

NEGATIVE CURVATURE IN LOCALLY REDUCIBLE ARTIN GROUPS

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ABSTRACT. In this paper, we define the 2-complete Artin complex and show that it is systolic for locally reducible Artin groups. The stabilizers of simplices in this complex are exactly the proper parabolic subgroups which are "2-complete." We use this systolicity to show that parabolic subgroups, with 2-completions that are not the whole Artin group, are weakly malnormal. This allows us to conclude that many locally reducible Artin groups are acylindrically hyperbolic.

1. INTRODUCTION

Geometric group theory uses geometric and topological methods to study finitely generated groups. If one considers the action of a group on a space satisfying certain geometric properties, one can often recover algebraic properties of the group. Artin groups, one class of particular interest, are closely related to the relatively well-studied Coxeter groups and mapping class groups. Much of the successful study of Artin groups has been through the action of the group on a negatively or nonpositively curved space. In this paper, we study a subclass known as locally reducible Artin groups by defining a new nonpositively curved complex and using it to show many Artin groups within this class are acylindrically hyperbolic.

Artin groups were introduced in the 1960's by Tits as an extension of Coxeter groups. There are many long-standing open questions and conjectures about Artin groups. For example, it is unknown whether Artin groups are torsion-free, have solvable word problem, or satisfy the famous $K(\pi, 1)$ -conjecture. Much progress has been made for particular subclasses of Artin groups, but as a whole they remain quite mysterious.

An **Artin group** is a group with a presentation of the form

$$\langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}} \rangle$$

for some finite set of generators $S = \{s_1, \dots, s_n\}$ where $m_{ij} \in \{2, 3, \dots, \infty\}$ for each $1 \leq i < j \leq n$. If $m_{ij} = \infty$, then s_i and s_j have no relation. We can encode the same information in a **defining graph**, denoted Γ , with vertices $V(\Gamma) = S$. If $m_{ij} < \infty$, then Γ has an edge between vertices s_i and s_j labeled by the integer m_{ij} . An Artin group with defining graph Γ is denoted A_Γ .

To each Artin group there is an associated **Coxeter group**, W_Γ , which is the quotient of A_Γ defined by adding the relations $s_i^2 = 1$ for all $1 \leq i \leq n$. If W_Γ is a finite group, we say that A_Γ is **of finite type**. Note that A_Γ will always be an infinite group.

For a subset of generators $T \subseteq S$, the associated **standard parabolic subgroup** of A_Γ is the subgroup generated by T . Van der Lek [36] proved that this subgroup is isomorphic to the Artin group determined by the full subgraph of Γ with vertices T , which we will denote A_T . Any conjugate of such a subgroup is called a **parabolic subgroup**.

Among the commonly studied subclasses of Artin groups, finite type Artin groups and right-angled Artin groups (those for which $m_{ij} = 2$ or ∞ for all i, j) have seen the most progress. In the finite type case, an Artin group admits a Garside structure, which provides normal forms and combinatorial structure for studying the group. Right-angled Artin groups are examples of graph products of groups. They have algorithmic properties and admit a CAT(0) cubical structure.

The class of 2-dimensional Artin groups has also seen significant progress. (The dimension of an Artin group is the maximum cardinality of a subset $T \subseteq V(\Gamma)$ such that A_T is finite type.) By groundbreaking work of Charney and Davis, 2-dimensional Artin groups are torsion-free and satisfy the $K(\pi, 1)$ -conjecture [9]. They are also known to have solvable word problem by work of Chermak [12]. More recently Cumplido, Martin, and Vaskou have developed a rich study of large-type Artin groups (those for which $m_{ij} \geq 3$ for all i, j) and have proven many long-standing conjectures about them: the intersection of a family of parabolic

subgroups is itself a parabolic subgroup [13], automorphisms are classified, and [38] the class of large-type Artin groups is rigid under isomorphism [27].

Our focus in this paper is the following class of Artin groups. They were introduced by Charney, who showed they satisfy both the $K(\pi, 1)$ -Conjecture and the Tits Conjecture [8].

Definition 1.1. An Artin group A_Γ is **locally reducible** if the only finite type triangles contained in Γ are of the form $2 - 2 - k$ for some $k \geq 2$.

Within this class of Artin groups, the only finite type parabolic subgroups which occur are cyclic subgroups, dihedral subgroups (i.e., A_T for $T = \{a, b\}$ where $m_{ab} \geq 3$), and direct products of cyclic and dihedral subgroups.

The notion of hyperbolic groups was introduced by Gromov in the 1980's [18], and sparked much of the field of geometric group theory as we know it today. The action of a group on a hyperbolic or nonpositively curved space has proven to be a very useful tool in the field, especially when the action is proper and cocompact. More recently, there have been many attempts at generalizing hyperbolic groups. In this paper, we focus on acylindrical hyperbolicity.

A group action of G on a metric space (X, d) is **acylindrical** if for every $\epsilon \geq 0$, there exist constants $R \geq 0$ and $N \geq 0$ such that for all points $x, y \in X$ such that $d(x, y) \geq R$, $|\{g \in G \mid d(x, gx) \leq \epsilon \text{ and } d(y, gy) \leq \epsilon\}| \leq N$. A group is **acylindrically hyperbolic** if it is not virtually cyclic and has an acylindrical action on a hyperbolic space.

Informally, acylindricity can be interpreted as a sort of properness on pairs of points that are far apart in the space. The original definition of acylindrical hyperbolicity goes back to Sela [34], which applies to group actions on trees. The more general and current definition is due to Bowditch [4] and was shown by Osin [33] to be equivalent to other conditions studied in [2, 21, 35].

