EMBEDDING PRODUCTS OF TREES INTO HIGHER RANK

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Abstract. We show that there exists a quasi-isometric embedding of the product of n copies of $\mathbb{H}^2_{\mathbb{R}}$ into any symmetric space of non-compact type of rank n, and there exists a bi-Lipschitz embedding of the product of n copies of the 3-regular tree T_3 into any thick Euclidean building of rank n with co-compact affine Weyl group. This extends a previous result of Fisher–Whyte. The proof is purely geometrical, and the result also applies to the non Bruhat–Tits buildings.

1. Introduction

Symmetric spaces of non-compact type and Euclidean buildings are important classes of non-positively curved metric spaces. They possess large symmetry groups and structures that often distinguish them from other spaces, and also from one another, even if one only considers the coarse geometry. The latter approach is part of Gromov's program to classify spaces and groups from their coarse geometry, and was partly motivated by a remarkable theorem of Mostow [Mos73]. Some of the well-known theorems in this direction are by Pansu [Pan89], Schwartz [Sch95, Sch96], Kleiner–Leeb [KL97], Eskin–Farb [EF97], Eskin [Esk98], and Drutu [Dru00].

Besides distinguishing these spaces, it is also interesting to study their relationships, especially which space is a totally geodesic subspace of another. While this question can be answered satisfactorily from the classification of semi-simple Lie groups and Lie triple systems, its coarse version is much more subtle. In other words, one might ask whether one space can be quasi-isometrically embedded into another one, or whether there is an obstruction to the existence of such an embedding. While there are many examples of isometric and quasi-isometric embeddings between rank one symmetric spaces, which can be constructed by a result of Bonk–Schramm [BS11], and examples of quasi-isometric embeddings from rank one into higher rank by Brady–Farb [BF98], see also [Lee00, Leu03] for Euclidean buildings, examples of embeddings between two spaces of equal and higher rank are very limited.

Recently, Fisher and Whyte [FW18] gave a sufficient condition for the existence of a quasi-isometric embedding between two symmetric spaces of non-compact type of equal rank. This condition is formulated in terms of the existence of a linear map between Cartan subalgebras preserving kernel of roots. A quasi-isometric embedding induced by this map is called an AN-map. They also provided examples of embeddings when their condition is held. In particular, they constructed a quasi-isometric embedding from the product of n copies of the real hyperbolic plane into the symmetric spaces of $\mathrm{SL}_{n+1}(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$. In [Ngu21], the second author gave a splitting decomposition of embeddings. Namely, any embedding

Mathematics Subject Classification : 20F65, 20F69, 30L05, 53C35.

TN is partially supported by Simons Travel Support for Mathematicians grants MPS-TSM-00002547.

between spaces of equal rank is close to a product of embeddings into irreducible targets. He further gave examples of quasi-isometric embeddings in rank 2 which are not AN-maps. In this paper, we generalize this approach by showing the following.

Theorem 1.1. *(see Theorem 1.4 for a more general statement)*

1. *If X is a thick Euclidean building of rank n with co-compact affine Weyl group, there exists a bi-Lipschitz embedding $T_3 \times \cdots \times T_3 \to X$ of the product of n copies of the 3-regular tree into X.*
2. *If X is a symmetric space of non-compact type of rank n, there exists a quasi-isometric embedding $\mathbb{H}^2_{\mathbb{R}} \times \cdots \times \mathbb{H}^2_{\mathbb{R}} \to X$ of the product of n copies of the real hyperbolic plane into X.*

Our approach can be regarded as a geometric AN-map, which is much more flexible than the one introduced by Fisher–Whyte, especially for Euclidean buildings. In particular, the embeddings of products of trees into thick Euclidean buildings also hold for the exotic ones. For example, the ones of type $\tilde{A}_2$ [Ron86, VM87, Bar00], and the ones whose Weyl group does not come from a root system [HKW10, BK12].

Since T_3 embeds quasi-isometrically into $\mathbb{H}^2_{\mathbb{R}}$, it follows that the product of n copies of T_3 embeds quasi-isometrically into any symmetric space of non-compact type of rank n. Moreover, by combining Theorem 1.1 with the quasi-isometric embeddings of Gromov-hyperbolic groups into products of binary trees [BDS07], we get exotic quasi-isometric embeddings of Gromov hyperbolic groups into Euclidean buildings and symmetric spaces.

Corollary 1.2. *Every Gromov hyperbolic group G admits a quasi-isometric embedding into any thick Euclidean building with co-compact affine Weyl group or symmetric space of non-compact type of rank $n+1$, where n is the topological dimension of $\partial_\infty G$.*

For example, $\mathbb{H}^n_{\mathbb{R}}$ embeds quasi-isometrically into any such Euclidean building of rank n. By combining with the quasi-isometric embeddings $\mathbb{H}^n_{\mathbb{R}} \to (\mathbb{H}^2_{\mathbb{R}})^{n-1}$ of [BF98] instead, we get

Corollary 1.3. *For any $n \geq 1$, $\mathbb{H}^{n+1}_{\mathbb{R}}$ embeds quasi-isometrically into any symmetric space of non-compact type of rank n.*

Finally, let us note that the quasi-isometric embedding of the product of copies of $\mathbb{H}^2_{\mathbb{R}}$ into a symmetric space can also obtained by an AN-map. This was pointed out to us by Yves Benoist, and we give a proof in the Appendix.

Main result.
We refer to Section 2.1 for the background material. Let X be a Euclidean building or a symmetric space of non-compact type of rank n, and let F_0 be a fixed apartment/maximal-flat. Given a wall $H \subset F_0$, i.e. a singular $(n-1)$-dimensional flat, we consider its cross section $CS(H)$, which can be seen as the set of $(n-1)$-flats in X which are parallel to H, and we define a projection map $X \to CS(H)$ (see Section 2). This projection map is defined by considering a suitable point η in the boundary of H, and assigning to each $x \in X$ the unique $(n-1)$-flat parallel to H to which the geodesic ray $[x, \eta)$ is strongly asymptotic, see Definition 2.5. We endow the cross sections with the Hausdorff distance. This projection map is a variation of the projection onto the space of strong asymptotic classes introduced by Leeb [Lee00].

If Δ is a subset of $\partial_T F_0$, we denote by X_Δ the union of all apartments/maximal-flats in X that contain Δ in their boundary at infinity.

We show that there exist walls $H_1, \dots, H_n$ in F_0, projection maps $\pi_i : X \to CS(H_i)$, and $\Delta \subset \partial_T F_0$ a suitable union of chambers such that the following map

$$\pi : X_\Delta \to CS(H_1) \times \cdots \times CS(H_n),$$

which is the restriction of the product map $\pi_1 \times \cdots \times \pi_n$ to X_Δ, satisfies the following.

Theorem 1.4.

(1) *If X is a Euclidean building, π is a bi-Lipschitz map. Moreover, the inclusion of X_Δ, equipped with the path-metric, in X is a bi-Lipschitz embedding, therefore it induces a bi-Lipschitz embedding from $CS(H_1) \times \cdots \times CS(H_n)$ into X.*

(2) *If X is a symmetric space of non-compact type, π is a quasi-isometry. Moreover, the inclusion of some δ-neighborhood of X_Δ, equipped with the path-metric, in X is a quasi-isometric embedding, therefore it induces a quasi-isometric embedding from $CS(H_1) \times \cdots \times CS(H_n)$ into X.*

The target space is equipped with the L^1 product metric. We refer to Section 4 for the construction of the walls H_i and Δ. Theorem 1.1 is an immediate consequence of Theorem 1.4:

Proof of Theorem 1.1. (1) If X is a thick Euclidean building of rank n with co-compact affine Weyl group, then all the cross sections are thick Euclidean buildings of rank 1, i.e. thick metric trees [Lee00]. Therefore there exists a bi-Lipchitz embedding $T_3 \to CS(H_i)$.
(2) When X is a symmetric space of non-compact type, the cross sections are rank one symmetric space of non-compact type [Ebe96, Chap. 2.20], therefore $\mathbb{H}^2_{\mathbb{R}} \to CS(H_i)$ isometrically. □

Remark 1.5. The constants of the bi-Lipchitz/quasi-isometric embeddings do not depend on the apartment/maximal-flat F_0 we started with. Moreover, F_0 is contained in the image of such embeddings.

Let us note that when X is a Euclidean building (resp. a symmetric space), the fact that the inclusion of X_Δ (resp. its δ-neighbordhood), in X is a bi-Lipschitz (resp. quasi-isometric) embedding, is a general result, as shown in step 4 of the proof of Theorem 1.4, and is true when Δ is any union of chambers in the boundary of some fixed apartment/maximal-flat F_0. For example, if Δ consists of only one chamber, then $X_\Delta = X$. If $\Delta = \partial_T F_0$, then $X_\Delta = F_0$. Obviously, the more chambers we add to Δ, the smaller X_Δ becomes. The subset Δ in Theorem 1.4 is the "smallest possible" for which the map π is injective (resp. quasi-injective).

About the proof. The proof of Theorem 1.4 will be done in four steps. If X is a Euclidean building (resp. a symmetric space), we will start by showing that π is a bi-Lipschitz embedding when restricted to a flat containing Δ at infinity. Then, in step 2, which represents the core of the proof, we show the general case, i.e. that it is a bi-Lipschitz (resp. quasi-isometric) embedding. In step 3, we show that it is surjective (resp. quasi-surjective), and finally that the inclusion of X_Δ (resp. a thickening of X_Δ), equipped with the path metric, in X is a bi-Lipschitz (resp. quasi-isometric) embedding.

Organisation of the paper. We start in Section 2.1 by recalling briefly the parallel sets and cross sections in symmetric spaces and Euclidean buildings. After giving some preliminary lemmas in Section 2.2, we define the projection map onto a cross section, to which Section 2.3 is devoted. We recall the generalized Iwasawa decomposition for semi-simple Lie groups in Section 2.4. In Section 3, we define the maximally distributed vertices in a spherical Coxeter complex. Finally, the main result is proved in Section 4.

Acknowledgements. We thank Yves Benoist for pointing out to us the AN-map for symmetric spaces. We are grateful to the Max-Planck Institute for Mathematics in Bonn for its financial support. The second author thanks Institut des Hautes Études Scientifiques, Institut Henri Poincaré and LabEx CARMIN for their support and hospitality.

2. Background and preliminary results

2.1. **Background.** We recall that if X,Y are two metric spaces, and $f : X \to Y$ a map,

(i) f is a *bi-Lipschitz embedding* if there exists $\lambda \geq 1$ such that for any $x, x' \in X$,
$$\frac{1}{\lambda} d_X(x, x') \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x').$$
If moreover f is surjective, it is called a *bi-Lipschitz equivalence.*

(ii) f is a *quasi-isometric embedding* if there exist $\lambda \geq 1$ and $C \geq 0$ such that for any $x, x' \in X$,
$$\frac{1}{\lambda} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + C.$$
If moreover there exists $M \geq 0$ such that for any $x \in X$ there exists $y \in Y$ such that $d_Y(f(x), y) \leq M$, f is called a *quasi-isometry.*

We refer to [BH13] for the background material on CAT(0) spaces, and to [Ebe96],[KL97], and [Lee00] for symmetric spaces and Euclidean buildings.

