To restart, or not to restart, that is the question

R. K. Singh*
Department of Physics, Bar-Ilan University, Ramat-Gan 5290002, Israel

The mean time taken by a Brownian particle to cover a distance L is $L^2/2D$, with D being its diffusion coefficient. We find that restarts increase this mean time. This is because restarts naturally introduce time overheads which need to be taken into account while addressing the escape properties. In addition, restarts impose a constraint on allowed overhead functions if they are to provide an advantage, that is, a reduced mean escape time. We explicitly study these constraints for Poisson and sharp restarts. Implementing an overhead function to control escape times means that the physical properties of the Brownian particle need to be appropriately modified. When restarts are non-instantaneous, an additional delay is introduced increasing the mean escape time further. Can restarts overcome the delays incurred due to finite return times? To restart, or not to restart, should be the question.

Introduction: Random processes under stochastic restarts have emerged as one of the most investigated topics in nonequilibrium statistical physics [1–3]. These studies have lead to a general consensus that the completion of a stochastic process can be expedited via restarts [4–11]. This has been notably seen in the field of computer science where restarting randomized algorithms [12, 13] leads to an improved performance [14]. In the realm of search processes restarts render mean search time finite [15–18]. The fundamental reason for this advantage is that intermittent restarts tend to prevent the trajectories from flying off to infinity, resulting in finite search times in infinite domains. However, in most cases of practical interest search often takes place in a bounded domain [19–24]. Does it mean that restarts possess a definitive advantage for search in finite domains? While the question has been addressed in previous studies [25–28], a few fundamental points pertaining to the practical significance of restarts for expediting search remain unaddressed. For example, if a protein is searching for a target DNA [29, 30] or a general search in cellular media [31, 32], is it viable for the searcher to continue its search solely under the effect of thermal fluctuations or employ some restart protocol? A straightforward answer to this question is that restarts can be employed as a means of expediting search if and only if a restart protocol actually expedites the completion of the stochastic process. For example, for escape over a fluctuating barrier it was shown in Ref. [28] that restart expedites escape only when the restart location is far from the potential well. However, escape over a fluctuating barrier involves an external dichotomous noise on top of which a restart strategy is employed, hence it becomes imperative to single out search processes with restarts and address the above question. Furthermore, search processes under restarts naturally give rise to time overheads, as it would take, for example, a finite amount of time to take the particle from the bottom of the potential well to the restart location. While a number of previous works have studied the effects of time overheads for search with restarts, like the Michaelis-Menten reaction scheme [5, 6, 33], inspection paradox [34], queues with restarts [35], search in a potential [36, 37], etc., overheads have been accounted for as a source of delay arising independent of the search process, not a natural consequence of the process itself. The importance of these questions cannot be overemphasized in light of the fact that controlling a search process is important to optimize search [38, 39].

We address the above questions using the simple example of a Brownian particle diffusing in one dimension as it is often the first model of choice to address stochastic search [40]. We ask a simple question: given that the mean time taken by a Brownian particle to cover a distance L is $\langle T\rangle = L^2/2D$, with D being the diffusion coefficient [41, 42], can we reduce this mean travel time employing restarts? To answer this question, consider a Brownian particle moving in an interval $[0, L]$ with a reflecting wall at $x = 0$ and an absorbing wall at $x = L$. Moreover, let us measure distance(s) in units of L and time(s) in units of L^2/D, thus reducing the motional quantities to dimensionless forms. As a result, our study is now reduced to a Brownian motion in the unit interval $[0, 1]$, with $\langle T\rangle = \frac{1}{2}$.

Restarts and associated overheads: If the restart location coincides with the initial location $x = 0$, then restarts delay escape [28]. This is because there's a current from every $x \in (0, 1)$ towards the origin, thus reducing the natural tendency of the particle to move away from $x = 0$. This implies that a necessary condition for restarts to be useful is that $x_0 \in (0, 1)$. For an appropriately chosen x_0, restarts can expedite escape to the absorbing wall by removing the trajectories which tend to move towards the origin. It is important to notice here that we are talking about an escape from $x = x_0$ to $x = 1$ and not for the full length of the unit interval, that is, from $x = 0$ to $x = 1$. So what happens to the motion from $x = 0$ to $x = x_0$? The particle cannot just vanish at the origin and reappear at x_0, there has to be some way for the particle to cover the sub-interval $[0, x_0]$. And it is this motion from $x = 0$ to $x = x_0$ which constitutes the *hidden cost* associated with restarts and needs to be taken into account in order to fully understand the effect of restarts on escape properties. Why does this hidden cost arise for a Brownian motion under restarts? Answer to this question lies in the fact that the mean time to cover the unit interval can be written as:

