GENERALIZATIONS OF NONCOMMUTATIVE NOETHER'S PROBLEM

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ABSTRACT. Noether's problem is a classical and very important problem in algebra. It is an intrinsecally interesting problem in invariant theory, but with far reaching applications in the sutdy of moduly spaces, PI-algebras, and the Inverse problem of Galois theory, among others. To obtain a noncommutative analogue of Noether's problem, one would need a significant skew field that shares a role similar to the field of ratioal functions. Given the importance of the Weyl fields due to Gelfand-Kirillov's Conjecture, in 2006 J. Alev and F. Dumas introduced what is nowdays called the noncommutative Noether's problem. Many papers in recent years [26], [21], [28], [65] have been dedicated to the subject. The aim of this article is to generalize the main result of [28] for more general versions of Noether's problem; and consider its analogue in prime characteristic.

Keywords: algebras of invariant differential opertors, algebras of crystalline differential operators, Weyl algebras, Noether's problem

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NOTIONS OF RATIONALITY

Let k denote an arbitrary base field.

The question of rationality of fields is an important one, whose study goes back to more than a century ago. One of the central problems in this area is the Lüroth problem: let F be a finite extension of k, with $k \subseteq F \subset k(x_1, \ldots, x_n)$. Is it true that, then, F is also a purely transcendental extension of k?

There is a nice geometric interpretation of this problem. Let X be a variety with k(X) = F. The embedding $F \subset k(x_1, \ldots, x_n)$ induces a dominat rational map $f : \mathbb{P}^n \to X$. Varieties with this property are called unirational, and varieties birationally equivalent with a \mathbb{P}^m are called rational. So geometrically Lüroth's problem asks: is every unirational variety rational?

Lüroth problem has a positive solution when n = 1, by Lüroth's Theorem (see, e.g., [33]), and when n = 2 and the base field is algebraically closed of characteristic 0, by Castelnuovo's rationality criterion (see, e.g., [67]). However, if the base field is not algebraically closed, counter-examples exists already for n = 2 even for $k = \mathbb{R}$. For instance, the field of fractions of

$$\mathbb{R}[x, y, z]/(x^2 + y^2 - z(z-1)(z-2))$$

is unirational but not a purely transcendental extension [22]. When n = 2, counter-examples also were found in algebraically closed fields of prime characteristic by Zariski [67]. When $n \ge 3$, counter-examples exists even for algebraically closed fields of zero characteristic [22].

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There is still another notion of rationality that is useful: let F be a finite extension of k. If for some indeterminates, $F(x_1, \ldots, x_r)$ is a purely transcendental extension of k, we say that F is stably-rational. Geometrically, a variety X is stably-rational if, for some m > 0, $X \times \mathbb{P}^m$ is rational. It should be noticed that rational \subsetneq stably-rational \subsetneq unirational. For this and other notions of rationality, see [17].

RATIONALITY OF THE FIELD OF INVARIANTS AND NOETHER'S PROBLEM

There is an important subcase of Lüroth's problem, which talks about the rationality of the ring of invariants of a purely transcendental extension. It was introduced and studied by Emmy Noether [49], [50]:

Noether's Problem: Let S_n acts by permutation of the variables in the rational field $k(x_1, \ldots, x_n)$. Let $G < S_n$ be a subgroup that permutes transitively the variables. When $k(x_1, \ldots, x_n)^G$ is a purely transcendental extension of k — with necessarily the same transcendence degree? Or, in other words, when $k(x_1, \ldots, x_n)^G \simeq k(x_1, \ldots, x_n)$?

Remark 0.1. A more appropriate name would be Noether's Conjecture, for she believed the above question to have a positive solution for all G.

Noether introduced this question thinking in applications to the Inverse problem in Galois theory. Consider $k = \mathbb{Q}$ (or any other Hilbertinian field, see, e.g. [35, Chapter 3]). If Noether's Problem has a positive solution for G, she showed that so has the Inverse problem of Galois theory: there is a Galois extension L of \mathbb{Q} such that $Gal(L, \mathbb{Q}) = G$.

We now introduce an useful terminology: let G be a finite group, and consider the field of rational functions $k(x_h)_{h\in G}$, with the variables indexed by the elements of G. There is a natural action by permutations of G on $k(x_h)_{h\in G}$: $g \in G$ sends x_h to x_{gh} . The field of invariants $k(x_h)_{h\in G}^{G}$ is denoted k(G).

Some important cases of positive solution to Noether's Problem are:

Theorem 0.2. (1) S_n acting on the rational function field in n-indeterminates. (2) \mathcal{A}_n acting on the rational function field in n-indeterminates, for n = 3, 4, 5. The case of alternating groups remains open for n > 5 [35].

- (3) S_n action on $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$, permuting x's and y's simultaneously [45]
- (4) The action of the quaternion group Q_8 in $k(x_1, x_2, x_3, x_4)$ [35]
- (5) Let G be a p-group, and char k = p. Then k(G) is a purely transcendental extension. [32]
- (6) G is a finite abelian group with exponent e such that either char k = 0 or it is coprime with e; and the field possesses a primitive e-th root of unity.
 [35]
- (7) Let n be a positive integer. Then $\mathbb{Q}(C_n)$ if and only if n divides

 $2^2.3^m.5^2.7^2.11.13.17.19.23.29.31.37.41.43.61.67.71$

 $m \ge 0.$ [53]

The first counter-examples to Noether's problem are due to Swan [64] and Vokresenskii [66], independently, over the rational numbers. Their counter-examples consisted of cyclic groups acting by permutation, the smallest one being C_{47} . Later Lenstra [40] classified all finite abelian groups of permutations for which Noether's problem has a positive solution; he found that the smallest group that can give a counter-example is C_8 . Finally, Saltman [57] obtained the first counterexamples over algebraically closed fields.

