# BI-LIPSCHITZ RIGIDITY OF DISCRETE SUBGROUPS 

RICHARD CANARY, HEE OH, AND ANDREW ZIMMER


#### Abstract

We obtain a bi-Lipschitz rigidity theorem for a Zariski dense discrete subgroup of a connected simple real algebraic group. As an application, we show that any Zariski dense discrete subgroup of a higher rank semisimple algebraic group $G$ cannot have a $C^{1}$-smooth slim limit set in $G / P$ for any non-maximal parabolic subgroup $P$.


## 1. Introduction

For $i=1,2$, let $G_{i}$ be a connected simple real algebraic group and $\Gamma_{i}$ a Zariski dense discrete subgroup of $G_{i}$. Let

$$
\rho: \Gamma_{1} \rightarrow \Gamma_{2}
$$

be an isomorphism. The classical rigidity problem searches for a condition on $\rho$ which guarantees that $\rho$ is algebraic, that is, it extends to a Lie group isomorphism $G_{1} \rightarrow G_{2}$.

If $\Gamma_{1}$ is a lattice in $G_{1}$ and either

- $G_{1}=G_{2}$ has rank one and is not locally isomorphic to $\operatorname{PSL}_{2}(\mathbb{R})$, or
- $G_{1}$ has higher rank,
then any isomorphism $\rho: \Gamma_{1} \rightarrow \Gamma_{2}$ is algebraic by celebrated theorems of Mostow, Prasad, and Margulis ([16], [17, [15]). On the other hand, there are very few rigidity theorems for non-lattice discrete subgroups, especially in higher rank. In this article, we provide a rigidity criterion $\rho: \Gamma_{1} \rightarrow \Gamma_{2}$ in terms of a $\rho$-boundary map between the limit sets of $\Gamma_{1}$ and $\Gamma_{2}$.

Since $\Gamma_{i}$ is Zariski dense, there exists a unique $\Gamma_{i}$-minimal subset $\Lambda_{i}$ in $\mathcal{F}_{i}=G_{i} / P_{i}$ for a parabolic subgroup $P_{i}$ of $G_{i}$, called the limit set. When both parabolic subgroups are maximal, our result takes the following simple form:

Theorem 1.1 (Bi-Lipschitz rigidity theorem I). Assume that $P_{1}$ and $P_{2}$ are maximal parabolic subgroups. Let $\rho: \Gamma_{1} \rightarrow \Gamma_{2}$ be an isomorphism. If there exists a bi-Lipschitz $\rho$-equivariant map $f: \Lambda_{1} \rightarrow \Lambda_{2}$, then $\rho$ extends to a Lie group isomorphism

$$
\bar{\rho}: G_{1} \rightarrow G_{2}
$$

which induces a diffeomorphism $\bar{f}: \mathcal{F}_{1} \rightarrow \mathcal{F}_{2}$ such that $\left.\bar{f}\right|_{\Lambda_{1}}=f$.

[^0]Recall that $f: \Lambda_{1} \rightarrow \Lambda_{2}$ is bi-Lipschitz if there exists $C \geq 1$ such that for all $\xi, \eta \in \Lambda_{1}$,

$$
\begin{equation*}
C^{-1} d_{\mathcal{F}_{1}}(\xi, \eta) \leq d_{\mathcal{F}_{2}}(f(\xi), f(\eta)) \leq C d_{\mathcal{F}_{1}}(\xi, \eta) \tag{1.1}
\end{equation*}
$$

where $d_{\mathcal{F}_{i}}$ is a Riemannian metric on $\mathcal{F}_{i}$ for $i=1,2$. Since any two Riemannian metrics on $\mathcal{F}_{i}$ are bi-Lipschitz equivalent to each other, this notion is well-defined. We note that there can be at most one $\rho$-equivariant map $f: \Lambda_{1} \rightarrow \Lambda_{2}$ [12, Lemma 4.5]. We emphasize that we do not require $f$ to be defined on all of $\mathcal{F}_{1}$, but only on $\Lambda_{1}$. For $G_{1}=G_{2}=\operatorname{SO}(n, 1)^{\circ}, n \geq 2$, Theorem 1.1] was proved by Tukia [29, Theorem D].
Remark 1.2. (1) The hypothesis that $G_{1}$ and $G_{2}$ are simple is necessary; see Remark 4.8.
(2) The global bi-Lipschitz hypothesis on $f$ can be replaced by the condition that $f$ is bi-Lipschitz on some non-empty open subset of $\Lambda_{1}$; see Lemma 4.9.
We now state a general version of Theorem 1.1 where $P_{1}$ and $P_{2}$ are arbitrary parabolic subgroups.

Theorem 1.3 (Bi-Lipschitz rigidity theorem II). Let $\rho: \Gamma_{1} \rightarrow \Gamma_{2}$ be an isomorphism. If there exists a bi-Lipschitz $\rho$-equivariant map $f: \Lambda_{1} \rightarrow \Lambda_{2}$, then $\rho$ extends to a Lie group isomorphism

$$
\bar{\rho}: G_{1} \rightarrow G_{2} .
$$

Moreover, there exists a parabolic subgroup $P_{2}^{\prime}$ of $G_{2}$ containing $P_{2}$ such that $\bar{\rho}\left(P_{1}\right) \subset P_{2}^{\prime}$ up to a conjugation and the smooth submersion $G_{1} / P_{1} \rightarrow$ $G_{2} / P_{2}^{\prime}$ induced by $\bar{\rho}$ coincides with the composition $\pi \circ f$ on $\Lambda_{1}$ where $\pi$ : $G_{2} / P_{2} \rightarrow G_{2} / P_{2}^{\prime}$ is the canonical factor map.


See Theorem 4.7 for a stronger version which relaxes the bi-Lipschitz condition to a $\kappa$-bi-Hölder condition for $\kappa>0$.

Remark 1.4. In general, $P_{2}^{\prime}$ is not the same as $P_{2}$. We use the theory of hyperconvex subgroups to construct a Zariski dense discrete subgroup of $\mathrm{SL}_{8}(\mathbb{R})$ which demonstrates this point in Proposition 6.1.

Theorem 1.3 also has consequences for the regularity of the limit set of $\Gamma$ in $G / P$ when $G$ is a higher rank semisimple real algebraic group and $P$ is a non-maximal parabolic subgroup.

Theorem 1.5 (Regularity of slim limit sets). Let $G$ be a connected semisimple real algebraic group of rank at least 2 and $P$ a non-maximal parabolic
subgroup of $G$. Any Zariski dense discrete subgroup of $G$ cannot have a slim limit set in $G / P$ which is a $C^{1}$-submanifold.

Note that any non-maximal parabolic subgroup $P$ is contained in at least two non-conjugate maximal parabolic subgroups of $G$. We call a subset $S \subset G / P$ slim if there exists a pair of non-conjugate maximal parabolic subgroups $P_{1}, P_{2}$ containing $P$ such that the canonical factor map $\pi_{i}: G / P \rightarrow G / P_{i}$ is injective on $S$ for $i=1,2$.


In particular, the limit set of any subgroup of a $P$-Anosov or relatively $P$ Anosov subgroup is always slim. More generally, if any two points in the limit set are in general position, then the limit set is slim.

The non-maximal hypothesis on $P$ in Theorem 1.5 is necessary, as there are many Zariski dense discrete subgroups of $\mathrm{PSL}_{n}(\mathbb{R}), n \geq 3$, whose limit sets are $C^{1}$-submanifolds of $\mathbb{P}\left(\mathbb{R}^{n}\right)$, e.g., images of Hitchin [14] and Benoist representations [2]. We remark that the limit sets of these examples are not $C^{2}$ as shown by Zimmer [32].

Remark 1.6. (1) When $G$ is of rank one, the limit set $\Lambda$ of a Zariski dense subgroup of $G$ is not a proper $C^{r}$-submanifold of $G / P$ where $r=1$ for $G=\mathrm{SO}(n, 1)^{\circ}$ and $r=2$ for other rank one groups ([30, Proposition 3.12 and Corollary 3.13]). In higher rank, there exists $0<r<\infty$, depending on $G$, such that $\Lambda$ is not a proper $C^{r}$-submanifold of $G / P$ for any parabolic subgroup $P$ [5, Lemma 2.11].
(2) Theorem 1.5 was previously established for images of Hitchin representations [26, Corollary 6.1] and for images of (1,1,2)-hyperconvex representation of a surface group [18, Corollary 7.7]. We also mention [6], 31, and [20 for related work on the regularity of the limit set for certain classes of subgroups of $G=\mathrm{SO}(d, 2), \mathrm{PSL}_{d}(\mathbb{R})$ and $\mathrm{SO}(p, q)$ respectively.

On the proofs. We deduce Theorem 1.3 and Theorem 1.5 from the following property of limit sets of a Zariski dense subgroup in higher rank:

Proposition 1.7. Let $G$ be a connected semisimple real algebraic group of rank at least 2. Let $Q_{1}$ and $Q_{2}$ be a pair of parabolic subgroups of $G$ such that there is no parabolic subgroup of $G$ containing $Q_{1}$ and a conjugate of $Q_{2}$ (e.g., a pair of non-conjugate maximal parabolic subgroups).

If $\Gamma<G$ is a Zariski dense discrete subgroup, then there is no $\Gamma$-equivariant bi-Lipchitz map between the limit sets of $\Gamma$ on $G / Q_{1}$ and $G / Q_{2}$.

Indeed, if $\rho$ in Theorem 1.3 does not extend to a Lie group isomorphism $G_{1} \rightarrow G_{2}$, then the following self-joining subgroup

$$
\begin{equation*}
\Gamma=(\mathrm{id} \times \rho)\left(\Gamma_{1}\right)=\left\{(g, \rho(g)): g \in \Gamma_{1}\right\} \tag{1.2}
\end{equation*}
$$

is a Zariski dense subgroup of the product $G=G_{1} \times G_{2}$. On the other hand, a bi-Lipschitz map $f$ as in Theorem 1.3 yields a bi-Lipschitz homeomorphism between the limit sets of the self-joining group $\Gamma$ in $G /\left(P_{1} \times G_{2}\right)$ and $G /\left(G_{1} \times P_{2}\right)$, which then gives a desired contradiction by Proposition 1.7. We mention the recent work [10] and [11] on related rigidity theorems which use the idea of self-joinings.

