Virial Theorem and Its Applications in Instability of Two-Phase Water-Wave

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Abstract

In this paper, we analyze the dynamics of two layers of immiscible, inviscid, incompressible, and irrotational fluids through a full nonlinear system. Our goal is to establish a virial theorem and prove the polynomial growth of slope and curvature of the interface over time when the fluid below is no denser than the one above. These phenomena, known as Rayleigh-Taylor instability and Kelvin-Helmholtz instability, will be proved for a broad class of regular initial data, including the case of 2D overlapping interface.

Contents

1 Introduction

1.1 Two-phase water-wave

We consider two layers in the domain $\mathbb{T}^d \times \mathbb{R}$ with interface assumed to be a graph

$$\Sigma(t) = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : y = \eta(t, x)\}, \tag{1.1}$$

where $x \in \mathbb{T}^d$, $y \in \mathbb{R}$ correspond to horizontal and vertical variable respectively. The dimension d is chosen to be 1 or 2 for physical relevance, while most of the results in this paper hold true for general dimensions. As a convention, we distinguish the quantities defined for upper and lower layer by adding superscript $-$ and $+$ respectively. For example, the upper and lower layers are denoted by

$$\begin{aligned} \Omega^-(t) &= \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : \eta(t, x) < y < H^-\}, \\ \Omega^+(t) &= \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : -H^+ < y < \eta(t, x)\}, \end{aligned} \tag{1.2}$$

where $H^\pm \in]0, +\infty]$ and the corresponding "bottoms" are

$$\Gamma^\pm = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : y = \mp H^\pm\}. \tag{1.3}$$

Since we are interested in inviscid, incompressible, and irrotational fluids, the velocity field $u^\pm$ satisfies

$$\begin{cases} \rho^\pm \left(\partial_t u^\pm + u^\pm \cdot \nabla_{x,y} u^\pm\right) + \nabla_{x,y}\left(P^\pm + \rho^\pm g y\right) = 0, & \text{in } \Omega^\pm(t), \\ \operatorname{div}_{x,y} u^\pm = 0, \quad \operatorname{curl}_{x,y} u^\pm = 0, & \text{in } \Omega^\pm(t), \end{cases} \tag{1.4}$$

where $\rho^\pm > 0$ is density, $P^\pm$ is the pressure related to surface tension (see (1.8) below), and $g > 0$ is the gravity acceleration. As a convention, we write the subscript x, y for derivatives involving both x and y, and ignore subscripts for those depending only on x.

In order to complete the description of the system, we need to determine the boundary conditions on $\Sigma(t)$ and $\Gamma^\pm$. Since the fluids cannot go through the outer boundary $\{y = \mp H^\pm\}$, one has

$$u^\pm \cdot n_b^\pm = 0, \quad \text{on } \Gamma^\pm, \tag{1.5}$$

where $n_b^\pm = (0, \mp 1)$ is the unit outward normal vector of the outer boundary $\Gamma^\pm$. Remark that this condition should be replaced by

$$u^\pm \to 0, \quad \text{as } y \to \pm\infty, \tag{1.6}$$

in the infinite depth case ($H^\pm = +\infty$). As for the interface $\Sigma(t)$, we assume that the normal component is continuous in absence of vacuum area, which is known as *kinematic boundary condition*,

$$\begin{cases} \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} u^\pm \cdot n, & \text{on } \Sigma(t), \\ u^+ \cdot n = u^- \cdot n, & \text{on } \Sigma(t), \end{cases} \tag{1.7}$$

where

$$n = \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\nabla\eta, 1)$$

is the unit normal vector of interface. We emphasize that generally the tangent component admits a jump at the interface, which could be a source of the instability. The last boundary condition is about the pressure $P^{\pm}$. In the presence of surface tension, one has

$$P^+ - P^- = \sigma\kappa, \quad \text{on } \Sigma(t), \tag{1.8}$$

where $\sigma > 0$ is the surface tension constant, and κ is the mean curvature :

$$\kappa(x) = -\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} \right). \tag{1.9}$$

To sum up, in this paper, we are interested in equation (1.4) on varying domain $\Omega^{\pm}(t)$ with boundary condition (1.5) or (1.6), (1.7), and (1.8). A reformulation of this system on a fixed domain will be established in Section 2. The generalized version for non-graph case is discussed in Section 5.

