# **BI-LIPSCHITZ RIGIDITY OF DISCRETE SUBGROUPS**

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ABSTRACT. We obtain a bi-Lipschitz rigidity theorem for a Zariski dense discrete subgroup of a connected simple real algebraic group. As an application, we show that any Zariski dense discrete subgroup of a higher rank semisimple algebraic group G cannot have a  $C^1$ -smooth slim limit set in G/P for any non-maximal parabolic subgroup P.

### 1. INTRODUCTION

For i = 1, 2, let  $G_i$  be a connected simple real algebraic group and  $\Gamma_i$  a Zariski dense discrete subgroup of  $G_i$ . Let

$$\rho: \Gamma_1 \to \Gamma_2$$

be an isomorphism. The classical rigidity problem searches for a condition on  $\rho$  which guarantees that  $\rho$  is algebraic, that is, it extends to a Lie group isomorphism  $G_1 \to G_2$ .

If  $\Gamma_1$  is a lattice in  $G_1$  and either

- $G_1 = G_2$  has rank one and is not locally isomorphic to  $PSL_2(\mathbb{R})$ , or
- $G_1$  has higher rank,

then any isomorphism  $\rho : \Gamma_1 \to \Gamma_2$  is algebraic by celebrated theorems of Mostow, Prasad, and Margulis ([16], [17], [15]). On the other hand, there are very few rigidity theorems for non-lattice discrete subgroups, especially in higher rank. In this article, we provide a rigidity criterion  $\rho : \Gamma_1 \to \Gamma_2$  in terms of a  $\rho$ -boundary map between the limit sets of  $\Gamma_1$  and  $\Gamma_2$ .

Since  $\Gamma_i$  is Zariski dense, there exists a unique  $\Gamma_i$ -minimal subset  $\Lambda_i$  in  $\mathcal{F}_i = G_i/P_i$  for a parabolic subgroup  $P_i$  of  $G_i$ , called the limit set. When both parabolic subgroups are maximal, our result takes the following simple form:

**Theorem 1.1** (Bi-Lipschitz rigidity theorem I). Assume that  $P_1$  and  $P_2$  are maximal parabolic subgroups. Let  $\rho : \Gamma_1 \to \Gamma_2$  be an isomorphism. If there exists a bi-Lipschitz  $\rho$ -equivariant map  $f : \Lambda_1 \to \Lambda_2$ , then  $\rho$  extends to a Lie group isomorphism

 $\bar{\rho}: G_1 \to G_2$ 

which induces a diffeomorphism  $\bar{f}: \mathcal{F}_1 \to \mathcal{F}_2$  such that  $\bar{f}|_{\Lambda_1} = f$ .

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Recall that  $f : \Lambda_1 \to \Lambda_2$  is bi-Lipschitz if there exists  $C \ge 1$  such that for all  $\xi, \eta \in \Lambda_1$ ,

(1.1) 
$$C^{-1}d_{\mathcal{F}_1}(\xi,\eta) \le d_{\mathcal{F}_2}(f(\xi),f(\eta)) \le Cd_{\mathcal{F}_1}(\xi,\eta)$$

where  $d_{\mathcal{F}_i}$  is a Riemannian metric on  $\mathcal{F}_i$  for i = 1, 2. Since any two Riemannian metrics on  $\mathcal{F}_i$  are bi-Lipschitz equivalent to each other, this notion is well-defined. We note that there can be at most one  $\rho$ -equivariant map  $f : \Lambda_1 \to \Lambda_2$  [12, Lemma 4.5]. We emphasize that we do not require f to be defined on all of  $\mathcal{F}_1$ , but only on  $\Lambda_1$ . For  $G_1 = G_2 = \mathrm{SO}(n, 1)^\circ$ ,  $n \geq 2$ , Theorem 1.1 was proved by Tukia [29, Theorem D].

- Remark 1.2. (1) The hypothesis that  $G_1$  and  $G_2$  are simple is necessary; see Remark 4.8.
  - (2) The global bi-Lipschitz hypothesis on f can be replaced by the condition that f is bi-Lipschitz on some non-empty open subset of  $\Lambda_1$ ; see Lemma 4.9.

We now state a general version of Theorem 1.1 where  $P_1$  and  $P_2$  are arbitrary parabolic subgroups.

**Theorem 1.3** (Bi-Lipschitz rigidity theorem II). Let  $\rho : \Gamma_1 \to \Gamma_2$  be an isomorphism. If there exists a bi-Lipschitz  $\rho$ -equivariant map  $f : \Lambda_1 \to \Lambda_2$ , then  $\rho$  extends to a Lie group isomorphism

$$\bar{\rho}: G_1 \to G_2.$$

Moreover, there exists a parabolic subgroup  $P'_2$  of  $G_2$  containing  $P_2$  such that  $\bar{\rho}(P_1) \subset P'_2$  up to a conjugation and the smooth submersion  $G_1/P_1 \rightarrow G_2/P'_2$  induced by  $\bar{\rho}$  coincides with the composition  $\pi \circ f$  on  $\Lambda_1$  where  $\pi : G_2/P_2 \rightarrow G_2/P'_2$  is the canonical factor map.

$$\begin{array}{ccc} \Lambda_1 & & \stackrel{f}{\longrightarrow} & \Lambda_2 \\ & & & & \downarrow \\ & & & & \downarrow \\ G_1/P_1 & \xrightarrow{\bar{\rho}} & G_2/P_2 \end{array}$$

See Theorem 4.7 for a stronger version which relaxes the bi-Lipschitz condition to a  $\kappa$ -bi-Hölder condition for  $\kappa > 0$ .

Remark 1.4. In general,  $P'_2$  is not the same as  $P_2$ . We use the theory of hyperconvex subgroups to construct a Zariski dense discrete subgroup of  $SL_8(\mathbb{R})$  which demonstrates this point in Proposition 6.1.

Theorem 1.3 also has consequences for the regularity of the limit set of  $\Gamma$  in G/P when G is a higher rank semisimple real algebraic group and P is a non-maximal parabolic subgroup.

**Theorem 1.5** (Regularity of slim limit sets). Let G be a connected semisimple real algebraic group of rank at least 2 and P a non-maximal parabolic subgroup of G. Any Zariski dense discrete subgroup of G cannot have a slim limit set in G/P which is a  $C^1$ -submanifold.

Note that any non-maximal parabolic subgroup P is contained in at least two non-conjugate maximal parabolic subgroups of G. We call a subset  $S \subset G/P$  slim if there exists a pair of non-conjugate maximal parabolic subgroups  $P_1, P_2$  containing P such that the canonical factor map  $\pi_i: G/P \to G/P_i$  is injective on S for i = 1, 2.



In particular, the limit set of any subgroup of a *P*-Anosov or relatively *P*-Anosov subgroup is always slim. More generally, if any two points in the limit set are in general position, then the limit set is slim.

The non-maximal hypothesis on P in Theorem 1.5 is necessary, as there are many Zariski dense discrete subgroups of  $PSL_n(\mathbb{R})$ ,  $n \geq 3$ , whose limit sets are  $C^1$ -submanifolds of  $\mathbb{P}(\mathbb{R}^n)$ , e.g., images of Hitchin [14] and Benoist representations [2]. We remark that the limit sets of these examples are not  $C^2$  as shown by Zimmer [32].

- Remark 1.6. (1) When G is of rank one, the limit set  $\Lambda$  of a Zariski dense subgroup of G is not a proper  $C^r$ -submanifold of G/P where r = 1 for  $G = SO(n, 1)^\circ$  and r = 2 for other rank one groups ([30, Proposition 3.12 and Corollary 3.13]). In higher rank, there exists  $0 < r < \infty$ , depending on G, such that  $\Lambda$  is not a proper  $C^r$ -submanifold of G/P for any parabolic subgroup P [5, Lemma 2.11].
  - (2) Theorem 1.5 was previously established for images of Hitchin representations [26, Corollary 6.1] and for images of (1, 1, 2)-hyperconvex representation of a surface group [18, Corollary 7.7]. We also mention [6], [31], and [20] for related work on the regularity of the limit set for certain classes of subgroups of G = SO(d, 2),  $PSL_d(\mathbb{R})$  and SO(p,q) respectively.

**On the proofs.** We deduce Theorem 1.3 and Theorem 1.5 from the following property of limit sets of a Zariski dense subgroup in higher rank:

**Proposition 1.7.** Let G be a connected semisimple real algebraic group of rank at least 2. Let  $Q_1$  and  $Q_2$  be a pair of parabolic subgroups of G such that there is no parabolic subgroup of G containing  $Q_1$  and a conjugate of  $Q_2$  (e.g., a pair of non-conjugate maximal parabolic subgroups).

If  $\Gamma < G$  is a Zariski dense discrete subgroup, then there is no  $\Gamma$ -equivariant bi-Lipchitz map between the limit sets of  $\Gamma$  on  $G/Q_1$  and  $G/Q_2$ .