If $\Gamma$ has a $C^{1}$-slim limit set in $G / P$ as in Theorem 1.5 and $P_{1}$ and $P_{2}$ are non-conjugate maximal parabolic subgroups containing $P$, we get a biLipschitz map between the limit sets of $\Gamma$ in $G / P_{1}$ and $G / P_{2}$ from the slimneess hypothesis. Therefore Proposition 1.7 implies Theorem 1.5

For the proof of Proposition 1.7, we relate the exponential contraction rates of loxodromic elements $\gamma \in \Gamma$ on $G / Q_{i}$ with the Jordan projections of the image of $\gamma$ under Tits representations of $G$. This part of the argument is motivated by earlier work of Zimmer [32, Section 8]. We then show that the bi-Lipschitz equivalence of the limit sets gives an obstruction to Benoist's theorem [1] on the non-empty interior property of the limit cone of a Zariski dense subgroup (see the proof of Proposition 4.3).

Acknowledgement. We would like to thank Dongryul Kim for helpful comments on a preliminary version of this paper.

## 2. Preliminaries

Unless mentioned otherwise, let $G$ be a connected semisimple real algebraic group throughout the paper. This means that $G$ is the identity component $\mathbf{G}(\mathbb{R})^{\circ}$ for a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. A parabolic $\mathbb{R}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is a proper algebraic subgroup defined over $\mathbb{R}$ such that the quotient $\mathbf{G} / \mathbf{P}$ is a projective algebraic variety. A parabolic subgroup $P$ of $G$ is of the form $\mathbf{P}(\mathbb{R})$ for a parabolic $\mathbb{R}$-subgroup $\mathbf{P}$ of $\mathbf{G}$; in this case, the quotient $G / P$ is equal to $(\mathbf{G} / \mathbf{P})(\mathbb{R})$ and is a real projective variety, called a $G$-boundary [3]. Any parabolic subgroup $P$ is conjugate to a unique standard parabolic subgroup of $G$, once we fix a root system associated to $G$.

To be precise, let $A$ be a maximal real split torus of $G$. The rank of $G$ is defined as the dimension of $A$. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of $G$ and $A$. Fix a positive Weyl chamber $\mathfrak{a}^{+} \subset \mathfrak{a}$ and set $A^{+}=\exp \mathfrak{a}^{+}$, and a maximal compact subgroup $K<G$ such that the Cartan decomposition $G=K A^{+} K$ holds. We denote by $M$ the centralizer of $A$ in $K$. For $g \in G$, we denote by $\mu(g)$ the Cartan projection of $g$, which is the unique element of $\mathfrak{a}^{+}$such that $g \in K \exp \mu(g) K$.

Any $g \in G$ can be written as the commuting product $g=g_{h} g_{e} g_{u}$ where $g_{h}$ is hyperbolic, $g_{e}$ is elliptic and $g_{u}$ is unipotent. The hyperbolic component
$g_{h}$ is conjugate to a unique element $\exp \lambda(g) \in A^{+}$and

$$
\begin{equation*}
\lambda(g) \in \mathfrak{a}^{+} \tag{2.1}
\end{equation*}
$$

is called the Jordan projection of $g$. When $\lambda(g) \in \operatorname{int} \mathfrak{a}^{+}, g \in G$ is called loxodromic in which case $g_{u}$ is necessarily trivial and $g_{e}$ is conjugate to an element $m \in M$.

Let $\Phi=\Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots and $\Pi$ the set of all simple roots given by the choice of $\mathfrak{a}^{+}$. The Weyl group $\mathcal{W}$ is given by $N_{K}(A) / M$ where $N_{K}(A)$ is the normalizer of $A$ in $K$.

Consider the real vector space $\mathrm{E}^{*}=\mathrm{X}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathrm{X}(A)$ is the group of all real characters of $A$ and let E be its dual. Denote by $(\cdot, \cdot)$ a $\mathcal{W}$-invariant inner product on E . We denote by $\left\{\omega_{\alpha}: \alpha \in \Pi\right\}$ the (restricted) fundamental weights of $\Phi$ defined by

$$
2 \frac{\left(\omega_{\alpha}, \beta\right)}{(\beta, \beta)}=c_{\alpha} \delta_{\alpha, \beta}
$$

where $c_{\alpha}=1$ if $2 \alpha \notin \Phi$ and $c_{\alpha}=2$ otherwise.
Fix an element $w_{0} \in N_{K}(A)$ of order 2 representing the longest Weyl element so that $\mathrm{Ad}_{w_{0}} \mathfrak{a}^{+}=-\mathfrak{a}^{+}$. The map

$$
\mathrm{i}=-\operatorname{Ad}_{w_{0}}: \mathfrak{a} \rightarrow \mathfrak{a}
$$

is called the opposition involution. It induces an involution of $\Phi$ preserving $\Pi$, for which we use the same notation i , so that $\mathrm{i}(\alpha)=\alpha \circ \mathrm{i}$ for all $\alpha \in \Phi$.

For a non-empty subset $\theta$ of $\Pi$, let $\mathfrak{a}_{\theta}=\cap_{\alpha \in \Pi-\theta} \operatorname{ker} \alpha$, and let $P_{\theta}$ denote a standard parabolic subgroup of $G$ corresponding to $\theta$; that is, $P_{\theta}=L_{\theta} N_{\theta}$ where $L_{\theta}$ is the centralizer of $\exp \mathfrak{a}_{\theta}$ and $N_{\theta}$ is the unipotent radical of $P_{\theta}$ which is generated by root subgroups associated to all positive roots which are not $\mathbb{Z}$-linear combinations of elements of $\Pi-\theta$. If $\theta=\Pi$, then $P=P_{\Pi}$ is a minimal parabolic subgroup. For a singleton $\theta=\{\alpha\}, P_{\alpha}$ is a maximal parabolic subgroup of $G$. Any parabolic subgroup $P$ is conjugate to a unique standard parabolic subgroup $P_{\theta}$ for some non-empty subset $\theta \subset \Pi$.

We consider the $\theta$-boundary:

$$
\mathcal{F}_{\theta}=G / P_{\theta} .
$$

We denote by $d_{\mathcal{F}_{\theta}}$ a Riemannian metric on $\mathcal{F}_{\theta}$. Let $P_{\theta}^{+}=w_{0} P_{\mathrm{i}(\theta)} w_{0}^{-1}$, which is the standard parabolic subgroup opposite to $P_{\theta}$ such that $P_{\theta} \cap P_{\theta}^{+}=L_{\theta}$. Hence $\mathcal{F}_{\mathrm{i}(\theta)}=G / P_{\mathrm{i}(\theta)}=G / P_{\theta}^{+}$. The $G$-orbit $\mathcal{F}_{\theta}^{(2)}=\left\{\left(g P_{\theta}, g w_{0} P_{\mathrm{i}(\theta)}\right): g \in\right.$ $G\}$ is the unique open $G$-orbit in $G / P_{\theta} \times G / P_{\theta}^{+}$under the diagonal $G$ action. Two elements $\xi \in \mathcal{F}_{\theta}$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position if $(\xi, \eta) \in \mathcal{F}_{\theta}^{(2)}$.

## 3. Contraction rates of loxodromic elements and Tits REPRESENTATIONS

The first part of the following theorem immediately follows as a special case of a theorem of Tits [25], and the second part is remarked in [1] and proved in [23].

Theorem 3.1 ([25, Theorem 7.2], [23, Lemma 2.13]). For each $\alpha \in \Pi$, there exists an irreducible representation $\rho_{\alpha}: G \rightarrow \mathrm{GL}\left(V_{\alpha}\right)$ whose highest (restricted) weight $\chi_{\alpha}$ is equal to $k_{\alpha} \omega_{\alpha}$ for some positive integer $k_{\alpha}$ and whose highest weight space is one-dimensional.

Moreover, all weights of $\rho_{\alpha}$ are $\chi_{\alpha}, \chi_{\alpha}-\alpha$ and weights of the form $\chi_{\alpha}-$ $\alpha-\sum_{\beta \in \Pi} n_{\beta} \beta$ with $n_{\beta}$ non-negative integers.

These representations are called Tits representations of $G$. Fix $\alpha \in \Pi$ and, as before, set $\mathcal{F}_{\alpha}=G / P_{\alpha}$. We denote by $V_{1}$ and $V_{2}$ the weight spaces of $\rho_{\alpha}$ for the highest weight $\chi_{\alpha}$ and the second highest weight $\chi_{\alpha}-\alpha$ respectively. We have $\operatorname{dim} V_{1}=1$ and $\operatorname{dim} V_{2} \geq 1$. If we set $\xi_{\alpha}=\left[P_{\alpha}\right] \in \mathcal{F}_{\alpha}$, the map $g \xi_{\alpha} \mapsto g V_{1}$ gives an embedding

$$
\begin{equation*}
\mathcal{F}_{\alpha} \rightarrow \mathbb{P}\left(V_{\alpha}\right) \tag{3.1}
\end{equation*}
$$

whose image is a closed subvariety. We may hence identify $\mathcal{F}_{\alpha}$ as a closed subvariety of $\mathbb{P}\left(V_{\alpha}\right)$. Let $\langle\cdot, \cdot\rangle_{\alpha}$ be a $K$-invariant inner product on $V_{\alpha}$ with respect to which $A$ is symmetric and we have the orthogonal weight space decomposition of $V_{\alpha}$. Using the norms on $V_{\alpha}$ and $\wedge^{2} V_{\alpha}$ induced by this inner product, we get a $K$-invariant Riemannian metric $d_{\alpha}$ on $\mathbb{P}\left(V_{\alpha}\right)$ :

$$
d_{\alpha}([v],[w])=\frac{\|v \wedge w\|}{\|v\|\|w\|} \quad \text { for }[v],[w] \in \mathbb{P}\left(V_{\alpha}\right) .
$$

Recall that an element $g \in G$ is loxodromic if there exist $a \in \operatorname{int} A^{+}$and $m \in M$ such that $g=h_{g} a m h_{g}^{-1}$ for some $h_{g} \in G$. The element $h_{g}$ is then uniquely determined modulo $A M$ and $\lambda(g)=\log a \in \operatorname{int} \mathfrak{a}^{+}$.

Let $\pi_{i}=\pi_{\alpha, i}: V_{\alpha} \rightarrow V_{i}$ be the orthogonal projection for $i=1,2$. Recall the following standard lemma:
Lemma 3.2. Let $g$ be a loxodromic element of $G$. For $\xi \in \mathcal{F}_{\alpha}$, we have $\pi_{1}\left(h_{g}^{-1} \xi\right) \neq 0$ if and only if $g^{n} \xi$ converges to $h_{g} \xi_{\alpha}$ as $n \rightarrow \infty$.