Let X is a complete CAT(0) space. We denote by ∂X its visual boundary, and we equip it with the *angular metric* $\angle$ defined, for any $\xi, \eta \in \partial X$, as
$$\angle(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta).$$
Let X be a Euclidean building or a symmetric space of non-compact type. We recall that its visual boundary ∂X, when equipped with the angular metric $\angle$, inherits a spherical building structure, and we denote it by $\partial_T X$. Let us note that we consider the angular metric $\angle$ and not the Tits metric on the boundary, i.e. the associated length metric[1].

The apartments of $\partial_T X$ are endowed with a structure of a spherical Coxeter complex [KL97, sect. 3.1]. A *spherical Coxeter complex* is a unit sphere S with a finite Weyl group $W <$ Isom(S) generated by reflections at *walls*, i.e. totally geodesic subspheres of codimension 1. A *singular sphere* $s \subset S$ is an intersection of walls.

The apartments in $\partial_T X$ correspond to boundaries of apartments/maximal-flats in X. Each apartment/maximal-flat $F \subset X$ is endowed with a structure of a Euclidean Coxeter complex [KL97, sect. 4.1]. A *Euclidean Coxeter complex* is a Euclidean space E with an affine Weyl

[1]When rank$(X) \geq 2$, these two metrics on ∂X coincide

group $W_{aff} < \mathrm{Isom}(E)$ generated by reflections at *walls*, i.e. affine subspaces of codimension 1, so that the image of W_{aff} in $\mathrm{Isom}(\partial_T E)$ is a finite reflection group. We call *flat* any totally geodesic Euclidean subspace of X. A *singular flat* in X is an intersection of walls. Finally, a *singular half-space* in X is a half apartment/maximal-flat bounded by a wall.

Remark 2.1. In the rest of the paper, by abuse of language, if X is a Euclidean building, we will also say maximal-flat to refer to its apartments.

Let s be a singular sphere in $\partial_T X$. The *parallel set* $P(s)$ of s is the union of all flats with boundary s. $P(s)$ is a convex subset of X and is isometric to the product

$$P(s) = \mathbb{R}^{\dim(s)+1} \times CS(s).$$

$CS(s)$ is called the *cross section* of s, and it can be seen as the set of flats with boundary s. When X is a Euclidean building (resp. a symmetric space of non-compact type) of rank n, $CS(s)$ is a Euclidean building (resp. a symmetric space of non-compact type) of rank $n - \dim(s) - 1$, see [Ebe96, Chap. 2.20],[Lee00, Sect. 3],[KL97, Sect. 4.8]. If F is a flat such that $\partial_T F = s$, we define $P(F) := P(s)$.

2.2. **Preliminary results.** Unless stated otherwise, X is either a symmetric space of non-compact type or a Euclidean building. Symmetric spaces are supposed of non-compact type.

If $x \in X$ and $\eta \in \partial_T X$, we denote by $[x, \eta)$ the geodesic ray from x to η. If $A \subset \partial_T X$, we denote by $[x, A)$ the cone $\bigcup \big\{[x, a), a \in A\big\}$.
If $A \subset X$ and $r \geq 0$, we denote by $N_r(A) = \{x \in X \text{ such that } d_X(x, A) \leq r\}$.

Lemma 2.2. *Let X be a symmetric space or a Euclidean building, α a geodesic in X with endpoints $\{\eta, \hat{\eta}\}$, and F a flat containing α. If η is an interior point of a top-dimensional cell of $\partial_T F$, then $P(\alpha) = P(F)$. In other words, $P(\{\eta, \hat{\eta}\}) = P(\partial_T F)$.*

Proof. Since $\alpha \subset F$, then $P(F) \subset P(\alpha)$.
Let $x \in P(\alpha)$, and let us denote by c the top-dimensional cell of $\partial_T F$ containing η in its interior. Let $\hat{c}$ be its opposite such that $\hat{\eta} \in \hat{c}$. Consider a maximal-flat E in X containing $\eta, \hat{\eta}$, so $c, \hat{c} \subset \partial_T E$ because $\eta, \hat{\eta}$ are interior points. Since s is the unique singular sphere spanned by η and $\hat{\eta}$, $s \in \partial_T E$ and x is contained in a flat $F' \subset E$ with boundary s. □

We say that a geodesic ray γ is *strongly asymptotic* to a subset $A \subset X$ if $d_X(\alpha(t), A) \xrightarrow[+\infty]{} 0$.

Lemma 2.3. *Let X be a symmetric space or a Euclidean building, s a singular sphere in $\partial_T X$, and η an interior point of a top-dimensional cell of s. For any $x \in X$, $[x, \eta)$ is strongly asymptotic to $P(s)$. Moreover, if X is a Euclidean building, there exists $T \geq 0$ such that for $t \geq T$, $[x, \eta)(t) \in P(s)$.*

Proof. Let $\hat{\eta}$ be the opposite of η in s.
• If X is a symmetric space, let α be a geodesic with endpoints $\eta, \hat{\eta}$. Consider the generalized Iwasawa decomposition, see Theorem 2.15, $\mathrm{Isom}_0(X) = K A_\eta N_\eta$ with respect to η and a point in α. Let γ be the geodesic containing x and η at $+\infty$, then there exists $a \in A_\eta$ and $n \in N_\eta$ such that $\gamma = an\alpha$. Note that $a\alpha$ is parallel to α so $a\alpha \subset P(\{\eta, \hat{\eta}\}) = P(s)$. Moreover, for any $t \in \mathbb{R}$

$$d_X(\gamma(t), a\alpha(t)) = d_X(an\alpha(t), a\alpha(t)) = d_X(n\alpha(t), \alpha(t)) \xrightarrow[+\infty]{} 0.$$

• If X is a Euclidean building: we denote the geodesic ray $[x,\eta)$ by γ. By the angle rigidity axiom [KL97, Sect. 4.1.2], $\angle_{\gamma(t)}(\eta,\hat{\eta})$ takes only finitely many values, and since $\angle_{\gamma(t)}(\eta,\hat{\eta}) \underset{+\infty}{\longrightarrow} \angle(\eta,\hat{\eta})=\pi$, there exists $T\geq 0$ such that if $t\geq T$ then $\angle_{\gamma(t)}(\eta,\hat{\eta})=\pi$, i.e. $\gamma(t)$ is in a geodesic joining η and $\hat{\eta}$. Therefore, for $t\geq T$, $\gamma(t)\in P(\{\eta,\hat{\eta}\})=P(s)$ by Lemma 2.2. □

An immediate consequence of Lemma 2.3 is the following.

Corollary 2.4. *If X be a symmetric space or a Euclidean building, s is a singular sphere in $\partial_T X$, and η an interior point of a top-dimensional cell of s, then for any $x\in X$ there exists a unique flat H with boundary s to which $[x,\eta)$ is strongly asymptotic.*

2.3. **The projection map.** Corollary 2.4 allows us to define a projection onto the cross section of a singular sphere via an interior point.

Definition 2.5. Let s be a singular sphere in $\partial_T X$, and η an interior point of a top-dimensional cell of s. We define the projection via η, $\pi: X\to CS(s)$, by assigning to x the unique flat with boundary s to which $[x,\eta)$ is strongly asymptotic.

Remark 2.6. Note that π depends of the top-dimensional cell of s, but does not depend on the choice of its interior point.

We endow $CS(s)$, viewed as the set of all flats with boundary s (hence all parallel), with the Hausdorff distance.

Lemma 2.7. *The map π is 1-Lipschitz.*

Proof. Let $x,x'\in X$, and $\gamma=[x,\eta)$ and $\gamma'=[x',\eta)$. For any $t\geq 0$,

$$d(\pi(x),\pi(x'))\leq d_X(\pi(x),\gamma(t))+d_X(\gamma(t),\gamma'(t))+d_X(\gamma'(t),\pi(x')).$$

By convexity of the distance function, $d_X(\gamma(t),\gamma'(t))\leq d_X(x,x')$, so when $t\to+\infty$ we get the result. □

Remark 2.8. As mentioned in the introduction, this projection is a variation of the projection onto the space of strong asymptotic classes introduced by Leeb [Lee00].

We end this section with some useful lemmas related to this projection map. Let us first recall that if X is a CAT(0) space, $x_0\in X$, and $\eta\in\partial X$, the Busemann function with respect to x_0 and η is the map $b: X\to\mathbb{R}$ such that for any $x\in X$,

$$b(x)=\lim_{t\to+\infty} d_X([x_0,\eta)(t),x)-t.$$

Lemma 2.9. *Let X be a complete CAT(0) space, $x\in X$, F a flat in X, and $\eta_1,\eta_2\in\partial_T F$ such that $\angle(\eta_1,\eta_2)<\pi$. If $[x,\eta_1)$ is strongly asymptotic to F and $\angle_x(\eta_1,\eta_2)=\angle(\eta_1,\eta_2)$, then for any interior point ξ of $\overline{\eta_1\eta_2}$, $[x,\xi)$ is strongly asymptotic to F.*

Proof. $\angle_x(\eta_1,\eta_2)=d_T(\eta_1,\eta_2)<\pi$ implies that $[x,\overline{\eta_1\eta_2})$ is a flat sector, see [BH13, Chap. 2 Cor. 9.9]. Let $\varepsilon>0$, there exists $x_1\in[x,\eta_1)$ such that $d_X(x_1,F)\leq\varepsilon$. By convexity of the distance function to F, $[x_1,\overline{\eta_1\eta_2})\subset N_\varepsilon(F)$. Since $[x,\overline{\eta_1\eta_2})$ is a flat sector, for any interior point ξ of $\overline{\eta_1\eta_2}$, $[x,\xi)$ enters in $[x_1,\overline{\eta_1\eta_2})$, therefore $d_X([x,\xi),F)\leq\varepsilon$. □

Corollary 2.10. *Let X be a complete CAT(0) space, F and F' two flats in X, $x \in F$, and $\eta \in \partial_T F \cap \partial_T F'$. If $[x, \eta)$ is strongly asymptotic to F', then for any interior point ξ of $\partial_T F \cap \partial_T F'$, $[x, \xi)$ is strongly asymptotic to F'.*

Proof. If F or F' is one-dimensional then the interior of $\partial_T F \cap \partial_T F'$ is empty. If not, let ξ be an interior point. If $\angle(\eta, \xi) = \pi$, there exists a geodesic $\gamma \subset F$ containing x and joining ξ and η. $[x, \eta)$ is strongly asymptotic to F' implies, by the flat stip theorem, that $\gamma \subset F'$. If $\angle(\eta, \xi) < \pi$ then, by convexity of $\partial_T F \cap \partial_T F'$, ξ lies in a segment $\overline{\eta\eta'}$ for some $\eta' \in \partial_T F \cap \partial_T F'$ such that $\angle(\eta_1, \eta_2) < \pi$. The conclusion follows from Lemma 2.9. $\square$