A cylindrically hyperbolic groups include all but finitely many mapping class groups [4, 28], $Out(F_n)$ for $n \geq 2$ [1], and many CAT(0) groups [35]. Certain classes of Artin groups are known to be acylindrically hyperbolic, including the following classes, provided the group is irreducible, not cyclic, and of sufficient rank:

- right-angled Artin groups [7, 33, 35]
- Artin groups whose defining graph is not a join [10, 11]
- XXL Artin groups (all $m_{ij} \geq 5$) [19]
- triangle-free Artin groups [24]
- Artin groups of euclidean type [6]
- 2-dimensional Artin groups [26, 37]

In this paper, we show that many locally reducible Artin groups are acylindrically hyperbolic. For A_Γ an arbitrary Artin group, define $\widehat{\Gamma}$ to be the graph obtained from Γ by deleting all edges not labeled by 2. We consider a single vertex of $\widehat{\Gamma}$ with no edges to be its own connected component. A parabolic subgroup of A_Γ is **2-complete** if it can be written as gA_Tg^{-1} for some $g \in A_\Gamma$ and some $T \subseteq V(\Gamma)$ where T is a union of connected components of $\widehat{\Gamma}$. A **2-completion** of a parabolic subgroup P is a 2-complete parabolic subgroup which contains P . Note that 2-completions are not unique (see Remark 3.1).

Theorem 1.2. Let A_Γ be an Artin group. If either

- (1) A_Γ is locally reducible and has a maximal finite-type subgroup which is dihedral and which has a 2-completion that is not all of A_Γ , or
- (2) A_Γ splits as an amalgamated product $A_{\Gamma_1} *_{A_{\Gamma_1 \cap \Gamma_2}} A_{\Gamma_2}$ such that A_{Γ_1} is locally reducible and there is a 2-completion of $A_{\Gamma_1 \cap \Gamma_2}$ which does not contain all of A_{Γ_1} ,

then A_Γ is acylindrically hyperbolic.

The following theorem is the key ingredient to this result. This allows us to apply work of Martin [25] and Minasyan and Osin [29] to conclude that many Artin groups which are locally reducible are acylindrically hyperbolic.

Theorem 1.3. Let A_Γ be a locally reducible Artin group such that $\widehat{\Gamma}$ has at least two connected components. Then any parabolic subgroup P of A_Γ which has a 2-completion that is not all of A_Γ is weakly malnormal (i.e., there exists $g \in G$ such that $|P \cap gPg^{-1}| < \infty$).

As is a common theme in the study of Artin groups, nonpositively curved geometric complexes are a key component of these results. For locally reducible Artin groups, it is known that the Deligne complex is CAT(0) [8]. In Section 3, we introduce the 2-complete Artin complex and show that it satisfies a combinatorial version of nonpositive curvature known as systolicity.

Intuitively, a geodesic metric space is said to be CAT(0) if its triangles are “at least as thin as” comparison triangles in \mathbb{R}^2 . In the case of a cubical complex with the standard Euclidean metric, there is a simple link condition to check that the complex is CAT(0). However, when dealing with a different metric on a cube complex, or in the setting of a simplicial complex, CAT(0)-ness can be quite hard to check, especially in high dimensions. Systolicity was introduced as an analogue of CAT(0)-ness for simplicial complexes by Januszkiewicz and Świątkowski [23] and independently by Haglund [20]. Systolicity shares many of the same nice consequences of CAT(0)-ness, and it can always be checked in a combinatorial way. See Section 4 for a precise definition. The following result is the key to proving Theorem 1.3.

Theorem 1.4. *Let A_Γ be a locally reducible Artin group. If there are at least three connected components in $\widehat{\Gamma}$, then the 2-complete Artin complex of A_Γ is systolic.*

We begin by recalling the definitions of the well-known Deligne and Artin complexes (Section 2). We then define the 2-complete Artin complex (Section 3) and show that it is systolic (Section 4). We prove Theorem 1.3 (Section 5) and apply theorems of Martin and Minasyan-Osin to show that many locally reducible Artin groups are acylindrically hyperbolic (Section 6). Finally, we discuss some potential future directions related to this work (Section 7).

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2. COMPLEXES ASSOCIATED TO ARTIN GROUPS

2.1. The Deligne Complex. The Deligne complex for an arbitrary Artin group A_Γ with generating set S was introduced by Charney and Davis [9], generalizing the work of Deligne [15], who defined such a complex for finite type Artin groups. Let $\mathcal{S}^f = \{T \subseteq S \mid A_T \text{ is finite type}\}$. We include $\emptyset \in \mathcal{S}^f$ with the standard convention that $A_\emptyset = \{1\}$. Note that \mathcal{S}^f is a partially ordered set with respect to inclusion.

Definition 2.1. The **Deligne complex** of A_Γ , denoted D_Γ , is the cubical complex whose vertices correspond to cosets gA_T for $g \in A_\Gamma$ and $T \in \mathcal{S}^f$ and whose k -dimensional cubes correspond to intervals of inclusion $[gA_R, gA_T]$ of length $k+1$.

The Artin group A_Γ acts by left multiplication on D_Γ . The stabilizer of a vertex gA_T is the parabolic subgroup gA_Tg^{-1} . A fundamental domain for the action is the subcomplex K_{D_Γ} of D_Γ spanned by vertices of the form A_T for $T \in \mathcal{S}^f$.