$$\langle T\rangle = \langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle, \tag{1}$$

where $\langle T_{0,x_0}\rangle = \frac{x_0^2}{2}$ is the mean time taken by the Brownian particle to cover the sub-interval $[0, x_0]$ and $\langle T_{x_0,1}\rangle = \frac{1-x_0^2}{2}$ is the mean time to go from $x = x_0$ to $x = 1$ [41, 42]. And whenever the motion is restarted at x_0, it directly affects the second term $\langle T_{x_0,1}\rangle$ in Eq. (1) while indirectly modifying $\langle T_{0,x_0}\rangle$. Moreover, restarts enter only once the Brownian particle reaches x_0. The time incurred in going from $x = 0$ to $x = x_0$, thus constitutes the overhead which needs to be taken into account when applying a restart protocol. As a result, in presence of restarts: $T_R = T_{0,x_0} + I(T_{x_0,1} \geq R)(R + T_R') + I(T_{x_0,1} < R)T_{x_0,1}$, where I is an indicator variable which takes value one when its argument is true and zero otherwise, R denotes the time of restart, and T_R is the completion time with restarts [9, 10, 35, 43]. Taking the expectation we have

$$\langle T_R\rangle = \frac{\langle T_{0,x_0}\rangle}{\langle I(T_{x_0,1} < R)\rangle} + \frac{\langle \min\{T_{x_0,1}, R\}\rangle}{\langle I(T_{x_0,1} < R)\rangle}. \tag{2}$$

In absence of any restarts $T_{x_0,1} < R$, thus reducing Eq. (2) to (1). Furthermore, overheads become more pronounced due to restarts (as $\langle I(T_{x_0,1} < R)\rangle < 1$). Physically, every time motion restarts, an overhead is incurred. Hence the advantage gained by restarts for motion from $x = x_0$ to $x = 1$, if any, should overcome time overheads incurred in going from $x = 0$ to $x = x_0$ in order to prove beneficial in such a way that $\langle T_R\rangle < \langle T\rangle$. Let us now investigate the conditions under which this inequality would hold true.

In Eq. (2) the first term accounts for the time overheads while the second term, $\langle \min\{T_{x_0,1}, R\}\rangle / \langle I(T_{x_0,1} < R)\rangle \equiv \langle T_{R,x_0,1}\rangle$ is the mean time to go from $x = x_0$ to $x = 1$ in presence of restarts [10]. As discussed above, $x_0 = 0 \Rightarrow \langle T_R\rangle \geq \langle T\rangle$. On the other hand, $\lim_{x_0\to 1^-}\langle T_{R,x_0,1}\rangle = 0$ which implies that $\lim_{x_0\to 1^-}\langle T_R\rangle = \lim_{x_0\to 1^-}\langle T_{0,x_0}\rangle/\langle I(T_{x_0,1} < R)\rangle = \langle T\rangle/\langle I(T_{x_0,1} < R)\rangle > \langle T_{x_0,1}\rangle$ since $\langle I(T_{x_0,1} < R)\rangle < 1$. This implies that for $x_0 \in \{0, 1\} \Rightarrow \langle T_R\rangle \geq \langle T\rangle$. Let us now study $\langle T_R\rangle$ for $x_0 \in (0, 1)$. As restarts expedite escape by removing trajectories moving away from the target (at $x = 1$) there would exist a $x_{0,c} \in (0, 1)$ such that $\lim_{x_0\to x_{0,c}^-}\langle T_{R,x_0,1}\rangle = (1 - x_{0,c}^2)/2 \Rightarrow \langle T_{R,x_0,1}\rangle > \langle T_{x_0,1}\rangle\ \forall\ x_0 < x_{0,c}$ and $\langle T_{R,x_0,1}\rangle < \langle T_{x_0,1}\rangle\ \forall\ x_0 > x_{0,c}$ [28], thus dissecting the unit interval such that $(0, 1) = (0, x_{0,c}) \cup (x_{0,c}, 1)$. The exact value of $x_{0,c}$ would, however, depend on the specific details of the distribution of R and the stochastic process (here Brownian motion). Now $x_0 < x_{0,c} \Rightarrow \langle T_R\rangle = \langle T_{0,x_0}\rangle/\langle I(T_{x_0,1} < R)\rangle + \langle T_{R,x_0,1}\rangle > (x_0^2/2)/\langle I(T_{x_0,1} < R)\rangle + (1-x_0^2)/2 > \langle T\rangle$. On the other hand, even though $\langle T_{R,x_0,1}\rangle < (1 - x_0^2)/2$ for $x_0 > x_{0,c}$, the fact that $\langle T_R\rangle\big|_{x_0=x_{0,c}} > \langle T\rangle$ and $\langle T_{R,x_0,1}\rangle$ is a monotonically decreasing function of x_0 implies that $\forall\ x_0 > x_{0,c},\ \langle T_R\rangle > \langle T\rangle$. In summary,

