|f(s)|^{\frac{1}{p}} \mathrm{~d} s \quad \text { by Lemma 3.2. }
\end{aligned}
$$

To sum up,

$$
f(t) \preccurlyeq f(0)+\left(1+\left\|\ell_{\nu_{1}}\right\|_{\infty}+M_{1}\left(\nu_{1}\right)\right) \int_{0}^{t}(T-s)^{-\frac{1}{2}}\left(f(s)+|f(s)|^{\frac{1}{p}}\right) \mathrm{d} s
$$

Assuming $p=1$, we get

$$
f(t) \preccurlyeq f(0)+\left(1+\left\|\ell_{\nu_{1}}\right\|_{\infty}+M_{1}\left(\nu_{1}\right)\right) \int_{0}^{t}(T-s)^{-\frac{1}{2}} f(s) \mathrm{d} s .
$$

By Grönwall's lemma,

$$
\begin{equation*}
\sup _{t \in \mathbb{T}} f(t) \preccurlyeq f(0) \exp \left\{2 \sqrt{T}\left(1+\left\|\ell_{\nu_{1}}\right\|_{\infty}+M_{1}\left(\nu_{1}\right)\right)\right\}, \tag{5.9}
\end{equation*}
$$

which implies the existence of the function $\Lambda$ as required in Theorem 2.3[1].

### 5.2 Weak and strong uniqueness of a solution

By (5.1),

$$
\mathrm{d} X_{t}^{k}=b^{k}\left(t, X_{t}^{k}\right) \mathrm{d} t+\sigma\left(t, X_{t}^{k}\right) \mathrm{d} B_{t}^{k} .
$$

Now we let $\nu:=\nu_{1}=\nu_{2}$. By (5.9), $\ell_{t}^{1}=\ell_{t}^{2}$ and $\mu_{t}^{1}=\mu_{t}^{2}$ for $t \in \mathbb{T}$. Then $\boldsymbol{b}:=b^{1}=b^{2}$. We consider the SDE

$$
\begin{equation*}
\mathrm{d} Y_{t}=\boldsymbol{b}\left(t, Y_{t}\right) \mathrm{d} t+\sigma\left(t, Y_{t}\right) \mathrm{d} B_{t}, \quad t \in \mathbb{T}, \tag{5.10}
\end{equation*}
$$

where the distribution of $Y_{0}$ is $\nu$. By [10, Theorem 1.1(1)], (5.10) is well-posed. On the other hand, $\left(X_{t}^{1}, t \in \mathbb{T}\right)$ and $\left(X_{t}^{2}, t \in \mathbb{T}\right)$ satisfy (5.10). It follows that (1.1) has both weak and strong uniqueness.

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