1.2 Virial theorem

The *virial theorem* describes the phenomenon that, in many physical systems, the (modified) kinetic and potential energy are equal in the time-averaged sense. Enunciated by Clausius [7] in 1870, it quickly became a powerful tool in physics [64, 48, 14]. The connection between virial theorem and numerous important models in classical physics has also been discovered in recent decades [25, 15, 49, 3]. In addition, the idea of this theorem could also be applied to some topics in biology [45] and economics [43].

In mathematics, virial theorem serves as an important method in the blow-up problem of PDEs. In the 1970s, Zakharov-Sobolev-Synakh [62] and Glassey [19] proved the virial theorem for nonlinear Schrödinger equation and deduced the existence of singular solution in defocusing settings. Such technique is also used by Merle [44], Sideris [50], Levine [42, 41], Keel-Tao [33], and Kenig-Merle [34]. Recently, Alazard-Zuily [3] proved the first virial theorem for water-wave equation and established the Rayleigh-Taylor instability for a large class of initial data, which will be discussed in the next part.

Additionally, the virial theorem is often expressed as *equipartition of energy* which asserts that the kinetic and potential energy are equal, asymptotically in time. This phenomenon itself has also attracted lots of attentions in the mathematics and physics community. We refer to [8] for the case of wave equation and Maxwell's equation, [18] for nonlinear wave equation, [12] for thermoelasticity, and [56] for more results on nonlinear wave equation.

Inspired by [3], we shall prove in this paper a virial theorem for the two-phase water-wave problem (1.4)-(1.8) and deduce polynomial growth for the majority of reasonable initial data. The result could be roughly stated as follows: for a sufficiently regular solution to (1.4)-(1.8), we can define a quantity $I(t)$, such that

$$\frac{d}{dt} I = \tilde{E}_k - E_p + R, \tag{1.10}$$

where $\tilde{E}_k$ is the modified kinetic energy, E_p is potential energy, and R is a non-negative reminder. The precised version will be given in Theorem 3.1.

1.3 Rayleigh-Taylor and Kelvin-Helmholtz instability

In the two-phase water-wave problem (1.4)-(1.8), if we assume the *Atwood number* A defined below,

$$A = \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-} < 0,$$

is negative, meaning that the denser fluid is placed above the other one, the system will become highly unstable, which is known as the *Rayleigh-Taylor Instability* (RTI), due to the pioneer work of Rayleigh [47] and Taylor [54]. A comprehensive physical description of this problem can be found in [35]. In the limiting case $A = 0$, or equivalently $\rho^+ = \rho^-$, the instability still exists, known as the *Kelvin-Helmholtz Instability* (KHI), which was firstly studied by Kelvin [55] and Helmholtz [26].

For 2D KHI problem, Lebeau proved in [39] that, once the interface has Hölder regularity $C^{1,\epsilon}$ with $\epsilon > 0$, the solution must be analytic in time and space. This result was also proved independently by Wu in [60], in which the interface is assumed to have only Lipschitz regularity. And for 2D RTI problem, due to [32], Hölder regularity $C^{1,\epsilon}$ of interface guarantees that the solution is smooth, or analytic in space with the additional assumption that the jump at interface is non-vanishing. Remark that the minimal regularity assumed for the interface should be Lipschitz in order to give a proper sense to the system (1.4)-(1.8). Therefore, instead of usual functional spaces like Sobolev or Hölder space, we should focus on the space of analytic functions. In addition, if the irrotational condition is eliminated, Ebin proved in [13] that both RTI and KHI problems are ill-posed (Hadamard sense) in Sobolev spaces. A similar result was also proved by Guo-Tice [21] for compressible fluids.

Note that, when vorticity and viscosity are considered, the well-posedness in Sobolev space is available, at least for small initial data, as in the case of water-wave problem [24, 22, 20]. For incompressible viscous fluids, in [58], the authors proved local well-posedness in high order Sobolev spaces for small data and their almost exponential decay. In compressible viscous setting, it is proved in [28, 29, 30] that the problem is locally well-posed for small initial data in high order Sobolev spaces and is exponentially stable near equilibrium when the surface tension is large enough.