Lemma 2.11. *Let X be a complete CAT(0) space, F a flat in X, $\eta \in \partial_T F$, and $x \in X$ such that $[x, \eta)$ is strongly asymptotic to F. Then $[x, \eta)$ is strongly asymptotic to some geodesic in F.*

Proof. Let b be a Busemann function with respect to η, and H the intersection of F with a level set of b, which is a hyperplane in F. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of points in F such that $d_X(f_n, [x, \eta)) \leq \frac{1}{n}$, and h_n be the projection of f_n on H (i.e. the intersection of H with the geodesic containing f_n and η). By convexity of the distance function,

$$d_X([x, \eta), [h_n, \eta)) = d_X([x, \eta), [f_n, \eta)) \leq d_X([x, \eta), f_n) \leq \frac{1}{n}.$$

Let $n, m \in \mathbb{N}$, and $z \in [x, \eta)$ such that $d_X(z, [f_n, \eta)) \leq \frac{2}{n}$ and $d_X(z, [f_m, \eta)) \leq \frac{2}{m}$. So for any $a \in [f_n, \eta)$ and $b \in [f_m, \eta)$,

$$d_X([f_n, \eta)), [f_m, \eta))) \leq d_X(a, z) + d_X(z, b)$$

Therefore,

$$d_X([f_n, \eta)), [f_m, \eta))) \leq d_X(z, [f_n, \eta)) + d_X(z, [f_m, \eta)) \leq \frac{2}{n} + \frac{2}{m}$$

Since $b(h_n) = b(h_m)$,

$$\begin{aligned} d_X(h_n, h_m) &= d_X([h_n, \eta), [h_m, \eta)) \\ &= d_X([f_n, \eta), [f_m, \eta)) \\ &\leq \frac{2}{n} + \frac{2}{m}. \end{aligned}$$

So $(h_n)_n$ is a Cauchy sequence in H, and there exists $h \in H$ such that $h_n \xrightarrow[+\infty]{} h$, and a same argument as before shows that $d_X([h, \eta), [x, \eta)) = 0$. So $[x, \eta)$ is strongly asymptotic to the extension, in F, of the geodesic ray $[h, \eta)$. $\square$

For the rest of this section, X is either a Euclidean building or a symmetric space.

Proposition 2.12. *Let F, F' be singular flats such that $\dim \partial_T F = \dim \partial_T F' = \dim(\partial_T F \cap \partial_T F')$, and let η be an interior point of a top-dimensional cell c of $\partial_T F \cap \partial_T F'$.*

1. *If there exists $x \in F$ such that $[x, \eta)$ is strongly asymptotic to F', then for any $y \in F$, $[y, \eta)$ is strongly asymptotic to F'.*
2. *Let s be a singular sphere in $\partial_T X$ containing c and $\dim s = \dim \partial_T F = \dim \partial_T F'$, and let $\pi : X \to CS(s)$ be the projection via η. Then for any $x, y \in F$, $\pi(x) = \pi(y)$. Moreover, if there exist $x \in F$ and $x' \in F'$ such that $\pi(x) = \pi(x')$, then $[x, \eta)$ is strongly asymptotic to F' (and $[x', \eta)$ is strongly asymptotic to F).*

Proof. (1) Let $\varepsilon > 0$. There exists $a \in [x, \eta)$ such that $d_X(a, F') \leq \varepsilon$. Let $z' \in F'$ such that $d_X(a, z') \leq \varepsilon$. Let us denote by c^{o} the interior of c. For any $\xi \in c^{\mathrm{o}}$, and any $t \geq 0$, by convexity of the distance function

$$d_X([a, \eta)(t), [z', \eta)(t)) \leq d_X(a, z') \leq \varepsilon.$$

So $[a, c^{\mathrm{o}}) \subset N_\varepsilon(F')$. Since c is a top-dimensional cell in $\partial_T F$ and $\eta \in c^{\mathrm{o}}$, for any $y \in F$, $[y, \eta)$ enters eventually in $[a, c^{\mathrm{o}})$. So

$$\lim_{t \to +\infty} d_X([y, \eta)(t), F') \leq \varepsilon.$$

This holds for any $\varepsilon > 0$, so $[y, \eta)$ is strongly asymptotic to F'.

(2) Let H be the singular flat with boundary s to which $[x, \eta)$ and $[x', \eta)$ are strongly asymptotic. By Lemma 2.11, there exist $z, z' \in H$ such that $[x, \eta)$ is strongly asymptotic to $[z, \eta)$ and $[x', \eta)$ is strongly asymptotic to $[z', \eta)$. In particular, $[z', \eta)$ is strongly asymptotic to $[x', \eta)$ and thus to F'. By (1), $[z, \eta)$ is also strongly asymptotic to F' since z and z' are both in H. $[x, \eta)$ is strongly asymptotic to $[z, \eta)$, so $[x, \eta)$ is strongly asymptotic to F'. The same argument show that $[x', \eta)$ is strongly asymptotic to F. □

Remark 2.13. The second point of Proposition 2.12 implies that under these conditions, if there exist $x \in F$ and $x' \in F'$ such that $\pi(x) = \pi(x')$, then $\pi(F) = \pi(F')$.

Corollary 2.14. *Let F, F' be singular flats such that $\dim \partial_T F = \dim \partial_T F' = \dim(\partial_T F \cap \partial_T F')$, and let η be an interior point of a top-dimensional cell c of $\partial_T F \cap \partial_T F'$.*

(1) *For any $x, y \in F$,*
$$d_X([x, \eta), F') = d_X([y, \eta), F').$$
(2) *Let s be a singular sphere in $\partial_T X$ containing c and $\dim s = \dim \partial_T F = \dim \partial_T F'$, and let $\pi : X \to CS(s)$ be the projection via η. For any $x \in F$ and $x' \in F'$,*
$$d(\pi(F), \pi(F')) = d(\pi(x), \pi(x')) = d_X([x, \eta), F') = d_X([x', \eta), F).$$

Proof. (1) If we denote by $s' = \partial_T F'$, by Corollary 2.4 there exists a flat F'' with boundary s', i.e. parallel to F', to which $[x, \eta)$ is strongly asymptotic. By Proposition 2.12, $[y, \eta)$ is also strongly asymptotic to F''. Therefore,

$$d_X([x, \eta), F') = d_X(F', F'') = d_X([y, \eta), F').$$

(2) Let $x \in F$ and $x' \in F'$. By Proposition 2.12, π is constant on F and on F', so $d(\pi(F), \pi(F')) = d(\pi(x), \pi(x'))$, that we will denote by D. This implies that $d_X(F, F') \geq D$ (π is 1-Lipschitz) and in particular $d_X([x, \eta), F') \geq D$. Let H (resp. H') be the flat with boundary s to which $[x, \eta)$ (resp. $[x', \eta)$) is strongly asymptotic. By Lemma 2.11, there exists a geodesic γ in H to which $[x, \eta)$ is strongly asymptotic. γ can be parameterized such that $d_X([x, \eta)(t), \gamma(t)) \underset{+\infty}{\longrightarrow} 0$. H and H' are parallel, so there exists a geodesic γ' in H', parallel to γ such that for any t, $d_X(\gamma(t), \gamma'(t)) = d_X(H, H') = D$. Moreover, $\pi(F') = H'$ and they have the same dimension, so there exists $x'' \in F'$ such that $[x'', \eta)$ is strongly asymptotic to γ' (by Lemma 2.11). So for any $t \geq 0$,

$$d_X([x, \eta)(t), [x', \eta)(t)) \leq d_X([x, \eta)(t), \gamma(t)) + d_X(\gamma(t), \gamma'(t)) + d_X(\gamma(t), [x', \eta)(t)).$$

Therefore $d_X([x, \eta), [x', \eta)) \leq D$, and in particular $d_X([x, \eta), F') \leq D$. □

2.4. **Generalized Iwasawa decomposition.** Let us recall briefly the generalized Iwasawa decomposition for symmetric spaces, and we refer to [Ebe96] for more details. Let X be a symmetric space of non-compact type, $G = \mathrm{Isom}_0(X)$, $\mathfrak{g}$ its Lie algebra, $x_0 \in X$ a basepoint, $K = \mathrm{Stab}_G(x_0)$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition with respect to x_0. Note that for any geodesic γ through x_0, there exists $Y \in \mathfrak{p}$ such that for any t, $\gamma(t) = \exp(tY).x_0$. Let us fix γ and Y, and consider $\mathfrak{a}$ a Cartan subspace of $\mathfrak{p}$ that contains Y, Φ the restricted root system of $\mathfrak{g}$ relative to $\mathfrak{a}$, and $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the restricted root space decomposition.

Let η be the point at $+\infty$ of γ, and let us denote $\mathfrak{a}_Y = \mathfrak{z}(Y)\cap\mathfrak{p}$, where $\mathfrak{z}(Y)$ is the centralizer of Y in $\mathfrak{g}$, $\mathfrak{n}_Y = \bigoplus_{\alpha\in\Phi,\alpha(Y)>0} \mathfrak{g}_\alpha$, $A_\eta = \exp(\mathfrak{a}_Y)$, and $N_\eta = \exp(\mathfrak{n}_Y)$.

Theorem 2.15 (Generalized Iwasawa decomposition)**.**

$$G = KA_\eta N_\eta.$$

Moreover, we have

(*i*) $A_\eta = A_\eta^{-1}$, *and it normalizes* N_η.
(*ii*) $A_\eta N_\eta$ *acts simply transitively on* X.
(*iii*) *For any* $a \in A_\eta$, $a\gamma$ *and* γ *are parallel, i.e.* $d_X(a.\gamma(t), \gamma(t))$ *is constant. For any* $n \in N_\eta$, $d_X(n.\gamma(t), \gamma(t)) \xrightarrow[+\infty]{} 0$.

We refer to [Ebe96, Chap. 2.19]. Note that A_η is not necessarily a subgroup: $\mathfrak{a}_Y$ is not necessarily a Lie subalgebra, unless η, i.e. Y, is regular. In this case $\mathfrak{a}_Y = \mathfrak{a}$, and we recover the usual Iwasawa decomposition.

3. Maximally distributed vertices in a spherical Coxeter complex

Let S be a spherical Coxeter complex, and $A \subset S$ a subset. If A has diameter $< \pi$, we denote by $\mathrm{Hull}(A)$ its convex hull in S. A hemisphere σ in S is called *singular* if its boundary sphere $\partial\sigma$ is a wall.
Let s be a singular sphere, and $\{\xi_i\}_i$ vertices in S. We say that the vertices $\{\xi_i\}_i$ span s if s is the smallest sphere (with respect to inclusion) that contains them.

Proposition 3.1. *Let S be a spherical Coxeter complex of dimension* $(n-1)$.