We will now define the Moussong metric for the Deligne complex. We will start by describing the Moussong metric in its original setting: the Davis complex [32]. This complex associated to the Coxeter group W_Γ is defined analogously to the Deligne complex. We consider the same set \mathcal{S}^f as above.

Definition 2.2. The **Davis complex** of W_Γ , denoted C_Γ , is the cubical complex whose vertices correspond to cosets wW_T for $w \in W_\Gamma$ and $T \in \mathcal{S}^f$ and whose k -dimensional cubes correspond to intervals of inclusion $[wW_R, wW_T]$ of length $k+1$.

Similarly to above, W_Γ acts by left multiplication on C_Γ and has a fundamental domain K_{C_Γ} spanned by vertices of the form W_T .

When W_Γ is finite, it can be viewed as a group of orthogonal transformations of \mathbb{R}^n . The generators $s \in S$ act as reflections across the walls of a simplicial cone $Z \subseteq \mathbb{R}^n$. There is a unique point $x_\emptyset \in Z \cap \mathbb{S}^{n-1}$ which is equidistant from each wall of Z . The convex hull X of the orbit of x_\emptyset under the action of W_Γ is isomorphic to the Davis complex C_Γ . The origin in \mathbb{R}^n corresponds to the vertex W_S in C_Γ and x_\emptyset in \mathbb{R}^n

corresponds to the vertex W_\emptyset in C_Γ . The fundamental domain K_{C_Γ} is identified with the intersection $X \cap Z$. This isomorphism gives a metric on C_Γ ; in this case C_Γ is called the **Coxeter cell** of W_Γ .

If W_Γ is infinite, we can cover C_Γ by the closed stars of vertices of the form wW_T for T maximal elements of \mathcal{S}^f . Since W_T is finite, a star corresponding to a vertex of this form is isomorphic to the Coxeter cell of W_T and can be given the metric of that Coxeter cell. These Coxeter cells together give a piecewise Euclidean metric on C_Γ , called the **Moussong metric**. Moussong proved that C_Γ with this metric is CAT(0) for all W_Γ [32].

For an Artin group A_Γ , the quotient map to its associated Coxeter group $A_\Gamma \rightarrow W_\Gamma$ induces an equivariant projection $D_\Gamma \rightarrow C_\Gamma$, which is an isomorphism when restricted to the fundamental domain K_{D_Γ} . We define the Moussong metric on the Deligne complex D_Γ to be the piecewise Euclidean metric defined on each translate of K_{D_Γ} by letting K_{D_Γ} be isometric to K_{C_Γ} .

Theorem 2.3. [8, Theorem 3.2] *If A_Γ is locally reducible, then D_Γ with the Moussong metric is CAT(0).*

2.2. The Artin Complex. When Charney and Davis defined the Deligne complex in [9], they also introduced the analogous Artin complex.

Definition 2.4. The **Artin complex** associated to A_Γ , denoted X_Γ , is the simplicial complex with vertices corresponding to cosets of the form $gA_{S \setminus \{t\}}$ for $g \in A_\Gamma$ and $t \in S$. A set of vertices spans a simplex if the corresponding cosets have nonempty intersection.

Cumplido, Martin, and Vaskou showed that the Artin complex is systolic for Artin groups of large type [13]. Using the consequences of this geometry, they show that the intersection of parabolic subgroups of a large-type Artin group is a parabolic subgroup, solving a long-standing conjecture. The Artin complex was also studied by Godelle and Paris, who in particular showed that it is a flag complex [17]. Their work has inspired a new approach to the $K(\pi, 1)$ -conjecture by Huang [22]. In this paper we define a modification of this complex, called the 2-complete Artin complex.

3. THE 2-COMPLETE ARTIN COMPLEX

Let A_Γ be an Artin group with defining graph Γ , and let v be a vertex in Γ . The **2-completion** of v is the full subgraph of Γ spanned by v together with all vertices which can be reached from v by an edge path labeled only by 2's.

Let $\mathcal{S}^{(2)} = \{T \subseteq S \mid \text{if } v \in T, \text{ then the 2-completion of } v \text{ is contained in } T\}$. This is a partially ordered set with respect to inclusion. If Γ is as in Example 3.3, then $T = \{a, b, c\} \in \mathcal{S}^{(2)}$ but $W = \{a, b, c, d\} \neq \mathcal{S}^{(2)}$.

Let $\widehat{\Gamma}$ be the subgraph of Γ which consists of all the vertices of Γ and the edges of Γ labeled by 2. Connected components of $\widehat{\Gamma}$ correspond to non-empty, minimal elements of $\mathcal{S}^{(2)}$. We consider a single vertex in Γ which has no edges labeled 2 to be its own connected component in $\widehat{\Gamma}$. (See Example 3.3.)

A subset $T \subseteq V(\Gamma)$ of generators is **2-complete** if it is an element of $\mathcal{S}^{(2)}$, i.e., if for every $v \in T$, all the vertices in the 2-completion of v are also in T . A parabolic subgroup is 2-complete if it can be written in the form gA_Tg^{-1} for some $g \in A_\Gamma$ and some 2-complete $T \subseteq V(\Gamma)$.

A **2-completion** of a parabolic subgroup P of A_Γ is a parabolic subgroup that is 2-complete and contains P .