Noether's problem continues to be extremely relevant to the Inverse problem in Galois theory [35]. For more information about permutation's Noether's problem, see [35] and [32].

Linear Noether's problem Let $G < GL_n(k)$ be a finite group acting linearly on $k(x_1, \ldots, x_n)$. Is $k(x_1, \ldots, x_n)^G$ a purely transcendental extension?

The first person to consider this kind of question, althought not in a sistematic way, was Burnside [14].

Here is an important list of cases of positive solution:

Theorem 0.3. (1) All permutation actions considered previously.

- (2) n = 1, G arbitrary.
- (3) n = 2, G arbitrary.
- (4) n = 3, G arbitrary, k algebrically closed of zero characteristic. [20]
- (5) By Chevalley-Shephard-Todd Theorem (see Theorem 3.11), whenever the natural representation of G is by a pseudo-reflection group and chark and |G| are coprime. [11], [20].

Again, as int the case of permutation groups, a positive solution of linear Noether's problem for a group G, shows that the Inverse problem in Galois theory has a positive solution for the same group, among other things [35]. In particular, this is be far the easiest way to show that pseudo-reflection groups give a positive solution to the Inverse problem.

Linear Noether's problem and the original Noether's problem are linked by the no-name lemma [52]: Let G be a finite group. There exists a faithful, finitedimensional, linear k-representation $G \hookrightarrow GL(V)$ such that $k(V)^G$ is a purely transcendental extension if and only if k(G) is stably-rational.

When we allow the action of infinite groups; that is, rational representations of a linear connected algebraic group G on a finite dimensional vector space V, the questions of the rationality of V/G or P(V)/G — which can also be understood as the rationality of the invariants of k(V), k(P(V)) — are related to the question of rationality of many moduli spaces [10]. As an example, if G is a connected solvable algebraic group over an algebraically closed field, and V is a rational representation, V/G is rational — the action of G stailizes a flag of V (by Lie-Kolchin theorem) and we may apply Miyata's theorem [45]. This problem, in this generality, has also many other important applications: see [17] and [19].

Now let's recall the following notion

Definition 0.4. A G-lattice is a faithful G-module M which is a finitely generated free abelian group.

Multiplicative Noether's Problem: Let M be a G-lattice. The group algebra k[M] is a ring of Laurient polynomials in rank M indeterminates, where G acts by algebra automorphisms. When is the invariant subfield $k(M)^G$ a purely transcendental extension?

Pherhaps the most spetacular application of multiplicative Noether's problem is due to Procesi [55]. He realized that the question of (stable)-rationality of the center of the division ring of fractions of the ring of $2 n \times n$ generic matrices is equivalent to the positive solution of particular case o multiplicaive Noether's problem. A nice simplification of his ideas can be found in [22]. The center is a purely trancendental extension for n = 2, 3, 4. For greater values of n, it is still an open problem [18].

There is a multiplicative analogue of the Chevalley-Shephard-Todd Thereom (see Theorem 3.11 below).

Definition 0.5. Let L be a G-lattice. We say that G is a reflection group if, with its induced action on the \mathbb{Q} -vector space $\mathbb{Q} \otimes_{\mathbb{Z}} L$, G acts as a reflection group.

Theorem 0.6. (Multiplicative Version of Chevalley-Shephard-Todd Theorem) Let L be a G-lattice, with chark and |G| coprime. The following are equivalent:

- (1) $k[L]^G$ is a regular ring.
- (2) k[L] is a finitely generated projective $k[L]^G$ -module.
- (3) k[L] is a finitely generated free $k[L]^G$ -module.
- (4) $k[L]^G$ is mixed Laurient polynomial ring.
- (5) G acts as a reflection group on L and $\mathbb{Z}[L]^G$ is a unique factorization domain.

Proof. [42, Theorem 7.1.1.]

Corollary 0.7. If L is a G-lattice that satisfy any of the above conditions, then $k(L)^G$ is a purely transcendental extension. In particular, $k[L]^G$ is polynomial if and only if L is isomorphic to the weight lattice of some reduced root system and G is the Weyl group.

Proof. [42, Corollary 7.1.2].

Apart from this result, we remark that, in practice, is very difficult to find groups G that satisfy the conditions of Theorem 0.6, and in general one needs computer assistance [41]. Nonetheless

Theorem 0.8. If rank L of the G-lattice is 1, 2 or $3, k(L)^G$ will always be a purely transcendental extension, independtly of G.

Proof. [32]

More on multiplicative Noether's problem can be found in the survey [32] and on the book [42].

In this paper we will consider the most general form for Noether's problem. We will use the terminology in [35]:

General Noether's problem Let G be any finite subgroup of automorphisms of $k(x_1, \ldots, x_n)$ whatsoever. When is $k(x_1, \ldots, x_n)^G$ a purely transcendental extension?

By Lüroth's Theorem, this problem has a positive solution for any field when n = 1, and by Castelnuovo rationality criterion, for n = 2 when k is algebraically closed and char k = 0

NONCOMMUTATIVE NOETHER'S PROBLEM

In the paper [3] J. Alev and F. Dumas introduced the following noncommutative analogue of Noether's problem, usually called just noncommutative Noether's problem. Let $char \mathbf{k} = 0$, $A_n(\mathbf{k})$ denote the Weyl algebras, and $F_n(\mathbf{k})$ denote their skew field of fractions, the Weyl fields. More generally, We write $A_{n,s}(\mathbf{k})$

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for $A_n(\mathsf{k}(x_1,\ldots,x_s))$, and $F_{n,s}(\mathsf{k})$ their skew field of fractions. For the sake of simplicity write $F_{0,s}(\mathsf{k}) = \mathsf{k}(x_1,\ldots,x_s)$.