(1) *There exist vertices* $\xi_1, \dots, \xi_n$ *in* S*:*
  (*i*) *which are not pairwise opposite, nor all contained in a wall;*
  (*ii*) *for any* $i = 1 \dots n$, $\{\xi_j\}_{j\neq i}$ *span a wall;*
  (*iii*) *if* σ *is a singular hemisphere in* S *containing them, then* $(n-1)$ *of them must lie in its boundary wall* $\partial\sigma$.

(2) *For* $i = 1, \dots, n$, *let* s_i *be the wall spanned by* $\{\xi_j\}_{j\neq i}$ *in* S. *Then* $\displaystyle\bigcap_{i=1}^{n} s_i = \emptyset$.

Proof. (1) The set of vertices satisfying (*i*) and (*ii*) is nonempty since it contains the vertices of a chamber. Take vertices $\xi_1, \dots, \xi_n$ satisfying (*i*) and (*ii*) such that $\mathrm{Hull}(\xi_1, \dots, \xi_n)$ contains the maximum number of chambers, and let us show that they satisfy (*iii*). Let σ be a singular hemisphere containing all these vertices, and suppose that the interior of σ contains more than one vertex. To simplify notations, suppose that $\xi_1, \dots, \xi_p \in \sigma\backslash\partial\sigma$, with $p \geq 2$. We use ξ_1 to "push" $\xi_2, \dots, \xi_p$ to $\partial\sigma$. For $i = 2 \dots p$, extend the geodesic segment

$\overline{\xi_1\xi_i}$ to $\overline{\xi_1\xi_i'}$, where ξ_i' is its first intersection with $\partial\sigma$. It is clear that the convex hull of $\xi_1, \xi_2', \dots, \xi_p', \xi_{p+1}, \dots, \xi_n$ is bigger than the initial one, and we claim that they still satisfy (i) and (ii), which leads to a contradiction.

Indeed, ξ_1 is clearly not opposite to any ξ_i'. If some ξ_i' is opposite to some ξ_j' (similarly if some ξ_i' is opposite to some ξ_j), then ξ_1, ξ_i, ξ_j lie in a same singular 1-sphere, therefore any $(n-1)$ vertices of $\xi_1, \dots, \xi_n$ that contain them span a sphere of dimension $< n-2$, which contradicts the fact that $\xi_1, \dots, \xi_n$ satisfy (ii). If $\xi_1, \xi_2', \dots, \xi_p', \xi_{p+1}, \dots, \xi_n$ are contained in a wall s, then their convex hull is in s, in particular $\xi_1, \dots, \xi_n \in s$, which contradicts (i). Therefore they still satisfy (i).

Finally, note that $\xi_2', \dots, \xi_p', \xi_{p+1}, \dots, \xi_n$ are all in $\partial\sigma$, which is a wall, and they do not span a smaller singular sphere, otherwise the convex hull would not be $(n-1)$-dimensional. Also, for any $k \in \{2, \dots, p\}$, $\xi_1, \xi_2', \dots, \xi_{k-1}', \xi_{k+1}', \dots, \xi_p', \xi_{p+1}, \dots, \xi_n$ are in the same wall that $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ span, and they cannot span a smaller sphere by the same argument. Similarly if one removes one of $\xi_{p+1}, \dots, \xi_n$. Therefore $\xi_1, \xi_2', \dots, \xi_p', \xi_{p+1}, \dots, \xi_n$ also satisfy (ii).

(2) Let us denote the opposites of $\xi_1, \dots, \xi_n$ in S by $\hat{\xi}_1, \dots, \hat{\xi}_n$. Note that, by the same argument as before, for any $\xi_{k_1}, \dots, \xi_{k_p}$, the singular sphere that they span is $(p-1)$-dimensional. In particular, for any $i \in \{1, \dots, n\}$, $\xi_i, \hat{\xi}_i \notin s_i$. Moreover, $s_i \cap s_j$ is spanned by $\{\xi_k\}_{k \neq i,j}$. So $\cap_{i \neq j} s_i = \{\xi_j, \hat{\xi}_j\}$, which are not in s_j. □

Remark 3.2. If the spherical Coxeter complex S is an apartment in $\partial_T X$, then the choice of the vertices $\xi_1, \dots, \xi_n$ no longer depends on S: if S' is another apartment containing them, then they also satisfy (1) and (2) of Proposition 3.1 in S'.

Definition 3.3. Let S be a spherical Coxeter complex of dimension $(n-1)$, and $\xi_1, \dots, \xi_n \in S$ vertices. We say that $\xi_1, \dots, \xi_n$ are *maximally distributed* if they satisfy the condition (1) in Proposition 3.1. See Figure 1 for some examples.

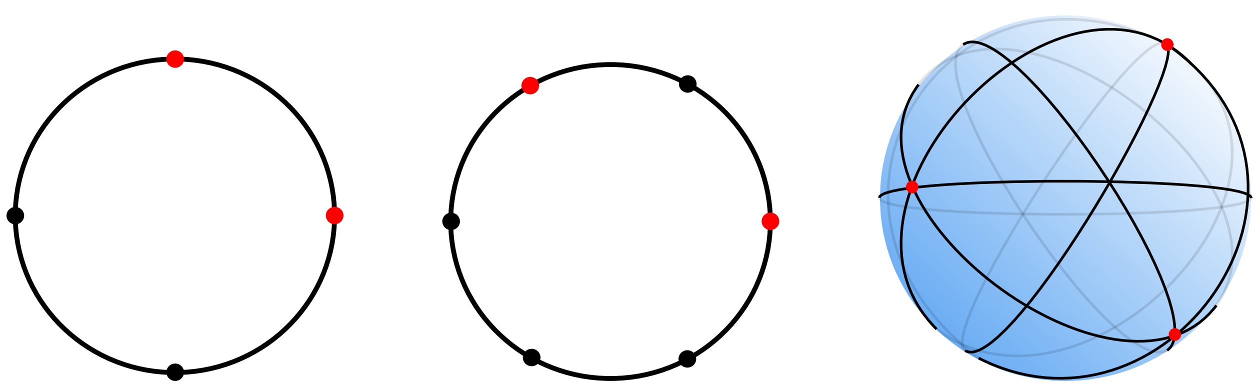

FIGURE 1. Maximally distributed vertices in $A_1 \times A_1$, A_2, and A_3.

Proposition 3.4. *Let X be a symmetric space or a Euclidean building, F a maximal-flat in X, and $\xi_1, \dots, \xi_n$ maximally distributed vertices in $\partial_T F$. For $i = 1, \dots, n$, let s_i be the wall in $\partial_T F$ spanned by $\{\xi_j\}_{j \neq i}$, and let η_i be an interior point of a top-dimensional cell of* $\mathrm{Hull}(\{\xi_j\}_{j\neq i}) \subset s_i$ *(see Figure 2 for an example in A_3). Let π_i be the projection map onto $CS(s_i)$ via η_i. Then for any maximal-flat F' containing all the ξ_i's at infinity, the fibers of π_i in F' are the $(n-1)$-flats with $\{\xi_j\}_{j \neq i}$ at infinity. In particular, π_i is constant on the $(n-1)$-flat containing $\{\xi_j\}_{j\neq i}$ at infinity.*

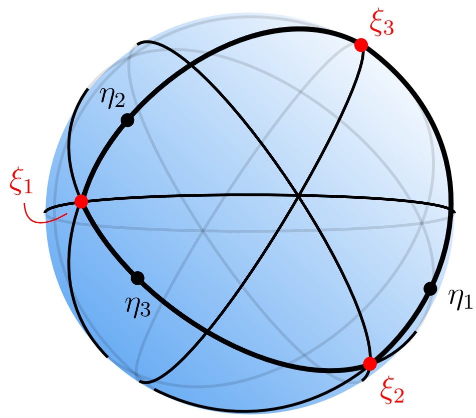

FIGURE 2

Proof. Suppose $i = 1$, let $z \in F'$, and let H' be the $(n-1)$-flat in F' containing z and containing $\xi_2, \dots \xi_n$ at infinity. Let us denote $H = \pi_1(z)$. η_1 is an interior point of a top-dimensional cell of $\partial_T H \cap \partial_T H'$, and $[z, \eta_1)$ is strongly asymptotic to H, so for any $y \in H'$ $[y, \eta_1)$ is also strongly asymptotic to H by Proposition 2.12, i.e. $\pi_1(H') = \pi_1(z)$.
If $z' \in F'$ is not contained in H', then $\pi_1(z') \neq \pi_1(z)$ by Corollary 2.14. □

4. Proof of the main result

Let us recall the setting. Let X be a Euclidean building or a symmetric space of non-compact type of rank n. Let F_0 be a maximal-flat, and $\xi_1, \dots, \xi_n \in \partial_T F_0$ maximally distributed vertices, see Definition 3.3. For all $i = 1, \dots, n$, let s_i be the wall in $\partial_T F_0$ spanned by $\{\xi_j\}_{j\neq i}$, and η_i an interior point of a top-dimensional cell of $\mathrm{Hull}(\{\xi_j\}_{j\neq i}) \subset s_i$.

We denote $\Delta = \mathrm{Hull}(\xi_1, \dots, \xi_n)$, and let X_Δ be the union of maximal-flats in X that contain Δ at infinity, and let π_i be the projection onto $CS(s_i)$ via η_i defined in Definition 2.5. Finally, let

$$\pi : X_\Delta \to CS(s_1) \times \cdots \times CS(s_n)$$

be the restriction of the product map $\pi_1 \times \cdots \times \pi_n$ to X_Δ. Each $CS(s_i)$ is equipped with the Hausdorff distance d_i, and we equip the product space with the L^1 product metric. We will show the following, which is a restatement of Theorem 1.4.

Theorem 4.1.

(1) *If X is a Euclidean building, π is a bi-Lipschitz map. Moreover, the inclusion of X_Δ, equipped with the path-metric, in X is a bi-Lipschitz embedding, therefore it induces a bi-Lipschitz embedding from $CS(s_1) \times \cdots \times CS(s_n)$ into X.*

(2) *If X is a symmetric space of non-compact type, π is a quasi-isometry. Moreover, the inclusion of some δ-neighborhood of X_Δ, equipped with the path-metric, in X is a quasi-isometric embedding, therefore it induces a quasi-isometric embedding from $CS(s_1) \times \cdots \times CS(s_n)$ into X.*

We refer to Section 1 for an overview of the proof, and its steps.

Throughout the proof, n is the rank of X. Since Theorem 1.4 is trivial in rank 1, we suppose $n \geq 2$. In particular, the angular $\angle$ and Tits metric d_T coincide in $\partial_T X$.

Step 1: π is a bi-Lipschitz embedding when restricted to a flat containing Δ at infinity (with uniform constants):

Let $F \subset X_\Delta$ be a maximal-flat in X containing Δ at infinity.

• *Substep 1: π is injective.*

Let $x, y \in F$ such that for all $i = 1, \ldots, n$, $\pi_i(x) = \pi_i(y)$. By Proposition 3.4, $\pi_i(x) = \pi_i(y)$ implies that x and y are in a same $(n-1)$-flat $H_i \subset F$ such that $\{\xi_j\}_{j \neq i} \subset \partial_T H_i$. The flats H_i intersect in a single point because $\cap_{i=1}^n \partial_T H_i = \emptyset$ by Proposition 3.1. Therefore $x = y$.