The point $y_{\alpha}^{g}:=h_{g} \xi_{\alpha} \in \mathcal{F}_{\alpha}$ is called the attracting fixed point of $g$.
Lemma 3.3. Let $g \in G$ be a loxodromic element and $\alpha \in \Pi$.
(1) For all $\xi \in \mathcal{F}_{\alpha}$ with $\pi_{1}\left(h_{g}^{-1} \xi\right) \neq 0$, we have

$$
-\alpha(\lambda(g)) \geq \limsup _{n \rightarrow \infty} \frac{1}{n} \log d_{\alpha}\left(g^{n} \xi, y_{\alpha}^{g}\right) .
$$

(2) For all $\xi \in \mathcal{F}_{\alpha}$ with $\pi_{1}\left(h_{g}^{-1} \xi\right) \neq 0$ and $\pi_{2}\left(h_{g}^{-1} \xi\right) \neq 0$, we have

$$
-\alpha(\lambda(g))=\lim _{n \rightarrow \infty} \frac{1}{n} \log d_{\alpha}\left(g^{n} \xi, y_{\alpha}^{g}\right) .
$$

Proof. It suffices to prove the claim when $h_{g}=e$, i.e., $g=a m \in A M$ with $\log a \in \operatorname{int} \mathfrak{a}^{+}$. Considering $\xi \in \mathcal{F}_{\alpha} \subset \mathbb{P}\left(V_{\alpha}\right)$, choose a vector $v \in V_{\alpha}$ representing $\xi$. List all distinct weights of $\rho_{\alpha}$ given by Theorem 3.1 as follows: $\chi_{1}=\chi_{\alpha}, \chi_{2}=\chi_{\alpha}-\alpha$, and $\chi_{i}=\chi_{\alpha}-\alpha-\beta_{i}, 3 \leq i \leq \ell$; in particular, $\beta_{i} \neq 0$ is a non-negative integral linear combinations of simple roots. Let $V_{i}$ denote the weight space corresponding to $\chi_{i}$ and write $v=v_{1}+v_{2}+\cdots+v_{\ell}$ so that $v_{i} \in V_{i}$ for each $1 \leq i \leq \ell$. Suppose that $\pi_{1}(\xi) \neq 0$, that is $v_{1} \neq 0$. We may then assume that $v_{1}$ is a unit vector relative to $\langle\cdot, \cdot\rangle_{\alpha}$. Since $M$ commutes with $A, M$ stabilizes each weight subspace, and in particular, $M v_{1}= \pm v_{1}$. Now

$$
g^{n} v=e^{n \chi_{\alpha}(\log a)} m^{n} v_{1}+e^{n\left(\chi_{\alpha}-\alpha\right)(\log a)} m^{n} v_{2}+\sum_{i=3}^{\ell} e^{n\left(\chi_{\alpha}-\alpha-\beta_{i}\right)(\log a)} m^{n} v_{i} .
$$

Hence the projection $p\left(g^{n} v\right)$ of $g^{n} v$ to the affine chart $\mathbb{A}=\left\{w \in V_{\alpha}\right.$ : $\left.\pi_{1}(w)=v_{1}\right\}$ is

$$
p\left(g^{n} v\right)=v_{1}+e^{-n \alpha(\log a)} m^{n} v_{2}^{\prime}+\sum_{i=3}^{\ell} e^{-n\left(\alpha+\beta_{i}\right)(\log a)} m^{n} v_{i}^{\prime}
$$

where $v_{i}^{\prime}= \pm v_{i}$, depending on the sign of $m^{n} v_{1}$. Note that $\lim g^{n} \xi=V_{1}$, and that the metric $d_{\alpha}$ on a neighborhood on $V_{1}$ in $\mathbb{P}\left(V_{\alpha}\right)$ is bi-Lipschitz equivalent to the metric $d$ on the affine chart $\mathbb{A}$, obtained by restricting the distance on $V_{\alpha}$ induced by $\langle\cdot, \cdot\rangle_{\alpha}$.

Since the weight spaces are orthogonal, we have

$$
d\left(p\left(g^{n} v\right), v_{1}\right)=e^{-n \alpha(\log a)}\left(\left\|v_{2}\right\|^{2}+\left\|w_{n}\right\|^{2}\right)^{1 / 2}
$$

where $w_{n}=\sum_{i=3}^{\ell} e^{-n \beta_{i}(\log a)} m^{n} v_{i}^{\prime}$ and $\|\cdot\|$ is the norm induced by $\langle\cdot, \cdot\rangle_{\alpha}$. Since $\log a \in \operatorname{int} \mathfrak{a}^{+}$and hence $\beta_{i}(\log a)>0$ for all $3 \leq i \leq \ell$, we have

$$
\lim _{n \rightarrow \infty} w_{n}=0 .
$$

First consider the case when $\pi_{2}(\xi)=0$, that is $v_{2}=0$. Since $\log \left\|w_{n}\right\|<0$ for all large $n$, we have

$$
\begin{aligned}
\limsup _{n \rightarrow \infty} \frac{1}{n} \log d_{\alpha}\left(g^{n} \xi, y_{\alpha}^{g}\right) & =\limsup _{n \rightarrow \infty} \frac{1}{n} \log d\left(p\left(g^{n} v\right), v_{1}\right) \\
& =\limsup _{n \rightarrow \infty} \frac{1}{n}\left(-n \alpha(\log a)+\log \left\|w_{n}\right\|\right) \leq-\alpha(\log a) .
\end{aligned}
$$

Now suppose that $\pi_{2}(\xi) \neq 0$, that is $v_{2} \neq 0$. Again since $w_{n} \rightarrow 0$, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \log d_{\alpha}\left(g^{n} \xi, y_{\alpha}^{g}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} \log d\left(p\left(g^{n} v\right), v_{1}\right)=-\alpha(\log a) .
$$

This finishes the proof.

## 4. Bi-Lipschitz Rigidity of discrete subgroups

Let $G$ be a connected semisimple real algebraic group and $X=G / K$ be the associated Riemannian symmetric space and fix $o=[K] \in X$.

We consider the following notion of convergence of a sequence in $G$ to an element of $\mathcal{F}_{\theta}=G / P_{\theta}$ for a non-empty subset $\theta \subset \Pi$.

For a sequence $g_{i} o \in X$ and $\xi \in \mathcal{F}_{\theta}$, we write $\lim g_{i} o=\xi$ and say $g_{i} o \in X$ converges to $\xi$ if
(1) $\min _{\alpha \in \theta} \alpha\left(\mu\left(g_{i}\right)\right) \rightarrow \infty$ as $i \rightarrow \infty$; and
(2) $\lim _{i \rightarrow \infty} \kappa_{g_{i}} P_{\theta}=\xi$ in $\mathcal{F}_{\theta}$ for some $\kappa_{g_{i}} \in K$ such that $g_{i} \in \kappa_{g_{i}} A^{+} K$.

Definition 4.1. Let $\Gamma<G$ be a discrete subgroup and let $\mathcal{F}=G / P$ for a parabolic subgroup $P$. Let $\theta \subset \Pi$ be a unique subset such that $P$ is conjugate to $P_{\theta}$ and hence $\mathcal{F}=\mathcal{F}_{\theta}$. The limit set of $\Gamma$ in $\mathcal{F}_{\theta}$ is then defined as the set of all accumulation points of $\Gamma(o)$ in $\mathcal{F}_{\theta}$ :

$$
\Lambda_{\theta}=\Lambda_{\theta}(\Gamma)=\left\{\lim \gamma_{i}(o) \in \mathcal{F}_{\theta}: \gamma_{i} \in \Gamma\right\} .
$$

It is a $\Gamma$-invariant closed subset of $\mathcal{F}_{\theta}$, which is non-empty provided $\Gamma$ contains a sequence $\gamma_{i}$ satisfying $\lim _{i \rightarrow \infty} \min _{\alpha \in \theta} \alpha\left(\mu\left(\gamma_{i}\right)\right)=\infty$. If $\Gamma$ is Zariski dense, $\Lambda_{\theta}$ is the unique $\Gamma$-minimal subset of $\mathcal{F}_{\theta}$ and can also be described as the set of all $\xi \in \mathcal{F}_{\theta}$ such that the Dirac measure $\delta_{\xi}$ is the weak limit of $\left(\gamma_{i}\right)_{*} \operatorname{Leb}_{\theta}$ for some sequence $\gamma_{i} \in \Gamma$ where $\operatorname{Leb}_{\theta}$ denotes the unique $K$ invariant probability measure on $\mathcal{F}_{\theta}([1],[21])$. Moreover, if $\Theta \subset \theta$, then $\Lambda_{\Theta}$ is equal to the image of $\Lambda_{\theta}$ under the canonical projection $\mathcal{F}_{\theta} \rightarrow \mathcal{F}_{\Theta}$, by minimality.

The limit cone of $\Gamma$ is defined as the smallest closed cone of $\mathfrak{a}^{+}$containing all Jordan projections of loxodromic elements of $\Gamma$.

Theorem 4.2 (Benoist [1). If $\Gamma<G$ is Zariski dense, its limit cone has non-empty interior in $\mathfrak{a}$.

For $\kappa>0$ and $\theta_{1}, \theta_{2} \subset \Pi$, a map $F: \Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$ is called $\kappa$-bi-Hölder if there exists $C>0$ such that for all $x, y \in \Lambda_{\theta_{1}}$

$$
\begin{equation*}
C^{-1} d_{\mathcal{F}_{\theta_{1}}}(x, y)^{\kappa} \leq d_{\mathcal{F}_{\theta_{2}}}(F(x), F(y)) \leq C d_{\mathcal{F}_{\theta_{1}}}(x, y)^{\kappa} \tag{4.1}
\end{equation*}
$$

where $d_{\mathcal{F}_{\theta_{i}}}$ is a Riemannian metric on $\mathcal{F}_{\theta_{i}}$ for $i=1,2$. Observe that if $\Gamma$ is Zariski dense, any $\Gamma$-equivariant $\kappa$-bi-Hölder map $\Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$ is a homeomorphism; the minimality of $\Lambda_{\theta_{2}}$ implies the surjectivity and the bi-Hölder property implies the injectivity. Therefore $F$ is $\kappa$-bi-Hölder if and only if $F$ is $\kappa$-Hölder and $F^{-1}$ is $\kappa^{-1}$-Hölder.