The point of view of noncommutative Noether's problem was influenced by the Gelfand-Kirillov conjecture [29], that had great influence in the study of enveloping algebras, but was eventually shown to be false in general. The Conecture said that given be an algebraic Lie algebra \mathfrak{g} , then the skew field of fractions of $U(\mathfrak{g})$ is isomorphic to $F_{n,s}(\mathsf{k})$ for adequate n, s. So the Weyl fields can be considered good noncommutative analogues of the field of rational functions. For more on the Conjecture, see [54].

Noncommutative Noether's problem Let G be a finite group acting linearly in $A_n(\mathsf{k})$, and hence on $F_n(\mathsf{k})$. When $F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})$?

Remark 0.9. One might wonder why we don't consider the possibility that $A_n(\mathsf{k})^G$ is isomorphic to $A_n(\mathsf{k})$. By a result of Alev and Polo [4], this is known to be impossible.

Remark 0.10. The question for infinite G was also considered. In this case a small modification on the statemente is necessary; see [3].

Remark 0.11. Invariants of the first Weyl field under other actions were considered in [2].

In [3] the problem was shown to have positive solution when n = 1, 2 and when the natural action of G decomposes in a direct sum of one dimensional G-modules. In [26] the problem was solved for the symmetric group with its permutation action, and as an consequence it was obtained the analogue of Gelfand-Kirillov Conjecture for finite W-algebras of type A. In [21] this fact was generalized for all complex reflection groups, and as corollary it was obtained the analogue of the Gelfand-Kirillov Conjecture for spherical subalgebras of rational Cherednik algebras and linear Galois algebras (which includes $U(\mathfrak{gl}_n)$, finite W-algebras of type A, OGZ algebra, [43]). In [28] it was proved that if a finite linear action gives a positive solution to linear Noether's problem, then the same action gives a positive solution of noncommutative Noether's problem. This result was used to show the validity of Gelfand-Kirillov Conjecture for spherical subalgebras of trigonometric Cherednik algebras in [59] and certain algebras in the paper [34]. In [65] a kind of the converse result was shown: if $k = \mathbb{C}$, and G is a finite group of linear automorphism defined over \mathbb{Z} and $F_n(\mathbb{C})^G \simeq F_n(\mathbb{C})$, then for algebraically closed fields k of characteristic big enough, $k(x_1, \ldots, x_n)^G$ is stably rational, and with this result counter-examples to noncommutative Noether's problem were found — in fact, the group actions were the same as Saltman's counter-examples to Noether's problem, for in these counterexamples, the invariant subfield is also not stably-rational. It remains an interesting open problem to determine if linear Noether's problem and its noncommutative analogue are equivalent, although we conjecture that this does not hold.

Our purpose in this paper is two-fold. The first is to generalize the noncommutative Noether's problem, and the second is to consider its version in prime characteristic. We will denote the usual (Grothendieck's) differential operator ring by \mathcal{D} and its crystalline version (from [9]) by \mathcal{D}_c .

The main theorem of [28] is Theorem 1.1: if $G < GL_n(k)$ is finite group of automorphisms that act linearly on $k(x_1, \ldots, x_n)$ and on the Weyl algebra $A_n(k)$ in the same way, a positive solution to linear Noether's problem implies a positive solution to noncommutative Noether's problem: $F_n(\mathbf{k})^G \simeq F_n(\mathbf{k})$. This theorem was proven using purely ring theoretical methods.

To generalize this result, we will need some basic affine algebraic geometry. Hence, from now on, we assume k algebraically closed.

Following the terminology in [51], we will call $B_n(\mathsf{k}) = \mathcal{D}(\mathsf{k}(x_1, \ldots, x_n)) = k(x_1, \ldots, x_n) \langle \partial_1, \ldots, \partial_n \rangle$. Any finite group G of automorphisms of $\mathsf{k}(x_1, \ldots, x_n)$ extends to a group of automorphisms of $B_n(\mathsf{k})$, and hence of $F_n(\mathsf{k})$

Theorem 0.12. In the context above, if general Noether's problem has a positive solution, then $F_n(k)^G \simeq F_n(k)$

We will offer also a direct, much simpler, proof of the next theorem, even it being just a corollary of the previous one.

Theorem 0.13. Let M be an G-lattice. Identify k[M] with $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{O}(\mathsf{k}^{\times n})$, where $\mathsf{k}^{\times n}$ is the n-torus. Then G acts on $\mathcal{D}(\mathsf{k}^{\times n})$, and hence on $F_n(\mathsf{k})$. If multiplicative Noether's problem has a positive solution, then $F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})$

Now we move to prime characteristic (and keep the field algebraically closed). The definition of the Weyl algebra by generators and relations make perfect sense in prime characteristic, and $A_n(k)$ is still a Noetherian domain (for more about the Weyl algebra in prime characteristic, see [56]. In particular, it is an Azumaya algebra over its center). In particular, its skew field of fractions, the Weyl fields $F_n(k)$, exists, although now they are finite dimensional over their centers. It has been stated explicitly in this author PhD thesis that he conjectured some form of noncommutative Noether's problem would make sense in prime characteristic.

However, Grothendieck's rings of differential operators in prime characteristic are not suitable for us, as they are not Noetherian or domains, and in particular, $A_n(k)$ and $\mathcal{D}(\mathbb{A}^n)$ are very different rings.

On the contrary, rings of crystalline differential operators have the desired properties: if X is an smooth affine variety, $\mathcal{D}_c(X)$ is a Noetherian domain [9]. We also have $A_n(\mathsf{k}) = \mathcal{D}_c(\mathbb{A}^n)$

In [28, Theorem 1.2] we proved that given two affine varieties X, Y, and a finite group G of automorphisms of X, if X/G is birationally equivalent to Y, then $Frac \mathcal{D}(X)^G \simeq Frac \mathcal{D}(Y)$.

We have an analogue for rings of crystalline differential operators.