Indeed, if  $\rho$  in Theorem 1.3 does not extend to a Lie group isomorphism  $G_1 \to G_2$ , then the following self-joining subgroup

(1.2) 
$$\Gamma = (\mathrm{id} \times \rho)(\Gamma_1) = \{(g, \rho(g)) : g \in \Gamma_1\}$$

is a Zariski dense subgroup of the product  $G = G_1 \times G_2$ . On the other hand, a bi-Lipschitz map f as in Theorem 1.3 yields a bi-Lipschitz homeomorphism between the limit sets of the self-joining group  $\Gamma$  in  $G/(P_1 \times G_2)$ and  $G/(G_1 \times P_2)$ , which then gives a desired contradiction by Proposition 1.7. We mention the recent work [10] and [11] on related rigidity theorems which use the idea of self-joinings.

If  $\Gamma$  has a  $C^1$ -slim limit set in G/P as in Theorem 1.5 and  $P_1$  and  $P_2$  are non-conjugate maximal parabolic subgroups containing P, we get a bi-Lipschitz map between the limit sets of  $\Gamma$  in  $G/P_1$  and  $G/P_2$  from the slimneess hypothesis. Therefore Proposition 1.7 implies Theorem 1.5.

For the proof of Proposition 1.7, we relate the exponential contraction rates of loxodromic elements  $\gamma \in \Gamma$  on  $G/Q_i$  with the Jordan projections of the image of  $\gamma$  under Tits representations of G. This part of the argument is motivated by earlier work of Zimmer [32, Section 8]. We then show that the bi-Lipschitz equivalence of the limit sets gives an obstruction to Benoist's theorem [1] on the non-empty interior property of the limit cone of a Zariski dense subgroup (see the proof of Proposition 4.3).

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### 2. Preliminaries

Unless mentioned otherwise, let G be a connected semisimple *real* algebraic group throughout the paper. This means that G is the identity component  $\mathbf{G}(\mathbb{R})^{\circ}$  for a semisimple algebraic group  $\mathbf{G}$  defined over  $\mathbb{R}$ . A parabolic  $\mathbb{R}$ -subgroup  $\mathbf{P}$  of  $\mathbf{G}$  is a proper algebraic subgroup defined over  $\mathbb{R}$ such that the quotient  $\mathbf{G}/\mathbf{P}$  is a projective algebraic variety. A parabolic subgroup P of G is of the form  $\mathbf{P}(\mathbb{R})$  for a parabolic  $\mathbb{R}$ -subgroup  $\mathbf{P}$  of  $\mathbf{G}$ ; in this case, the quotient G/P is equal to  $(\mathbf{G}/\mathbf{P})(\mathbb{R})$  and is a real projective variety, called a G-boundary [3]. Any parabolic subgroup P is conjugate to a unique standard parabolic subgroup of G, once we fix a root system associated to G.

To be precise, let A be a maximal real split torus of G. The rank of G is defined as the dimension of A. Let  $\mathfrak{g}$  and  $\mathfrak{a}$  respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$  and set  $A^+ = \exp \mathfrak{a}^+$ , and a maximal compact subgroup K < G such that the Cartan decomposition  $G = KA^+K$  holds. We denote by M the centralizer of A in K. For  $g \in G$ , we denote by  $\mu(g)$  the Cartan projection of g, which is the unique element of  $\mathfrak{a}^+$  such that  $g \in K \exp \mu(g)K$ .

Any  $g \in G$  can be written as the commuting product  $g = g_h g_e g_u$  where  $g_h$  is hyperbolic,  $g_e$  is elliptic and  $g_u$  is unipotent. The hyperbolic component

 $g_h$  is conjugate to a unique element  $\exp \lambda(g) \in A^+$  and

(2.1) 
$$\lambda(g) \in \mathfrak{a}^+$$

is called the Jordan projection of g. When  $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$ ,  $g \in G$  is called *loxodromic* in which case  $g_u$  is necessarily trivial and  $g_e$  is conjugate to an element  $m \in M$ .

Let  $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$  denote the set of all roots and  $\Pi$  the set of all simple roots given by the choice of  $\mathfrak{a}^+$ . The Weyl group  $\mathcal{W}$  is given by  $N_K(A)/M$ where  $N_K(A)$  is the normalizer of A in K.

Consider the real vector space  $\mathsf{E}^* = \mathsf{X}(A) \otimes_{\mathbb{Z}} \mathbb{R}$  where  $\mathsf{X}(A)$  is the group of all real characters of A and let  $\mathsf{E}$  be its dual. Denote by  $(\cdot, \cdot)$  a  $\mathcal{W}$ -invariant inner product on  $\mathsf{E}$ . We denote by  $\{\omega_{\alpha} : \alpha \in \Pi\}$  the (restricted) fundamental weights of  $\Phi$  defined by

$$2\frac{(\omega_{\alpha},\beta)}{(\beta,\beta)} = c_{\alpha}\delta_{\alpha,\beta}$$

where  $c_{\alpha} = 1$  if  $2\alpha \notin \Phi$  and  $c_{\alpha} = 2$  otherwise.

Fix an element  $w_0 \in N_K(A)$  of order 2 representing the longest Weyl element so that  $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$ . The map

$$\mathbf{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

is called the opposition involution. It induces an involution of  $\Phi$  preserving  $\Pi$ , for which we use the same notation i, so that  $i(\alpha) = \alpha \circ i$  for all  $\alpha \in \Phi$ .

For a non-empty subset  $\theta$  of  $\Pi$ , let  $\mathfrak{a}_{\theta} = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$ , and let  $P_{\theta}$  denote a standard parabolic subgroup of G corresponding to  $\theta$ ; that is,  $P_{\theta} = L_{\theta}N_{\theta}$ where  $L_{\theta}$  is the centralizer of  $\exp \mathfrak{a}_{\theta}$  and  $N_{\theta}$  is the unipotent radical of  $P_{\theta}$ which is generated by root subgroups associated to all positive roots which are not  $\mathbb{Z}$ -linear combinations of elements of  $\Pi - \theta$ . If  $\theta = \Pi$ , then  $P = P_{\Pi}$ is a minimal parabolic subgroup. For a singleton  $\theta = {\alpha}, P_{\alpha}$  is a maximal parabolic subgroup of G. Any parabolic subgroup P is conjugate to a unique standard parabolic subgroup  $P_{\theta}$  for some non-empty subset  $\theta \subset \Pi$ .

We consider the  $\theta$ -boundary:

$$\mathcal{F}_{\theta} = G/P_{\theta}.$$

We denote by  $d_{\mathcal{F}_{\theta}}$  a Riemannian metric on  $\mathcal{F}_{\theta}$ . Let  $P_{\theta}^{+} = w_0 P_{i(\theta)} w_0^{-1}$ , which is the standard parabolic subgroup opposite to  $P_{\theta}$  such that  $P_{\theta} \cap P_{\theta}^{+} = L_{\theta}$ . Hence  $\mathcal{F}_{i(\theta)} = G/P_{i(\theta)} = G/P_{\theta}^{+}$ . The *G*-orbit  $\mathcal{F}_{\theta}^{(2)} = \{(gP_{\theta}, gw_0P_{i(\theta)}) : g \in G\}$  is the unique open *G*-orbit in  $G/P_{\theta} \times G/P_{\theta}^{+}$  under the diagonal *G*action. Two elements  $\xi \in \mathcal{F}_{\theta}$  and  $\eta \in \mathcal{F}_{i(\theta)}$  are said to be in general position if  $(\xi, \eta) \in \mathcal{F}_{\theta}^{(2)}$ .

# 3. Contraction rates of loxodromic elements and Tits representations

The first part of the following theorem immediately follows as a special case of a theorem of Tits [25], and the second part is remarked in [1] and proved in [23].

**Theorem 3.1** ([25, Theorem 7.2], [23, Lemma 2.13]). For each  $\alpha \in \Pi$ , there exists an irreducible representation  $\rho_{\alpha} : G \to \operatorname{GL}(V_{\alpha})$  whose highest (restricted) weight  $\chi_{\alpha}$  is equal to  $k_{\alpha}\omega_{\alpha}$  for some positive integer  $k_{\alpha}$  and whose highest weight space is one-dimensional.

Moreover, all weights of  $\rho_{\alpha}$  are  $\chi_{\alpha}$ ,  $\chi_{\alpha} - \alpha$  and weights of the form  $\chi_{\alpha} - \alpha - \sum_{\beta \in \Pi} n_{\beta}\beta$  with  $n_{\beta}$  non-negative integers.

These representations are called Tits representations of G. Fix  $\alpha \in \Pi$  and, as before, set  $\mathcal{F}_{\alpha} = G/P_{\alpha}$ . We denote by  $V_1$  and  $V_2$  the weight spaces of  $\rho_{\alpha}$ for the highest weight  $\chi_{\alpha}$  and the second highest weight  $\chi_{\alpha} - \alpha$  respectively. We have dim  $V_1 = 1$  and dim  $V_2 \geq 1$ . If we set  $\xi_{\alpha} = [P_{\alpha}] \in \mathcal{F}_{\alpha}$ , the map  $g\xi_{\alpha} \mapsto gV_1$  gives an embedding

(3.1) 
$$\mathcal{F}_{\alpha} \to \mathbb{P}(V_{\alpha})$$

whose image is a closed subvariety. We may hence identify  $\mathcal{F}_{\alpha}$  as a closed subvariety of  $\mathbb{P}(V_{\alpha})$ . Let  $\langle \cdot, \cdot \rangle_{\alpha}$  be a *K*-invariant inner product on  $V_{\alpha}$  with respect to which *A* is symmetric and we have the orthogonal weight space decomposition of  $V_{\alpha}$ . Using the norms on  $V_{\alpha}$  and  $\wedge^2 V_{\alpha}$  induced by this inner product, we get a *K*-invariant Riemannian metric  $d_{\alpha}$  on  $\mathbb{P}(V_{\alpha})$ :

$$d_{\alpha}([v], [w]) = \frac{\|v \wedge w\|}{\|v\| \|w\|} \quad \text{for } [v], [w] \in \mathbb{P}(V_{\alpha}).$$

Recall that an element  $g \in G$  is *loxodromic* if there exist  $a \in \text{int } A^+$  and  $m \in M$  such that  $g = h_g am h_g^{-1}$  for some  $h_g \in G$ . The element  $h_g$  is then uniquely determined modulo AM and  $\lambda(g) = \log a \in \text{int } \mathfrak{a}^+$ .