Proposition 1.7 follows from the following for $\kappa=1$ :
Proposition 4.3. Let $\Gamma<G$ be Zariski dense. Let $\theta_{1}$ and $\theta_{2}$ be disjoint non-empty subsets of $\Pi$. Then for any $\kappa>0$, there exists no $\Gamma$-equivariant $\kappa$-bi-Hölder map $F: \Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$.
Proof. For simplicity, we write $\Lambda_{i}=\Lambda_{\theta_{i}}$ and $d_{\theta_{i}}=d_{\mathcal{F}_{\theta_{i}}}$. Let $F: \Lambda_{1} \rightarrow \Lambda_{2}$ be a $\Gamma$-equivariant homeomorphsim. Fix $\kappa>0$. Since $\theta_{1} \cap \theta_{2}=\emptyset$, the union
$\bigcup_{\alpha_{1} \in \theta_{1}, \alpha_{2} \in \theta_{2}} \operatorname{ker}\left(\kappa \alpha_{1}-\alpha_{2}\right)$ is a finite union of hyperplanes of $\mathfrak{a}$. Therefore by Theorem 4.2, $\Gamma$ contains a loxodromic element $\gamma$ such that

$$
\left\{\kappa \cdot \alpha(\lambda(\gamma)): \alpha \in \theta_{1}\right\} \cap\left\{\alpha(\lambda(\gamma)): \alpha \in \theta_{2}\right\}=\emptyset .
$$

For each $i=1,2$, let $\alpha_{i} \in \theta_{i}$ be such that

$$
\begin{equation*}
\alpha_{i}(\lambda(\gamma))=\min \left\{\alpha(\lambda(\gamma)): \alpha \in \theta_{i}\right\} . \tag{4.2}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\kappa \cdot \alpha_{1}(\lambda(\gamma)) \neq \alpha_{2}(\lambda(\gamma)) . \tag{4.3}
\end{equation*}
$$

Claim: If $F^{-1}$ is $\kappa^{-1}$-Hölder, then

$$
\begin{equation*}
\alpha_{2}(\lambda(\gamma)) \leq \kappa \cdot \alpha_{1}(\lambda(\gamma)) . \tag{4.4}
\end{equation*}
$$

By replacing $\Gamma$ by a suitable conjugate, we may also assume that $\gamma=$ $a m \in \Gamma$ with $a \in \operatorname{int} A^{+}$and $m \in M$. For each $i=1,2$, let $y_{i}=y_{\alpha_{i}}^{\gamma}$ denote the attracting fixed point of $\gamma$ in $\mathcal{F}_{i}$; we have $y_{i} \in \Lambda_{i}$. As $\Gamma$ is Zariski dense, $\Lambda_{i}$ is Zariski dense in $\mathcal{F}_{i}$ for each $i=1,2$. Let $\pi_{\alpha, 1}$ and $\pi_{\alpha, 2}$ be as in Lemmas 3.2 and 3.3 for each $\alpha \in \Pi$. Since the set

$$
\mathcal{O}=\left\{\xi \in \mathcal{F}_{1}: \pi_{\alpha, 1}(\xi) \neq 0, \pi_{\alpha, 2}(\xi) \neq 0 \text { for all } \alpha \in \theta_{1}\right\}
$$

is a Zariski open subset of $\mathcal{F}_{1}$, the intersection $\mathcal{O} \cap \Lambda_{1}$ is a non-empty open subset of $\Lambda_{1}$. As $F$ is a homeomorphism, the image $F\left(\mathcal{O} \cap \Lambda_{1}\right)$ is a non-empty open subset of $\Lambda_{2}$. Since $Z=\left\{\xi \in \mathcal{F}_{2}: \gamma^{n} \xi \nrightarrow y_{2}\right.$ as $\left.n \rightarrow \infty\right\}$ is a proper Zariski closed subset of $\mathcal{F}_{2}$ by Lemma 3.2, $F\left(\mathcal{O} \cap \Lambda_{1}\right)$ cannot be contained in $Z$; otherwise it would imply that $\Lambda_{2}$ is contained in a proper Zariski closed subset by the $\Gamma_{2}$-minimality of $\Lambda_{2}$, which contradicts the Zariski density of $\Gamma_{2}$. Therefore there exists an element $\xi \in \mathcal{O} \cap \Lambda_{1}$ such that $\lim _{n \rightarrow \infty} \gamma^{n} F(\xi)=y_{2}$. By the equivariance and continuity of $F$, we have

$$
\begin{equation*}
F\left(y_{1}\right)=\lim F\left(\gamma^{n} \xi\right)=\lim \gamma^{n} F(\xi)=y_{2} . \tag{4.5}
\end{equation*}
$$

Let $i=1,2$. Since $P_{\theta_{i}}=\bigcap_{\alpha \in \theta_{i}} P_{\alpha}$, we have a diagonal embedding

$$
\mathcal{F}_{i}=G / P_{\theta_{i}} \rightarrow \prod_{\alpha \in \theta_{i}} \mathbb{P}\left(V_{\alpha}\right)
$$

via the product of the maps in (3.1). Consider the metric $d_{i}$ on $\mathcal{F}_{i}$ obtained as the restriction of $\sum_{\alpha \in \theta_{i}} d_{\alpha}$ to $\mathcal{F}_{i}$ : for $\eta=g P_{\theta_{1}}$ and $\eta^{\prime}=g^{\prime} P_{\theta_{2}}$ with g, $g^{\prime} \in G$,

$$
d_{i}\left(\eta, \eta^{\prime}\right)=\sum_{\alpha \in \theta_{i}} d_{\alpha}\left(\eta, \eta^{\prime}\right)
$$

where $d_{\alpha}\left(\eta, \eta^{\prime}\right):=d_{\alpha}\left(g V_{\alpha, 1}, g^{\prime} V_{\alpha, 1}\right)$ where $V_{\alpha, 1}$ is the highest weight line of $\rho_{\alpha}$ as in (3.1). Since $d_{i}$ is bi-Lipschitz equivalent to a Riemannian metric on $\mathcal{F}_{i}$, we have that $F^{-1}:\left(\Lambda_{2}, d_{2}\right) \rightarrow\left(\Lambda_{1}, d_{1}\right)$ is $\kappa^{-1}$-Hölder.

Since $\xi \in \mathcal{O}$ and $\lim \gamma^{n} F(\xi)=y_{2}$, we have by Lemma 3.3 that

$$
-\alpha(\lambda(\gamma))=\lim \frac{1}{n} \log d_{\alpha}\left(\gamma^{n} \xi, y_{1}\right) \quad \text { for each } \alpha \in \theta_{1}
$$

and

$$
-\alpha(\lambda(\gamma)) \geq \lim \sup \frac{1}{n} \log d_{\alpha}\left(\gamma^{n} F(\xi), y_{2}\right) \text { for each } \alpha \in \theta_{2}
$$

Since $d_{\alpha_{1}}\left(\eta, \eta^{\prime}\right) \leq d_{1}\left(\eta, \eta^{\prime}\right), d_{2}\left(\eta, \eta^{\prime}\right) \leq \# \theta_{2} \max _{\alpha \in \theta_{2}} d_{\alpha}\left(\eta, \eta^{\prime}\right)$, and $F^{-1}$ is $\kappa^{-1}$-Hölder, we have

$$
\begin{align*}
-\alpha_{1}(\lambda(\gamma)) & =\lim \frac{1}{n} \log d_{\alpha_{1}}\left(\gamma^{n} \xi, y_{1}\right)  \tag{4.6}\\
& \leq \lim \frac{1}{n} \log d_{1}\left(\gamma^{n} \xi, y_{1}\right) \\
& \leq \kappa^{-1} \lim \sup \frac{1}{n} \log d_{2}\left(F\left(\gamma^{n} \xi\right), F\left(y_{1}\right)\right) \\
& =\kappa^{-1} \lim \sup \frac{1}{n} \log d_{2}\left(\gamma^{n} F(\xi), y_{2}\right) \\
& =\kappa^{-1} \max _{\alpha \in \theta_{2}} \lim \sup \frac{1}{n} \log d_{\alpha}\left(\gamma^{n} F(\xi), y_{2}\right) \\
& \leq-\kappa^{-1} \min _{\alpha \in \theta_{2}} \alpha(\lambda(\gamma))=-\kappa^{-1} \alpha_{2}(\lambda(\gamma)) .
\end{align*}
$$

This implies that $\alpha_{2}(\lambda(\gamma)) \leq \kappa \alpha_{1}(\lambda(\gamma))$, proving the claim.
By switching the role of $\theta_{1}$ and $\theta_{2}$, this claim then implies that if $F$ is $\kappa$-Hölder, then $\alpha_{1}(\lambda(\gamma)) \leq \kappa^{-1} \alpha_{2}(\lambda(\gamma))$. Therefore if $F$ is $\kappa$-bi-Hölder, then $\kappa \cdot \alpha_{1}(\lambda(\gamma))=\alpha_{2}(\lambda(\gamma))$, contradicting (4.3). This finishes the proof.

The proof of Proposition 4.3 shows the following as well:
Proposition 4.4. Let $\Gamma<G$ be Zariski dense and let $\theta_{1}, \theta_{2} \subset \Pi$ be nonempty disjoint subsets. Suppose that $\Lambda_{\theta_{1}}$ and $\Lambda_{\theta_{2}}$ are $C^{1}$-submanifolds of $\mathcal{F}_{\theta_{1}}$ and $\mathcal{F}_{\theta_{2}}$ respectively. If $F: \Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$ is a $\Gamma$-equivariant homeomorphism, $F$ cannot be $C^{1}$ with non-vanishing Jacobian at any $\xi \in \mathcal{A}$, where $\mathcal{A} \subset \Lambda_{\theta_{1}}$ is the set of all attracting fixed points of loxodromic elements $\gamma \in \Gamma$ such that $\left\{\alpha(\lambda(\gamma)): \alpha \in \theta_{1}\right\} \cap\left\{\alpha(\lambda(\gamma)): \alpha \in \theta_{2}\right\}=\emptyset$.

Proof. Let $\gamma \in \Gamma$ be as above. For each $i=1,2$, let $y_{i} \in \Lambda_{\theta_{i}}$ be the attracting fixed point of $\gamma$. Then $F\left(y_{1}\right)=y_{2}$ by (4.5). Suppose that $F$ is $C^{1}$ at $y_{1}$, and the Jacobian of $F$ at $y_{1}$ is not zero. Then $F^{-1}$ is also $C^{1}$ at $y_{2}$. Using the exponential maps and the Taylor series expansion of $F$, we get that there exist $c \geq 1$ and an open neighborhood $U$ of $y_{1}$ in $\Lambda_{\theta_{1}}$ such that for all $y \in U$,

$$
\begin{equation*}
c^{-1} d_{1}\left(y, y_{1}\right) \leq d_{2}\left(F(y), F\left(y_{1}\right)\right) \leq c d_{1}\left(y, y_{1}\right) . \tag{4.7}
\end{equation*}
$$

Let $\alpha_{i} \in \theta_{i}$ be as in (4.2). Without loss of generality, we may assume $\alpha_{1}(\lambda(\gamma))<\alpha_{2}(\lambda(\gamma))$ by switching the indexes if necessary. On the other hand, using (4.7), the computation (4.6) gives $\alpha_{2}(\lambda(\gamma)) \leq \alpha_{1}(\lambda(\gamma))$, which yields a contradiction.