• *Substep 2: π is a bi-Lipschitz embedding.*

The goal is to show that there exists $\alpha > 0$ such that for any $x, y \in F$ $d_X(x, y) \leq \alpha \sum_{i=1}^n d_i(\pi_i(x), \pi_i(y))$.

Let us denote the opposites of $\xi_1, \ldots, \xi_n$ in $\partial_T F$ by $\hat{\xi_1}, \ldots, \hat{\xi_n}$. Let us start by noting that for all $i = 1, \ldots, n$, $\pi_i(F)$ is a constant speed path in $CS(s_i)$. Also, by Proposition 3.4, when moving in F along a geodesic going to ξ_i, only π_i changes, the rest of the projections are constant. Therefore, to go from x to y, we start from x by following a geodesic with endpoints $\{\xi_1, \hat{\xi_1}\}$ until we equalize π_1, then we do the same for the other directions. After equalizing all π_i's, by injectivity in the previous substep, we would have reached y.

Let $\theta_i = d_T(\xi_i, s_i)$, and let x, y be in a same geodesic with endpoints $\xi_i, \hat{\xi_i}$. If $\theta_i = \pi/2$, $d_X(x, y) = d_i(\pi_i(x), \pi_i(y))$.
If $\theta_i < \pi/2$, $d_X(x, y) = \alpha_i \, d_i(\pi_i(x), \pi_i(y))$, where $\alpha_i = \frac{1}{\tan(\theta_i)}$. Let us denote $\alpha = \max_i \alpha_i$. By concatenating such paths, we have $\forall x, y \in F$, $d_X(x, y) \leq \alpha \sum_{i=1}^n d_i(\pi_i(x), \pi_i(y))$. Note that α is independent of the maximal-flat F, and the previous inequality holds whenever x and y are in a same maximal-flat containing Δ at infinity.

Remark 4.2. Let us note that the maximal distribution of the vertices $\xi_1, \ldots, \xi_n$ is not needed for the injectivity of π inside a same flat containing Δ at infinity. We only needed that the walls spanned by $\{\xi_j\}_{j \neq i}$, for any i, have trivial intersection. So the result still holds in a same flat if Δ consisted of a single chamber for example. However, the injectivity fails without the the maximal distribution of the vertices if one considers different maximal flats with Δ at infinity.

Step 2: π is a bi-Lipschitz (resp. quasi-isometric) embedding:

• *Substep 1: π is injective (resp. quasi-injective).*

By quasi-injective, we mean that for any δ big enough, there exists $D \geq 0$ such that for any $x, y \in X_\Delta$, if $d_i(\pi_i(x), \pi_i(y)) \leq \delta$ for all $i = 1, \dots, n$ then $d_X(x, y) \leq D$.
The proof for Euclidean buildings and for symmetric spaces is similar, but, for the sake of clarity, we will treat them separately.

X is a Euclidean building: Let $x, y \in X_\Delta$ such that $\pi_i(x) = \pi_i(y)$ for $i = 1, \dots, n$. If they lie in a same maximal-flat containing Δ at infinity, then we are done by the previous step. If not, let F_x and F_y be maximal-flats such that $x \in F_x$, $y \in F_y$ and $\Delta \subset \partial_T F_x \cap \partial_T F_y$. $\Delta \subset \partial_T F_x \cap \partial_T F_y$, so F_x and F_y share a chamber at infinity and must intersect [KL97, Lemma 4.6.5]. Moreover, by [KL97, Cor. 4.4.6], their intersection is a Weyl polyhedron, i.e. an intersection of singular half-spaces $\{M_i\}_{i \in I}$ of F_x, so $F_x \cap F_y = \cap_{i \in I} M_i$. Note that for all $i \in I$, $\Delta \subset \partial_T M_i$. By assumption, $x \notin F_y$, so it is not inside one of these singular half-spaces. Let us denote it by M, and let H be its boundary wall. For every $i = 1, \dots, n$, $\xi_i \in \partial_T M$, and $\partial_T M$ is a singular hemisphere in $\partial_T F_x$. Since $\xi_1, \dots, \xi_n$ are maximally distributed, by Proposition 3.1, $\exists i \in \{1, \dots, n\}$ such that for all $j \neq i$, $\xi_j \in \partial_T H$, and in particular $\eta_i \in \partial_T H$. With loss of generality, we suppose $i = 1$. This implies that $[x, \eta_1)$ stays parallel to H (because $\eta_1 \in \partial_T H$), and therefore never enters in M, see Figure 3:

$$d_X([x, \eta_1), H) = d_X(x, H) > 0.$$

On the other hand, $\pi_1(x) = \pi_1(y)$ implies, by Proposition 2.12, that $[x, \eta_1)$ is strongly

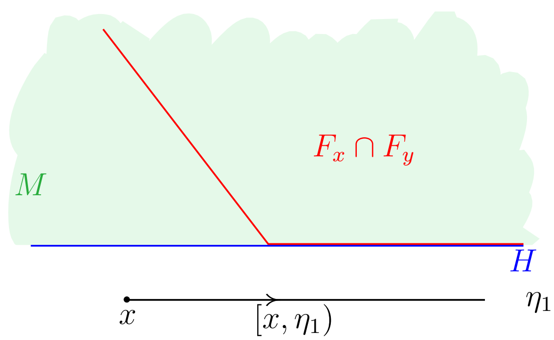

FIGURE 3. $[x, \eta_1)$ staying outside of M.

asymptotic to F_y. However, $F_x \cap F_y \subset M$ and H is convex so

$$0 = d_X([x, \eta_1), F_y) = d_X([x, \eta_1), F_x \cap F_y) \geq d_X([x, \eta_1), M) = d_X([x, \eta_1), H) > 0.$$

We get a contradiction. So the condition $\pi_i(x) = \pi_i(y)$ for $i = 1, \dots, n$ implies that x and y lie in same maximal-flat containing Δ at infinity, and we are done.

X is a symmetric space: let $x, y \in X_\Delta$ such that $d_i(\pi_i(x), \pi_i(y)) \leq 1$ for any $i = 1, \dots, n$. Let F_x and F_y be maximal-flats such that $x \in F_x$, $y \in F_y$, and $\Delta \subset \partial_T F_x \cap \partial_T F_y$. In particular, F_x and F_y share a chamber at infinity so $d_X(F_x, F_y) = 0$. The following lemma is a special case of [Esk98, Lemma B.1] (see also [Mos73, Chapter 7]).

Lemma 4.3. *There exist constants λ_0 and λ depending only on X such that the following holds: if F_1 and F_2 are maximal-flats in X with $d_X(F_1, F_2) = 0$, then for any $\delta \geq \lambda_0$, there exist singular convex polyhedrons P and P' in F_1 (i.e. intersections of singular half-spaces in F_1) such that $d_{Haus}(P, P') \leq \lambda\delta$ and*

$$P' \subset F_1 \cap N_\delta(F_2) \subset P.$$

Let us fix $\delta > \lambda_0$, and let P and P' be the singular convex polyhedrons in F_x such that $d_{Haus}(P, P') \leq \lambda\delta$, $P' \subset F_x \cap N_\delta(F_y) \subset P$, and let $(M_i)_{i\in I}$ be the singular half-spaces in F_x such that $P = \cap_{i\in I} M_i$. Note that for any $i \in I$, $\Delta \subset \partial_T M_i$. Let us show that the condition $d_i(\pi_i(x), \pi_i(y)) \leq \delta$ for any $i = 1, \dots, n$ implies that $x \in P$. If $x \in M_i$ for any $i \in I$, we are done. If not, let us denote this singular half-space by M, and let H be its wall. Again, $\xi_1, \dots, \xi_n \in \partial_T M$ are maximally distributed, so $\exists i \in \{1, \dots, n\}$ such that for all $j \neq i$, $\xi_j \in \partial_T H$, and in particular $\eta_i \in \partial_T H$. We assume again that $i = 1$. So $[x, \eta_1)$ stays parallel to H: $d_X(x, H) = d_X([x, \eta_1), H)$.
On the other hand, by Corollary 2.14, $d_1(\pi_1(x), \pi_1(y)) \leq \delta$ implies that $d_X([x, \eta_1), F_y) \leq \delta$. Since $[x, \eta_1) \subset F_x$, we get

$$d\left([x, \eta_1), F_x \cap N_\delta(F_y)\right) = 0.$$

$F_x \cap N_\delta(F_y) \subset P \subset M$, so $d\left([x, \eta_1), M\right) = 0$. However, $x \notin M$ and H is the boundary wall of M, so

$$d\left([x, \eta_1), M\right) = d\left([x, \eta_1), H\right) = d_X(x, H).$$

Therefore $x \in H$, and we get a contradiction.

Hence, the condition $d_i(\pi_i(x), \pi_i(y)) \leq \delta$ for all $i = 1, \dots, n$ implies that $x \in P$. Therefore $x \in N_\lambda(F_x \cap N_{\lambda\delta}(F_y))$, and in particular, $x \in N_{\lambda+\lambda\delta}(F_y)$. Let $y' \in F_y$ such that $d_X(x, y') \leq \lambda + \lambda\delta$. The projections are 1-Lipschitz, so for any i, $d_i(\pi_i(x), \pi_i(y')) \leq \lambda + \lambda\delta$. y and y' lie in a same flat, so by step 1

$$d_X(y, y') \leq \alpha \sum_{i=1}^{n} d_i(\pi_i(y), \pi_i(y')).$$

Therefore

$$\begin{aligned}
d_X(x, y) &\leq d_X(x, y') + d_X(y', y) \\
&\leq \lambda + \lambda\delta + \alpha \sum_{i=1}^{n} d_i(\pi_i(y), \pi_i(y')) \\
&\leq \lambda + \lambda\delta + \alpha \left(\sum_{i=1}^{n} d_i(\pi_i(y), \pi_i(x)) + \sum_{i=1}^{n} d_i(\pi_i(x), \pi_i(y')) \right) \\
&\leq \lambda + \lambda\delta + \alpha \left(0 + n(\lambda + \lambda\delta)\right) \\
&\leq (1 + \alpha n)(\lambda + \lambda\delta).
\end{aligned} \tag{4.1}$$

Note that $D = (1 + \alpha n)(\lambda + \lambda\delta)$ depends only on X.

• *Substep 2: π is a bi-Lipschitz (resp. quasi-isometric) embedding*

Again, for the sake of clarity, let us treat the Euclidean buildings and symmetric spaces separately.