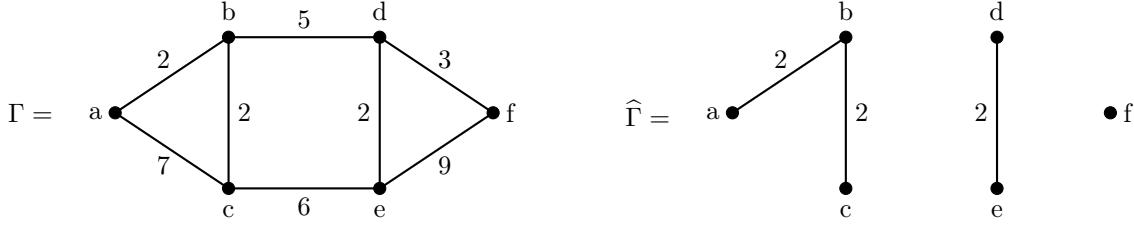
Remark 3.1. A parabolic subgroup can have many distinct 2-completions. For a standard parabolic subgroup A_T , there is a canonical choice: A_R where R is the union of the 2-completions of the vertices of T . However in the case of a general (non-standard) parabolic subgroup, there is not a canonical choice of 2-completion since a parabolic subgroup may be conjugate to more than one standard parabolic.

Definition 3.2. The **2-complete Artin complex**, \widehat{X}_Γ , is the following simplicial complex of groups:

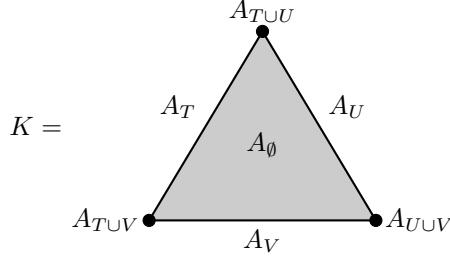
- there is a vertex for every left coset of a standard parabolic subgroup of the form $A_{\Gamma \setminus T}$ for T some single connected component of $\widehat{\Gamma}$, and
- a collection of vertices spans a simplex if the associated cosets have collective nonempty intersection.

Let K denote the fundamental domain of \widehat{X}_Γ , i.e., the simplex corresponding to standard parabolic subgroups of A_Γ .

Example 3.3. We will now show an example of $\widehat{\Gamma}$ and K , given the following defining graph Γ .



Label the connected components in $\widehat{\Gamma}$ as $T = \{a, b, c\}$, $U = \{d, e\}$, and $V = \{f\}$. The 2-complete Artin complex \widehat{X}_Γ has the following fundamental domain:



Observation 3.4. The 2-complete proper parabolic subgroups of A_Γ are exactly the stabilizers of simplices of \widehat{X}_Γ .

We say a vertex of \widehat{X}_Γ has **type** $\Gamma \setminus T$ if it corresponds to coset of the form $gA_{\Gamma \setminus T}$. Likewise, the type of each simplex in \widehat{X}_Γ is given by the subset of Γ corresponding to its associated coset. Equivalently, the type of a simplex is determined by the intersection of the types of the vertices of that simplex. Note that each top-dimensional simplex corresponds to a coset of the form gA_\emptyset .

Proposition 3.5 (Basic Properties of \widehat{X}_Γ). Let A_Γ be an Artin group.

- (1) If $\widehat{\Gamma}$ has at least two connected components, then \widehat{X}_Γ is connected.
- (2) If $\widehat{\Gamma}$ has at least three connected components, then \widehat{X}_Γ is simply connected.
- (3) The link in \widehat{X}_Γ of a simplex of type U is isomorphic to the 2-complete Artin complex $\widehat{X}_{\Gamma'}$ where Γ' is the subgraph of Γ spanned by the vertices of U .

Proof. The 2-complete Artin complex \widehat{X}_Γ is a simplicial complex of groups in the language of Bridson-Haefliger [5]. The fundamental domain K has vertices of the form $A_{\Gamma \setminus T}$ for T a single connected component of $\widehat{\Gamma}$, and edges between vertices corresponding to nonempty intersection. The morphisms associated to the inclusion of faces of K are the natural inclusions of the respective Artin groups. In this language, \widehat{X}_Γ is the development of K , in the sense of [5].

(1): If there are at least two connected components of $\widehat{\Gamma}$, then the fundamental domain K of \widehat{X}_Γ is at least one-dimensional and is connected. Furthermore, the Artin group A_Γ is generated by the collection of parabolic subgroups which are stabilizers of simplices of \widehat{X}_Γ . By [5, Prop 12.20], \widehat{X}_Γ is connected.

(2): If there are at least three connected components of $\widehat{\Gamma}$, then by [5, Prop 12.20 (4)], \widehat{X}_Γ is simply connected.

(3): By [5, Construction 12.24], we can describe the link of a simplex in \widehat{X}_Γ as the development of an appropriate subcomplex of groups. \square

4. THE 2-COMPLETE ARTIN COMPLEX IS SYSTOLIC

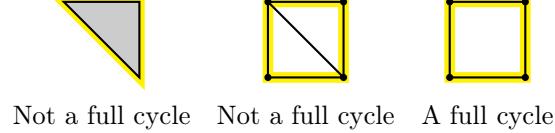
In this section we show that for a locally reducible Artin group A_Γ , where $\widehat{\Gamma}$ has at least three connected components, the associated 2-complete Artin complex \widehat{X}_Γ is systolic. Let X be a simplicial complex. A subcomplex $Y \subseteq X$ is **full** if any simplex of X spanned by a set of vertices of Y is a simplex of Y . A **full**

cycle $\gamma \subseteq X$ is a cycle that is full as a subcomplex of X . See below for some examples. The **systole** of a simplicial complex X is

$$\text{sys}(X) = \min\{|\gamma| \mid \gamma \text{ is a full cycle in } X\}$$

For a simplicial complex X , $\text{sys}(X) \geq 3$, and if there are no full cycles in X then $\text{sys}(X) = \infty$. A simplicial complex X is **locally k -large** if $\text{sys}(\text{lk}_X(Y)) \geq k$ for all simplices $Y \subseteq X$. X is **k -large** if it is locally k -large and $\text{sys}(X) \geq k$.