$$\langle T_R\rangle \geq \langle T\rangle\ \forall\ x_0 \in [0, 1]. \tag{3}$$

This is the main result of this paper and it implies that restarts always delay the mean travel time from one point to another. It is for this same reason that restarts delay escape over a fluctuating barrier when the restart location lies at the bottom of the potential well [28]. Furthermore, the above result holds true for any distribution of restart times R. More importantly, the above analysis for Brownian motion is straightforwardly extended to any random walk traversing the unit interval and satisfying Eq. (1), provided $\langle T_{0,x_0}\rangle$ is a monotonically increasing function of x_0 while $\langle T_{x_0,1}\rangle$ is a monotonically decreasing function of x_0.

Coming back to (3), $\langle T_R\rangle \geq \langle T\rangle$ not because restarts are beneficial only over a sub-interval but that they tend to enhance the effect of time overheads incurred in bringing the particle from the origin to the restart location. The question now is, can we modify the mean overhead time $\langle T_{0,x_0}\rangle$ rendering the escape under restarts beneficial, that is, $\langle T_R\rangle \leq \langle T\rangle$? We only require $\langle T_{0,x_0}\rangle$ to be a monotonically increasing function of x_0 such that $\lim_{x_0\to 0^+}\langle T_{0,x_0}\rangle = 0$, as larger distances take longer time on an average. Under these constraints, the requirement $\langle T_R\rangle \leq \langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle$ alongwith Eq. (2) implies

$$\langle T_{0,x_0}\rangle \leq \langle T_{x_0,1}\rangle \frac{\langle I(T_{x_0,1} < R)\rangle}{1 - \langle I(T_{x_0,1} < R)\rangle} - \frac{\langle \min\{T_{x_0,1}, R\}\rangle}{1 - \langle I(T_{x_0,1} < R)\rangle}. \tag{4}$$

The above inequality describes the set of allowed overhead functions $\langle T_{0,x_0}\rangle$ such that restarts are beneficial. The RHS of the above inequality provides an upper bound for $\langle T_{0,x_0}\rangle$ and let us denote it by $U(x_0)$. We now study in detail the upper bound $U(x_0)$ in (4) for specific restart protocols.

Poisson and sharp restarts: Poisson restarts are characterized by an exponential distribution of restart times, $P_r(R) = re^{-rR}$ with r being the rate of restart. As a result, $\langle I(T_{x_0,1} < R)\rangle = \tilde{F}(x_0, r)$ where $\tilde{F}(x_0, r) = \int_0^\infty dt\ e^{-rt}F(x_0, t)$ is the Laplace transform of the first passage time distribution for escape from $x = 1$ starting at $x = x_0$. Similarly, $\langle \min\{T_{x_0,1}, R\}\rangle = \frac{1-\tilde{F}(x_0,r)}{r}$. As a result, for Poisson restarts:

$$\langle T^r_{0,x_0}\rangle \leq U^r(x_0) = \frac{1 - x_0^2}{2} \times \frac{\tilde{F}(x_0, r)}{1 - \tilde{F}(x_0, r)} - \frac{1}{r}. \tag{5}$$

On the other hand, for sharp restarts the inter-restart times are a fixed, hence $P_\tau(R) = \delta(R - \tau)$. As a result, $\langle I(T_{x_0,1} < R)\rangle = \int_0^\tau dt\ F(x_0, t)$ and $\langle \min\{T_{x_0,1}, R\}\rangle =$