Let  $\pi_i = \pi_{\alpha,i} : V_{\alpha} \to V_i$  be the orthogonal projection for i = 1, 2. Recall the following standard lemma:

**Lemma 3.2.** Let g be a loxodromic element of G. For  $\xi \in \mathcal{F}_{\alpha}$ , we have  $\pi_1(h_q^{-1}\xi) \neq 0$  if and only if  $g^n\xi$  converges to  $h_g\xi_{\alpha}$  as  $n \to \infty$ .

The point  $y_{\alpha}^{g} := h_{g} \xi_{\alpha} \in \mathcal{F}_{\alpha}$  is called the attracting fixed point of g.

**Lemma 3.3.** Let  $g \in G$  be a loxodromic element and  $\alpha \in \Pi$ .

(1) For all  $\xi \in \mathcal{F}_{\alpha}$  with  $\pi_1(h_q^{-1}\xi) \neq 0$ , we have

$$-\alpha(\lambda(g)) \ge \limsup_{n \to \infty} \frac{1}{n} \log d_{\alpha}(g^n \xi, y_{\alpha}^g).$$

(2) For all  $\xi \in \mathcal{F}_{\alpha}$  with  $\pi_1(h_g^{-1}\xi) \neq 0$  and  $\pi_2(h_g^{-1}\xi) \neq 0$ , we have

$$-\alpha(\lambda(g)) = \lim_{n \to \infty} \frac{1}{n} \log d_{\alpha}(g^n \xi, y^g_{\alpha}).$$

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Proof. It suffices to prove the claim when  $h_g = e$ , i.e.,  $g = am \in AM$ with  $\log a \in \operatorname{int} \mathfrak{a}^+$ . Considering  $\xi \in \mathcal{F}_\alpha \subset \mathbb{P}(V_\alpha)$ , choose a vector  $v \in V_\alpha$ representing  $\xi$ . List all distinct weights of  $\rho_\alpha$  given by Theorem 3.1 as follows:  $\chi_1 = \chi_\alpha, \chi_2 = \chi_\alpha - \alpha$ , and  $\chi_i = \chi_\alpha - \alpha - \beta_i, 3 \leq i \leq \ell$ ; in particular,  $\beta_i \neq 0$  is a non-negative integral linear combinations of simple roots. Let  $V_i$ denote the weight space corresponding to  $\chi_i$  and write  $v = v_1 + v_2 + \cdots + v_\ell$ so that  $v_i \in V_i$  for each  $1 \leq i \leq \ell$ . Suppose that  $\pi_1(\xi) \neq 0$ , that is  $v_1 \neq 0$ . We may then assume that  $v_1$  is a unit vector relative to  $\langle \cdot, \cdot \rangle_\alpha$ . Since Mcommutes with A, M stabilizes each weight subspace, and in particular,  $Mv_1 = \pm v_1$ . Now

$$g^{n}v = e^{n\chi_{\alpha}(\log a)}m^{n}v_{1} + e^{n(\chi_{\alpha}-\alpha)(\log a)}m^{n}v_{2} + \sum_{i=3}^{\ell}e^{n(\chi_{\alpha}-\alpha-\beta_{i})(\log a)}m^{n}v_{i}.$$

Hence the projection  $p(g^n v)$  of  $g^n v$  to the affine chart  $\mathbb{A} = \{ w \in V_\alpha : \pi_1(w) = v_1 \}$  is

$$p(g^{n}v) = v_{1} + e^{-n\alpha(\log a)}m^{n}v_{2}' + \sum_{i=3}^{\ell} e^{-n(\alpha+\beta_{i})(\log a)}m^{n}v_{i}'$$

where  $v'_i = \pm v_i$ , depending on the sign of  $m^n v_1$ . Note that  $\lim g^n \xi = V_1$ , and that the metric  $d_{\alpha}$  on a neighborhood on  $V_1$  in  $\mathbb{P}(V_{\alpha})$  is bi-Lipschitz equivalent to the metric d on the affine chart  $\mathbb{A}$ , obtained by restricting the distance on  $V_{\alpha}$  induced by  $\langle \cdot, \cdot \rangle_{\alpha}$ .

Since the weight spaces are orthogonal, we have

$$d(p(g^n v), v_1) = e^{-n\alpha(\log a)} \left( \|v_2\|^2 + \|w_n\|^2 \right)^{1/2}$$

where  $w_n = \sum_{i=3}^{\ell} e^{-n\beta_i(\log a)} m^n v'_i$  and  $\|\cdot\|$  is the norm induced by  $\langle \cdot, \cdot \rangle_{\alpha}$ . Since  $\log a \in \operatorname{int} \mathfrak{a}^+$  and hence  $\beta_i(\log a) > 0$  for all  $3 \le i \le \ell$ , we have

$$\lim_{n \to \infty} w_n = 0.$$

First consider the case when  $\pi_2(\xi) = 0$ , that is  $v_2 = 0$ . Since  $\log ||w_n|| < 0$  for all large n, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log d_{\alpha}(g^{n}\xi, y_{\alpha}^{g}) = \limsup_{n \to \infty} \frac{1}{n} \log d(p(g^{n}v), v_{1})$$
$$= \limsup_{n \to \infty} \frac{1}{n} (-n\alpha(\log a) + \log ||w_{n}||) \le -\alpha(\log a).$$

Now suppose that  $\pi_2(\xi) \neq 0$ , that is  $v_2 \neq 0$ . Again since  $w_n \to 0$ , we have

$$\lim_{n \to \infty} \frac{1}{n} \log d_{\alpha}(g^n \xi, y^g_{\alpha}) = \lim_{n \to \infty} \frac{1}{n} \log d(p(g^n v), v_1) = -\alpha(\log a).$$

This finishes the proof.

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#### 4. BI-LIPSCHITZ RIGIDITY OF DISCRETE SUBGROUPS

Let G be a connected semisimple real algebraic group and X = G/K be the associated Riemannian symmetric space and fix  $o = [K] \in X$ .

We consider the following notion of convergence of a sequence in G to an element of  $\mathcal{F}_{\theta} = G/P_{\theta}$  for a non-empty subset  $\theta \subset \Pi$ .

For a sequence  $g_i o \in X$  and  $\xi \in \mathcal{F}_{\theta}$ , we write  $\lim g_i o = \xi$  and say  $g_i o \in X$  converges to  $\xi$  if

(1)  $\min_{\alpha \in \theta} \alpha(\mu(g_i)) \to \infty$  as  $i \to \infty$ ; and

(2)  $\lim_{i\to\infty} \kappa_{g_i} P_{\theta} = \xi$  in  $\mathcal{F}_{\theta}$  for some  $\kappa_{g_i} \in K$  such that  $g_i \in \kappa_{g_i} A^+ K$ .

**Definition 4.1.** Let  $\Gamma < G$  be a discrete subgroup and let  $\mathcal{F} = G/P$  for a parabolic subgroup P. Let  $\theta \subset \Pi$  be a unique subset such that P is conjugate to  $P_{\theta}$  and hence  $\mathcal{F} = \mathcal{F}_{\theta}$ . The limit set of  $\Gamma$  in  $\mathcal{F}_{\theta}$  is then defined as the set of all accumulation points of  $\Gamma(o)$  in  $\mathcal{F}_{\theta}$ :

$$\Lambda_{\theta} = \Lambda_{\theta}(\Gamma) = \{\lim \gamma_i(o) \in \mathcal{F}_{\theta} : \gamma_i \in \Gamma\}.$$

It is a  $\Gamma$ -invariant closed subset of  $\mathcal{F}_{\theta}$ , which is non-empty provided  $\Gamma$  contains a sequence  $\gamma_i$  satisfying  $\lim_{i\to\infty} \min_{\alpha\in\theta} \alpha(\mu(\gamma_i)) = \infty$ . If  $\Gamma$  is Zariski dense,  $\Lambda_{\theta}$  is the unique  $\Gamma$ -minimal subset of  $\mathcal{F}_{\theta}$  and can also be described as the set of all  $\xi \in \mathcal{F}_{\theta}$  such that the Dirac measure  $\delta_{\xi}$  is the weak limit of  $(\gamma_i)_*$  Leb<sub> $\theta$ </sub> for some sequence  $\gamma_i \in \Gamma$  where Leb<sub> $\theta$ </sub> denotes the unique *K*invariant probability measure on  $\mathcal{F}_{\theta}$  ([1], [21]). Moreover, if  $\Theta \subset \theta$ , then  $\Lambda_{\Theta}$  is equal to the image of  $\Lambda_{\theta}$  under the canonical projection  $\mathcal{F}_{\theta} \to \mathcal{F}_{\Theta}$ , by minimality.