Remark 4.5. It would be interesting to know whether $\mathcal{A}$ can be replaced by the set of all conical limit points of $\Gamma$ in Proposition 4.4. A point $\xi=g P_{\theta_{1}}$ is $\Gamma$-conical if $\lim \sup \Gamma g\left(K \cap P_{\theta_{1}}\right) A^{+} \neq \emptyset$, that is, there exists a sequence
$\gamma_{i} \in \Gamma, a_{i} \in A^{+}$and $m_{i} \in K \cap P_{\theta_{i}}$ such that $\gamma_{i} g m_{i} a_{i}$ converges (see [13, Lemma 5.4] for an equivalent definition in terms of shadows).

This question is inspired by a related result for $G=\mathrm{SO}(n+1,1)^{\circ}$. Tukia [27] showed that if $f: \mathbb{S}^{n} \rightarrow \mathbb{S}^{n}$ is a homeomorphism which conjugates a discrete subgroup $\Gamma_{1}$ of $G$ to another discrete group $\Gamma_{2}$ and has a nonvanishing Jacobian at a conical limit point of $\Gamma_{1}$, then $\Gamma_{1}$ is conjugate to $\Gamma_{2}$ (see also [7 for an extension of this result to other rank one groups). For a related result for $(1,1,2)$-hyperconvex groups, see [18, Corollary 7.5].

In the rest of this section, let $G_{i}$ be a connected simple real algebraic group and $\theta_{i}$ be a non-empty set of simple roots of $G_{i}$ for $i=1,2$. Let $\Gamma_{i}<G_{i}$ be a Zariski dense discrete subgroup and $\Lambda_{\theta_{i}}$ denote the limit set of $\Gamma_{i}$ in $\mathcal{F}_{i}=G_{i} / P_{\theta_{i}}$.

Lemma 4.6. [12, Lemma 4.5] For any isomorphism $\rho: \Gamma_{1} \rightarrow \Gamma_{2}$, there exists at most one $\rho$-equivariant continuous map $f: \Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$.

Indeed, $f$ must send the attracting fixed point of any loxodromic element $\gamma$ to that of $\rho(\gamma)$ whenever $\rho(\gamma)$ is loxodromic. Since the set of attracting fixed points of loxodromic elements is dense in $\Lambda_{\theta_{1}}$ by the Zariski density hypothesis on $\Gamma_{1}[1]$ and $f$ is continuous, this determines the map $f$.

Theorem 1.3 is a special case of the following theorem for $\kappa=1$ :
Theorem 4.7. Suppose that there exists a $\rho$-equivariant $\kappa$-bi-Hölder map $f: \Lambda_{\theta_{1}} \rightarrow \mathcal{F}_{2}$ for some $\kappa>0$. Then $\rho$ extends to a Lie group isomorphism $\bar{\rho}: G_{1} \rightarrow G_{2}$. Moreover, there exists a non-empty subset $\Theta_{2} \subset \theta_{2}$ such that $\bar{\rho}$ maps $P_{\theta_{1}}$ into a conjugate of $P_{\Theta_{2}}$ and the smooth submersion $G_{1} / P_{\theta_{1}} \rightarrow$ $G_{2} / P_{\Theta_{2}}$ induced by $\bar{\rho}$ coincides with the composition $\pi \circ f$ on $\Lambda_{\theta_{1}}$ where $\pi: G_{2} / P_{\theta_{2}} \rightarrow G_{2} / P_{\Theta_{2}}$ is the canonical factor map.

Proof. Let $G=G_{1} \times G_{2}$. Define the following self-joining subgroup

$$
\Gamma=(\operatorname{id} \times \rho)\left(\Gamma_{1}\right)=\left\{(\gamma, \rho(\gamma)): \gamma \in \Gamma_{1}\right\}<G .
$$

Note that $P_{1}:=P_{\theta_{1}} \times G_{2}$ and $P_{2}:=G_{1} \times P_{\theta_{2}}$ are parabolic subgroups of $G$. The maps $g_{1} P_{\theta_{1}} \mapsto\left(g_{1}, e\right) P_{1}$ and $g_{2} P_{\theta_{2}} \mapsto\left(e, g_{2}\right) P_{2}$ define diffeomorphisms between $G_{1} / P_{\theta_{1}}$ and $G_{2} / P_{\theta_{2}}$ with $G / P_{1}$ and $G / P_{2}$ respectively. Moreover, under this identification, the limit set $\Lambda_{\theta_{i}}$ of $\Gamma_{i}$ in $G_{i} / P_{\theta_{i}}$ corresponds to the limit set $\Lambda_{i}$ of the self-joining $\Gamma$ in $G / P_{i}$ for each $i=1,2$.

Since $f$ is a $\rho$-equivariant continuous embedding of $\Lambda_{\theta_{1}}$ into $G / P_{\theta_{2}}$, its image is a $\Gamma_{2}$-invariant compact subset. Since $\Lambda_{\theta_{1}}$ is a $\Gamma_{1}$-minimal subset, the image $f\left(\Lambda_{\theta_{1}}\right)$ is also a $\Gamma_{2}$-minimal subset. Therefore $f\left(\Lambda_{\theta_{1}}\right)=\Lambda_{\theta_{2}}$ and hence we have a $\Gamma$-equivariant bijection $f: \Lambda_{1} \rightarrow \Lambda_{2}$ which is $\kappa$-bi-Hölder.

Since $P_{1}$ and $P_{2}$ are parabolic subgroups corresponding to disjoint subsets of simple roots of $G$, Proposition 4.3 implies that $\Gamma$ cannot be Zariski dense in $G$. Since both $G_{1}$ and $G_{2}$ are simple, the non-Zariski density of the selfjoining group $\Gamma$ implies that $\rho$ extends to a Lie group isomorphism $\bar{\rho}: G_{1} \rightarrow$ $G_{2}$ (cf. [4]).

Since $\bar{\rho}\left(P_{\theta_{1}}\right)$ must be a parabolic subgroup of $G_{2}$, there exists $g \in G_{2}$ such that $\bar{\rho}\left(P_{\theta_{1}}\right)=g P_{\theta_{0}} g^{-1}$ where $\theta_{0}$ is a non-empty subset of some simple roots of $G_{2}$. We claim $\theta_{0} \cap \theta_{2} \neq \emptyset$. By replacing $\rho$ by inn $(g) \circ \rho$ where $\operatorname{inn}(g): G_{2} \rightarrow G_{2}$ is the conjugation by $g$, we may assume without loss of generality that $g=e$. The isomorphism $\bar{\rho}$ induces a diffeomorphism $\tilde{\Phi}: G_{1} / P_{\theta_{1}} \rightarrow G_{2} / P_{\theta_{0}}$ given by $\tilde{\Phi}\left(g_{1} P_{\theta_{1}}\right)=\bar{\rho}\left(g_{1}\right) P_{\theta_{0}}$. Denote by $\Lambda_{\theta_{0}}$ the limit set of $\Gamma_{2}$ in $G_{2} / P_{\theta_{0}}$. Since $\left.\bar{\rho}\right|_{\Gamma_{1}}=\rho$ and hence $\tilde{\Phi}$ is $\rho$-equivariant, we have $\tilde{\Phi}\left(\Lambda_{\theta_{1}}\right)=\Lambda_{\theta_{0}}$. Then the composition $F:=f \circ \tilde{\Phi}^{-1}$ restricted to $\Lambda_{\theta_{0}}$ yields a $\kappa$-bi-Hölder map between $\Lambda_{\theta_{0}}$ and $\Lambda_{\theta_{2}}$. Since $\tilde{\Phi}^{-1}$ is $\rho^{-1}$-equivariant and $f$ is $\rho$-equivariant, $F$ is $\Gamma_{2}$-equivariant. So by applying Proposition 4.3 one more time, we obtain $\theta_{0} \cap \theta_{2} \neq \emptyset$. Setting $\Theta_{2}=\theta_{0} \cap \theta_{2}$, since $P_{\theta_{0}}$ and $P_{\theta_{2}}$ are subgroups of $P_{\Theta_{2}}$, we get a map $\Phi:=G_{1} / P_{\theta_{1}} \rightarrow G_{2} / P_{\Theta_{2}}$ by composing $\tilde{\Phi}$ with the canonical factor map $G_{1} / P_{\theta_{0}} \rightarrow G_{2} / P_{\Theta_{2}}$. The last claim $\Phi=\pi \circ f$ on $\Lambda_{\theta_{1}}$ follows from Lemma 4.6. This finishes the proof.

Remark 4.8. The hypothesis that $G_{1}$ and $G_{2}$ are simple is necessary in Theorem 4.7. For example, consider a discrete Zariski dense subgroup $\Gamma$ of a simple algebraic group $G$ with a discrete faithful representation $\rho: \Gamma \rightarrow G$ which does not extend to $G$. Then $\Gamma_{\rho}=(\mathrm{id} \times \rho)(\Gamma)$ is Zariski dense in $G$ and the map $\gamma \rightarrow(\gamma, \rho(\gamma))$ gives an isomorphism $\Gamma \rightarrow \Gamma_{\rho}$. On the other hand, for any parabolic subgroup $P$ of $G$, the isomorphism $G / P \simeq(G \times G) /(P \times G)$ provides an equivariant bi-Lipschitz bijection the limit set of $\Gamma$ in $G / P$ and the limit set of $\Gamma_{\rho}$ in $(G \times G) /(P \times G)$.

We note that the global bi-Hölder condition in Proposition 4.3 and Theorem 4.7 can be relaxed to a local bi-Hölder condition by the following lemma.