Definition 4.1. A simplicial complex X is **systolic** if it is connected, simply-connected, and locally 6-large.



Theorem 4.2. [5, Theorem 2.11] *The sum of the interior angles of a quadrilateral in a CAT(0) space cannot be greater than 2π .*

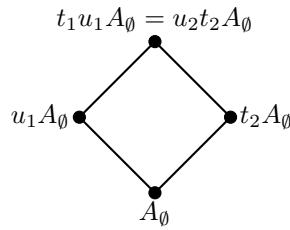
Lemma 4.3. [8, Lemma 5.1] *Let A_Γ be a locally reducible Artin group. For any $T \subseteq S$, the natural inclusion $D_T \hookrightarrow D_\Gamma$ is an isometric embedding, and hence the image is convex.*

Proposition 4.4. *Let A_Γ be locally reducible, and let $R \subseteq V(\Gamma)$ correspond to a finite type clique in Γ . If $|R| \geq 3$, then R must be contained in a single connected component of $\widehat{\Gamma}$.*

Proof. Choose $r \in R$. For any vertices $u, v \in R \setminus \{r\}$, the collection $\{r, u, v\}$ spans a finite-type triangle in Γ . Since A_Γ is locally reducible, this triangle has type $2 - 2 - k$. Thus u and v must lie in the 2-completion of r , and hence all three vertices must be in the same connected component of $\widehat{\Gamma}$. \square

Lemma 4.5. *Let A_Γ be a locally reducible Artin group such that $\widehat{\Gamma}$ has at least two connected components. Let T and U be two distinct connected components of $\widehat{\Gamma}$. If $t_1, t_2 \in A_T$ and $u_1, u_2 \in A_U$ are pairs of (not necessarily distinct) elements, then $t_1 u_1 \neq u_2 t_2$.*

Proof. Suppose by way of contradiction that there are elements $t_1, t_2 \in A_T$ and $u_1, u_2 \in A_U$ such that $t_1 u_1 = u_2 t_2$. Then there is a quadrilateral in the Deligne complex D_Γ determined by vertices $A_\emptyset, u_1 A_\emptyset, t_1 u_1 A_\emptyset = u_2 t_2 A_\emptyset, t_2 A_\emptyset$, and the geodesics between them.



The geodesics forming the sides of this quadrilateral are not edges, and do not even need to be edge paths. However since D_Γ is CAT(0) by Theorem 2.3, the geodesics are unique.

We will now consider the measure of each interior angle of the quadrilateral. The goal is to show that such a quadrilateral cannot exist using Theorem 4.2.

By Lemma 4.3, the two sides $[A_\emptyset, u_1 A_\emptyset]$ and $[A_\emptyset, t_2 A_\emptyset]$ are contained in the Deligne complex associated to A_U (denoted D_U) and the Deligne complex associated to A_T (denoted D_T), respectively. Hence the angle between the two sides, i.e., the interior angle of the quadrilateral at A_\emptyset , is bounded below by the angle at A_\emptyset between D_U and D_T as embedded subcomplexes of D_Γ .

Let v_s denote the vertex in $\text{lk}_{D_\Gamma}(A_\emptyset)$ corresponding to the edge $(A_\emptyset, A_{\{s\}})$ in D_Γ . The angle between D_U and D_T in D_Γ is determined by the distance between vertices v_t and v_u in $\text{lk}_{D_\Gamma}(A_\emptyset)$ for all $t \in T$ and $u \in U$.

Note that the 1-skeleton of $\text{lk}_{D_\Gamma}(A_\emptyset)$ is isomorphic to the defining graph Γ . Indeed, there is one vertex in $\text{lk}_{D_\Gamma}(A_\emptyset)$ for each edge in D_Γ of the form $(A_\emptyset, A_{\{s\}})$, and hence for each generator $s \in V(\Gamma)$. And there is an edge between two vertices v_r and v_s in $\text{lk}_{D_\Gamma}(A_\emptyset)$ if and only if there is a 2-dimensional cube in D_Γ with

vertices A_\emptyset , $A_{\{r\}}$, $A_{\{s\}}$, and $A_{\{r,s\}}$, which happens precisely when $A_{\{r,s\}}$ is a finite type Artin group, i.e., there is an edge between r and s in Γ .

Let $t \in T$ and $u \in U$ be arbitrary. By Proposition 4.4, for any generator $s \in V(\Gamma)$, it must be that $A_{\{s,t,u\}}$ is not finite type. Hence in $lk_{D_\Gamma}(A_\emptyset)$, there are no n -dimensional simplices between v_t and v_u for $n \geq 2$, and so the distance between v_t and v_u is determined by the 1-skeleton of $lk_{D_\Gamma}(A_\emptyset)$. For an edge (r, s) in Γ with label m_{rs} , the corresponding edge (v_r, v_s) in $lk_{D_\Gamma}(A_\emptyset)$ has length $\pi - \frac{\pi}{m_{rs}}$ with respect to the Moussong metric. Since $m_{tu} \geq 3$, the distance between v_t and v_u is at least $\frac{2\pi}{3}$.

Therefore, the interior angle of the above quadrilateral at vertex A_\emptyset is at least $\frac{2\pi}{3}$.

We can repeat this argument for vertices $u_1 A_\emptyset$ and $t_2 A_\emptyset$ by translating each by u_1^{-1} and t_2^{-1} , respectively, which preserves angle measures.