The limit cone of  $\Gamma$  is defined as the smallest closed cone of  $\mathfrak{a}^+$  containing all Jordan projections of loxodromic elements of  $\Gamma$ .

**Theorem 4.2** (Benoist [1]). If  $\Gamma < G$  is Zariski dense, its limit cone has non-empty interior in  $\mathfrak{a}$ .

For  $\kappa > 0$  and  $\theta_1, \theta_2 \subset \Pi$ , a map  $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$  is called  $\kappa$ -bi-Hölder if there exists C > 0 such that for all  $x, y \in \Lambda_{\theta_1}$ 

(4.1) 
$$C^{-1}d_{\mathcal{F}_{\theta_1}}(x,y)^{\kappa} \le d_{\mathcal{F}_{\theta_2}}(F(x),F(y)) \le Cd_{\mathcal{F}_{\theta_1}}(x,y)^{\kappa}$$

where  $d_{\mathcal{F}_{\theta_i}}$  is a Riemannian metric on  $\mathcal{F}_{\theta_i}$  for i = 1, 2. Observe that if  $\Gamma$  is Zariski dense, any  $\Gamma$ -equivariant  $\kappa$ -bi-Hölder map  $\Lambda_{\theta_1} \to \Lambda_{\theta_2}$  is a homeomorphism; the minimality of  $\Lambda_{\theta_2}$  implies the surjectivity and the bi-Hölder property implies the injectivity. Therefore F is  $\kappa$ -bi-Hölder if and only if F is  $\kappa$ -Hölder and  $F^{-1}$  is  $\kappa^{-1}$ -Hölder.

Proposition 1.7 follows from the following for  $\kappa = 1$ :

**Proposition 4.3.** Let  $\Gamma < G$  be Zariski dense. Let  $\theta_1$  and  $\theta_2$  be disjoint non-empty subsets of  $\Pi$ . Then for any  $\kappa > 0$ , there exists no  $\Gamma$ -equivariant  $\kappa$ -bi-Hölder map  $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$ .

*Proof.* For simplicity, we write  $\Lambda_i = \Lambda_{\theta_i}$  and  $d_{\theta_i} = d_{\mathcal{F}_{\theta_i}}$ . Let  $F : \Lambda_1 \to \Lambda_2$  be a  $\Gamma$ -equivariant homeomorphism. Fix  $\kappa > 0$ . Since  $\theta_1 \cap \theta_2 = \emptyset$ , the union

 $\bigcup_{\alpha_1 \in \theta_1, \alpha_2 \in \theta_2} \ker(\kappa \alpha_1 - \alpha_2) \text{ is a finite union of hyperplanes of } \mathfrak{a}.$  Therefore by Theorem 4.2,  $\Gamma$  contains a loxodromic element  $\gamma$  such that

$$\{\kappa \cdot \alpha(\lambda(\gamma)) : \alpha \in \theta_1\} \cap \{\alpha(\lambda(\gamma)) : \alpha \in \theta_2\} = \emptyset.$$

For each i = 1, 2, let  $\alpha_i \in \theta_i$  be such that

(4.2) 
$$\alpha_i(\lambda(\gamma)) = \min\{\alpha(\lambda(\gamma)) : \alpha \in \theta_i\}.$$

Note that

(4.3) 
$$\kappa \cdot \alpha_1(\lambda(\gamma)) \neq \alpha_2(\lambda(\gamma)).$$

**Claim:** If  $F^{-1}$  is  $\kappa^{-1}$ -Hölder, then

(4.4) 
$$\alpha_2(\lambda(\gamma)) \le \kappa \cdot \alpha_1(\lambda(\gamma)).$$

By replacing  $\Gamma$  by a suitable conjugate, we may also assume that  $\gamma = am \in \Gamma$  with  $a \in \text{int } A^+$  and  $m \in M$ . For each i = 1, 2, let  $y_i = y_{\alpha_i}^{\gamma}$  denote the attracting fixed point of  $\gamma$  in  $\mathcal{F}_i$ ; we have  $y_i \in \Lambda_i$ . As  $\Gamma$  is Zariski dense,  $\Lambda_i$  is Zariski dense in  $\mathcal{F}_i$  for each i = 1, 2. Let  $\pi_{\alpha,1}$  and  $\pi_{\alpha,2}$  be as in Lemmas 3.2 and 3.3 for each  $\alpha \in \Pi$ . Since the set

$$\mathcal{O} = \{\xi \in \mathcal{F}_1 : \pi_{\alpha,1}(\xi) \neq 0, \pi_{\alpha,2}(\xi) \neq 0 \text{ for all } \alpha \in \theta_1\}$$

is a Zariski open subset of  $\mathcal{F}_1$ , the intersection  $\mathcal{O} \cap \Lambda_1$  is a non-empty open subset of  $\Lambda_1$ . As F is a homeomorphism, the image  $F(\mathcal{O} \cap \Lambda_1)$  is a non-empty open subset of  $\Lambda_2$ . Since  $Z = \{\xi \in \mathcal{F}_2 : \gamma^n \xi \not\to y_2 \text{ as } n \to \infty\}$  is a proper Zariski closed subset of  $\mathcal{F}_2$  by Lemma 3.2,  $F(\mathcal{O} \cap \Lambda_1)$  cannot be contained in Z; otherwise it would imply that  $\Lambda_2$  is contained in a proper Zariski closed subset by the  $\Gamma_2$ -minimality of  $\Lambda_2$ , which contradicts the Zariski density of  $\Gamma_2$ . Therefore there exists an element  $\xi \in \mathcal{O} \cap \Lambda_1$  such that  $\lim_{n\to\infty} \gamma^n F(\xi) = y_2$ . By the equivariance and continuity of F, we have

(4.5) 
$$F(y_1) = \lim F(\gamma^n \xi) = \lim \gamma^n F(\xi) = y_2.$$

Let i = 1, 2. Since  $P_{\theta_i} = \bigcap_{\alpha \in \theta_i} P_{\alpha}$ , we have a diagonal embedding

$$\mathcal{F}_i = G/P_{\theta_i} \to \prod_{\alpha \in \theta_i} \mathbb{P}(V_\alpha)$$

via the product of the maps in (3.1). Consider the metric  $d_i$  on  $\mathcal{F}_i$  obtained as the restriction of  $\sum_{\alpha \in \theta_i} d_{\alpha}$  to  $\mathcal{F}_i$ : for  $\eta = gP_{\theta_1}$  and  $\eta' = g'P_{\theta_2}$  with  $g, g' \in G$ ,

$$d_i(\eta, \eta') = \sum_{\alpha \in \theta_i} d_\alpha(\eta, \eta')$$

where  $d_{\alpha}(\eta, \eta') := d_{\alpha}(gV_{\alpha,1}, g'V_{\alpha,1})$  where  $V_{\alpha,1}$  is the highest weight line of  $\rho_{\alpha}$  as in (3.1). Since  $d_i$  is bi-Lipschitz equivalent to a Riemannian metric on  $\mathcal{F}_i$ , we have that  $F^{-1}: (\Lambda_2, d_2) \to (\Lambda_1, d_1)$  is  $\kappa^{-1}$ -Hölder.

Since  $\xi \in \mathcal{O}$  and  $\lim \gamma^n F(\xi) = y_2$ , we have by Lemma 3.3 that

$$-\alpha(\lambda(\gamma)) = \lim \frac{1}{n} \log d_{\alpha}(\gamma^{n}\xi, y_{1}) \quad \text{for each } \alpha \in \theta_{1}$$

and

$$-\alpha(\lambda(\gamma)) \ge \limsup \frac{1}{n} \log d_{\alpha}(\gamma^{n} F(\xi), y_{2}) \text{ for each } \alpha \in \theta_{2}.$$

Since  $d_{\alpha_1}(\eta, \eta') \leq d_1(\eta, \eta')$ ,  $d_2(\eta, \eta') \leq \#\theta_2 \max_{\alpha \in \theta_2} d_\alpha(\eta, \eta')$ , and  $F^{-1}$  is  $\kappa^{-1}$ -Hölder, we have

$$(4.6) \qquad -\alpha_1(\lambda(\gamma)) = \lim \frac{1}{n} \log d_{\alpha_1}(\gamma^n \xi, y_1) \\ \leq \lim \frac{1}{n} \log d_1(\gamma^n \xi, y_1) \\ \leq \kappa^{-1} \limsup \frac{1}{n} \log d_2(F(\gamma^n \xi), F(y_1)) \\ = \kappa^{-1} \limsup \frac{1}{n} \log d_2(\gamma^n F(\xi), y_2) \\ = \kappa^{-1} \max_{\alpha \in \theta_2} \limsup \frac{1}{n} \log d_\alpha(\gamma^n F(\xi), y_2) \\ \leq -\kappa^{-1} \min_{\alpha \in \theta_2} \alpha(\lambda(\gamma)) = -\kappa^{-1} \alpha_2(\lambda(\gamma)).$$

This implies that  $\alpha_2(\lambda(\gamma)) \leq \kappa \alpha_1(\lambda(\gamma))$ , proving the claim.