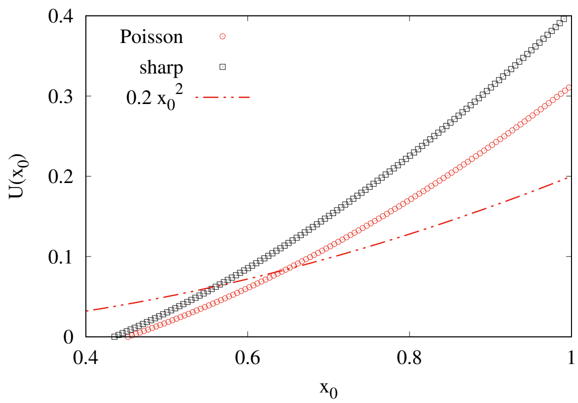

FIG. 1. Upper bound of the overhead $U(x_0)$ for Poisson and sharp restart protocols following the right hand side of inequalities (5) and (6) respectively for Brownian motion in the unit interval. Rate of Poisson restart $r = 1$ and time for sharp restart $\tau = 1$.

$\int_0^\tau dt\, q(x_0, t)$, where $q(x_0, t) = \int_t^\infty du\, F(x_0, u)$ is the survival probability. Using these results in (4) we have, for sharp restarts:

$$\langle T^\tau_{0,x_0}\rangle \le U^\tau(x_0) = \frac{1-x_0^2}{2} \times \frac{1-q(x_0,\tau)}{q(x_0,\tau)} - \frac{\int_0^\tau dt\, q(x_0,t)}{q(x_0,\tau)}. \tag{6}$$

Now for Brownian motion in $[0, 1]$, $\tilde{F}(x_0, r) = \cosh(\sqrt{r}x_0)/\cosh(\sqrt{r})$ whereas $q(x_0, t) = 2\sum_{n=0}^\infty \frac{(-1)^n}{\lambda_n}\cos(\lambda_n x_0)e^{-\lambda_n^2 t}$ with $\lambda_n = (n + 1/2)\pi$ [41, 42]. Using these results we study the upper bounds $U(x_0)$ and find that $U(x_0) = 0$ for x_0 less than a critical value, depending on the restart protocol (see Fig. 1). Moreover, $U^\tau(x_0) \ge U^r(x_0)$ and for a wider range of restart locations. In other words, even when Poisson restarts prove detrimental to escape (region above red curve in Fig. 1), sharp restarts still work such that $\langle T^\tau_R\rangle \le \langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle$ (the region between the black and red curves in Fig. 1). Furthermore, changing the value of the restart rates r or $1/\tau$ does not result in a qualitative modification of the overhead functions $U(x_0)$, that is, it remains a monotonically increasing function of x_0 beyond some critical point (dependent on r or $1/\tau$).

Escape under restarts: The upper bounds $U^r(x_0)$ and $U^\tau(x_0)$ impose additional constraint on the overhead. As a representative example, we choose $\langle T_{0,x_0}\rangle = x_0^2/5$ (see red line in Fig. 1). The solution of $\langle T_{0,x_0}\rangle = U(x_0)$ defines the critical location $x_{0,c}$ for a given restart protocol such that for $x_0 > x_{0,c}$, $\langle T_R\rangle \le \langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle$. In addition, time overheads tend to reduce the domain of applicability of restarts, that is, $x_{0,c}$ shifts towards right. This is easily seen, for example, for Poisson restarts where $x_{0,c}$ is determined by the condition $\mathrm{CV}^2(x_{0,c}) = 1$ [10], with $\mathrm{CV}^2 = [\langle T^2_{x_0,1}\rangle - \langle T_{x_0,1}\rangle^2]/\langle T_{x_0,1}\rangle^2$ being the coefficient of variation of $F(x_0, t)$. For Brownian motion in

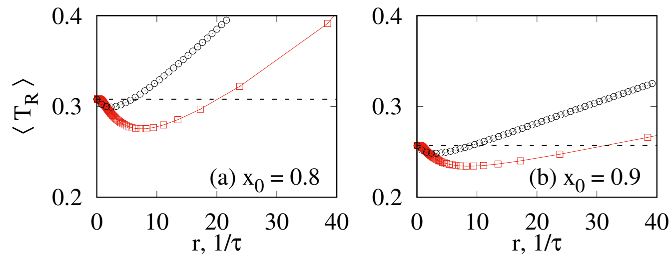

FIG. 2. Mean escape time $\langle T_R\rangle$ under restarts for Poisson ($\circ$) and sharp restart ($\square$) protocols in presence of time overheads with $\langle T_{0,x_0}\rangle = x_0^2/5$ as a function of the restart rate r or $1/\tau$ calculated using Eq. (2). The black dashed line represents $\langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle$.