Theorem 0.14. Let k be an algebraically closed field of prime characteristic. If X is an smooth affine variety and G a finite group of automorphisms of it, such that X/G is birationally equivalent to an affine smooth variety Y, then $Frac \mathcal{D}_c(X)^G \simeq Frac \mathcal{D}_c(Y)$.

Remark 0.15. Notice that, unlike the characteristic 0 case, in prime characteristic we must restrict attention to smooth varieties.

Corollary 0.16. Let G be a finite group of automorphisms of $k[x_1, \ldots, x_n]$. If $k(x_1, \ldots, x_n)^G$ is a purely transcendental extension, then $F_n(k)^G \simeq F_n(k)$

So, just like in characteristic 0 case, in prime characteristic a positive solution to Noether's problem implies a positive solution to its noncommutative analogue.

We also have

Corollary 0.17. Let G be a finite group of automorphisms of $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. If $k(x_1, \ldots, x_n)^G$ is a purely transcendental extension, then $F_n(k)^G \simeq F_n(k)$.

This last corollary includes multiplicative Noether's problem.

Our next result is the version in prime charcteristic on the Gelfand-Kirillov Conjecture for rational Cherednik algebras (rational Cherednik algebras in prime characteristic were studied on a number of places, for instance, [12]).

Theorem 0.18. Assume $2|W| \in k^{\times}$. Let $U_{1,c}(h, W)$ be a spherical subalgebra of a rational Cherednik algebra. Then $Frac U_{1,c}(h, W) \simeq F_n(k)$, where $n = \dim h$

We also have a generalization of [59, Theorem 3.14] to prime characteristic.

Theorem 0.19. Let k be an algebraically closed field of prime characteristic. Let X be an smooth affine variety and G a finite group of automorphisms of it such that X/G is birrationally equivalent to a smooth affine variety Y. Then $\mathcal{O}(T^*X)^G$ has a field of fractions which is isomorphic as a Poisson field to the fraction field of $\mathcal{O}(T^*Y)$.

With this result, we can consider J. Baudry's Poisson Noether's problem [5] in prime characteristic.

Let (V, ω) be a symplectic vector space, $n = \dim V$ and call its Poisson function field as $\mathcal{P}_n(\mathsf{k})$. Let X be a Poisson variety. We call X Poisson rational if $\mathsf{k}(X)$ is isomorphic to $\mathcal{P}_n(\mathsf{k})$, $n = \dim X$, as a Poisson field. Hence Poisson rationality is a refinement of the notion of rationality in the class of Poisson varieties.

Poisson Noether's problem asks: let (V, ω) a symplectic vector space and G a finite group of symplectomorphisms. When is V/G Poisson rational?

Our contribution to this problem in prime characteristic will be Theorem 3.12.

Remark 0.20. Other noncommutative analogues of Noether's problem can be found in [39] for free skew fields, and in [25], [23], [31] for the skew field of tensor products of quantum planes.

RATIONALIY OF RINGS OF DIFFERENTIAL OPERATORS

In this section the base field is algebraically closed of zero characteristic.

Definition 0.21. Let X be an irreducible variety.

- (1) $\mathcal{D}(X)$ is called rational if $Frac \mathcal{D}(X) \simeq F_m(k)$ for some m.
- (2) $\mathcal{D}(X)$ is called stably-rational if there exists an n and m such that $Frac(\mathcal{D}(X) \otimes A_n(\mathbf{k})) \simeq F_m(k)$.
- (3) $\mathcal{D}(X)$ is called unirational if there is an embedding $\mathcal{D}(X)$ into some $F_m(k)$, for adequate m.

Theorem 0.22. (1) If X is rational, then $\mathcal{D}(X)$ is rational. More precisely, if X is birationally equivalent to \mathbb{A}^n , $Frac \mathcal{D}(X) \simeq F_n(k)$.

- (2) If X is stably rational, then $\mathcal{D}(X)$ is stably rational. More precisely, if $X \times \mathbb{A}^n$ is rational, then $Frac(\mathcal{D}(X) \otimes A_n(\mathsf{k})) \simeq F_m(\mathsf{k})$, with $m = \dim X + n$
- (3) if X is unirational, then $\mathcal{D}(X)$ is unirational. More precisely, if there is a rational dominant map $\mathbb{A}^n \to X$, then we have and embedding of $Frac \mathcal{D}(X)$ into $F_n(k)$. If $Frac \mathcal{D}(X)$ embedds into $F_m(k)$, then necessarily $m \ge \dim X$

1. Preliminaries

Suppose initially that k is an arbitrary field. Lets recall some definitions.

Definition 1.1. The n-th Weyl algebra $A_n(k)$ is the algebra generated by generators, $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to relations

$$[x_i, x_j] = [y_i, y_j] = 0, [y_i, x_j] = \delta_{ij}, i, j = 1, \dots, n$$

In case *char* $\mathbf{k} = 0$, the Weyl algebras are simple Noetherian domains with center \mathbf{k} [51]. In case of positive characteristic, $A_n(\mathbf{k})$ again is Noetherian domain, but now the algebra is a finite module over its center $\mathbf{k}[x_1^p, \ldots, x_n^p, y_1^p, \ldots, x_n^p]$ [56].

The Weyl algebras are related to rings of differential operators, introduced by Grothendieck [30]:

Definition 1.2. Let A be a k-algebra. Set

$$\mathcal{D}_0(A) = \{ \theta \in End_k A | [\theta, a] = 0, \forall a \in A \} \simeq A,$$

and, inductively,

$$\mathcal{D}_{i}(A) = \{ \theta \in End_{\mathsf{k}} A | [\theta, a] \in \mathcal{D}_{i-1}(A), \forall a \in A \}.$$

The ring of differential operators on A is $\mathcal{D}(A) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(A)$.