By switching the role of  $\theta_1$  and  $\theta_2$ , this claim then implies that if F is  $\kappa$ -Hölder, then  $\alpha_1(\lambda(\gamma)) \leq \kappa^{-1}\alpha_2(\lambda(\gamma))$ . Therefore if F is  $\kappa$ -bi-Hölder, then  $\kappa \cdot \alpha_1(\lambda(\gamma)) = \alpha_2(\lambda(\gamma))$ , contradicting (4.3). This finishes the proof.

The proof of Proposition 4.3 shows the following as well:

**Proposition 4.4.** Let  $\Gamma < G$  be Zariski dense and let  $\theta_1, \theta_2 \subset \Pi$  be nonempty disjoint subsets. Suppose that  $\Lambda_{\theta_1}$  and  $\Lambda_{\theta_2}$  are  $C^1$ -submanifolds of  $\mathcal{F}_{\theta_1}$ and  $\mathcal{F}_{\theta_2}$  respectively. If  $F : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$  is a  $\Gamma$ -equivariant homeomorphism, F cannot be  $C^1$  with non-vanishing Jacobian at any  $\xi \in \mathcal{A}$ , where  $\mathcal{A} \subset \Lambda_{\theta_1}$ is the set of all attracting fixed points of loxodromic elements  $\gamma \in \Gamma$  such that  $\{\alpha(\lambda(\gamma)) : \alpha \in \theta_1\} \cap \{\alpha(\lambda(\gamma)) : \alpha \in \theta_2\} = \emptyset$ .

Proof. Let  $\gamma \in \Gamma$  be as above. For each i = 1, 2, let  $y_i \in \Lambda_{\theta_i}$  be the attracting fixed point of  $\gamma$ . Then  $F(y_1) = y_2$  by (4.5). Suppose that F is  $C^1$  at  $y_1$ , and the Jacobian of F at  $y_1$  is not zero. Then  $F^{-1}$  is also  $C^1$  at  $y_2$ . Using the exponential maps and the Taylor series expansion of F, we get that there exist  $c \geq 1$  and an open neighborhood U of  $y_1$  in  $\Lambda_{\theta_1}$  such that for all  $y \in U$ ,

(4.7) 
$$c^{-1}d_1(y,y_1) \le d_2(F(y),F(y_1)) \le cd_1(y,y_1).$$

Let  $\alpha_i \in \theta_i$  be as in (4.2). Without loss of generality, we may assume  $\alpha_1(\lambda(\gamma)) < \alpha_2(\lambda(\gamma))$  by switching the indexes if necessary. On the other hand, using (4.7), the computation (4.6) gives  $\alpha_2(\lambda(\gamma)) \leq \alpha_1(\lambda(\gamma))$ , which yields a contradiction.

Remark 4.5. It would be interesting to know whether  $\mathcal{A}$  can be replaced by the set of all *conical* limit points of  $\Gamma$  in Proposition 4.4. A point  $\xi = gP_{\theta_1}$ is  $\Gamma$ -conical if  $\limsup \Gamma g(K \cap P_{\theta_1})A^+ \neq \emptyset$ , that is, there exists a sequence

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 $\gamma_i \in \Gamma$ ,  $a_i \in A^+$  and  $m_i \in K \cap P_{\theta_i}$  such that  $\gamma_i g m_i a_i$  converges (see [13, Lemma 5.4] for an equivalent definition in terms of shadows).

This question is inspired by a related result for  $G = SO(n+1,1)^{\circ}$ . Tukia [27] showed that if  $f : \mathbb{S}^n \to \mathbb{S}^n$  is a homeomorphism which conjugates a discrete subgroup  $\Gamma_1$  of G to another discrete group  $\Gamma_2$  and has a nonvanishing Jacobian at a conical limit point of  $\Gamma_1$ , then  $\Gamma_1$  is conjugate to  $\Gamma_2$ (see also [7] for an extension of this result to other rank one groups). For a related result for (1, 1, 2)-hyperconvex groups, see [18, Corollary 7.5].

In the rest of this section, let  $G_i$  be a connected simple real algebraic group and  $\theta_i$  be a non-empty set of simple roots of  $G_i$  for i = 1, 2. Let  $\Gamma_i < G_i$  be a Zariski dense discrete subgroup and  $\Lambda_{\theta_i}$  denote the limit set of  $\Gamma_i$  in  $\mathcal{F}_i = G_i/P_{\theta_i}$ .

**Lemma 4.6.** [12, Lemma 4.5] For any isomorphism  $\rho : \Gamma_1 \to \Gamma_2$ , there exists at most one  $\rho$ -equivariant continuous map  $f : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$ .

Indeed, f must send the attracting fixed point of any loxodromic element  $\gamma$  to that of  $\rho(\gamma)$  whenever  $\rho(\gamma)$  is loxodromic. Since the set of attracting fixed points of loxodromic elements is dense in  $\Lambda_{\theta_1}$  by the Zariski density hypothesis on  $\Gamma_1$  [1] and f is continuous, this determines the map f.

Theorem 1.3 is a special case of the following theorem for  $\kappa = 1$ :

**Theorem 4.7.** Suppose that there exists a  $\rho$ -equivariant  $\kappa$ -bi-Hölder map  $f: \Lambda_{\theta_1} \to \mathcal{F}_2$  for some  $\kappa > 0$ . Then  $\rho$  extends to a Lie group isomorphism  $\bar{\rho}: G_1 \to G_2$ . Moreover, there exists a non-empty subset  $\Theta_2 \subset \theta_2$  such that  $\bar{\rho}$  maps  $P_{\theta_1}$  into a conjugate of  $P_{\Theta_2}$  and the smooth submersion  $G_1/P_{\theta_1} \to G_2/P_{\Theta_2}$  induced by  $\bar{\rho}$  coincides with the composition  $\pi \circ f$  on  $\Lambda_{\theta_1}$  where  $\pi: G_2/P_{\theta_2} \to G_2/P_{\Theta_2}$  is the canonical factor map.

*Proof.* Let  $G = G_1 \times G_2$ . Define the following self-joining subgroup

$$\Gamma = (\mathrm{id} \times \rho)(\Gamma_1) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma_1\} < G.$$

Note that  $P_1 := P_{\theta_1} \times G_2$  and  $P_2 := G_1 \times P_{\theta_2}$  are parabolic subgroups of G. The maps  $g_1P_{\theta_1} \mapsto (g_1, e)P_1$  and  $g_2P_{\theta_2} \mapsto (e, g_2)P_2$  define diffeomorphisms between  $G_1/P_{\theta_1}$  and  $G_2/P_{\theta_2}$  with  $G/P_1$  and  $G/P_2$  respectively. Moreover, under this identification, the limit set  $\Lambda_{\theta_i}$  of  $\Gamma_i$  in  $G_i/P_{\theta_i}$  corresponds to the limit set  $\Lambda_i$  of the self-joining  $\Gamma$  in  $G/P_i$  for each i = 1, 2.

Since f is a  $\rho$ -equivariant continuous embedding of  $\Lambda_{\theta_1}$  into  $G/P_{\theta_2}$ , its image is a  $\Gamma_2$ -invariant compact subset. Since  $\Lambda_{\theta_1}$  is a  $\Gamma_1$ -minimal subset, the image  $f(\Lambda_{\theta_1})$  is also a  $\Gamma_2$ -minimal subset. Therefore  $f(\Lambda_{\theta_1}) = \Lambda_{\theta_2}$  and hence we have a  $\Gamma$ -equivariant bijection  $f : \Lambda_1 \to \Lambda_2$  which is  $\kappa$ -bi-Hölder.