Lemma 4.9. Keep the notation as in Theorem 4.7 but assume $G_{1}$ and $G_{2}$ are semisimple, not just simple. Let $f: \Lambda_{\theta_{1}} \rightarrow \Lambda_{\theta_{2}}$ be a $\rho$-equivariant homeomorphism which is $\kappa$-bi-Hölder on some non-empty open subset $U$ of $\Lambda_{\theta_{1}}$ for some $\kappa>0$. Then $f$ is $\kappa$-bi-Hölder globally.
Proof. Let $\Lambda_{i}=\Lambda_{\theta_{i}}$ for $i=1,2$. Since $\Lambda_{1}$ is $\Gamma_{1}$-minimal, $\Lambda_{1}=\Gamma_{1} U$ and hence, by compactness, we have $\Lambda_{1}$ is a finite union of $\gamma_{k} U$ for some $\gamma_{1}, \cdots, \gamma_{n} \in \Gamma_{1}$. If $f$ is not $\kappa$-Hölder globally, by the compactness of $\Lambda_{1}$, we have a sequence $\xi_{i} \rightarrow \xi$ and $\eta_{i} \rightarrow \eta$ such that

$$
\begin{equation*}
\frac{d_{\mathcal{F}_{2}}\left(f\left(\xi_{i}\right), f\left(\eta_{i}\right)\right)}{d_{\mathcal{F}_{1}}\left(\xi_{i}, \eta_{i}\right)^{\kappa}} \rightarrow \infty . \tag{4.8}
\end{equation*}
$$

Since $\mathcal{F}_{2}$ is compact, we have $d_{\mathcal{F}_{1}}\left(\xi_{i}, \eta_{i}\right) \rightarrow 0$. Therefore, for some $1 \leq$ $k \leq n, \xi_{i}, \eta_{i} \in \gamma_{k} U$ for all $i$. Noting that the action of each element of $g_{i} \in G_{i}$ on $\mathcal{F}_{i}$ is a diffeomorphism for $i=1,2$, we can let $L$ be the maximum of the bi-Lipschitz constants of $\gamma_{k}$ on $\mathcal{F}_{1}$ and of $\rho\left(\gamma_{k}\right)$ on $\mathcal{F}_{2}$. Now we have $d_{\mathcal{F}_{2}}\left(f\left(\xi_{i}\right), f\left(\eta_{i}\right)\right) \leq L d_{\mathcal{F}_{2}}\left(f\left(\gamma_{k}^{-1} \xi_{i}, \gamma_{k}^{-1} \eta_{i}\right)\right)$ and $d_{\mathcal{F}_{1}}\left(\xi_{i}, \eta_{i}\right) \geq$ $\left.L^{-1} d_{\mathcal{F}_{1}}\left(\gamma_{k}^{-1} \xi_{i}, \gamma_{k}^{-1} \eta_{i}\right)\right)$. Since $f$ is $\kappa$-Hölder on $U$, it follows that the ratio in (4.8) is bounded, yielding a contradiction. This shows that $f$ is $\kappa$-Hölder
globally. Similarly by considering $f^{-1}$, we can show that $f^{-1}$ is $\kappa^{-1}$-Hölder globally.

Theorem 1.1 is now a special case of the following corollary of Theorem 4.7 together with Lemma 4.9

Corollary 4.10. Let $\alpha_{i}$ be a simple root of $G_{i}$ for $i=1,2$. Suppose that there exists a $\rho$-equivariant bijection $f: \Lambda_{\alpha_{1}} \rightarrow \Lambda_{\alpha_{2}}$ which is $\kappa$-bi-Hölder on some non-empty open subset of $\Lambda_{\alpha_{1}}$ for some $\kappa>0$. Then $\kappa=1$ and $\rho$ extends to a Lie group isomorphism $\bar{\rho}: G_{1} \rightarrow G_{2}$ which induces a diffeomorphism $\bar{f}: G_{1} / P_{\alpha_{1}} \rightarrow G_{2} / P_{\alpha_{2}}$ such that $\left.\bar{f}\right|_{\Lambda_{1}}=f$.

Note that the conclusion $\kappa=1$ follows since $\bar{f}$ is diffeomorphism and hence bi-Lipschitz.

Remark 4.11. In general, we cannot replace $f$ bi-Lipschitz by Lipschitz in Theorem [1.1. For example, let $\Gamma$ be a Schottky subgroup of $\mathrm{SL}_{2}(\mathbb{R})$ generated by two loxodromic elements $a, b$. Then for any $N \geq 2$, the representation $\rho$ of $\Gamma$ into $\mathrm{SL}_{2}(\mathbb{R})$ given by $a \mapsto a^{N}$ and $b \mapsto b^{N}$ induces an equivariant homeomorphism $\Lambda \rightarrow \Lambda$ which is Lipschitz, but not bi-Lipschitz. Clearly, $\rho$ does not extend to $\mathrm{SL}_{2}(\mathbb{R})$.

On the other hand, we have the following corollary of the proof of Theorem 4.7 where $f$ is required only to be Lipschitz under an extra hypothesis on the Hausdorff dimension of limit sets. In the statement below, a Möbius transformation is the extension of any isometry of $\mathbb{H}^{n+1}$ to its boundary $\mathbb{S}^{n}=\partial \mathbb{H}^{n+1}$.

Corollary 4.12. For $i=1,2$, let $\Gamma_{i}$ be a convex cocompact Zariski dense subgroup of $G_{i}=\mathrm{SO}^{\circ}\left(n_{i}+1,1\right), n_{i} \geq 1$. Let $\Lambda_{i} \subset \mathbb{S}^{n_{i}}$ be the limit set of $\Gamma_{i}$. Suppose that the Hausdorff dimension of $\Lambda_{1}$ is equal to the Hausdorff dimension of $\Lambda_{2}$. Let $f: \Lambda_{1} \rightarrow \Lambda_{2}$ be a $\rho$-equivariant homeomorphism which is Lipschitz on some non-empty open subset of $\Lambda_{1}$. Then $\rho$ extends to a Lie group isomorphism of $G_{1} \rightarrow G_{2}$ and $f$ extends to a Möbius transformation of $\mathbb{S}^{n}$ for $n=n_{1}=n_{2}$.

Proof. By the proof of Lemma 4.9, $f$ is Lipschitz on all of $\Lambda_{1}$. Let $\Gamma:=$ (id $\times \rho)\left(\Gamma_{1}\right)$ be the self-joining subgroup of $G=G_{1} \times G_{2}$. For $i=1,2$, let $\alpha_{i}$ be the simple root of $G=G_{1} \times G_{2}$ from the $i$-th factor. Then for any loxodromic element $g=(\gamma, \rho(\gamma)) \in G, \alpha_{1}(\lambda(g))$ and $\alpha_{2}(\lambda(g))$ are equal to $\lambda(\gamma)$ and $\lambda(\rho(\gamma))$ respectively. Suppose that $\Gamma$ is Zariski dense in $G$. The proof of Proposition 4.3 for $\Gamma$ shows that if there exists a loxodromic element $g=(\gamma, \rho(\gamma)) \in \Gamma$ such that $\alpha_{1}(\lambda(g))>\alpha_{2}(\lambda(g))$, then $f: \Lambda_{1} \rightarrow \Lambda_{2}$ cannot be Lipschitz. On the other hand, if $\Lambda_{1}$ and $\Lambda_{2}$ have the same Hausdorff dimension, the middle direction $(1,1) \in \mathfrak{a} \simeq \mathbb{R}^{2}$ is always contained in the interior of the limit cone of $\Gamma$ by [9, Corollary 4.2]. Note that when $\Gamma_{i}$ are cocompact lattices and $n_{1}=n_{2}=2$, 9, Corollary 4.2] is due to Thurston [24]. Therefore, the desired element $g \in \Gamma$ can always be found. This
implies that $\Gamma$ cannot be Zariski dense in $G$. As before, this implies the conclusion.

## 5. Slim limit sets of $G / P$ for $P$ non-maximal

Let $\Gamma$ be a Zariski dense subgroup of a connected semisimple real algebraic group $G$. Fix a subset $\theta \subset \Pi$ with $\# \theta \geq 2$. Recall from the introduction that a subset $S \subset \mathcal{F}_{\theta}$ is called slim if there exists a pair of distinct elements $\alpha_{1}$ and $\alpha_{2}$ of $\theta$ such that the limit set $\Lambda_{\theta}$ injects to $G / P_{\alpha_{1}}$ and $G / P_{\alpha_{2}}$ under the canonical projection map $\mathcal{F}_{\theta} \rightarrow G / P_{\alpha_{i}}$ for $i=1,2$.

In this section we prove the following theorem.
Theorem 5.1. If $\# \theta \geq 2$ and $\Lambda_{\theta}$ is a slim subset of $\mathcal{F}_{\theta}$, then no non-empty open subset $U$ of $\Lambda_{\theta}$ is contained in a proper $C^{1}$-submanifold of $\mathcal{F}_{\theta}$.

We first prove the following lemma which connects Theorem 5.1 with Proposition 4.3.
Lemma 5.2. Let $\theta_{0} \subset \theta \subset \Pi$. Suppose that $\Lambda_{\theta}$ is a $C^{1}$-submanifold of $\mathcal{F}_{\theta}$ and that the canonical projection $\mathcal{F}_{\theta} \rightarrow \mathcal{F}_{\theta_{0}}$ is injective on $\Lambda_{\theta}$. Then $\Lambda_{\theta_{0}}$ is a $C^{1}$-submanifold of $\mathcal{F}_{\theta_{0}}$ and $f_{\theta_{0}}: \Lambda_{\theta} \rightarrow \Lambda_{\theta_{0}}$ is a $\Gamma$-equivariant diffeomorphism.

Proof. For simplicity, we write $\Lambda=\Lambda_{\theta}$. We suppose that $\Lambda$ is a $C^{1}$ submanifold of $\mathcal{F}_{\theta}$. Since the projection $\mathcal{F}_{\theta} \rightarrow \mathcal{F}_{\theta_{0}}$ given by $f\left(g P_{\theta}\right)=g P_{\theta_{0}}$ is a smooth map, its restriction $f: \Lambda \rightarrow \mathcal{F}_{\theta_{0}}$ is a $C^{1}$ map which is also injective by hypothesis. We claim that there exists a point $x \in \Lambda$ where $d f_{x}: T_{x} \Lambda \rightarrow T_{f(x)} \mathcal{F}_{\theta_{0}}$ is injective. Pick a point $x \in \Lambda$ which maximizes $\operatorname{rank} d f_{y}, y \in \Lambda$. Then there exists a neighborhood of $x$ in $\Lambda$ where $d f$ has constant rank. Then if $r:=\operatorname{rank} d f_{x}$, there exist local coordinates near $x$ where

$$
f\left(x^{1}, \ldots, x^{m}\right)=\left(x^{1}, \ldots, x^{r}, 0, \ldots, 0\right)
$$

Since $f$ is injective, we must have $r=m$ and hence $d f_{x}$ is injective.
Now the set $\left\{x \in \Lambda: d f_{x}\right.$ is injective $\}$ is open and $\Gamma$-invariant. Since $\Gamma$ acts minimally on $\Lambda$, this set must be all of $\Lambda$. Thus $f$ is an immersion. Since $f$ is an injective immersion and $\Lambda$ is compact, $f$ is a $C^{1}$-embedding. Hence $f$ is a diffeomorphism onto its image, which is $\Lambda_{\theta_{0}}$. In particular, $\Lambda_{\theta_{0}}$ is a $C^{1}$-submanifold of $\mathcal{F}_{\theta_{0}}$.