For an affine variety X, we call the ring of differential operators in X, $\mathcal{D}(X) = \mathcal{D}(\mathcal{O}(X))$. In case *char* $\mathbf{k} = 0$, $\mathcal{D}(X)$ is an Ore domain [28], and if moreover X is smooth, $\mathcal{D}(X)$ is simple and left and right Noetherian [51]. Moreover, by the definition, $\mathcal{D}(X)$ comes with a filtration such that the associated graded algebra is $\mathcal{O}(T^*X)$ (for smooth X) ([30], [51]). We have $A_n(\mathbf{k}) = \mathcal{D}(\mathbb{A}^n)$.

In caracteristic 0 and affine regular domains A, there is an alternative definition of differential operators that coincides with the Grothendieck's one: $\mathcal{D}(A)$ can be described as the subalgebra of $End_k A$ generated by A and $Der_k A$ [51, 15.5.5]. For arbitrary A, if we denote this definition as $\Delta(A)$, we have that just $htat\Delta(A) \subset$ $\mathcal{D}(A)$, and the inclusion may be proper if A is not regular. In fact, the famous Nakai Conjecture [51] says that $\Delta A = \mathcal{D}(A)$ if and only if A is a finitely generated regular domain. Notice that some patological behavior can happen if A is not finitely generated. In a recent preprint [48], an example of regular domain A but not finitely generated is given such that $Der_k A = 0$ and the natural inclusion map $A \to \mathcal{D}(A)$ is an isomorphism.

For smooth varieties in char $\mathbf{k} = p > 0$, as shown first by Smith [61], the correct definition of ring of differential operators is the Grothedieck's one. However, the situation changes drastically in positive characteristic. For instance, denoting by P_n the polynomial algebra in *n*-indeterminates, $\mathcal{D}(P_n)$ is not Noetherian, not finitely generated, and has a lot of zero-divisors and nilpotent elements. Its Gelfand-Kirillov dimension also gives the "wrong" number: *n*, instead of 2n. However, the algebra is still simple [6]. In particular, $A_n(\mathbf{k})$ and $\mathcal{D}(\mathbb{A}^n)$ are non-isomorphic rings.

It is clear that the classical notion of differential operator will lead us nowhere in our task to considering the noncommutative Noether's problem in prime characteristic — they are not even domains. The solution is found in changing the technology: we will need a modified version of differential operators introduced in [9] (a particular case of a more general definition in [8]): crystaline differential operators.

Definition 1.3. Let k be an algebraically closed field of prime characteristic, X an smooth affine variety. $\mathcal{D}_c(X)$, the ring of crystalline differential operators on X, is generated by $\mathcal{O}(X)$ and $Der_k\mathcal{O}(X)$ subject to the relations

$$f.\partial = f\partial, \ \partial.f - f.\partial = \partial(f),$$

$$\partial \partial \partial' - \partial' \partial \partial = [\partial, \partial'], f \in \mathcal{O}(X), \partial, \partial' \in Der_k \mathcal{O}(X).$$

Remark 1.4. The notion of crystalline differential operators works only for smooth varieties.

We have now that $\mathcal{D}_c(\mathbb{A}^n) \simeq A_n(\mathsf{k})$.

 $\mathcal{D}_c(X)$ has a natural filtration: $\mathcal{D}_c^0(X) = \mathcal{O}(X), \ \mathcal{D}_c^i(X) = \mathcal{D}_c^i i - 1(X) + Der_k \mathcal{O}(X) \cdot \mathcal{D}_c^i i - 1(X).$

Proposition 1.5. [9]

- (1) $gr \mathcal{D}_c(X) = \mathcal{O}(T^*X).$
- (2) The Poisson algebra structure on $\mathcal{O}(T^*X)$ induced by the filtered quantization by $\mathcal{D}_c(X)$ coincides with the usual Poisson algebra structure from the standard symplectic form on T^*X .

Since the associated graded algebra is a finitely generated Noetherian domain, by usual filtered techniques [51], we have that $\mathcal{D}_c(X)$ is a finitely generated left and right Noetherian domain. Hence we can study skew fields of fractions.

Moreover, we have ([51]):

Proposition 1.6. $GK\mathcal{D}_c(X) = 2\dim X, \ gl.dim \mathcal{D}_c(X) \leq \dim X, \ \mathcal{KD}_c(X) \leq \dim X.$

Remark 1.7. Proposition 1.5 is well known for fields of zero characteristic and usual differential operators.

2. Proof of characteristic 0 results

Before we move to questions about the caracteristic prime case, let's discuss the results we have in characteristic 0.

2.1. Generalizations of noncommutive Noether's problem. In this section we will work over algebraically closed fields.

We want to use this result to prove Theorem 0.12. However, our automorphism group comes from $k(x_1, \ldots, x_n)$, and not from an affine variety — so we can't use [28, Theorem 1.2]. But we are going to show that there always exists an open affine variety $X \subset \mathbb{A}^n$ such that the group of birational automorphisms on \mathbb{A}^n restricts to biregular automorphism on X, and X/G is still rational. Clearly, then, $Frac \mathcal{O}(X)^G \simeq k(x_1, \ldots, x_n)^G \simeq k(x_1, \ldots, x_n)$

Lemma 2.1. Let G be a finite group of birational automorphisms of an affine variety X. Then there is a G-invariant affine open subset U where G acts by biregular automorphisms.

Proof. For each $g \in G$ there is an affine open subset $U_g \subset X$ such that $g: U_g \to g(U_g)$ is a biregular isomorphism.

Calling $U = \bigcap_{g \in G} U_g$, we have that $g : U \to g(U)$ is a biregular isomorphism for every $g \in G$. U is again affine, since finite intersection of affine open subsets is again affine.

Setting $W = \bigcap_{g \in G} g(U)$, we have that W is G-invariant, affine, and each $g \in G$ is a biregular isomorphism of W onto itself.