Since  $P_1$  and  $P_2$  are parabolic subgroups corresponding to disjoint subsets of simple roots of G, Proposition 4.3 implies that  $\Gamma$  cannot be Zariski dense in G. Since both  $G_1$  and  $G_2$  are simple, the non-Zariski density of the selfjoining group  $\Gamma$  implies that  $\rho$  extends to a Lie group isomorphism  $\bar{\rho}: G_1 \to$  $G_2$  (cf. [4]). Since  $\bar{\rho}(P_{\theta_1})$  must be a parabolic subgroup of  $G_2$ , there exists  $g \in G_2$ such that  $\bar{\rho}(P_{\theta_1}) = gP_{\theta_0}g^{-1}$  where  $\theta_0$  is a non-empty subset of some simple roots of  $G_2$ . We claim  $\theta_0 \cap \theta_2 \neq \emptyset$ . By replacing  $\rho$  by  $\operatorname{inn}(g) \circ \rho$  where  $\operatorname{inn}(g) : G_2 \to G_2$  is the conjugation by g, we may assume without loss of generality that g = e. The isomorphism  $\bar{\rho}$  induces a diffeomorphism  $\tilde{\Phi} : G_1/P_{\theta_1} \to G_2/P_{\theta_0}$  given by  $\tilde{\Phi}(g_1P_{\theta_1}) = \bar{\rho}(g_1)P_{\theta_0}$ . Denote by  $\Lambda_{\theta_0}$  the limit set of  $\Gamma_2$  in  $G_2/P_{\theta_0}$ . Since  $\bar{\rho}|_{\Gamma_1} = \rho$  and hence  $\tilde{\Phi}$  is  $\rho$ -equivariant, we have  $\tilde{\Phi}(\Lambda_{\theta_1}) = \Lambda_{\theta_0}$ . Then the composition  $F := f \circ \tilde{\Phi}^{-1}$  restricted to  $\Lambda_{\theta_0}$ yields a  $\kappa$ -bi-Hölder map between  $\Lambda_{\theta_0}$  and  $\Lambda_{\theta_2}$ . Since  $\tilde{\Phi}^{-1}$  is  $\rho^{-1}$ -equivariant and f is  $\rho$ -equivariant, F is  $\Gamma_2$ -equivariant. So by applying Proposition 4.3 one more time, we obtain  $\theta_0 \cap \theta_2 \neq \emptyset$ . Setting  $\Theta_2 = \theta_0 \cap \theta_2$ , since  $P_{\theta_0}$ and  $P_{\theta_2}$  are subgroups of  $P_{\Theta_2}$ , we get a map  $\Phi := G_1/P_{\theta_1} \to G_2/P_{\Theta_2}$  by composing  $\tilde{\Phi}$  with the canonical factor map  $G_1/P_{\theta_0} \to G_2/P_{\Theta_2}$ . The last claim  $\Phi = \pi \circ f$  on  $\Lambda_{\theta_1}$  follows from Lemma 4.6. This finishes the proof.  $\Box$ 

Remark 4.8. The hypothesis that  $G_1$  and  $G_2$  are simple is necessary in Theorem 4.7. For example, consider a discrete Zariski dense subgroup  $\Gamma$  of a simple algebraic group G with a discrete faithful representation  $\rho: \Gamma \to G$ which does not extend to G. Then  $\Gamma_{\rho} = (\mathrm{id} \times \rho)(\Gamma)$  is Zariski dense in G and the map  $\gamma \to (\gamma, \rho(\gamma))$  gives an isomorphism  $\Gamma \to \Gamma_{\rho}$ . On the other hand, for any parabolic subgroup P of G, the isomorphism  $G/P \simeq (G \times G)/(P \times G)$ provides an equivariant bi-Lipschitz bijection the limit set of  $\Gamma$  in G/P and the limit set of  $\Gamma_{\rho}$  in  $(G \times G)/(P \times G)$ .

We note that the global bi-Hölder condition in Proposition 4.3 and Theorem 4.7 can be relaxed to a local bi-Hölder condition by the following lemma.

**Lemma 4.9.** Keep the notation as in Theorem 4.7 but assume  $G_1$  and  $G_2$  are semisimple, not just simple. Let  $f : \Lambda_{\theta_1} \to \Lambda_{\theta_2}$  be a  $\rho$ -equivariant homeomorphism which is  $\kappa$ -bi-Hölder on some non-empty open subset U of  $\Lambda_{\theta_1}$  for some  $\kappa > 0$ . Then f is  $\kappa$ -bi-Hölder globally.

*Proof.* Let  $\Lambda_i = \Lambda_{\theta_i}$  for i = 1, 2. Since  $\Lambda_1$  is  $\Gamma_1$ -minimal,  $\Lambda_1 = \Gamma_1 U$ and hence, by compactness, we have  $\Lambda_1$  is a finite union of  $\gamma_k U$  for some  $\gamma_1, \dots, \gamma_n \in \Gamma_1$ . If f is not  $\kappa$ -Hölder globally, by the compactness of  $\Lambda_1$ , we have a sequence  $\xi_i \to \xi$  and  $\eta_i \to \eta$  such that

(4.8) 
$$\frac{d_{\mathcal{F}_2}(f(\xi_i), f(\eta_i))}{d_{\mathcal{F}_1}(\xi_i, \eta_i)^{\kappa}} \to \infty$$

Since  $\mathcal{F}_2$  is compact, we have  $d_{\mathcal{F}_1}(\xi_i, \eta_i) \to 0$ . Therefore, for some  $1 \leq k \leq n, \ \xi_i, \eta_i \in \gamma_k U$  for all *i*. Noting that the action of each element of  $g_i \in G_i$  on  $\mathcal{F}_i$  is a diffeomorphism for i = 1, 2, we can let L be the maximum of the bi-Lipschitz constants of  $\gamma_k$  on  $\mathcal{F}_1$  and of  $\rho(\gamma_k)$  on  $\mathcal{F}_2$ . Now we have  $d_{\mathcal{F}_2}(f(\xi_i), f(\eta_i)) \leq Ld_{\mathcal{F}_2}(f(\gamma_k^{-1}\xi_i, \gamma_k^{-1}\eta_i))$  and  $d_{\mathcal{F}_1}(\xi_i, \eta_i) \geq L^{-1}d_{\mathcal{F}_1}(\gamma_k^{-1}\xi_i, \gamma_k^{-1}\eta_i))$ . Since f is  $\kappa$ -Hölder on U, it follows that the ratio in (4.8) is bounded, yielding a contradiction. This shows that f is  $\kappa$ -Hölder

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globally. Similarly by considering  $f^{-1}$ , we can show that  $f^{-1}$  is  $\kappa^{-1}$ -Hölder globally.

Theorem 1.1 is now a special case of the following corollary of Theorem 4.7 together with Lemma 4.9:

**Corollary 4.10.** Let  $\alpha_i$  be a simple root of  $G_i$  for i = 1, 2. Suppose that there exists a  $\rho$ -equivariant bijection  $f : \Lambda_{\alpha_1} \to \Lambda_{\alpha_2}$  which is  $\kappa$ -bi-Hölder on some non-empty open subset of  $\Lambda_{\alpha_1}$  for some  $\kappa > 0$ . Then  $\kappa = 1$  and  $\rho$  extends to a Lie group isomorphism  $\bar{\rho} : G_1 \to G_2$  which induces a diffeomorphism  $\bar{f} : G_1/P_{\alpha_1} \to G_2/P_{\alpha_2}$  such that  $\bar{f}|_{\Lambda_1} = f$ .

Note that the conclusion  $\kappa = 1$  follows since  $\bar{f}$  is diffeomorphism and hence bi-Lipschitz.

Remark 4.11. In general, we cannot replace f bi-Lipschitz by Lipschitz in Theorem 1.1. For example, let  $\Gamma$  be a Schottky subgroup of  $\operatorname{SL}_2(\mathbb{R})$  generated by two loxodromic elements a, b. Then for any  $N \geq 2$ , the representation  $\rho$  of  $\Gamma$  into  $\operatorname{SL}_2(\mathbb{R})$  given by  $a \mapsto a^N$  and  $b \mapsto b^N$  induces an equivariant homeomorphism  $\Lambda \to \Lambda$  which is Lipschitz, but not bi-Lipschitz. Clearly,  $\rho$ does not extend to  $\operatorname{SL}_2(\mathbb{R})$ .

On the other hand, we have the following corollary of the proof of Theorem 4.7 where f is required only to be Lipschitz under an extra hypothesis on the Hausdorff dimension of limit sets. In the statement below, a Möbius transformation is the extension of *any* isometry of  $\mathbb{H}^{n+1}$  to its boundary  $\mathbb{S}^n = \partial \mathbb{H}^{n+1}$ .

**Corollary 4.12.** For i = 1, 2, let  $\Gamma_i$  be a convex cocompact Zariski dense subgroup of  $G_i = SO^{\circ}(n_i + 1, 1), n_i \geq 1$ . Let  $\Lambda_i \subset \mathbb{S}^{n_i}$  be the limit set of  $\Gamma_i$ . Suppose that the Hausdorff dimension of  $\Lambda_1$  is equal to the Hausdorff dimension of  $\Lambda_2$ . Let  $f : \Lambda_1 \to \Lambda_2$  be a  $\rho$ -equivariant homeomorphism which is Lipschitz on some non-empty open subset of  $\Lambda_1$ . Then  $\rho$  extends to a Lie group isomorphism of  $G_1 \to G_2$  and f extends to a Möbius transformation of  $\mathbb{S}^n$  for  $n = n_1 = n_2$ .

Proof. By the proof of Lemma 4.9, f is Lipschitz on all of  $\Lambda_1$ . Let  $\Gamma := (\mathrm{id} \times \rho)(\Gamma_1)$  be the self-joining subgroup of  $G = G_1 \times G_2$ . For i = 1, 2, let  $\alpha_i$  be the simple root of  $G = G_1 \times G_2$  from the *i*-th factor. Then for any loxodromic element  $g = (\gamma, \rho(\gamma)) \in G$ ,  $\alpha_1(\lambda(g))$  and  $\alpha_2(\lambda(g))$  are equal to  $\lambda(\gamma)$  and  $\lambda(\rho(\gamma))$  respectively. Suppose that  $\Gamma$  is Zariski dense in G. The proof of Proposition 4.3 for  $\Gamma$  shows that if there exists a loxodromic element  $g = (\gamma, \rho(\gamma)) \in \Gamma$  such that  $\alpha_1(\lambda(g)) > \alpha_2(\lambda(g))$ , then  $f : \Lambda_1 \to \Lambda_2$  cannot be Lipschitz. On the other hand, if  $\Lambda_1$  and  $\Lambda_2$  have the same Hausdorff dimension, the middle direction  $(1, 1) \in \mathfrak{a} \simeq \mathbb{R}^2$  is always contained in the interior of the limit cone of  $\Gamma$  by [9, Corollary 4.2]. Note that when  $\Gamma_i$  are cocompact lattices and  $n_1 = n_2 = 2$ , [9, Corollary 4.2] is due to Thurston [24]. Therefore, the desired element  $g \in \Gamma$  can always be found. This

implies that  $\Gamma$  cannot be Zariski dense in G. As before, this implies the conclusion.