Proof of Theorem 5.1, By the hypothesis on the slimness of $\Lambda_{\theta}$, there exists a pair of distinct elements $\alpha_{1}$ and $\alpha_{2}$ of $\theta$ such that $\Lambda_{\theta}$ injects to $G / P_{\alpha_{1}}$ and $G / P_{\alpha_{2}}$.

Suppose on the contrary that some non-empty open subset $U$ of $\Lambda_{\theta}$ is contained in some $C^{1}$-submanifold. Since $\Lambda_{\theta}$ is $\Gamma$-minimal, we have that for any $\xi \in \Lambda_{\theta}, \Gamma \xi$ is dense, so $\gamma \xi \in U$ for some $\gamma \in \Gamma$. Since $\xi \in \gamma^{-1} U$, it follows that $\Lambda_{\theta}$ is a $C^{1}$-submanifold of $\mathcal{F}_{\theta}$. By Lemma 5.2, we have $\Gamma$-equivariant diffeomorphisms $f_{\alpha_{i}}: \Lambda_{\theta} \rightarrow \Lambda_{\alpha_{i}}$ for each $i=1,2$. Hence
$f_{\alpha_{2}} \circ f_{\alpha_{1}}^{-1}: \Lambda_{\alpha_{1}} \rightarrow \Lambda_{\alpha_{2}}$ is a $\Gamma$-equivariant diffeomorphism, contradicting Proposition 4.3. This finishes the proof.

Remark 5.3. We remark that Proposition 4.3 implies that if $\Lambda$ is a slim subset of $G / P$, then there exists a maximal parabolic subgroup $Q$ containing $P$ such that the projection $G / P \rightarrow G / Q$ restricted to $\Lambda$ is not bi-Lipschitz.

Antipodal groups. Theorem 1.5 applies to the class of $P$-antipodal discrete subgroups of $G$, which contains any subgroup of a $P$-Anosov or a relatively $P$-Anosov subgroup. To define an antipodality, we recall that a parabolic subgroup $P$ is called reflexive if its conjugacy class contains a parabolic subgroup $P^{\prime}$ opposite to $P$, that is, $P \cap P^{\prime}$ is a common Levi subgroup of both $P$ and $P^{\prime}$. For example, a minimal parabolic subgroup of $G$ is always reflexive. For a parabolic subgroup $P$, let $P_{\text {reflexive }}$ be the largest reflexive parabolic subgroup contained in $P$. If $P=P_{\theta}$, then $P_{\text {reflexive }}=P_{\theta \cup i}(\theta)$.

Definition 5.4. A discrete subgroup $\Gamma$ is called $P$-antipodal if its limit set in $G / P_{\text {reflexive }}$ is antipodal in the sense that any two distinct points are in general position.

If a discrete subgroup $\Gamma$ is $P$-antipodal, then its limit set on $G / P$ injects to $G / P^{\prime}$ for any $P^{\prime}$ containing $P$ [13, Lemma 9.5]. Hence if $\Gamma$ is $P$-antipodal for a non-maximal parabolic subgroup $P$, then its limit set is a slim subset of $G / P$. Therefore the following corollary is a special case of Theorem 1.5,

Corollary 5.5. Let $G$ be a connected semisimple real algebraic group of rank at least 2 and $P$ a non-maximal parabolic subgroup of $G$. The limit set of a Zariski dense $P$-antipodal subgroup of $G$ cannot be a $C^{1}$-submanifold of $G / P$.

Note that there are many slim limit sets which are not antipodal (e.g., the limit set of a self-joining group defined in (1.2)).

## 6. An EXAMPLE

In this final section, we construct an example of a Zariski dense discrete subgroup of $\mathrm{SL}_{8}(\mathbb{R})$ which explains the necessity of introducing $P_{2}^{\prime}$ in the conclusion of Theorem 1.3 in the case when $P_{2}$ is not maximal. The examples we construct are Borel-Anosov and ( $1,1,2$ )-hyperconvex subgroups of $\mathrm{SL}_{8}(\mathbb{R})$.

We begin by setting up some notation. For any $d \geq 2$, let $A$ be the diagonal subgroup of $\mathrm{SL}_{d}(\mathbb{R})$ consisting of diagonal elements with positive entries so that $\mathfrak{a}$ and $\mathfrak{a}^{+}$can respectively be identified with $\mathfrak{a}=\left\{\left(u_{1}, \cdots, u_{d}\right)\right.$ : $\left.\sum_{k=1}^{d} u_{k}=0\right\}$ and $\mathfrak{a}^{+}=\left\{\left(u_{1}, \cdots, u_{d}\right) \in \mathfrak{a}: u_{1} \geq \cdots \geq u_{d}\right\}$. For $1 \leq k \leq$ $d-1$, let

$$
\alpha_{k}\left(\left(u_{1}, \cdots, u_{d}\right)\right)=u_{k}-u_{k+1} ;
$$

then $\Pi=\left\{\alpha_{k}: 1 \leq k \leq d-1\right\}$ is the set of all simple roots. For any $g \in \mathrm{SL}_{d}(\mathbb{R})$, its Jordan projection $\lambda(g) \in \mathfrak{a}^{+}$satisfies

$$
\alpha_{k}(\lambda(g))=\log \frac{\lambda_{k}(g)}{\lambda_{k+1}(g)}
$$

where $\lambda_{1}(g) \geq \cdots \geq \lambda_{d}(g)$ are the absolute values of the eigenvalues of $g$. Also, for $\theta \subset \Pi$, the boundary $\mathcal{F}_{\theta}=\mathrm{SL}_{d}(\mathbb{R}) / P_{\theta}$ coincides with the partial flag manifold consisting of flags with subspaces of dimensions $\left\{k: \alpha_{k} \in \theta\right\}$.

Let $\Delta$ be a hyperbolic group and denote by $\partial \Delta$ its Gromov boundary. Recall from [8] that a representation $\rho: \Delta \rightarrow \mathrm{SL}_{d}(\mathbb{R})$ is $\left\{\alpha_{k}\right\}$-Anosov if there exist constants $c, C>0$ so that for all $\gamma \in \Delta$,

$$
\alpha_{k}(\lambda(\rho(\gamma)) \geq c|\gamma|-C
$$

where $|\gamma|$ is the minimal word length of $\gamma$ with respect to a fixed finite generating set of $\Delta$. If $\rho$ is $\left\{\alpha_{k}\right\}$-Anosov, it admits a pair of unique continuous equivariant embeddings $\xi_{\rho}^{k}: \partial \Delta \rightarrow \operatorname{Gr}_{k}\left(\mathbb{R}^{d}\right)$ and $\xi_{\rho}^{d-k}: \partial \Delta \rightarrow \operatorname{Gr}_{d-k}\left(\mathbb{R}^{d}\right)$. Furthermore, the image of $\left(\xi_{\rho}^{k}, \xi_{\rho}^{d-k}\right)$ coincides with the limit set of $\rho(\Delta)$ in $\mathcal{F}_{\left\{\alpha_{k}, \alpha_{d-k}\right\}}$. We say that $\rho$ is Borel-Anosov if it is $\left\{\alpha_{k}\right\}$-Anosov for all $1 \leq k \leq d-1$. The image of a Borel-Anosov representation is called a Borel Anosov subgroup.

A representation $\rho: \Delta \rightarrow \operatorname{SL}_{d}(\mathbb{R})$ is (1,1,2)-hyperconvex if it is $\left\{\alpha_{1}, \alpha_{2}\right\}$ Anosov and for all distinct $x, y, z \in \partial \Delta$,

$$
\xi_{\rho}^{1}(x) \oplus \xi_{\rho}^{1}(y) \oplus \xi_{\rho}^{d-2}(z)=\mathbb{R}^{d}
$$

Both being $\left\{\alpha_{k}\right\}$-Anosov and being (1,1,2)-hyperconvex are open conditions in the representation variety (see [19, Proposition 6.2]).

Proposition 6.1. There exists a Zariski dense discrete subgroup $\Gamma<\mathrm{SL}_{8}(\mathbb{R})$ which admits an equivariant Lipschitz bijection $\Lambda_{\alpha_{3}} \rightarrow \Lambda_{\alpha_{1}}$. Moreover, $\Gamma$ is Borel-Anosov, (1,1,2)-hyperconvex, and the projection map $p: \Lambda_{\left\{\alpha_{1}, \alpha_{3}\right\}} \rightarrow$ $\Lambda_{\alpha_{3}}$ is a bi-Lipschitz bijection.

Theorem 1.3 in this case applies with $f=p^{-1}, P_{1}=P_{\alpha_{3}}, P_{2}=P_{\left\{\alpha_{1}, \alpha_{3}\right\}}$ and $P_{2}^{\prime}=P_{\alpha_{3}}$.

Let $\Delta=\left\langle a_{1}, a_{2}\right\rangle$ be the free group with two generators $a_{1}, a_{2}$. Let $N \geq 2$. Let $\tau_{1}: \Delta \rightarrow \mathrm{SL}_{2}(\mathbb{R})$ be a convex cocompact representation and $\tau_{2}: \Delta \rightarrow$ $\mathrm{SL}_{2}(\mathbb{R})$ be defined so that $\tau_{2}\left(a_{i}\right)=\tau_{1}\left(a_{i}\right)^{N}$ for $i=1,2$. We may choose $N$ large enough that

$$
\frac{\alpha_{1}\left(\lambda\left(\tau_{2}(\gamma)\right)\right)}{\alpha_{1}\left(\lambda\left(\tau_{1}(\gamma)\right)\right)} \geq 4 \quad \text { for all non-trivial } \quad \gamma \in \Delta .
$$

Let $\iota: \mathrm{SL}_{2}(\mathbb{R}) \rightarrow \mathrm{SL}_{4}(\mathbb{R})$ be an irreducible representation, which is unique up to conjugations. Then each $\rho_{i}=\iota \circ \tau_{i}$ is a positive representation and hence Borel Anosov and (1,1,2)-hyperconvex [19, Corollary 6.13]. One easily checks that $\frac{\alpha_{1}\left(\lambda\left(\rho_{2}(\gamma)\right)\right)}{\alpha_{1}\left(\lambda\left(\rho_{1}(\gamma)\right)\right)} \geq 4$ for all non-trivial $\gamma \in \Delta$. Then a theorem of Tsouvalas [26, Theorem 1.9] implies that $f_{\rho_{1}, \rho_{2}}=\xi_{\rho_{2}}^{1} \circ\left(\xi_{\rho_{1}}^{1}\right)^{-1}$ is 4-Hölder.