Example 2.2. Let $G < GL_n(\mathbb{Z})$ act on $k(x_1, \ldots, x_n)$ as follows:

$$g.x_j = \prod_{i=1}^n x_i^{a_{ij}}, g = (a_{ij})_{i,j=1,\dots,n} \in GL_n(\mathbb{Z}), \ j = 1,\dots,n.$$

These are precisely the group actions that arise in the multiplicative Noether's problem mentioned in the Introduction. G corresponds to a finite group of birational automorphisms of the affine space \mathbb{A}^n . A open affine subset U as in the previous lemma is $\operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = k^{\times n}$, the n torus. This is the case because G acts on $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by algebra automorphisms and on $k^{\times n}$ by biregular automorphisms.

Lemma 2.3. Suppose X an affine variety, G a group acting on it, such that X/G is rational. Then $Frac \mathcal{D}(X/G) \simeq F_n(k)$, $n = \dim X$.

Proof. Let $S = \mathcal{O}(X)^G \setminus \{0\}$. $Frac \mathcal{D}(X/G) = Frac \mathcal{D}(X/G)_S = Frac \mathcal{D}(\mathcal{O}_S^G)$, where we used [46, Proposition 1.8] and that $\mathcal{O}(X/G) = \mathcal{O}(X)^G$. Since X/G is rational, $\mathcal{O}(X)_S^G \simeq \mathsf{k}(x_1, \ldots, x_n)$. Hence $Frac \mathcal{D}(X/G) \simeq Frac B_n(\mathsf{k}) = F_n(\mathsf{k})$. \Box

The proof of the following lemma work in any characteristic.

Lemma 2.4. Let G a finite group acting on an affine variety X. Then there is an open subset $U \subset X$ where G restricts to a free action. Moreover, there is no loss of generality in assuming U affine.

Proof. For each $g \neq 1$ in G call $Y_g = \{x \in X | g(x) = x\}$. Y_g is closed being the inverse, uder the map $(id, g) : X \to X \times X$, of the diagonal. Since G is finite, $Y = \bigcup Y_g$ is a closed subset of X in the Zarisk topology, and $U = X \setminus Y$ is an open set where G acts freely.

Theorem 2.5. Let X be a variety and G a group acting freely on it. Then $\mathcal{D}(X)^G \simeq \mathcal{D}(X/G)$

Proof. [15, Theorem 3.7(1)]

Now we return to our original situation. We have a group G of birational automorphisms of \mathbb{A}^n such that $\mathsf{k}(x_1,\ldots,x_n)^G \simeq \mathsf{k}(x_1,\ldots,x_n)$.

By Lemmas 2.1 and 2.4, there is an open affine subset $U \subset \mathbb{A}^n$ such that $G|_U$ acts freely by biregular automorphisms on U, and U/G is rational.

We have $F_n(\mathbf{k})^G = Frac \mathcal{D}(U)^G$. By Theorem 2.5, $Frac \mathcal{D}(U)^G \simeq Frac \mathcal{D}(U/G)$, which by Lemma 2.3, is isomorphic to $F_n(\mathbf{k})$. Hence Theorem 0.12 is proved.

As promised, we now give a simpler proof for Theorem 0.13. By hypothesis, we have a finite group G of biregular automorphisms of the torus $\mathsf{k}^{\times n}$, such that $\mathsf{k}^{\times n}/G$ is rational. By [28, Theorem 1.2], $Frac \mathcal{D}(\mathsf{k}^{\times n})^G = F_n(\mathsf{k})^G \simeq F_n(\mathsf{k})$.

2.2. Rationality for rings of differential operators. Let's prove items 1, 2 and 3 of Theorem 0.22.

First we recall a well-known fact about rings of differential operators, which can be found, for instance, in [46]

Proposition 2.6. Let A a finitely generated algebra which is also a domain. Let S be a multiplicatively closed subset of A. Then the localizations of $\mathcal{D}(A)$ in the left and on the righ by S exists, and they are isomorphic to $\mathcal{D}(A)_S$

Lemma 2.7. Let A be an Ore domain and S any denominator set on A. $Frac A = Frac A_S$.

Proof. Proof of item 1

If X is rational, call by S its set of non-null elements, and by A its ring of regular functions. Clearly $A_S \simeq \mathsf{k}(x_1, \ldots, x_n)$. , where n is the Krull dimension of A. $Frac \mathcal{D}(A) = Frac \mathcal{D}(A)_S = Frac \mathcal{D}(A_S) = Frac \mathcal{D}(\mathsf{k}(x_1, \ldots, x_n)) = Frac B_n(\mathsf{k}) = Frac F_n(\mathsf{k})$.

Let's know prove item 3.

Proof. **Proof of item 3** If X is unirational, we have an embedding of its function field F into $k(x_1, \ldots, x_n)$, where $n \ge dimX$ by transcendence degree considerations. Taking differential operators preserve the embedding $\mathcal{D}(F) \subset B_n(\mathsf{k})$.

Using lemma 2.7, $Frac \mathcal{D}(X) = Frac \mathcal{D}(F) \subset F_n(\mathsf{k})$, as taking the skew-field of fractions preserves embeddings. Finally, if $Frac \mathcal{D}(X)$ embedds in some $F_n(\mathsf{k})$, by [24, Theorem 10], $\dim X \leq n$.

The hardest one to prove is the second item. So we will need some preparation.

Proposition 2.8. Let A be an affine integral domain. There exists an element c in A such that localization A_c is regular

Proof. [51, 15.2.10]

Let X be an affine variety, and $A = \mathcal{O}(X)$. Let c be one of the elements c of the previous proposition such that A_c is regular. I will denote by $X_c = Spec A_c$; X_c is a smooth open affine subvariety of X.