# 5. Slim limit sets of G/P for P non-maximal

Let  $\Gamma$  be a Zariski dense subgroup of a connected semisimple real algebraic group G. Fix a subset  $\theta \subset \Pi$  with  $\#\theta \geq 2$ . Recall from the introduction that a subset  $S \subset \mathcal{F}_{\theta}$  is called *slim* if there exists a pair of distinct elements  $\alpha_1$  and  $\alpha_2$  of  $\theta$  such that the limit set  $\Lambda_{\theta}$  injects to  $G/P_{\alpha_1}$  and  $G/P_{\alpha_2}$  under the canonical projection map  $\mathcal{F}_{\theta} \to G/P_{\alpha_i}$  for i = 1, 2.

In this section we prove the following theorem.

**Theorem 5.1.** If  $\#\theta \ge 2$  and  $\Lambda_{\theta}$  is a slim subset of  $\mathcal{F}_{\theta}$ , then no non-empty open subset U of  $\Lambda_{\theta}$  is contained in a proper C<sup>1</sup>-submanifold of  $\mathcal{F}_{\theta}$ .

We first prove the following lemma which connects Theorem 5.1 with Proposition 4.3.

**Lemma 5.2.** Let  $\theta_0 \subset \theta \subset \Pi$ . Suppose that  $\Lambda_{\theta}$  is a  $C^1$ -submanifold of  $\mathcal{F}_{\theta}$  and that the canonical projection  $\mathcal{F}_{\theta} \to \mathcal{F}_{\theta_0}$  is injective on  $\Lambda_{\theta}$ . Then  $\Lambda_{\theta_0}$  is a  $C^1$ -submanifold of  $\mathcal{F}_{\theta_0}$  and  $f_{\theta_0} : \Lambda_{\theta} \to \Lambda_{\theta_0}$  is a  $\Gamma$ -equivariant diffeomorphism.

Proof. For simplicity, we write  $\Lambda = \Lambda_{\theta}$ . We suppose that  $\Lambda$  is a  $C^1$ submanifold of  $\mathcal{F}_{\theta}$ . Since the projection  $\mathcal{F}_{\theta} \to \mathcal{F}_{\theta_0}$  given by  $f(gP_{\theta}) = gP_{\theta_0}$ is a smooth map, its restriction  $f : \Lambda \to \mathcal{F}_{\theta_0}$  is a  $C^1$  map which is also injective by hypothesis. We claim that there exists a point  $x \in \Lambda$  where  $df_x : T_x\Lambda \to T_{f(x)}\mathcal{F}_{\theta_0}$  is injective. Pick a point  $x \in \Lambda$  which maximizes rank  $df_y, y \in \Lambda$ . Then there exists a neighborhood of x in  $\Lambda$  where df has constant rank. Then if  $r := \operatorname{rank} df_x$ , there exist local coordinates near xwhere

$$f(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

Since f is injective, we must have r = m and hence  $df_x$  is injective.

Now the set  $\{x \in \Lambda : df_x \text{ is injective}\}$  is open and  $\Gamma$ -invariant. Since  $\Gamma$  acts minimally on  $\Lambda$ , this set must be all of  $\Lambda$ . Thus f is an immersion. Since f is an injective immersion and  $\Lambda$  is compact, f is a  $C^1$ -embedding. Hence f is a diffeomorphism onto its image, which is  $\Lambda_{\theta_0}$ . In particular,  $\Lambda_{\theta_0}$ is a  $C^1$ -submanifold of  $\mathcal{F}_{\theta_0}$ .

**Proof of Theorem 5.1.** By the hypothesis on the slimness of  $\Lambda_{\theta}$ , there exists a pair of distinct elements  $\alpha_1$  and  $\alpha_2$  of  $\theta$  such that  $\Lambda_{\theta}$  injects to  $G/P_{\alpha_1}$  and  $G/P_{\alpha_2}$ .

Suppose on the contrary that some non-empty open subset U of  $\Lambda_{\theta}$  is contained in some  $C^1$ -submanifold. Since  $\Lambda_{\theta}$  is  $\Gamma$ -minimal, we have that for any  $\xi \in \Lambda_{\theta}$ ,  $\Gamma\xi$  is dense, so  $\gamma\xi \in U$  for some  $\gamma \in \Gamma$ . Since  $\xi \in \gamma^{-1}U$ , it follows that  $\Lambda_{\theta}$  is a  $C^1$ -submanifold of  $\mathcal{F}_{\theta}$ . By Lemma 5.2, we have  $\Gamma$ -equivariant diffeomorphisms  $f_{\alpha_i} : \Lambda_{\theta} \to \Lambda_{\alpha_i}$  for each i = 1, 2. Hence

 $f_{\alpha_2} \circ f_{\alpha_1}^{-1} : \Lambda_{\alpha_1} \to \Lambda_{\alpha_2}$  is a  $\Gamma$ -equivariant diffeomorphism, contradicting Proposition 4.3. This finishes the proof.

Remark 5.3. We remark that Proposition 4.3 implies that if  $\Lambda$  is a slim subset of G/P, then there exists a maximal parabolic subgroup Q containing P such that the projection  $G/P \to G/Q$  restricted to  $\Lambda$  is not bi-Lipschitz.

Antipodal groups. Theorem 1.5 applies to the class of P-antipodal discrete subgroups of G, which contains any subgroup of a P-Anosov or a relatively P-Anosov subgroup. To define an antipodality, we recall that a parabolic subgroup P is called reflexive if its conjugacy class contains a parabolic subgroup P' opposite to P, that is,  $P \cap P'$  is a common Levi subgroup of both P and P'. For example, a minimal parabolic subgroup of G is always reflexive. For a parabolic subgroup P, let  $P_{\text{reflexive}}$  be the largest reflexive parabolic subgroup contained in P. If  $P = P_{\theta}$ , then  $P_{\text{reflexive}} = P_{\theta \cup i(\theta)}$ .

**Definition 5.4.** A discrete subgroup  $\Gamma$  is called *P*-antipodal if its limit set in  $G/P_{\text{reflexive}}$  is antipodal in the sense that any two distinct points are in general position.

If a discrete subgroup  $\Gamma$  is *P*-antipodal, then its limit set on G/P injects to G/P' for any P' containing P [13, Lemma 9.5]. Hence if  $\Gamma$  is *P*-antipodal for a non-maximal parabolic subgroup P, then its limit set is a slim subset of G/P. Therefore the following corollary is a special case of Theorem 1.5.

**Corollary 5.5.** Let G be a connected semisimple real algebraic group of rank at least 2 and P a non-maximal parabolic subgroup of G. The limit set of a Zariski dense P-antipodal subgroup of G cannot be a  $C^1$ -submanifold of G/P.

Note that there are many slim limit sets which are not antipodal (e.g., the limit set of a self-joining group defined in (1.2)).

### 6. An example

In this final section, we construct an example of a Zariski dense discrete subgroup of  $SL_8(\mathbb{R})$  which explains the necessity of introducing  $P'_2$  in the conclusion of Theorem 1.3 in the case when  $P_2$  is not maximal. The examples we construct are Borel-Anosov and (1,1,2)-hyperconvex subgroups of  $SL_8(\mathbb{R})$ .

We begin by setting up some notation. For any  $d \ge 2$ , let A be the diagonal subgroup of  $\operatorname{SL}_d(\mathbb{R})$  consisting of diagonal elements with positive entries so that  $\mathfrak{a}$  and  $\mathfrak{a}^+$  can respectively be identified with  $\mathfrak{a} = \{(u_1, \dots, u_d) : \sum_{k=1}^d u_k = 0\}$  and  $\mathfrak{a}^+ = \{(u_1, \dots, u_d) \in \mathfrak{a} : u_1 \ge \dots \ge u_d\}$ . For  $1 \le k \le d-1$ , let

$$\alpha_k((u_1,\cdots,u_d)) = u_k - u_{k+1};$$

then  $\Pi = \{\alpha_k : 1 \leq k \leq d-1\}$  is the set of all simple roots. For any  $g \in \mathrm{SL}_d(\mathbb{R})$ , its Jordan projection  $\lambda(g) \in \mathfrak{a}^+$  satisfies

$$\alpha_k(\lambda(g)) = \log \frac{\lambda_k(g)}{\lambda_{k+1}(g)}$$

where  $\lambda_1(g) \geq \cdots \geq \lambda_d(g)$  are the absolute values of the eigenvalues of g. Also, for  $\theta \subset \Pi$ , the boundary  $\mathcal{F}_{\theta} = \mathrm{SL}_d(\mathbb{R})/P_{\theta}$  coincides with the partial flag manifold consisting of flags with subspaces of dimensions  $\{k : \alpha_k \in \theta\}$ .