In analytical framework, it is possible to establish local well-posedness theory for KHI and RTI. For example, in [53, 52], the authors used a Cauchy-Kowalewski theorem to prove that if the initial data could be extended to a complex strip of width $r_0 > 0$, then there exists a unique solution in the space formed by analytic functions in a strip of width $r \in]0, r_0[$, and the existence time is of size $r_0 - r$. This result has only been proved for small initial data, which is not essential in 2D torus case due to Xie [61] by pulling the system to unit disc via conformal mapping. To sum up, only analytic solutions are reasonable in RTI and KHI problem. Therefore, in this paper, we shall consider the solutions belonging to the same space of analytic functions as in papers mentioned above, which will be clarified in Appendix A.

The existing results on these nonlinear problems are mainly based on growing mode or maximal regularity [27, 23, 57, 29, 31, 46, 59]. Even if they could be applied to more general contexts (compressible, rotational, viscous, etc), these methods give merely the existence of unstable perturbation near zero. From the physical nature of this problem, one should expect the blow-up for most (regular) initial data, which could be extremely challenging due to the full nonlinearity of the system.

The virial theorem introduced in previous part provides a possible solution. As in the work of Alazard-Zuily [3], once we are able to prove the virial identity (1.10), we may deduce, for example in the absence of surface tension, from the fact $E_p = \frac{Ag}{2} \int \eta^2 dx \leqslant 0$ that

$$I(t) \geqslant I(0) + Ct, \quad \forall t \in [0, T],$$

where $I(0)$ is determined only by initial data and $C > 0$ is a function of the preserved total energy. Furthermore, any quantity controlling I, such as $\|\nabla \eta\|_{L^\infty}$, admits a growth rate from this inequality. The main advantage of using the virial theorem is that the growing rate depends on the total energy rather than the profile of initial data, allowing us to deal with a wide class of data instead of special perturbations. The rigorous statement of results and their proof will be given in Section 4. The generalization for non-graph case is left in Section 5.

1.4 Plan of the paper and conventions

In Section 2, we give an equivalent form of the system (1.4)-(1.8) defined on a fixed domain $\mathbb{T}^d$ and present the Hamiltonian structure in terms of new variables. A virial theorem for the reformulated system is stated and proved in Section 3, which yields the RTI precised in Section 4, while the KHI will also be proved in exactly the same way. In Section 5, we will consider a general case where the interface is not necessarily a graph, but a non-self-intersecting curve. Under such an assumption, an analogue of instability results in the previous section will be established. The regularity of solutions studied in this paper is clarified in Appendix A. Some identities used in the sections above are collected in Appendix B and a formal justification of Hamiltonian formulation stated in Section 2 is given in Appendix C. In Appendix D, we shall prove some properties of normal geodesic coordinate used in Section 5.

In the rest of this paper, we say that a solution $U(t)$ is regular, if

$$U \in C([0, T]; H^m), \quad U(0) \in H^m, \tag{1.11}$$

where $T > 0$ is the lifespan and H^m is the usual Sobolev space for some $m \gg 1$.

2 Reformulation and Hamiltonian structure

In this section, we will rewrite the system (1.4)-(1.8) as equations on $\mathbb{T}^d$ and give the Hamiltonian formulation together with the conservation laws. Recall that we are only interested in regular solutions, ensuring that all the calculations below make sense.

2.1 Topological assumptions and scalar potential

From the physical nature of the RTI or KHI problem, we may assume that the domain $\Omega^\pm$ of fluid has the same first homology group as $\mathbb{T}^d$. Rigorously, if $y_0^\pm \in \mathbb{R}$ is such that $\mathbb{T}^d \times \{y_0^\pm\}$ is included in $\Omega^\pm(0)$, we shall assume that the map from $H_1(\mathbb{T}^d)$ to $H_1(\Omega^\pm)$, induced by the injection $i^\pm : x \to (x, y_0^\pm)$ from $\mathbb{T}^d$ to $\Omega^\pm(0)$, is an isomorphism, where H_1 denotes the first homology group. Consequently, this assumption will hold true for all time t since $\Omega^\pm(t)$ is diffeomorphic to $\Omega^\pm(0)$ by the flow. Note that this assumption is evident when the initial interface $\Sigma(0)$ is a graph, which is the main case we are interested in.