Proof. **Proof of item 2** Let $c \in \mathcal{O}(X)$ be such that X_c is smooth. First notice that we can localize $\mathcal{D}(X) \otimes A_n(\mathsf{k})$ by $c \otimes 1$ beucase this element act ad-nilpotently [38, Thm 4.9]. So $\mathcal{D}(X_c) \otimes A_n(\mathsf{k})$ is a localization of of $\mathcal{D}(X) \otimes A_n(\mathsf{k})$. Suppose that $X \times \mathbb{A}^n$ is birational, say, to \mathbb{A}^m .; then the same holds for $X_c \otimes \mathbb{A}^n$. By transcendence degree considerations, $\dim X + n = m$. Both X_c and \mathbb{A}^n are smooth affine varieties with finite Krull dimension, so using [8, Lemma 2.5], we have that $\mathcal{D}(X_c) \otimes A_n(\mathsf{k})$ is isomorphic to $\mathcal{D}(X_c \times \mathbb{A}^n)$. Using Theorem 0.22 item 1, we obtain that $Frac \mathcal{D}(X_c \times \mathbb{A}^n) = F_m(\mathsf{k})$. In fact, m must be $\dim X + n$, by [24, Theorem 8].

3. The situation in prime characetristic

In this section k is an algebraically closed field of prime characteristic.

Proposition 3.1. Let X and Y be smooth affine varieties with $k(X) \simeq k(Y)$. Then $Frac \mathcal{D}_c(X) \simeq Frac \mathcal{D}_c(Y)$.

Proof. Analogous to the case of zero characteristic (see remark below). The ring of crystalline differential operators sheafifies [9]; that is, just as usual rings of differential operators, it is compatible with localization. \Box

Remark 3.2. For characteristic 0 and usual rings of differential operators the above is [51, 15.1.25].

If G is a finite group of automorphisms of a smooth variety X then we can define an action on $\mathcal{D}_c(X)$ by defining the action on the generators as follows:

$$g.f = g(f), g.\partial = g \circ \partial \circ g^{-1}, g \in G, f \in \mathcal{O}(X), \partial \in Der_k\mathcal{O}(X).$$

Theorem 3.3. Let X be an smooth affine variety and G be a finite groups of automorphisms of X that acts freely. Then

$$\mathcal{D}_c(X)^G \simeq \mathcal{D}_c(X/G)$$

Proof. Since G acts freely on X, the projection $\pi: X \to X/G$ is étale ([47], §II.7). Hence the induced map on the tangent bundles $d\pi: T_X \to \pi^* T_{X/G}$ is an isomoprhism. Hence we also have that $Sym_{\mathcal{O}(X)}T_X \simeq \pi^* Sym_{\mathcal{O}(X/G)}T_{X/G}$ and clearly we have $\mathcal{O}(X)^G \simeq \mathcal{O}(X/G)$. By the PBW theorem (which is just Proposition 1.5), the result follows.

Remark 3.4. Notice that the analogue result holds in characteristic zero: Theorem 2.5.

Proof of Theorem 0.14 Since G is finite, there is an open subset U of X such that the restriction of the action of G to U is free. Without loss of generality, we can assume U affine.

Hence by Theorem 3.3:

$$(1) \mathcal{D}_c(U)^G \simeq \mathcal{D}_c(U/G).$$

Now, k(X) = k(U), hence by Proposition 3.1 (2) $Frac\mathcal{D}_c(X)^G = Frac\mathcal{D}_c(U)^G$. Also k(U/G) = k(Y) (as X/G and so U/G is birationally equivalent to Y). So again by Proposition 3.1 (3) $Frac\mathcal{D}_c(U/G) \simeq Frac\mathcal{D}_c(Y)$. Combining (1), (2), (3) we have

$$Frac \mathcal{D}_c(X)^G \simeq Frac \mathcal{D}_c(Y)$$

as desired.

I repeat now the corollary that says, essentially: Noethers problem implies noncommutative Noether's problem in prime characteristc

Corollary 3.5. Let G be a finite group of automorphisms of $k[x_1, \ldots, x_n]$. If $k(x_1, \ldots, x_n)^G$ is a purely transcendental extension, then $F_n(k)^G \simeq F_n(k)$

Remark 3.6. We did not impose that G acts linearly.

3.1. Rational Cherednik algebras in prime characterisc. The theory of rational Cherednik algebras over fields of prime characteristic parallel the one over \mathbb{C} developed in [27]. We follow [12].

Let k be an algebraically closed field of odd characteristic. Let (V, ω) be an even dimensional vector space V with a non-degenerated sympletic form ω . A finite subgroup of SP(V) is called a symplectic reflection group if it is generated by symplectic reflections, which are symplectic isomorphisms g such that rank 1 - g = 2.

Let Γ be a symplectic reflection group, S the set of symplectic reflections, $c : S \to k$ invariant under conjugation. Let $t \in k$. We allways assume

$$2|\Gamma| \in \mathsf{k}^{\times}$$
.

The symplectic reflection algebra $H_{t,c}$ is the quotient of $T(V) * \Gamma$ by the relations

$$[x,y] = t\omega(x,y) - \sum_{s \in S} c(s)\omega_s(x,y),$$

where $x, y \in V$ and ω_s is the skew-symmetric form with radical ker(I-s) and coincides with ω in Im(I-s).

Let W be a pseudo-reflection group with representation h. W acts on $V = h \oplus h^*$, which has a W-invariant symplectic form $\omega((u, f), (x, g)) = g(u) - f(x)$. W becomes in this way a symplectic reflection group, with the set of pseudo-reflections corresponding to the symplectic reflections. Let $0 \neq t \in k$, $c : S \to k$ invariant under conjugation. The symplectic reflection algebra $H_{t,c}$ is then called a rational Cherednik algebra and denoted $H_{t,c}(h, W)$.

Let $e = 1/|W| \sum_{w \in W}$ be an indempotent in $H_{t,c}(h, W)$. The spherical subalgebra $U_{t,c}(h, W)$ is $eH_{t,c}(h, W)e$ (with unit e).

Proposition 3.7. The spherical subalgebra is a finitely generated Noetherian domain [12, Theorem 3.1].