Let  $\Delta$  be a hyperbolic group and denote by  $\partial \Delta$  its Gromov boundary. Recall from [8] that a representation  $\rho : \Delta \to \operatorname{SL}_d(\mathbb{R})$  is  $\{\alpha_k\}$ -Anosov if there exist constants c, C > 0 so that for all  $\gamma \in \Delta$ ,

$$\alpha_k(\lambda(\rho(\gamma)) \ge c|\gamma| - C$$

where  $|\gamma|$  is the minimal word length of  $\gamma$  with respect to a fixed finite generating set of  $\Delta$ . If  $\rho$  is  $\{\alpha_k\}$ -Anosov, it admits a pair of unique continuous equivariant embeddings  $\xi_{\rho}^k : \partial \Delta \to \operatorname{Gr}_k(\mathbb{R}^d)$  and  $\xi_{\rho}^{d-k} : \partial \Delta \to \operatorname{Gr}_{d-k}(\mathbb{R}^d)$ . Furthermore, the image of  $(\xi_{\rho}^k, \xi_{\rho}^{d-k})$  coincides with the limit set of  $\rho(\Delta)$ in  $\mathcal{F}_{\{\alpha_k, \alpha_{d-k}\}}$ . We say that  $\rho$  is Borel-Anosov if it is  $\{\alpha_k\}$ -Anosov for all  $1 \leq k \leq d-1$ . The image of a Borel-Anosov representation is called a Borel Anosov subgroup.

A representation  $\rho : \Delta \to \mathrm{SL}_d(\mathbb{R})$  is (1, 1, 2)-hyperconvex if it is  $\{\alpha_1, \alpha_2\}$ -Anosov and for all distinct  $x, y, z \in \partial \Delta$ ,

$$\xi^1_{\rho}(x) \oplus \xi^1_{\rho}(y) \oplus \xi^{d-2}_{\rho}(z) = \mathbb{R}^d.$$

Both being  $\{\alpha_k\}$ -Anosov and being (1, 1, 2)-hyperconvex are open conditions in the representation variety (see [19, Proposition 6.2]).

**Proposition 6.1.** There exists a Zariski dense discrete subgroup  $\Gamma < SL_8(\mathbb{R})$ which admits an equivariant Lipschitz bijection  $\Lambda_{\alpha_3} \to \Lambda_{\alpha_1}$ . Moreover,  $\Gamma$  is Borel-Anosov, (1, 1, 2)-hyperconvex, and the projection map  $p : \Lambda_{\{\alpha_1, \alpha_3\}} \to \Lambda_{\alpha_3}$  is a bi-Lipschitz bijection.

Theorem 1.3 in this case applies with  $f = p^{-1}$ ,  $P_1 = P_{\alpha_3}$ ,  $P_2 = P_{\{\alpha_1,\alpha_3\}}$ and  $P'_2 = P_{\alpha_3}$ .

Let  $\Delta = \langle a_1, a_2 \rangle$  be the free group with two generators  $a_1, a_2$ . Let  $N \geq 2$ . Let  $\tau_1 : \Delta \to \operatorname{SL}_2(\mathbb{R})$  be a convex cocompact representation and  $\tau_2 : \Delta \to \operatorname{SL}_2(\mathbb{R})$  be defined so that  $\tau_2(a_i) = \tau_1(a_i)^N$  for i = 1, 2. We may choose N large enough that

$$\frac{\alpha_1(\lambda(\tau_2(\gamma)))}{\alpha_1(\lambda(\tau_1(\gamma)))} \ge 4 \quad \text{for all non-trivial} \quad \gamma \in \Delta.$$

Let  $\iota : \operatorname{SL}_2(\mathbb{R}) \to \operatorname{SL}_4(\mathbb{R})$  be an irreducible representation, which is unique up to conjugations. Then each  $\rho_i = \iota \circ \tau_i$  is a positive representation and hence Borel Anosov and (1, 1, 2)-hyperconvex [19, Corollary 6.13]. One easily checks that  $\frac{\alpha_1(\lambda(\rho_2(\gamma)))}{\alpha_1(\lambda(\rho_1(\gamma)))} \ge 4$  for all non-trivial  $\gamma \in \Delta$ . Then a theorem of Tsouvalas [26, Theorem 1.9] implies that  $f_{\rho_1,\rho_2} = \xi^1_{\rho_2} \circ (\xi^1_{\rho_1})^{-1}$  is 4-Hölder. Let  $\Phi_0 : \Delta \to SL_8(\mathbb{R})$  denote the representation given by the direct sum  $\Phi_0 = \rho_1 \oplus \rho_2$ . One checks that

 $\lambda_1(\rho_2(\gamma)) > \lambda_2(\rho_2(\gamma)) > \lambda_1(\rho_1(\gamma)) > \dots > \lambda_4(\rho_1(\gamma)) > \lambda_3(\rho_2(\gamma)) > \lambda_4(\rho_2(\gamma))$ for all non-trivial  $\gamma \in \Delta$  and that  $\Phi_0$  is Borel Anosov with limit maps given by

$$\zeta_0^k(x) = \begin{cases} \{0\} \oplus \xi_{\rho_2}^k(x) & \text{if } k = 1, 2\\ \xi_{\rho_1}^{k-2}(x) \oplus \xi_{\rho_2}^2(x) & \text{if } k = 3, 4, 5\\ \mathbb{R}^4 \oplus \xi_{\rho_2}^{k-4}(x) & \text{if } k = 6, 7. \end{cases}$$

Then, the fact that  $f_{\rho_1,\rho_2}$  is 4-Hölder implies that  $\zeta_0^1 \circ (\zeta_0^3)^{-1}$  is also 4-Hölder. In particular,  $\zeta_0^1 \circ (\zeta_0^3)^{-1} : \Lambda_{\alpha_3}(\Phi_0(\Delta)) \to \Lambda_{\alpha_1}(\Phi_0(\Delta))$  is Lipschitz.

However,  $\Phi_0(\Delta)$  is not Zariski dense. Since  $\Delta$  is the free group on two generators, there exists an arbitrary small deformation  $\Phi : \Delta \to SL_8(\mathbb{R})$ of  $\Phi_0$  which is Borel Anosov with Zariski dense image. Arguing exactly as in [31, Section 9], one can show that  $\Phi_0$  and  $\wedge^3 \Phi_0$  are both (1, 1, 2)hyperconvex. Therefore, we may assume that  $\Phi$  and  $\wedge^3 \Phi$  are both (1, 1, 2)hyperconvex.

One may then use standard techniques (cf. [31]) to show that if  $\Phi$  is sufficiently close to  $\Phi_0$ , then

$$\frac{2}{3} \leq \frac{\alpha_1(\lambda(\Phi(\gamma)))}{\alpha_1(\lambda(\Phi_0(\gamma)))} \leq \frac{3}{2} \quad \text{and} \quad \frac{2}{3} \leq \frac{\alpha_1(\lambda(\wedge^3 \Phi(\gamma)))}{\alpha_1(\lambda(\wedge^3 \Phi_0(\gamma)))} \leq \frac{3}{2}$$

for all non-trivial  $\gamma \in \Delta$ . Let  $\zeta = (\zeta^k)$  be the limit map of  $\Phi(\Delta)$  and  $\hat{\zeta}_0^1 : \partial \Delta \to \Lambda_{\alpha_1}(\wedge^3 \Phi_0(\Delta))$  and  $\hat{\zeta}^1 : \partial \Delta \to \Lambda_{\alpha_1}(\wedge^3 \Phi(\Delta))$  be limit maps of  $\wedge^3 \Phi_0$  and  $\wedge^3 \Phi$ . One may again apply Tsouvalas's theorem [26, Theorem 1.9] to conclude that  $\zeta^1 \circ (\zeta_0^1)^{-1}$  and  $\hat{\zeta}_0^1 \circ (\hat{\zeta}^1)^{-1}$  are  $\frac{2}{3}$ -Hölder. There is a  $C^1$ -equivariant identification of  $\Lambda_{\alpha_1}(\wedge^3 \Phi_0(\Delta))$  with  $\Lambda_{\alpha_3}(\Phi_0(\Delta))$  and an analogous identification for  $\Phi$ , so we may conclude that  $\zeta_0^3 \circ (\zeta^3)^{-1}$  is  $\frac{2}{3}$ -Hölder. Now set

$$\Gamma := \Phi(\Delta) < \mathrm{SL}_8(\mathbb{R}).$$

Then the limit map

$$\zeta^1 \circ (\zeta^3)^{-1} = \left(\zeta^1 \circ (\zeta_0^1)^{-1}\right) \circ \left(\zeta_0^1 \circ (\zeta_0^3)^{-1}\right) \circ \left(\zeta_0^3 \circ (\zeta^3)^{-1}\right)$$

is a  $\frac{16}{9}$ -Hölder and hence yields a Lipschitz map from  $\Lambda_{\alpha_3}$  to  $\Lambda_{\alpha_1}$ . Since  $\Gamma$  is Borel Anosov, the projection map  $\Lambda_{\{\alpha_1,\alpha_3\}} \to \Lambda_{\alpha_3}$  is now a bi-Lipschitz homeomorphism. This proves Proposition 6.1.

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