<u>X is a Euclidean building:</u> let $x, y \in X_\Delta$. If they lie in a same maximal-flat containing Δ at infinity, we are done by step 1. If not, let F_x and F_y be maximal-flats such that $x \in F_x$, $y \in F_y$ and $\Delta \subset \partial_T F_x \cap \partial_T F_y$. $\pi_i([x, \xi_i))$ and $\pi_i([y, \xi_i))$ are not necessarily equal, but they still share a ray. Indeed, since $CS(s_i)$ is a Euclidean building of dimension 1, i.e. a metric tree, $[x, \xi_i)$ and $[y, \xi_i)$ stay at bounded distance near $+\infty$, and π_i is 1-Lipschitz, then $\pi_i([x, \xi_i))$ and $\pi_i([y, \xi_i))$ share a ray. We have two cases, either one of $\pi_i([x, \xi_i))$, $\pi_i([y, \xi_i))$ is a subset of the other. Or there is a branching and they form a tripod. Let us start from $i = 1$.

If we are in the first case, i.e. if $\pi_1([y,\xi_1)) \subset \pi_1([x,\xi_1))$, we can start from x and follow $[x,\xi_1)$ until we equalize π_1. In other words, $\exists x_1 \in [x,\xi_1)$ such that $\pi_1(x_1) = \pi_1(y)$. Since $x_1 \in [x,\xi_1)$, $\forall i \neq 1$, $\pi_i(x_1) = \pi_i(x)$. And $x, x_1 \in [x,\xi_1) \subset F_x$ so $d_X(x,x_1) \leq \alpha d_1(\pi_1(x),\pi_1(x_1)) = \alpha d_1(\pi_1(x),\pi_1(y))$. We denote by $y_1 = y$, and move to $i=2$ to equalize π_2. If $\pi_1([x,\xi_1)) \subset \pi_1([y,\xi_1))$, we take $y_1 \in [y,\xi_1)$ such that $\pi_1(y_1) = \pi_1(x)$, and denote $x_1 = x$.

If we are in the second case and there was a branching, let $x_1 \in [x,\xi_1)$ and $y_1 \in [y,\xi_1)$ such that $\pi_1(x_1) = \pi_1(y_1)$ is the branching point. Since $x, x_1 \in [x,\xi_1) \subset F_x$ and $y, y_1 \in [y,\xi_1) \subset F_y$, we have

$$\begin{aligned} d_X(x,x_1) &\leq \alpha\, d_1(\pi_1(x),\pi_1(x_1)), \\ d_X(y,y_1) &\leq \alpha\, d_1(\pi_1(y),\pi_1(y_1)). \end{aligned} \tag{4.2}$$

$\pi_1(x_1) = \pi_1(y_1)$ is the branching point, so $d_1(\pi_1(x),\pi_1(y)) = d_1(\pi_1(x),\pi_1(x_1)) + d_1(\pi_1(y_1),\pi_1(y))$ and 4.2 implies that

$$d_X(x,x_1) + d_X(y,y_1) \leq \alpha\, d_1(\pi_1(x),\pi_1(y)). \tag{4.3}$$

We repeat this process by starting from x_1, y_1 and we equalize π_2. We get at the end a path $x = x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n = y_n \to y_{n-1} \to \cdots \to y_1 \to y_0 = y$. Note that $x_n = y_n$ by injectivity because we've equalized all π_i. By the triangle inequality and by 4.3:

$$d_X(x,y) \leq \sum_{i=1}^{n} d_X(x_{i-1},x_i) + d_X(y_{i-1},y_i) \leq \sum_{i=1}^{n} \alpha\, d_i(\pi_i(x),\pi_i(y)).$$

X is a symmetric space: Let $x, y \in X_\Delta$. If they lie in a same maximal-flat containing Δ at infinity, we are done by step 1. If not, let F_x and F_y be maximal-flats such that $x \in F_x$, $y \in F_y$ and $\Delta \subset \partial_T F_x \cap \partial_T F_y$. $\pi_1([x,\xi_1))$ and $\pi_1([y,\xi_1))$ are again geodesic rays with the same point at $+\infty$ in $CS(s_1)$, which is a rank one symmetric space of non-compact type. The difference with the building case is that the rays no longer share a ray. To overcome this, we need the following lemma.

Lemma 4.4. *Let X be a symmetric space of non-compact type. For any regular point $\eta \in \partial_T X$, there exists $\delta > 0$ such that if $\mathrm{Isom}_0(X) = KAN$ is an Iwasawa decomposition with respect to η, and $x, y \in X$ are in a same N-orbit, then*

$$d_X\left([x,\eta)(d), [y,\eta)(d)\right) \leq \delta,$$

where $d = d_X(x,y)$.

For a proof, see for example the proof of [Leu00, Lemma 4]. As an application, since in rank one the stabilizer of any point acts transitively on the boundary, δ does not depend on η. If $\eta \in \partial_T X$, we denote by b_η a Busemann function with respect to η and some base point in X. Note that if $x, y \in X$, $b_\eta(x) - b_\eta(y)$ does not depend on the basepoint. So, we have the following.

Corollary 4.5 (see Figure 4)**.** *Let X be a rank one symmetric space of non-compact type. There exists δ such that for any $\eta \in \partial_T X$ and for any $x, y \in X$ the following holds:*
If $b_\eta(x) - b_\eta(y) \geq 0$, let $x' \in [x,\eta)$ such that $b_\eta(x') = b_\eta(y)$, let $x'' \in [x',\eta)$ such that $d_X(x',x'') = d_X(x',y)$, and $y'' \in [y,\eta)$ such that $d_X(y,y'') = d_X(x',y)$. Then the path

$$x \to x'' \to y'' \to y,$$

where each arrow is a geodesic segment, has total length $\leq 3\, d_X(x,y) + \delta$.

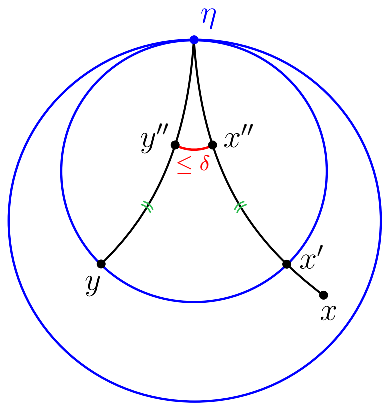

FIGURE 4. The quasi-geodesic $x \to x'' \to y'' \to y$.

Proof. Let B be the horoball whose boundary horosphere contains y, and let $p : X \to B$ be the projection. $p(x) = x'$ so

$$d_X(x, x') \leq d_X(x, y). \tag{4.4}$$

B is convex so p is 1-Lipschitz [BH13, Chap. 2 Prop. 2.4], and we have

$$d_X(x', y) = d_X(p(x), p(y)) \leq d_X(x, y). \tag{4.5}$$

$b_\eta(x') = b_\eta(y)$ and X is rank one so x' and y are in the same N-orbit for some Iwasawa decomposition. By Lemma 4.4, $d_X(x'', y'') \leq \delta$, and the path $x' \to x'' \to y'' \to y$ has length $\leq 2d_X(x', y) + \delta$. We conclude by using 4.4 and 4.5 in $d_X(x, y) \leq d_X(x, x') + d_X(x', y)$. ☐

Let us go back to the proof. Let $\delta > 0$ be as in Corollary 4.5, that works for all the cross sections $CS(s_i)$, for $i = 1, \dots, n$. Let us start from $i = 1$.

Let $x_1 \in [x, \xi_1)$ and $y_1 \in [y, \xi_1)$ such that, as in Corollary 4.5, the path, in $CS(s_1)$, $\pi_1(x) \to \pi_1(x_1) \to \pi_1(y_1) \to \pi_1(y)$ has length $\leq 3\, d_1(\pi_1(x), \pi_1(y)) + \delta$.

$$d_1(\pi_1(x), \pi_1(x_1)) + d_1(\pi_1(x_1), \pi_1(y_1)) + d_1(\pi_1(y_1), \pi_1(y)) \leq 3\, d_1(\pi_1(x), \pi_1(y)) + \delta.$$

$x, x_1 \in [x, \xi_1) \subset F_x$, and $y, y_1 \in [y, \xi_1) \subset F_y$, so

$$\begin{aligned} d_X(x, x_1) &\leq \alpha\, d_1(\pi_1(x), \pi_1(x_1)), \\ d_X(y, y_1) &\leq \alpha\, d_1(\pi_1(y), \pi_1(y_1)). \end{aligned}$$

Therefore,

$$\begin{aligned} d_X(x, x_1) + d_X(y, y_1) &\leq \alpha\, (d_1(\pi_1(x), \pi_1(x_1)) + d_1(\pi_1(y), \pi_1(y_1))) \\ &\leq 3\, \alpha\, d_1(\pi_1(x), \pi_1(y)) + \alpha\delta. \end{aligned} \tag{4.6}$$

We repeat this process by starting from x_1, y_1. We get at the end a path $x = x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n \to y_n \to y_{n-1} \to \cdots \to y_1 \to y_0 = y$. Let us note that, unline in the building case, $x_n \neq y_n$. However, since $\pi(x_n) = \pi(y_n)$ for all i, by the quasi-injectivity in the previous step,

$$d_X(x_n, y_n) \leq D,$$

where D is the constant in 4.1. By the triangle inequality and by 4.6:

$$\begin{aligned} d_X(x,y) &\leq \sum_{i=1}^{n} \big(d_X(x_{i-1},x_i) + d_X(y_{i-1},y_i)\big) + d_X(x_n,y_n) \\ &\leq \sum_{i=1}^{n} \big(3\,\alpha\, d_i(\pi_i(x),\pi_i(y)) + \alpha\delta\big) + D \\ &\leq 3\alpha \sum_{i=1}^{n} \big(d_i(\pi_i(x),\pi_i(y))\big) + (n\alpha\delta + D). \end{aligned}$$

Step 3: π is surjective (resp. quasi-surjective):

By quasi-surjective, we mean that if X is a symmetric space and if $(H'_1,\ldots,H'_n) \in CS(s_1) \times \cdots \times CS(s_n)$, there exists $x \in X_\Delta$ such that for any $i = 1,\ldots,n$, $d_i(\pi_i(x), H'_i)) \leq 1$.

Let us start by the following observation.