It follows that the sum of interior angles in the quadrilateral at vertices $u_1 A_\emptyset$, A_\emptyset , and $t_2 A_\emptyset$, is at least 2π . By Theorem 4.2, it is impossible for the fourth vertex of this quadrilateral to exist. Hence we have reached a contradiction, and it must be that $t_1 u_1 \neq u_2 t_2$. \square

Lemma 4.6. *Let A_Γ be a locally reducible Artin group. If $\widehat{\Gamma}$ has exactly two connected components, then \widehat{X}_Γ is 6-large.*

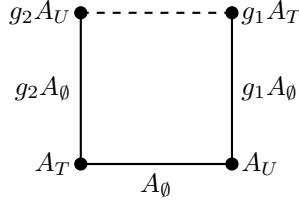
Proof. If there are exactly two connected components of $\widehat{\Gamma}$, the fundamental domain K of \widehat{X}_Γ is a 1-dimensional simplex. By construction, there will be no higher-dimensional simplices in \widehat{X}_Γ . Thus \widehat{X}_Γ is 6-large if $\text{sys}(\widehat{X}_\Gamma) \geq 6$.

Let T and U be the subsets of generators corresponding to the two connected components of $\widehat{\Gamma}$. Any edge in \widehat{X}_Γ will have exactly one vertex of type T and one vertex of type U , since every edge in the complex is some translate of K . In particular, any cycle in \widehat{X}_Γ will have even length.

We need to show that $\text{sys}(\widehat{X}_\Gamma) \geq 6$. By the preceding paragraph, this is equivalent to showing that there are no full cycles of length 4 in \widehat{X}_Γ . Consider a path of length 3 in \widehat{X}_Γ . By translating, we can assume that the fundamental domain K is the middle edge in the path.

Since the local group of K is A_\emptyset , the local groups of the two edges adjacent to K in the path are of the form $g_1 A_\emptyset$ and $g_2 A_\emptyset$ for some $g_1, g_2 \in A_\Gamma$.

Since \widehat{X}_Γ has an edge between $g_2 A_U$ and A_T , there is a coset representative of $g_2 A_U$ which is in A_T . Without loss of generality, assume g_2 is this coset representative. Similarly, by considering the edge between A_U and $g_1 A_T$, we may assume $g_1 \in A_U$.



If there was an edge connecting the two endpoints of this path, the cosets $g_2 A_U$ and $g_1 A_T$ would have nonempty intersection, implying there are elements $u \in A_U$ and $t \in A_T$ such that $g_2 u = g_1 t$. But this contradicts Lemma 4.5, so the endpoints of the path cannot be connected by an edge. Thus there are no full cycles of length 4 in \widehat{X}_Γ . Hence if $\widehat{\Gamma}$ has two connected components, then \widehat{X}_Γ is 6-large. \square

The following proposition of Januszkiewicz-Swiatkowski [23], specialized for our case, will be used in the last step of our proof of Theorem 1.4. Recall that a simplicial complex is systolic if it is connected, simply-connected, and locally 6-large.

Proposition 4.7. [23, Prop 1.4] *If X is a systolic simplicial complex, then X is 6-large.*

Theorem 1.4. *Let A_Γ be a locally reducible Artin group. If there are at least three connected components in $\widehat{\Gamma}$, then \widehat{X}_Γ is systolic.*

Proof. Let A_Γ be a locally reducible Artin group and suppose $\widehat{\Gamma}$ has at least three connected components. By Proposition 3.5, the complex \widehat{X}_Γ is connected and simply connected. To show that \widehat{X}_Γ is systolic, we must show that it is locally 6-large. We will use induction on the number of connected components n of $\widehat{\Gamma}$.

Start by assuming that $n = 3$. To show that \widehat{X}_Γ is locally 6-large, we need to show that $\text{sys}(Lk_{\widehat{X}_\Gamma}(g\Delta_T)) \geq 6$ for any simplex $g\Delta_T$. Choose an arbitrary simplex $g\Delta_T$ of type T , that is, choose a simplex in the orbit of a face of the fundamental domain K which has local group A_T . By Proposition 3.5, $Lk_{\widehat{X}_\Gamma}(g\Delta_T)$ is isomorphic to the 2-complete Artin complex associated with Artin group A_T .

By construction, T is a set of standard generators determined by some collection of connected components of $\widehat{\Gamma}$. By the definition of \widehat{X}_Γ , T cannot contain all of the standard generators: it must be missing all generators from at least one connected component of $\widehat{\Gamma}$. Consider the subgraph Z of Γ consisting only of the vertices associated to generators in T . Let \widehat{Z} be the analogue of $\widehat{\Gamma}$, i.e., the graph with vertex set T and all edges between vertices which were labeled 2 in Γ . Since the generators in at least one connected component of $\widehat{\Gamma}$ had to be omitted from T , \widehat{Z} has strictly fewer connected components than $\widehat{\Gamma}$.

Let k be the number of connected components of \widehat{Z} . By the assumption that $n = 3$, we have $0 \leq k < 3$. If $k = 2$, then by Lemma 4.6, $\widehat{X}_Z \simeq Lk_{\widehat{X}_\Gamma}(g\Delta_T)$ is 6-large. If $k = 0$ or 1, then $\widehat{X}_Z \simeq Lk_{\widehat{X}_\Gamma}(g\Delta_T)$ has no embedded full cycles, so $\text{sys}(Lk_{\widehat{X}_\Gamma}(g\Delta_T)) = \infty$. Hence when $n = 3$, \widehat{X}_Γ is locally 6-large and therefore is systolic.