Proof. **Proof of Theorem** 0.22

By [12, Theorem 4.5 and Remark 4.6], there is a W invariant element $\delta \in \mathsf{k}[h]$ such that $H_{t,c}(h,W)\delta^{-1} \simeq \mathcal{D}(h^{reg}) * W$. $e\mathcal{D}_c(h_{reg}) * We \simeq \mathcal{D}_c(h_{reg})^W$, hence $U_{t,c}(h,W)\delta^{-1} \simeq \mathcal{D}_c(h_{reg})^W$. W acts freely in h_{req} ; hence may then use Theorem 3.3. We have $\mathcal{D}_c(h_{reg})^W \simeq \mathcal{D}_c(h_{reg}/W)$.

By assumption, W is a pseudo-reflection group and |W| is coprime to *char* k. Chevalley-Shephard-Todd Theorem also hold in this situation (Theorem 3.11). Hence h_{reg}/W is an affine rational variety and so by Theorem 0.14, the skew field of fractions indeed is $F_n(\mathbf{k})$, $n = \dim h$.

3.2. Contangent-bundle.

Theorem 3.8. Let X be an smooth affine variety, G a finite group that acts freely on it. Then T * (X/G) is isomorphic, as a Poisson variety, to (T * X)/G.

Proof. It follows the same steps as Theorem 3.3, but simpler, since we are now at the level of contagente bundles \Box

Remark 3.9. In characteristic 0 the above result was shown in [59, Theorem 3.4].

Proposition 3.10. Let X and Y be birationally equivalent smooth affine varieties. Then $k(T^*X)$ and $k(T^*Y)$ are isomorphic as Poisson fields.

Proof. The proof of [59, Proposition 3.12] works on any characteristic.

proof of Theorem 0.19

Let U be an affine open subset of X where G acts freely. As k(X) = k(U), (1) $Frac \mathcal{O}(T^*X)^G \simeq Frac \mathcal{O}(T^*U)^G$, as Poisson fields, by Proposition 3.10. By Theorem 3.8, (2) $Frac \mathcal{O}(T^*U)^G \simeq Frac \mathcal{O}(T^*(U/G))$ as Poisson fields. As X/G is birational to Y, U/G is also, and hence (3) $Frac \mathcal{O}(T^*(U/G)) \simeq \mathcal{O}(T^*Y)$ as Poisson fields, by Proposition 3.10 again. Combining (1), (2), (3), we obtain our result.

Now we will use this theorem for Poisson-Noether's Problem. It is convenient to recall Chevalley-Shephard-Todd-(Serre) Theorem:

Theorem 3.11. Let V be a finite dimensional vector space, G < GL(V) a finite group whose order is not divisible by chark. Then $S(V^*)^G$ is again a polynomial algebra if and only if G is a pseudo-reflection group.

Proof. [11].

If the characteristic of the field divides the order of the group, Serre ([60]) has shown that for $S(V^*)^G$ to be a polynomial algebra, a necessary condition is that Gis a pseudo-reflection group; but it is not sufficient (see a counter-example in [36, 19-2])

In characteristic 0, the pseudo-reflection groups have long been classified using the classification of Shephard and Todd for complex reflection groups and Clark-Ewing Theorem (see, e.g., [36, Chapter 15]).

More recently the irreducible pseudo-reflection groups were classified over any characteristic, by Kantor, Wagner, Zaleskii and Serezkin, and those for which the invariants of the polynomial algebra ara again polynomial by Kemper and Malle (see both aspects of the classification in [37]).

Let k be an algebraically closed field of prime characteristic, and G an irreducible pseudo-reflection group. We will call the group K-M if it is in Kemper and Maller list.

Theorem 3.12. Let G < GL(h) be a K-M group. Then if we make G act diagonally on $h \oplus h^*$, with its canonical symplectic form, $(h \oplus h^*)/G$ is Poisson rational.

Proof. Like our work in the \mathbb{C} case ([59]) we may use Theorem 0.19. In our situation, X = h. h/G is clearly birationally equivalent to Y = h — they are in fact isomorphic. Finally, recall that $T^*h = h \oplus h^*$. Applying Theorem 0.19 we obtain our desired result.

Appendix

In the paper [27], Etingof and Ginzburg conjectured (see Proposition 17.6* of their paper) an analogue of the Gelfand-Kirillov Conjecture for all spherical subalgebras (at t = 1) of symplectic reflection algebras. Namely, Let $A_1(V)$ be the Weyl algebra of the symplectic vector space (V, ω) at t = 1, and Γ the symplectic reflection group. Let $n = \dim V$. Then the skew field of fractions should be $F_n(\mathbb{C})^{\Gamma}$. When Γ is a complex reflection group, the conjecture is true by the Dunkl embedding, and the skew field of fractions is in fact isomorphic to $F_n(\mathbb{C})$ ([21]). The other non-expecptional family of symplectic reflection groups on Cohen's classification [16] are the so called wreath product type $G \wr S_n$, G a finite subgroup of $SL_2(\mathbb{C})$, which are classified [63]. If $\Gamma = G \wr S_n$, it is folkore that $F_n(\mathbb{C})^{\Gamma} \simeq F_n(\mathbb{C})$. We offer a proof of this fact. First notice that if G is a group of automorphisms of a ring R and H a normal subgroup of G, $R^G = (R^H)^{G/H}$. So when Γ is of wreath product type, we have that $F_n(\mathbb{C})^{\Gamma} = ((F_1(\mathbb{C})^G)^{\otimes n})^{S_n}$. By a result from [1], $F_1(\mathbb{C})^G \simeq F_1(\mathbb{C})$. Hence $F_n(\mathbb{C})^{\Gamma} \simeq F_n(\mathbb{C})^{S_n} \simeq F_n(\mathbb{C})$. In the last isomorphism we used [26, Theorem 4.1].

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