In $2D$ and $3D$ cases ($d = 1, 2$), we clarify that, under the hypothesis (2.2) below, the velocity field $u^\pm$ admits a scalar potential $\phi^\pm$ in the sense that

$$\nabla_{x,y}\phi^\pm = u^\pm, \quad \text{in } \Omega^\pm(t).$$

Such $\phi^\pm$ exists if and only if the field $u^\pm$ is conservative, namely, for all closed smooth curves $\gamma^\pm : \mathbb{S}^1 \to \Omega^\pm$,

$$\int_{\gamma^\pm} u^\pm \cdot dL := \int_{\mathbb{S}^1} u^\pm(\gamma^\pm(\theta))\gamma^\pm_\theta(\theta)d\theta = 0. \tag{2.1}$$

If $\gamma^\pm$ is homotopic to a singleton, (2.1) follows from the irrotational condition. Therefore, it remains to check this equality for the bases of the first homology group $H_1(\Omega^\pm)$:

$$\gamma_j^\pm = (\gamma_j^{\pm,1}, ..., \gamma_j^{\pm,d}, \gamma_j^{\pm,y}) : \mathbb{S}^1 \to \Omega^\pm, \quad j = 1, ..., d$$

where $\gamma_j^{\pm,k}(\theta) = \delta_{jk}\theta$ for $k = 1, ..., d$, and $\gamma_j^{\pm,y}(\theta) = y_0^\pm \in \mathbb{R}$ such that $\gamma_j^\pm$ is a curve inside $\Omega^\pm$. Generally, (2.1) is not true for $\gamma^\pm = \gamma_j^\pm$, but this property can be preserved by the equation (1.4), which implies

$$\begin{aligned}
\frac{d}{dt}\int_{\gamma_j^\pm} u^\pm \cdot dL &= \int_{\gamma_j^\pm} u_t^\pm \cdot dL \\
&= -\int_{\gamma_j^\pm} \left(u^\pm \cdot \nabla_{x,y} u^\pm + \nabla_{x,y}\left(\frac{P^\pm}{\rho^\pm} + \rho^\pm g y\right)\right) \cdot dL \\
&= -\int_{\mathbb{S}^1} u^\pm \cdot \nabla_{x,y} u^{\pm,j} d\theta.
\end{aligned}$$

In $2D$ or $3D$ case, rotation-free condition of $u^\pm$ is equivalent to

$$\nabla_{x,y} u^{\pm,j} = \partial_{x_j} u^\pm.$$

Thus, we have

$$\frac{d}{dt}\int_{\gamma_j^\pm} u^\pm \cdot dL = -\int_{\mathbb{S}^1} u^\pm \cdot \partial_{x_j} u^\pm d\theta = -\int_{\gamma_j^\pm} \nabla_{x,y}|u^\pm|^2 \cdot dL = 0.$$

That is to say, (2.1) holds for all time t if and only if it is true for $\gamma^\pm = \gamma_j^\pm$ initially.

In this paper, we assume that the initial velocity field $u_0^\pm$ satisfies

$$\int_{\gamma_j^\pm} u_0^\pm \cdot dL = 0, \quad \forall j = 1, ..., d, \tag{2.2}$$

and then the potential $\phi^\pm$ defined below exists all the time,

$$\Delta_{x,y}\phi^\pm = 0, \quad \nabla_{x,y}\phi^\pm = u^\pm, \quad \text{in } \Omega^\pm(t); \quad \partial_y \phi^\pm = 0, \quad \text{on } \Gamma^\pm. \tag{2.3}$$

Note that the Poisson equation for $\phi^\pm$ follows from the null divergence of $u^\pm$, while the boundary condition on $\Gamma^\pm$ is no more than a restatement of (1.5).