Now assume that $n > 3$, i.e., $\widehat{\Gamma}$ has more than 3 connected components. Also assume that every 2-complete Artin complex \widehat{X}_Z where \widehat{Z} has $k < n$ connected components is 6-large. Indeed, if $k = 0, 1$, or 2 , then we have shown \widehat{X}_Z is 6-large. If $3 \leq k < n$, then by induction we assume \widehat{X}_Z is systolic and by Proposition 4.7 it is 6-large. It remains to show that \widehat{X}_Γ is locally 6-large.

Consider an arbitrary simplex $h\Delta_U$ of \widehat{X}_Γ of type U . By Proposition 3.5, we know that $Lk_{\widehat{X}_\Gamma}(h\Delta_U)$ is isomorphic to the 2-complete Artin complex associated with Artin group A_U . By construction, U cannot contain generators from all the connected components of $\widehat{\Gamma}$, so as a 2-complete Artin complex $Lk_{\widehat{X}_\Gamma}(h\Delta_U)$ is associated with strictly fewer than n connected components. So by the induction hypothesis, $Lk_{\widehat{X}_\Gamma}(h\Delta_U)$ is 6-large. Since our choice of simplex was arbitrary, \widehat{X}_Γ is locally 6-large.

Hence if there are at least three connected components of $\widehat{\Gamma}$, \widehat{X}_Γ is systolic. \square

5. WEAKLY MALNORMAL

In this section, we will use the systolicity of the 2-complete Artin complex and the following well-known result from systolic geometry to prove that most parabolic subgroups of locally reducible Artin groups are weakly malnormal. (See, e.g., [13, Lemma 14].)

Lemma 5.1. *Let G be a group acting without inversions on a systolic simplicial complex X . If $H \leq G$ fixes two vertices of X , then H fixes pointwise every combinatorial geodesic between them.*

A combinatorial geodesic between two vertices is a minimal-length edge path. Note that in a systolic simplicial complex, there could be many distinct combinatorial geodesics between two given vertices.

Definition 5.2. Let G be a group. A subgroup H of G is **weakly malnormal** if there exists $g \in G$ such that $|H \cap gHg^{-1}| < \infty$.

Theorem 1.3. *Let A_Γ be a locally reducible Artin group such that $\widehat{\Gamma}$ has at least two connected components. Then any parabolic subgroup P of A_Γ which has a 2-completion that is not all of A_Γ is weakly malnormal.*

Proof. Let V be an arbitrary connected component of $\widehat{\Gamma}$. We will start by showing that the parabolic subgroup $A_{\Gamma \setminus V}$ is weakly malnormal. Our goal is to construct an element $g \in A_\Gamma$ such that $|A_{\Gamma \setminus V} \cap gA_{\Gamma \setminus V}g^{-1}| < \infty$; in fact, we will show this intersection is trivial.

Recall that by the construction of \widehat{X}_Γ , there is a vertex v in the fundamental domain K whose local group is $A_{\Gamma \setminus V}$. Furthermore, there is a face of K of dimension $(\dim K - 1)$ “opposite” to v whose local group is A_V . Specifically, this face is determined by vertices associated to parabolic subgroups of the form $A_{\Gamma \setminus T}$ for $T \neq V$. Since v is the only vertex of K not contained in this face, v is adjacent to every vertex in the face corresponding to A_V .

Let the generators of V be labeled $\{v_1, \dots, v_\ell\}$ and let $g = v_1 \dots v_\ell$. We will look at the translate of K by this element g . By construction, g setwise fixes the face of K corresponding to A_V and translates vertex v (corresponding to $A_{\Gamma \setminus V}$) to the vertex $w = gv$ corresponding to $gA_{\Gamma \setminus V}$. Similarly to v , w is connected to all vertices in the face of K corresponding to parabolic subgroups of the form $A_{\Gamma \setminus T}$ for $T \neq V$. Hence the combinatorial distance between vertices v and w is 2.

In the case that $\widehat{\Gamma}$ has three or more connected components, any choice of vertex of K on the face A_V corresponds to a unique combinatorial geodesic between v and w . By Theorem 1.4, \widehat{X}_Γ is systolic. Thus a subgroup that fixes two vertices must pointwise fix every combinatorial geodesic between them by Lemma 5.1. Since $\text{Stab}(v) = A_{\Gamma \setminus V}$ and $\text{Stab}(w) = gA_{\Gamma \setminus V}g^{-1}$, every combinatorial geodesic between these vertices is fixed by $\text{Stab}(v) \cap \text{Stab}(w) = A_{\Gamma \setminus V} \cap gA_{\Gamma \setminus V}g^{-1}$. Hence $A_{\Gamma \setminus V} \cap gA_{\Gamma \setminus V}g^{-1}$ pointwise fixes all the vertices of K , so it must pointwise fix K itself. The local group of K is A_\emptyset and thus has trivial stabilizer, so $A_{\Gamma \setminus V} \cap gA_{\Gamma \setminus V}g^{-1}$ must be trivial.

In the case that $\widehat{\Gamma}$ has exactly two connected components, K is an edge, and the face of K corresponding to A_V is itself a vertex. In that case we have an edge path of length 2: $v = A_{\Gamma \setminus V}$ to A_V to $w = gA_{\Gamma \setminus V}$. By Lemma 4.6, \widehat{X}_Γ is one-dimensional and 6-large. Thus this path of length 2 from v to w must be the unique combinatorial geodesic between v and w , else \widehat{X}_Γ would have a cycle of length 4. Since this geodesic is unique, it is fixed by $\text{Stab}(v) \cap \text{Stab}(w)$. Then as above, $A_{\Gamma \setminus V} \cap gA_{\Gamma \setminus V}g^{-1}$ must pointwise fix K , and so the intersection must be trivial.