Let $\Phi_{0}: \Delta \rightarrow \mathrm{SL}_{8}(\mathbb{R})$ denote the representation given by the direct sum $\Phi_{0}=\rho_{1} \oplus \rho_{2}$. One checks that
$\lambda_{1}\left(\rho_{2}(\gamma)\right)>\lambda_{2}\left(\rho_{2}(\gamma)\right)>\lambda_{1}\left(\rho_{1}(\gamma)\right)>\cdots>\lambda_{4}\left(\rho_{1}(\gamma)\right)>\lambda_{3}\left(\rho_{2}(\gamma)\right)>\lambda_{4}\left(\rho_{2}(\gamma)\right)$
for all non-trivial $\gamma \in \Delta$ and that $\Phi_{0}$ is Borel Anosov with limit maps given by

$$
\zeta_{0}^{k}(x)= \begin{cases}\{0\} \oplus \xi_{\rho_{2}}^{k}(x) & \text { if } k=1,2 \\ \xi_{\rho_{1}}^{k-2}(x) \oplus \xi_{\rho_{2}}^{2}(x) & \text { if } k=3,4,5 . \\ \mathbb{R}^{4} \oplus \xi_{\rho_{2}}^{k-4}(x) & \text { if } k=6,7 .\end{cases}
$$

Then, the fact that $f_{\rho_{1}, \rho_{2}}$ is 4-Hölder implies that $\zeta_{0}^{1} \circ\left(\zeta_{0}^{3}\right)^{-1}$ is also 4-Hölder. In particular, $\zeta_{0}^{1} \circ\left(\zeta_{0}^{3}\right)^{-1}: \Lambda_{\alpha_{3}}\left(\Phi_{0}(\Delta)\right) \rightarrow \Lambda_{\alpha_{1}}\left(\Phi_{0}(\Delta)\right)$ is Lipschitz.

However, $\Phi_{0}(\Delta)$ is not Zariski dense. Since $\Delta$ is the free group on two generators, there exists an arbitrary small deformation $\Phi: \Delta \rightarrow \operatorname{SL}_{8}(\mathbb{R})$ of $\Phi_{0}$ which is Borel Anosov with Zariski dense image. Arguing exactly as in [31, Section 9], one can show that $\Phi_{0}$ and $\wedge^{3} \Phi_{0}$ are both ( $1,1,2$ )hyperconvex. Therefore, we may assume that $\Phi$ and $\wedge^{3} \Phi$ are both ( $1,1,2$ )hyperconvex.

One may then use standard techniques (cf. [31) to show that if $\Phi$ is sufficiently close to $\Phi_{0}$, then

$$
\frac{2}{3} \leq \frac{\alpha_{1}(\lambda(\Phi(\gamma)))}{\alpha_{1}\left(\lambda\left(\Phi_{0}(\gamma)\right)\right)} \leq \frac{3}{2} \quad \text { and } \quad \frac{2}{3} \leq \frac{\alpha_{1}\left(\lambda\left(\wedge^{3} \Phi(\gamma)\right)\right)}{\alpha_{1}\left(\lambda\left(\wedge^{3} \Phi_{0}(\gamma)\right)\right)} \leq \frac{3}{2}
$$

for all non-trivial $\gamma \in \Delta$. Let $\zeta=\left(\zeta^{k}\right)$ be the limit map of $\Phi(\Delta)$ and $\hat{\zeta}_{0}^{1}: \partial \Delta \rightarrow \Lambda_{\alpha_{1}}\left(\wedge^{3} \Phi_{0}(\Delta)\right)$ and $\hat{\zeta}^{1}: \partial \Delta \rightarrow \Lambda_{\alpha_{1}}\left(\wedge^{3} \Phi(\Delta)\right)$ be limit maps of $\wedge^{3} \Phi_{0}$ and $\Lambda^{3} \Phi$. One may again apply Tsouvalas's theorem [26, Theorem 1.9] to conclude that $\zeta^{1} \circ\left(\zeta_{0}^{1}\right)^{-1}$ and $\hat{\zeta}_{0}^{1} \circ\left(\hat{\zeta}^{1}\right)^{-1}$ are $\frac{2}{3}$-Hölder. There is a $C^{1}$-equivariant identification of $\Lambda_{\alpha_{1}}\left(\wedge^{3} \Phi_{0}(\Delta)\right)$ with $\Lambda_{\alpha_{3}}\left(\Phi_{0}(\Delta)\right)$ and an analogous identification for $\Phi$, so we may conclude that $\zeta_{0}^{3} \circ\left(\zeta^{3}\right)^{-1}$ is $\frac{2}{3}$ Hölder. Now set

$$
\Gamma:=\Phi(\Delta)<\operatorname{SL}_{8}(\mathbb{R}) .
$$

Then the limit map

$$
\zeta^{1} \circ\left(\zeta^{3}\right)^{-1}=\left(\zeta^{1} \circ\left(\zeta_{0}^{1}\right)^{-1}\right) \circ\left(\zeta_{0}^{1} \circ\left(\zeta_{0}^{3}\right)^{-1}\right) \circ\left(\zeta_{0}^{3} \circ\left(\zeta^{3}\right)^{-1}\right)
$$

is a $\frac{16}{9}$-Hölder and hence yields a Lipschitz map from $\Lambda_{\alpha_{3}}$ to $\Lambda_{\alpha_{1}}$. Since $\Gamma$ is Borel Anosov, the projection map $\Lambda_{\left\{\alpha_{1}, \alpha_{3}\right\}} \rightarrow \Lambda_{\alpha_{3}}$ is now a bi-Lipschitz homeomorphism. This proves Proposition 6.1,

## References

[1] Y. Benoist. Propriétés asymptotiques des groupes linéaires. Geom. Funct. Anal., 7(1):1-47, 1997.
[2] Y. Benoist. Convexes divisibles I. In Algebraic groups and arithmetic, Tata Inst. Fund. Res. Stud. Math. 17 (2004), 339-374.
[3] A. Borel. Linear algebraic groups. Graduate Texts in Math., Vol 126, Springer.
[4] F. Dalbo and I. Kim. A criterion of conjugacy for Zariski dense subgroups. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 8, 647-650.
[5] S. Edwards, M. Lee, and H. Oh. Anosov groups: local mixing, counting, and equidistribution. Geometry 6 Topology 27 (2023), 513-573.
[6] O. Glorieux and D. Monclair. Regularity of limit sets of AdS quasi-Fuchsian groups. Preprint, arXiv:1809.10639.
[7] N. Ivanov. Action of Möbius transformations on homeomorphisms: stability and rigidity. Geom. Func. Anal., 6:79-119, 1996.
[8] F. Kassel and R. Potrie. Eigenvalue gaps for hyperbolic groups and semigroups. J. Mod. Dyn. 18:161-208, 2022.
[9] D. Kim, Y. Minsky and H. Oh. Tent property of the growth indicator functions and applications. Geom. Dedicata 218 (2024), Paper No: 14
[10] D. Kim and H. Oh. Rigidity of Kleinian groups via self-joinings. Inventiones Mathematicae, Vol 234, issue 3 (2023)
[11] D. Kim and H. Oh. Rigidity of Kleinian groups via self-joinings: measure theoretic criterion. Preprint (arXiv:2302.03552)
[12] D. Kim and H. Oh. Conformal measure rigidity of representations via self-joinings. Preprint, arXiv:2302.03539.
[13] D. Kim, H. Oh and Y. Wang. Properly discontinuous actions, growth indicators and conformal measures for transverse subgroups. Preprint, arXiv:2306.06846.
[14] F. Labourie. Anosov flows, surface groups and curves in projective space. Invent. Math., 165:51-114, 2006.
[15] G. A. Margulis. Discrete subgroups of semisimple Lie groups. Ergeb. Math. Grenzgeb. 17, Springer, 1991.
[16] G. Mostow. Strong rigidity of locally symmetric spaces. Annals of Mathematics Studies, No. 78. Princeton University Press, 1973.
[17] G. Prasad. Strong rigidity of Q-rank 1 lattices. Invent. Math., 21:255-286, 1973.
[18] B. Pozzetti and A. Sambarino. Metric properties of boundary maps, Hilbert entropy and non-differentiability. Preprint, arXiv:2310.07373.
[19] B. Pozzetti, A. Sambarino, and A. Wienhard. Conformality for a robust class of non-conformal attractors. J. Reine Angew. Math. 774:1-51, 2021.
[20] B. Pozzetti, A. Sambarino, and A. Wienhard. Anosov representations with Lipschitz limit set. Geom.Top., 27:3303-3360, 2023.
[21] J.-F. Quint. Mesures de Patterson-Sullivan en rang supérieur. Geom. Funct. Anal., 12(4):776-809, 2002.
[22] J.-F. Quint. L'indicateur de croissance des groupes de Schottky Ergodic theory and dynamical systems. 23 (2003), 249-272
[23] I. Smilga. Proper affine actions in non-swinging representations. Groups Geom. Dyn., 12(2):449-528, 2018.
[24] W. Thurston. Minimal stretch maps between hyperbolic surfaces. Preprint, arXiv: 980103.
[25] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. J. Reine Angew. Math., 247:196-220, 1971.
[26] K. Tsouvalas. The Hölder exponent of Anosov limit maps. Preprint, arXiv:2306.15823
[27] P. Tukia. Differentiability and rigidity of Möbius groups. Invent. Math., 82:557-578, 1985.
[28] P. Tukia. A rigidity theorem for Möbius groups. Invent. Math. 97, no. 2, 405-431, 1989.
[29] P. Tukia. Homomorphisms of constant stretch between Möbius groups. Comment. Math. Helvetici. 66, 151-167 1991
[30] D. Winter. Mixing of frame flow for rank one locally symmetric spaces and measure classification. Israel J. Math. 210 (2015), no. 1, 467-507.
[31] T. Zhang and A. Zimmer. Regularity of limit sets of Anosov representations. Preprint, arXiv:1903.11021
[32] A. Zimmer. Projective Anosov representations, convex cocompact actions and rigidity. J. Differential Geometry, 119 (2021) 513-586.

Department of Mathematics, University of Michigan, Ann Arbor, Mi
Email address: canary@umich.edu
Department of Mathematics, Yale University, New Haven, CT 06511 and Korea Institute for Advanced Study, Seoul

Email address: hee.oh@yale.edu
Department of Mathematics, University of Wisconsin-Madison, Madison, WI

Email address: amzimmer2@wisc.edu


[^0]:    Canary is partially supported by the NSF grant No. DMS-2304636. Oh is partially supported by the NSF grant No. DMS-1900101. Zimmer is partially supported by a Sloan research fellowship and NSF grant No. DMS-2105580.