Lemma 4.6. *For any $i = 1,\ldots,n$, $P(s_i) \subset X_\Delta$.*

Proof. Let $i \in \{1,\ldots,n\}$, and let H be the flat in F_0 with boundary s_i. Let σ be the singular hemisphere of $\partial_T F_0$ bounded by s_i and containing ξ_i, and let m be its center. Therefore, $m \in \partial_\infty CS(s_i)$.
Now let $H' \in CS(s_i)$. Then any geodesic in $CS(s_i)$ that contains it and contains m at infinity corresponds to a maximal-flat F in X whose boundary contains s_i and m. By convexity, $\partial_T F$ contains σ and therefore contains ξ_i. Recall that for any $j \neq i$, $\xi_j \in s_i \subset \partial_T F$. We conclude that $\Delta \subset \partial_T F$. ☐

Let $x \in X_\Delta$, and F be the flat that contains x such that $\Delta \subset \partial_T F$. Let us denote $\pi_i(x) = H_i$ for any i. Let $H'_1 \in CS(s_1)$.
Since $\pi_1(x) = H_1$, there exists $x_1 \in H_1$ such that $[x,\eta_1)$ is strongly asymptotic to $[x_1,\eta_1)$. As in Lemma 4.6, let m_1 be the center of the singular hemisphere σ_1 bounded by s_1 and containing ξ_1. By considering, in $CS(s_1)$, two geodesic rays containing H_1 and m_1 at infinity (resp. H'_1 and m'_1), there exist two flats F_1 and F'_1 such that $H_1 \subset F_1$, $H'_1 \subset F'_1$, and $\sigma_1 \subset \partial_T F_1 \cap \partial_T F'_1$. Moreover, since for any regular point μ in σ_1, $[x,\mu)$ is strongly asymptotic to F'_1, it also holds, by Corollary 2.10, for any interior point of σ_1, and in particular for ξ_1. By Lemma 2.11, there exists $x'_1 \in F'_1$ such that $[x_1,\xi_1)$ is strongly asymptotic to $[x'_1,\xi_1)$. H'_1 is transverve to $[x'_1,\xi_1)$, so x'_1 can be taken in H'_1. Let us denote the opposites of ξ_1 and m_1 in F'_1 by ξ'_1 and m'_1. see Figure 5 for the building case.
For the rest of the proof, let us treat the building and symmetric space cases separately.

X is a Euclidean building: Let us show that there exists $z \in X_\Delta$ such that $\pi_1(z) = H'_1$ and for any $i \neq 1$ $\pi_i(z) = H_i$. By repeating the process, this completes the proof.

Claim 4.7. $[x,\xi_1)$ enters in a maximal-flat F' satisfying $\Delta \cup \{\xi'_1\} \subset \partial_T F'$.

Proof of the claim. Let H be the $(n-1)$-flat in F satisfying $\pi_1(H) = \pi_1(x)$ (see Proposition 3.4), and let $s = \partial_T H$. Note that $\xi_2,\ldots,\xi_n \in s$, therefore s_1 and s both contain the top-dimensional cell that contains η_1. By [Lee00, Lemma 3.5], $CS(s)$ and $CS(s_1)$ have the same boundary, in particular $m_1, m'_1 \in \partial CS(s)$. Since $\pi_1([x,\xi_1))$ is a parametrization of the

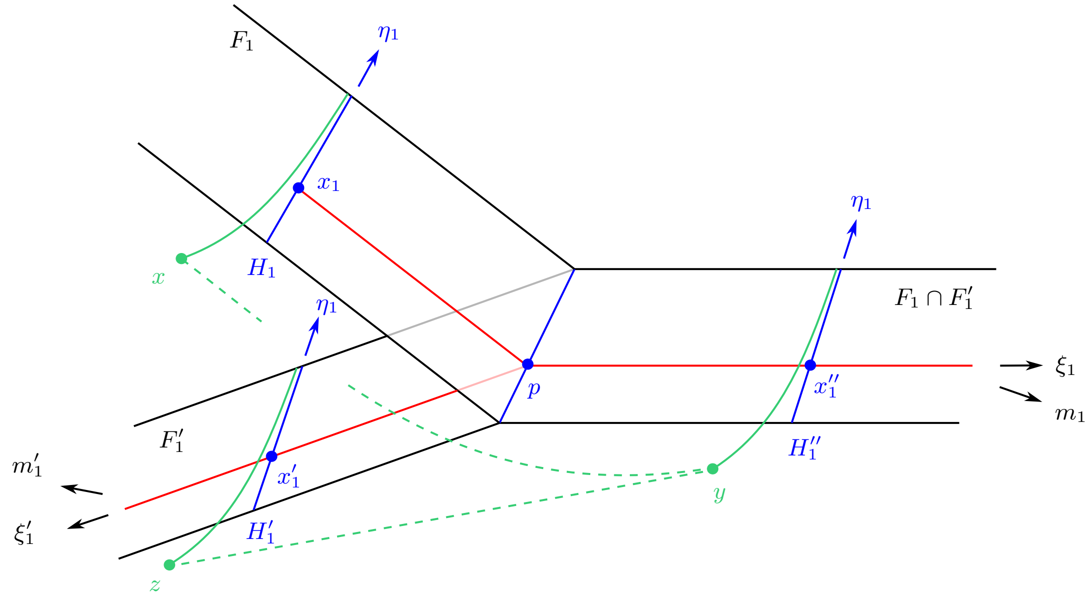

FIGURE 5

geodesic ray $[\pi_1(x), m_1)$ in $CS(s)$, it enters in the geodesic joining m'_1 and m_1. Such a geodesic in $CS(s)$ corresponds to a maximal-flat F' that contains s,m_1, and m'_1 in its boundary. By convexity, as in Lemma 4.6, $\xi_1, \xi'_1 \in \partial_T F'$. $\square$

Since $\pi_1(x) = \pi_1(x_1)$, it follows that $\pi_1([x, \xi_1)) = \pi_1([x_1, \xi_1))$. Let p be the branching point of $[x_1, \xi_1)$ and $[x'_1, \xi_1)$. There exists $y \in [x_1, \xi_1)$ such that $\pi_1(y) \in \pi_1([p, \xi_1))$, and y is contained in a flat F' that satisfies Claim 4.7.

Let us denote $\pi_1(y) = H''_1$, and let $x''_1 \in H''_1 \cap [x_1, \xi_1)$ such that $[y, \eta_1)$ is strongly asymptotic to $[x''_1, \eta_1)$. Now we will move backwards in F' towards ξ'_1. $\pi_1(y) = \pi_1(x''_1)$, so $\pi_1([y, \xi'_1)) = \pi_1([x''_1, \xi'_1))$. Since $x'_1 \in [x''_1, \xi'_1)$, there exists $z \in [y, \xi'_1)$ such that $\pi_1(z) = \pi_1(x'_1) = H'_1$, see Figure 5. $y \in F'$ and $\xi'_1 \in \partial_T F'_1$ so $z \in F' \subset X_\Delta$. Moreover, in the path $x \to y \to z$, we followed geodesics pointing towards ξ_1 so $\pi_2, \dots, \pi_n$ are constant along the path.

X is a symmetric space: we will show that for any $\varepsilon > 0$, there exists $z \in X_\Delta$ such that $d_1(\pi_1(z), H'_1) \leq \varepsilon$, and for any $i \neq 1$, $d_i(\pi_i(z), H_i) \leq \varepsilon$.

We have the same setting except that F_1 and F'_1 do not share a singular half-space (if they do then they are equal). As in Claim 4.7, there exists a maximal-flat F' such that $\Delta \cup \{\xi'_1\} \subset \partial_T F'$, and to which $[x, \xi_1)$ is strongly asymptotic. The proof is similar. As in the building case, $\pi_1([x, \xi_1)) = \pi_1([x_1, \xi_1))$. Note also that $[x_1, \xi_1)$ and $[x'_1, \xi_1)$ are strongly asymptotic, so their images by π_1 are also strongly asymptotic (π_1 is 1-Lipschitz). In particular $d\big(\pi_1([x, \xi_1)(t)), \pi_1([x'_1, \xi_1))\big) \xrightarrow[+\infty]{} 0$. To sum up

$$
\begin{aligned}
&[x, \xi_1) \text{ is strongly asymptotic to } F', \\
&\pi_1([x, \xi_1)) \text{ is strongly asymptotic to } \pi_1([x'_1, \xi_1)).
\end{aligned}
$$

So there exists $y \in [x, \xi_1)$ such that $y \in N_\varepsilon(F')$ and $d_1(\pi_1(y), \pi_1([x'_1, \xi_1))) \leq \varepsilon$. Let $y' \in F'$ such that $d_X(y, y') \leq \varepsilon$, and let $x''_1 \in [x'_1, \xi_1)$ such that $d_1(\pi_1(y), \pi_1(x''_1)) \leq \varepsilon$. So $d_1(\pi_1(y'), \pi_1(x''_1)) \leq 2\varepsilon$.

Now that $y' \in F'$ which is in X_Δ and contains ξ_1' in its boundary, we can move towards it while staying in X_Δ. $d_1(\pi_1(y'), \pi_1(x_1'')) \leq 2\varepsilon$, so the geodesic rays $\pi_1([y', \xi_1'))$ and $\pi_1([x_1'', \xi_1'))$, in $CS(s_1)$, are at Hausdorff distance $\leq 2\varepsilon$. $H_1' \in \pi_1([x_1'', \xi_1'))$ so there exists $z \in [y', \xi_1')$ such that $d_1(\pi_1(z), H_1') \leq 2\varepsilon$.

Note that in both paths $x \to y$ and $y' \to z$, only π_1 changes so for any $i \neq 1$, $\pi_i(z) = \pi_i(y')$ which is at distance $\leq \varepsilon$ from $\pi_i(y) = \pi_i(x) = H_i$. So z satisfies

$$\begin{aligned} &d_1(\pi_1(z), H_1') \leq 2\varepsilon, \\ &d_1(\pi_i(z), H_i) \leq \varepsilon, \text{ for any } i \neq 1. \end{aligned}$$

By repeating the process for $i \neq 1$, we have shown that: for any $\varepsilon > 0$, if $H_i \in CS(s_i)$ for $i = 1, \ldots, n$, there exists $x' \in X_\Delta$ such that for any i, $d_i(\pi_i(x'), H_i)) \leq 2n\varepsilon$. By taking ε small enough, this completes the proof of the quasi-surjectivity.

Remark 4.8. Let us note that π is also surjective for symmetric spaces, but the proof is tedious and the quasi-surjectivity is enough for our purpose.

Remark 4.9. Note also that the maximal distribution of the vertices $\xi_1, \ldots, \xi_n$ is not needed for the surjectivity of π. We only needed that for any i, $\xi_i \notin s_i$ and $\xi_i \in s_j$ for any $j \neq i$, so that one can move in the direction of ξ_i and only changing π_i.

Step 4: $X_\Delta \to X$ is a bi-Lipschitz (resp. quasi-isometric) embedding:

Let us threat the two cases separately.

<u>X is a Euclidean building:</u> let us prove a stronger result, from which the proof immediately follows.

Proposition 4.10. *Let η be an interior point of a chamber C in $\partial_T X$. There exists a constant $\lambda > 0$ such that for any maximal-flats F_1 and F_2 such that $\eta \in \partial_T F_1 \cap \partial_T F_2$, the following holds:*
If $x \in F_1$ and $y \in F_2$, then there exists a path in $F_1 \cup F_2$ from x to y of length $\leq \lambda d_X(x, y)$.

Let us first prove the following lemma, which is the equivalent of Lemma 4.4 for buildings.

Lemma 4.11. *Let η be an interior point of a chamber C in $\partial_T X$. There exists $\beta \geq 0$ such that the following holds:*
For any $x, y \in X$, if $[x, \eta)$ is strongly asymptotic to $[y, \eta)$ and $b_\eta(x) = b_\eta(y)$, then

$$[x, \eta)(\beta d) = [y, \eta)(\beta d),$$

where $d = d_X(x, y)$. In other words, the branching point of $[x, \eta)$ and $[y, \eta)$ is at distance $\leq \beta d_X(x, y)$ from x and y.