Hence in both cases, $A_{\Gamma \setminus V}$ is weakly malnormal. Thus any subgroup of $A_{\Gamma \setminus V}$ is weakly malnormal, and any conjugate is also weakly malnormal. Hence if P is a parabolic subgroup with a 2-completion that is not all of A_Γ , then P is weakly malnormal. \square

6. ACYLINDRICAL HYPERBOLICITY

The following are theorems of Martin and Minasyan and Osin, respectively, which we will use to apply our result about weakly malnormal subgroups and show that certain locally reducible Artin groups are acylindrically hyperbolic.

Theorem 6.1. [25, Theorem B] Let X be a CAT(0) simplicial complex, with an action by simplicial isomorphisms of a group G . Assume there exists a vertex v of X such that:

- (1) The orbits of G on the link $Lk_X(v)$ are unbounded for the associated angular metric.
- (2) $\text{Stab}(v)$ is weakly malnormal in G .

Then G is either virtually cyclic or acylindrically hyperbolic.

Theorem 6.2. [29, Corollary 2.2] Let G split as an amalgamated product: $G = A *_C B$ such that $A \neq C \neq B$ and C is weakly malnormal in G . Then G is either virtually cyclic or acylindrically hyperbolic.

Theorem 1.2. Let A_Γ be an Artin group. If either

- (1) A_Γ is locally reducible and has a maximal finite-type subgroup which is dihedral and which has a 2-completion that is not all of A_Γ , or
- (2) A_Γ splits as an amalgamated product $A_{\Gamma_1} *_{A_{\Gamma_1 \cap \Gamma_2}} A_{\Gamma_2}$ such that A_{Γ_1} is locally reducible and there is a 2-completion of $A_{\Gamma_1 \cap \Gamma_2}$ which does not contain all of A_{Γ_1} ,

then A_Γ is acylindrically hyperbolic.

Proof. Let A_Γ an Artin group.

(1): We assume that A_Γ is locally reducible and has a maximal finite-type subgroup which is dihedral and which has a 2-completion that is not all of A_Γ . Thus Γ contains at least one edge, say $[a, b]$, with $m_{ab} \geq 3$, such that $A_{\{a,b\}}$ is a maximal finite-type standard parabolic subgroup (i.e., $A_{\{a,b\}}$ is not contained in a larger finite-type parabolic subgroup) and there is a 2-completion of $A_{\{a,b\}}$ that is not all of A_Γ .

Since A_Γ is locally reducible, the Deligne complex D_Γ associated to A_Γ is CAT(0) by [8]. Consider $Lk_{D_\Gamma}(A_{\{a,b\}})$. There is a path in the link given by the sequence $\{g_i A_\emptyset\}_{i=0}^\infty$ where $g_i = ababa\dots$ is a word of length i . By [37, Lemma 4.5] the length of g_i is unbounded, and by Theorem 1.3, $A_{\{a,b\}}$ is weakly malnormal. Hence by Theorem 6.1, A_Γ is acylindrically hyperbolic.

(2): Now we assume that A_Γ splits as an amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_{\Gamma_1 \cap \Gamma_2}} A_{\Gamma_2}$ such that A_{Γ_1} is locally reducible and there is a 2-completion of $A_{\Gamma_1 \cap \Gamma_2}$ which does not contain all of A_{Γ_1} . By Theorem 1.3,

$A_{\Gamma_1 \cap \Gamma_2}$ is weakly malnormal in A_{Γ_1} , and hence is weakly malnormal in A_Γ . By Theorem 6.2, A_Γ is either virtually cyclic or acylindrically hyperbolic. Since A_Γ is an amalgamated product, it is not virtually cyclic, hence it is acylindrically hyperbolic. \square

Remark 6.3. One may wonder whether the assumption that $\widehat{\Gamma}$ has two connected components is necessary. There are certainly counterexamples to acylindrical hyperbolicity when $\widehat{\Gamma}$ has one connected component. Take, for example, an Artin group which splits as a direct product. Then $\widehat{\Gamma}$ has one connected component, and A_Γ is not acylindrically hyperbolic.

7. FUTURE DIRECTIONS

We can ask whether the intersection of a collection of parabolic subgroups of a particular Artin group is itself a parabolic subgroup. This is another question that is open in general, but is conjectured to be true for all Artin groups. It has been proven to be true for several classes including finite-type Artin groups [14], right-angled Artin groups (Artin groups for which $m_{ij} = 2$ or ∞ for all i, j) [16], large-type Artin groups (Artin groups for which $m_{ij} \geq 3$ for all i, j) [13], and 2-dimensional, (2,2)-free Artin groups [3]. It is also known to be true for the intersection of two parabolic subgroups of an FC-type Artin group, when at least one of the parabolic subgroups is finite-type [31] and [30].

In many cases when the affirmative answer is known, the solution has come from the study of one or more geometric spaces with some property of non-positive curvature (e.g. hyperbolic, CAT(0), systolic). In this case, when stabilizers of vertices (or simplices, cubes, etc.) are parabolic subgroups, the geometric properties of the action help greatly in the study of their intersection. Examples include the Deligne complex and the Artin complex (see Section 2).

Question 7.1. *For A_Γ a locally reducible Artin group, is the intersection of an arbitrary collection of 2-complete parabolic subgroups itself a parabolic subgroup?*

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