the unit interval $\mathrm{CV}^2(x_0) = \frac{2(1+x_0^2)}{3(1-x_0^2)} \Rightarrow x_{0,c} = 1/\sqrt{5}$, in absence of any overheads. It is also evident from Fig. 1 that when overheads are present, we have $x_{0,c} > 1/\sqrt{5}$ (see the solution of $U^r(x_0) = x_0^2/5$ in Fig. 1). The case of sharp restarts can be similarly addressed. We see from Fig. 2 that $\langle T_R\rangle$ exhibits a non-monotonic behavior for the two restart protocols, with the dashed line representing $\langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle = \frac{x_0^2}{5} + \frac{1-x_0^2}{2}$. More importantly, $\langle T_R\rangle < \langle T_{0,x_0}\rangle + \langle T_{x_0,1}\rangle$ for appropriately chosen rate of restart. It is also evident from Fig. 2 that $\langle T^\tau_R\rangle < \langle T^r_R\rangle$. Even though choosing any other function for $\langle T_{0,x_0}\rangle$ would result in quantitative modifications, we believe that the relation $\langle T^\tau_R\rangle \le \langle T^r_R\rangle$ would hold true independent of the specific nature of the overhead $\langle T_{0,x_0}\rangle$ [35], though we do not furnish any proof in support of this assertion.

Implementing overheads and finite time restarts: As seen above, modifying the overhead function $\langle T_{0,x_0}\rangle$ leads to a reduction of $\langle T_R\rangle$. But what does it physically mean by the phrase "modifying the overhead function"? This is a very important question, particularly in light of the fact that a proper physical basis for modifying $\langle T_{0,x_0}\rangle$ would provide a better handle to control search under restarts. In order to answer this question, we numerically study the Langevin equation

$$\dot{x}(t) = \eta(t), \tag{7}$$

under sharp restarts. Here $\eta(t)$ is a Gaussian white noise with mean zero and correlation $\langle \eta(t)\eta(t')\rangle = 2D_0\delta(t-t')$, and D_0 is a dimensionless quantity. Physically, D_0 represents a multiplying factor modifying the diffusion coefficient of the Brownian particle. As a result, the mean time for the Brownian particle to go from the origin to x_0 is $x_0^2/2D_0$ and for $D_0 = 5/2$ reduces to $\langle T_{0,x_0}\rangle = x_0^2/5$ (the overhead function studied in Fig. 2). In other words, if the Brownian particle covers the sub-interval $[0, x_0]$ with a modified diffusion coefficient, this could lead to a reduced time overhead. We implement this as follows: we numerically solve Eq. (7) for a particle starting at

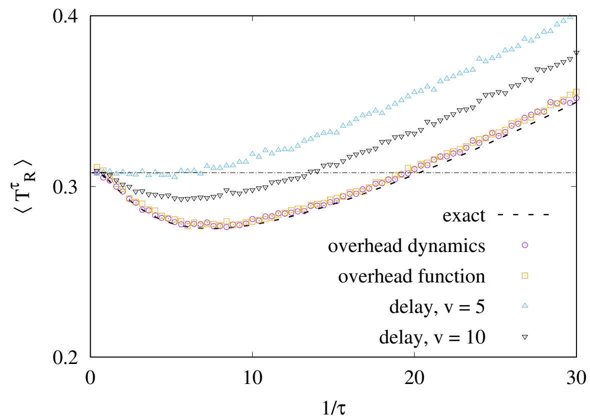

FIG. 3. Mean escape time $\langle T_R^\tau \rangle$ to cover the unit interval under sharp restarts taking into account time overheads and delay associated with restart. Time overheads arising due to a modified diffusion coefficient ($\circ$) and using an overhead function ($\square$) give identical results. The black dashed line follows Eq. 2 with restart location $x_{0,R} = 0.8$. Bringing back the particle to $x_{0,R}$ is at a fixed speed v further delays the mean time to cover the unit interval. The dot-dashed line represents the mean time $\langle T \rangle = \frac{x_{0,R}^2}{5} + \frac{1-x_{0,R}^2}{2}$ in absence of any restarts (see Eq. (1)).