2.2 Craig-Sulem-Zakharov formulation

In order to reformulate the system as equations on a fixed domain $\mathbb{T}^d$, instead of studying $u^{\pm}$ (or equivalently $\phi^{\pm}$), it is more convenient to consider

$$\psi^{\pm}(x) = \phi^{\pm}(x,y)|_{y=\eta\mp 0} := \lim_{y\to\eta\mp 0} \phi^{\pm}(x,y), \tag{2.4}$$

which, due to (1.4), is governed by

$$\rho^{\pm}\left(\partial_t\psi^{\pm} + g\eta + N^{\pm}\right) + P^{\pm}|_{y=\eta\mp 0} = 0, \quad \text{on } \mathbb{T}^d \tag{2.5}$$

where

$$N^{\pm} = B^{\pm}V^{\pm}\cdot\nabla\eta + \frac{|V^{\pm}|^2 - (B^{\pm})^2}{2}, \tag{2.6}$$

$$B^{\pm} = \phi_y^{\pm}|_{y=\eta\mp 0}, \tag{2.7}$$

$$V^{\pm} = \nabla\phi^{\pm}|_{y=\eta\mp 0}. \tag{2.8}$$

Recall that ∇ stands for the derivative in horizontal variable x. To derive (2.5), we initially replace $u^{\pm}$ by $\nabla_{x,y}\phi^{\pm}$ in equation (1.4), and $\Delta_{x,y}\phi^{\pm} = 0$ yields

$$\nabla_{x,y}\left(\rho^{\pm}\left(\partial_t\phi^{\pm} + \frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}\right) + P^{\pm} + \rho^{\pm}gy\right) = 0, \quad \text{in } \Omega^{\pm}(t).$$

This deduces the famous *Bernoulli's equation*,

$$\rho^{\pm}\left(\partial_t\phi^{\pm} + \frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}\right) + P^{\pm} + \rho^{\pm}gy = 0, \quad \text{in } \Omega^{\pm}(t). \tag{2.9}$$

Remark that the right hand side of (2.9) should be a constant depending only on time t. For simplicity, this constant is absorbed in the definition of $P^{\pm}$, which has no impact in all the equations concerned. In terms of B, V defined above, the restriction of equation (2.9) on interface reads

$$\rho^{\pm}\left(\partial_t\phi^{\pm}|_{y=\eta\mp 0} + g\eta + \frac{(B^{\pm})^2 + |V^{\pm}|^2}{2}\right) + P^{\pm}|_{y=\eta\mp 0} = 0,$$

which implies (2.5) since, by definition (2.14) and kinematic boundary condition (1.7),

$$\begin{aligned}
\partial_t\psi^{\pm} =& \partial_t\phi^{\pm}|_{y=\eta\mp 0} + \partial_y\phi^{\pm}|_{y=\eta\mp 0}\eta_t \\
=& \partial_t\phi^{\pm}|_{y=\eta\mp 0} + B^{\pm}(-\nabla\eta, 1)\cdot u^{\pm}|_{y=\eta\mp 0} \\
=& \partial_t\phi^{\pm}|_{y=\eta\mp 0} + B^{\pm}(-\nabla\eta, 1)\cdot(V^{\pm}, B^{\pm}) = \partial_t\phi^{\pm}|_{y=\eta\mp 0} - B^{\pm}V^{\pm}\cdot\nabla\eta + (B^{\pm})^2.
\end{aligned}$$

By combing (2.3) with (1.5) and (2.4), it is possible to recover from $\psi^{\pm}$ the potential $\phi^{\pm}$ by solving a Poisson equation :

$$\begin{cases} \Delta_{x,y}\phi^{\pm} = 0, & \text{in } \Omega^{\pm}(t), \\ \phi_y^{\pm} = 0, & \text{on } \Gamma^{\pm}, \\ \psi^{\pm} = \phi^{\pm}|_{y=\eta\mp 0}. \end{cases} \tag{2.10}$$

As a consequence, one may define the *Dirichlet-to-Neumann operator* as :

$$G^{\pm}(\eta)\psi^{\pm} := \sqrt{1+|\nabla\eta|^2}\, n\cdot\nabla_{x,y}\phi^{\pm}|_{y=\eta\mp 0}, \tag{2.11}$$

The boundary condition (1.7) on $\Sigma(t)$ then becomes

$$
\begin{cases} \partial_t \eta = G^{\pm}(\eta)\psi^{\pm}, & \text{on } \mathbb{T}^d, \\ G^{+}(\eta)\psi^{+} = G^{-}(\eta)\psi^{-} & \text{on } \mathbb{T}^d, \end{cases} \tag{2.12}
$$