Proof. Let z be the branching point of $[x, \eta)$ and $[y, \eta)$. Note that $b_\eta(x) = b_\eta(y)$ implies that $d_X(x, z) = d_X(y, z)$. Note also that, since η is a regular point, z is the entering point of $[x, \eta)$ in the cone $[y, C)$. Let H be the wall in F_2 by which $[x, \eta)$ enters in $[y, C)$. Then

$$\angle_z(x, y) = 2 d_T(\eta, \partial_T H).$$

Since η is an interior point of C,

$$d_T(\eta, \partial_T H) \geq d_T(\eta, \partial C).$$

Let $\theta = 2\, d_T(\eta, \partial C)$, which does not depend on x and y. By considering the geodesic triangle $[x,y]$,$[y,z]$,$[z,x]$, (see [BH13, Chap.2 Ex 1.9])

$$\begin{aligned} d_X(x,y)^2 &\geq d_X(x,z)^2 + d_X(y,z)^2 - 2 d_X(x,z) d_X(y,z) \cos(\angle_z(x,y)) \\ &\geq 2(1-\cos(\theta)) d_X(x,z)^2. \end{aligned}$$

Therefore,

$$d_X(x,z) \leq \frac{1}{\sqrt{2(1-\cos(\theta))}}\, d_X(x,y),$$

and the path $x \to z \to y$ has length $\leq \frac{2}{\sqrt{2(1-\cos(\theta))}}\, d_X(x,y)$. □

Proof of Proposition 4.10. Without loss of generality, suppose $b_\eta(x) - b_\eta(y) \geq 0$.

• If $[x, \eta)$ is strongly asymptotic to $[y, \eta)$, let $x' \in [x, \eta)$ such that $b_\eta(x') = b_\eta(y)$, and let z be the branching point. By Lemma 4.11,

$$d_X(x', z) = d_X(y, z) \leq \beta\, d_X(x', y).$$

x' is the projection of x onto the horoball, centered at η, and whose boundary horocycle contains x' and y, so

$$d_X(x, x') \leq d_X(x, y).$$

This projection is 1-Lipschitz, so

$$d_X(x', y) \leq d_X(x, y).$$

Therefore, the path $x \to x' \to z \to y$ has length

$$d_X(x, x') + d_X(x', z) + d_X(z, y) \leq d_X(x, y) + 2\beta d_X(x', y) \leq (2\beta + 1) d_X(x, y).$$

• If not, let $y_1 \in F_2$ such that $[x, \eta)$ is strongly asymptotic to $[y_1, \eta)$ and $b_\eta(y_1) = b_\eta(y)$. Let z be the branching point of $[x, \eta)$ and $[y_1, \eta)$, and $z' \in [y, \eta)$ such that $b_\eta(z') = b_\eta(z)$. We consider the path $x \to z \to y_1 \to y$. By the first case,

$$d_X(x, z) + d_X(z, y_1) \leq (2\beta + 1) d_X(x, y).$$

$b_\eta(y_1) = b_\eta(y)$, $b_\eta(z') = b_\eta(z)$, and they are all in F_2, so

$$d_X(y, y_1) = d_X(z', z) = d_X(p(y), p(x)) \leq d_X(x, y),$$

where p is the projection onto the horoball centered at η, and whose boundary horocycle contains z' and z. We conclude that this path has length $\leq (2\beta + 2) d_X(x, y)$. □

<u>X is a symmetric space:</u> let us prove that for some $\delta > 0$, $N_\delta(X_\Delta)$, equipped with the path metric, embeds quasi-isometrically in X. To do so, as in the building case, let us prove the following result, which can be seen as a generalization of Corollary 4.5, and from which the proof follows.

Proposition 4.12. *Let η be an interior point of a chamber C in $\partial_T X$. There exist constants $\delta, \lambda, K > 0$ such that for any maximal-flats F_1 and F_2 such that $\eta \in \partial_T F_1 \cap \partial_T F_2$, the following holds: if $x \in F_1$ and $y \in F_2$, then there exists a path in $N_\delta(F_1 \cup F_2)$ from x to y of length $\leq \lambda d_X(x, y) + K$.*

Proof. Let δ be the constant of Lemma 4.4 and, without loss of generality, we suppose $b_\eta(x) - b_\eta(y) \geq 0$.

• If $[x, \eta)$ is strongly asymptotic to $[y, \eta)$, Lemma 4.4 implies that there exists a path in $N_\delta([x, \eta) \cup [y, \eta)) \subset N_\delta(F_1 \cup F_2)$ of length $\leq 3\, d_X(x, y) + \delta$.

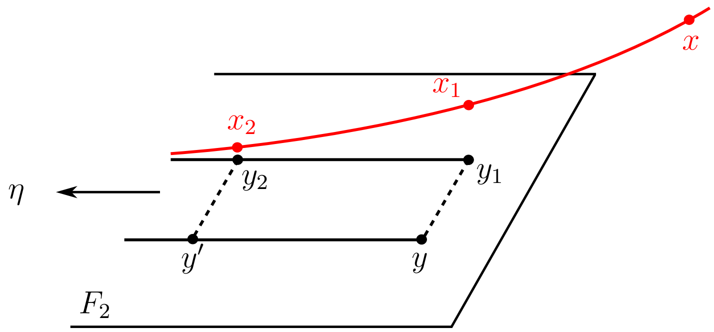

FIGURE 6. Path from x to y in $N_\delta(X_\Delta)$.

• If not, let us consider the following points: $y_1 \in F_2$ such that $[x, \eta)$ is strongly asymptotic to $[y_1, \eta)$ and $b_\eta(y_1) = b_\eta(y)$, $x_1 \in [x, \eta)$ such that $b_\eta(x_1) = b_\eta(y_1)$, $x_2 \in [x_1, \eta)$ such that $d_X(x_1, x_2) = d_X(x_1, y_1)$, $y_2 \in [y_1, \eta)$ such that $d_X(y_1, y_2) = d_X(x_1, y_1)$, and $y' \in [y, \eta)$ such that $b_\eta(y') = b_\eta(x_2) = b_\eta(y_2)$, see Figure 6.

The first case implies that the path $x \to x_1 \to x_2 \to y_2 \to y_1$, which is in $N_\delta(F_1 \cup F_2)$, has length $\leq 3d_X(x, y_1) + \delta$. And, as in the building case, $d_X(x, y_1) \leq d_X(x, y)$, and $d_X(y_1, y) = d_X(y_2, y') \leq d_X(x, y)$. We conclude that the path $x \to x_1 \to x_2 \to y_2 \to y_1 \to y$ has length $\leq 4d_X(x, y) + \delta$. $\square$

This completes the proof of Theorem 1.4.

5. Appendix

In this appendix, we show that the quasi-isometric embedding of the product of n copies of $\mathbb{H}^2_{\mathbb{R}}$ into any symmetric space of non-compact type of rank n can also be obtained as an AN-map. The idea of the proof was communicated to the first author by Yves Benoist.

Let us recall the following theorem due to Fisher–Whyte [FW18, Theorem 1.5].

Theorem. *Let G_1 and G_2 be semisimple Lie groups of equal rank with Iwasawa decompositions $G_i = K_i A_i N_i$. Every injective homomorphism $A_1 N_1 \to A_2 N_2$ is a quasi-isometric embedding.*

Let X be a symmetric space of non-compact type of rank n, and $G = KAN$ an Iwasawa decomposition, where $G = \mathrm{Isom}_0(X)$. To show that there exists a quasi-isometric embedding from the product of n copies of the real hyperbolic plane into X, we need to show that there exists a subgroup of AN isomorphic to the product of n copies of the affine group

$$\left\{ \begin{pmatrix} e^t & se^t \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t, s \in \mathbb{R} \right\}.$$

We use the notations from Section 2.4: Let $\mathfrak{a}$ be a Cartan subspace such that $A = \exp(\mathfrak{a})$, and $\mathfrak{n} = \bigoplus_{\alpha\in\Phi,\alpha(Y)>0}\mathfrak{g}_\alpha$ the sum of positive root spaces with respect to some regular vector $Y \in \mathfrak{a}$ such that $N = \exp(\mathfrak{n})$. Let us show that it is enough to find linearly independent positive roots $\alpha_1, \dots, \alpha_n$ such that for any $i \neq j$, $\alpha_i + \alpha_j$ is not a root.

For any $i = 1, \dots, n$, take $Z_i \in \mathfrak{g}_{\alpha_i}\backslash\{0\}$. Since for any $i \neq j$, $\alpha_i + \alpha_j$ is not a root, e^{Z_i} and e^{Z_j} commute. Therefore

$$\prod_{i=1}^{n} \exp(\mathbb{R}.Z_i) \simeq \mathbb{R}^n,$$

on which A acts diagonally. Indeed, since $\alpha_1, \dots, \alpha_n$ are linearly independent, pick, for any $i = 1, \dots, n$, $X_i \in \bigcap_{j\neq i}\ker\alpha_j$ such that $\alpha_i(X_i) = 2$. So for any $i \neq j$, e^{X_i} and e^{Z_j} commute:

$$e^{X_i}e^{Z_j} = e^{X_i}e^{Z_j}e^{-X_i}e^{X_i} = \exp(e^{\alpha_j(X_i)}Z_j)e^{X_i} = e^{Z_j}e^{X_i}.$$

Therefore, for any $t_1, \dots, t_n, s_1, \dots, s_n \in \mathbb{R}$,

$$H := \exp\left(\sum_{i=1}^{n} t_i X_i\right)\exp\left(\sum_{j=1}^{n} s_j Z_j\right) = \prod_{i=1}^{n} e^{t_i X_i}e^{s_i Z_i}.$$

It is easy to check that for any $i = 1, \dots, n$,

$$\left\{e^{tX_i}e^{sZ_i}\,\middle|\, t, s \in \mathbb{R}\right\} \simeq \left\{\begin{pmatrix} e^t & se^t \\ 0 & e^{-t}\end{pmatrix}\,\middle|\, t, s \in \mathbb{R}\right\}.$$

Therefore H is the desired subgroup of AN which isomorphic to the product of n copies of the affine group.

Finding the roots $\alpha_1, \dots, \alpha_n$: We recall that there exists a natural order on the set of positive roots: given two roots α and β, $\alpha \leq \beta$ iff $\beta - \alpha$ is a non-negative linear combination of simple roots. We refer to [Bou81, Chap.6] for more details. We start by taking α_1 the biggest positive root, and let α_2 be the biggest positive root than is not in span$\{\alpha_1\}$. $\alpha_1 + \alpha_2$ is not a root, otherwise it would be bigger than α_1. Take α_3 the biggest positive root that is not in span$\{\alpha_1, \alpha_2\}$. Again, and by the same argument, $\alpha_1 + \alpha_3$ and $\alpha_2 + \alpha_3$ are not roots. We conclude by induction.

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