$x = 0$ with $D_0 = 5/2$ and once the particle reaches $x_{0,R}$, we start applying the sharp restart protocol with time τ and the associated overhead $T_{0,x_{0,R}}$. The restart location is now $x_{0,R}$. The instance of first crossing the absorbing wall at $x = 1$ provides us the first passage time T_R^τ with overheads coming from a dynamical process. Alternatively, we can solve the Brownian motion under restarts with the particle at $t = 0$ at $x = x_{0,R}$ and every time the motion restarts, add an overhead time $\langle T_{0,x_{0,R}} \rangle = x_{0,R}^2/5$. The fact that these two methods provide identical results justifies the physical basis of time overheads $T_{0,x_{0,R}}$ arising due to a modified dynamics in the sub-interval $[0, x_{0,R}]$. Numerical estimation of $\langle T_R^\tau \rangle$ corroborates our assertion (see Fig. 3). The example of modifying overheads choosing a quadratic overhead function serves as a proof of concept for alternative measures like $\langle T_{0,x_{0,R}} \rangle \sim x_{0,R}$ (the particle moving ballistically to cover the sub-interval $[0, x_{0,R}]$), etc.

While getting a handle on the overhead function $\langle T_{0,x_{0,R}} \rangle$ is important to control escape under restarts, a practical constraint needs to be taken into account- it takes a finite amount of time to bring the particle from $x \in (0, 1)$ back to its restart location $x_{0,R}$. We implement this by taking the particle back to $x_{0,R}$ at a fixed speed, thus introducing a time delay $T_D = |x - x_{0,R}|/v$ [44], further exacerbating the situation due to the overheads. We solve Eq. (7) in presence of overheads and implement sharp restarts by bringing the particle back to $x_{0,R}$ at a speed v. We see from Fig. 3 that T_D increases $\langle T_R^\tau \rangle$ as compared to the case with instantaneous restarts. Furthermore, the range of restart times τ such that $\langle T_R^\tau \rangle < \langle T \rangle$ is reduced. And if the time to relocate the particle to $x_{0,R}$ is very high (a low value of v), it can lead to a situation in which the advantage brought about by restarts is lost, resulting in $\langle T_R^\tau \rangle \geq \langle T \rangle$ (for example, $v = 5$ in Fig. 3).

Discussion: Extensive research on stochastic restarts over the past decade or so have shown that if the rate of restart is appropriately chosen, then restarts can expedite the completion of a random search process. In this paper we revisit this thought by studying the simple example of a Brownian particle moving in the unit interval and ask one question: can restarts reduce the mean time taken by a Brownian particle to go from one end of the unit interval to another? We find the answer in the negative. In other words, restarts always delay the mean time to cover the whole interval. The reason for this certain delay lies in the fact that a restart location somewhere in the interval naturally introduces time overheads which become more pronounced under restarts. We find that if these overheads come from the same process (here Brownian motion, but true for any stochastic process following Eq. (1)), then restarts certainly delay the escape. The situation is not as bad as it seems, as introduction of restarts impose constraints on the set of allowed overheads and if those constraints are respected, restarts do tend to reduce the mean time. The downside is that the dynamical properties of the Brownian particle, say its diffusion coefficient, needs to be modified. Furthermore, if restarts are non-instantaneous, then the mean time is further increased, and might lead to a situation in which restarts prove detrimental to escape.

While it may seem simple from a theoretical point of view, modifying the dynamical properties like the diffusion coefficient is an added cost to the already costly affair of restarts [45]. Hence, application of restarts in a search problem should be thought over thoroughly in terms of associated costs and advantages gained. On one hand we a have bare random walk searching for a target with a mean search time, and on the other hand we have the full machinery of controlling dynamical properties and restart strategies applied to the random walk to get a better control on the search times. For example, a higher diffusion coefficient can be realized by increasing the temperature, but doing it over and over again every time the motion is restarted may require a much more precision in control than would be actually needed for a search in absence of restarts. Hence, to restart, or not to restart, is the question which should be answered in terms of costs incurred and advantages gained before we decide whether to restart or not.

* rksinghmp@gmail.com

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