Till now, we have managed to rewrite the full system as equations of $\psi^{\pm}$, η, and $P^{\pm}$ over $\mathbb{T}^d$

$$
\begin{cases} \partial_t \eta = G^{\pm}(\eta)\psi^{\pm}, \\ G^{+}(\eta)\psi^{+} = G^{-}(\eta)\psi^{-}, \\ \rho^{\pm}\left(\partial_t \psi^{\pm} + g\eta + N^{\pm}\right) + P^{\pm}|_{y=\eta\mp 0} = 0, \\ P^{+}|_{y=\eta-0} - P^{-}|_{y=\eta+0} = \sigma\kappa. \end{cases} \tag{2.13}
$$

For further simplification, we introduce the following notation : for quantities $f^{\pm}$ defined in $\Omega^{\pm}(t)$,

$$
[f] := f^{+}|_{y=\eta-0} - f^{-}|_{y=\eta+0}.
$$

We turn to study the new variable

$$
\psi := \underline{\rho}^{+}\psi^{+} - \underline{\rho}^{-}\psi^{-}, \tag{2.14}
$$

with

$$
\underline{\rho}^{\pm} := \frac{\rho^{\pm}}{\rho^{+} + \rho^{-}}. \tag{2.15}
$$

Moreover, we define the unified Dirichlet-to-Neumann operator :

$$
G(\eta) := G^{-}(\eta)\left(\underline{\rho}^{+}G^{-}(\eta) - \underline{\rho}^{-}G^{+}(\eta)\right)^{-1} G^{+}(\eta). \tag{2.16}
$$

In view of the fact that $\pm G^{\pm}(\eta)$ is linear, strictly positive and self-adjoint, the operator above is well-defined, at least for regular solutions, linear, strictly positive and self-adjoint. A detailed study of $G(\eta)$ is given by Lannes [37] in the framework of Sobolev space. By observing that

$$
G(\eta)\psi = G^{\pm}(\eta)\psi^{\pm},
$$

we can finally state the formulation to be studied in this paper :

$$
\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g'\eta + [\underline{\rho}N] + \sigma'\kappa = 0, \end{cases} \tag{2.17}
$$

where

$$
g' := [\underline{\rho}]g, \quad \sigma' = \frac{\sigma}{\rho^{+} + \rho^{-}} \tag{2.18}
$$

are *reduced gravity* and *reduced surface tension constant*, respectively, and $[\underline{\rho}]$ is exactly the *Atwood number* A.

Remark 2.1. *The idea of turning to variables $(\eta, \psi^{\pm})$ originates from the work of Zakharov [63] and Craig-Sulem [11]. The formulation (2.17) for two fluids was initially given by Benjamin-Bridges [5], where the authors also established the Hamiltonian structure w.r.t. (η, ψ), which will be presented in the following paragraph. In the presence of vorticity, Castro-Lannes [6] also derive a formal Hamiltonian formulation for the triple (η, ψ, ω), where ψ is defined by projection and ω is the vorticity.*

2.3 Hamiltonian structure and conservation laws

In [5], the authors discovered the Hamiltonian structure of (2.17):

$$\begin{cases} \partial_t \eta = \dfrac{\delta \mathcal{H}}{\delta \psi}, \\ \partial_t \psi = -\dfrac{\delta \mathcal{H}}{\delta \eta}, \end{cases} \tag{2.19}$$

where the Hamiltonian is defined as

$$\mathcal{H} = \sum_{\pm} \frac{1}{2} \iint_{\Omega^\pm} \underline{\rho}^\pm |u^\pm|^2 dxdy + \frac{g'}{2} \int \eta^2 dx + \sigma' \int \left(\sqrt{1+|\nabla \eta|^2} - 1 \right) dx. \tag{2.20}$$

One may find in [5, 36, 10] the justification of (2.19), while a formal proof is also available in the Appendix C.

By observing that the mass Hamiltonian M defined in (2.21) satisfies $\{\mathcal{H}, M\} = 0$, we have the following conservation laws:

$$M = \int \eta dx, \qquad \text{(Mass)} \tag{2.21}$$

$$E = \mathcal{H} = E_k + E_p, \qquad \text{(Energy)} \tag{2.22}$$

where

$$E_k = \sum_{\pm} \frac{1}{2} \iint_{\Omega^\pm} \underline{\rho}^\pm |u^\pm|^2 dxdy = \frac{1}{2} \int \psi G(\eta) \psi dx, \tag{2.23}$$

$$E_p = \frac{g'}{2} \int \eta^2 dx + \sigma' \int \left(\sqrt{1+|\nabla \eta|^2} - 1 \right) dx \tag{2.24}$$

are kinetic and potential energy respectively, where the second formula for kinetic energy is proved in Proposition B.2.

In the rest of this paper, one may assume without loss of generality that

$$\int \eta dx = 0. \tag{2.25}$$

Furthermore, in the infinite depth case, one may check easily that

$$\frac{d}{dt} \int \psi dx = 0.$$

In fact, from (2.17), we have

$$\frac{d}{dt} \int \psi dx = -g' \int \eta dx - \int [\underline{\rho} N] dx - \sigma' \int \kappa dx = \underline{\rho}^- \int N^- dx - \underline{\rho}^+ \int N^+ dx = 0,$$

where the last equality is a consequence of the following identities:

$$\begin{aligned} \nabla \cdot \int_{-\infty}^{\eta} \nabla \phi^+ \phi_y^+ dy =& \nabla \eta \cdot V^+ B^+ + \int_{-\infty}^{\eta} \Delta \phi^+ \phi_y^+ dy + \int_{-\infty}^{\eta} \nabla \phi^+ \cdot \nabla \phi_y^+ dy \\ =& \nabla \eta \cdot V^+ B^+ - \int_{-\infty}^{\eta} \phi_{yy}^+ \phi_y^+ dy + \int_{-\infty}^{\eta} \nabla \phi^+ \cdot \nabla \phi_y^+ dy \end{aligned}$$

$$=\nabla\eta\cdot V^+B^+ + \int_{-\infty}^{\eta}\partial_y\left(\frac{|\nabla\phi^+|^2-\left(\phi_y^+\right)^2}{2}\right)dy = N^+,$$

and similarly

$$\nabla\cdot\int_{\eta}^{+\infty}\nabla\phi^-\phi_y^- dy = -\nabla\eta\cdot V^-B^- + \int_{\eta}^{+\infty}\partial_y\left(\frac{|\nabla\phi^-|^2-\left(\phi_y^-\right)^2}{2}\right)dy = -N^-.$$

Therefore, in the infinite depth case, we shall also assume without loss of generality that

$$\int\psi dx = 0. \tag{2.26}$$

3 Virial theorem

In this section, we state and prove a *virial theorem* for the system (2.17).

Theorem 3.1. *Let* (η,ψ) *be a regular solution to* (2.17)*, then we have*

$$\frac{1}{2}\frac{d}{dt}\int\eta\psi dx = \tilde{E}_k - E_p + R, \tag{3.1}$$

where

$$\tilde{E}_k := \sum_{\pm}\iint_{\Omega^\pm}\underline{\rho}^\pm\left(\frac{1}{4}|\nabla\phi^\pm|^2+\frac{3}{4}(\phi_y^\pm)^2\right)dxdy \tag{3.2}$$

is modified kinetic energy and the reminder R *reads*

$$R = \sum_{\pm}\frac{\underline{\rho}^\pm H^\pm}{4}\int_{\Gamma^\pm}|\nabla\phi^\pm|^2dx + \sigma'\int\frac{\left(\sqrt{1+|\nabla\eta|^2}-1\right)^2}{2\sqrt{1+|\nabla\eta|^2}}dx. \tag{3.3}$$

Proof. Due to equation (2.17), one has immediately

$$\frac{d}{dt}\int\eta\psi dx = \int\eta_t\psi dx + \int\eta\psi_t dx = \int\psi G(\eta)\psi dx - g'\int\eta^2dx - \int\eta[\underline{\rho}N]dx - \sigma'\int\kappa\eta dx.$$

A direct calculus, which is precised in Proposition B.2, gives

$$\int\psi G(\eta)\psi dx = \sum_{\pm}\iint_{\Omega^\pm}\underline{\rho}^\pm|u^\pm|^2dxdy.$$

By plugging it into the calculation above together with the definition (1.9) of κ

$$\kappa(x) = -\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right),$$

one obtains

$$\frac{d}{dt}\int\eta\psi dx = \sum_{\pm}\underline{\rho}^\pm\iint_{\Omega^\pm}|u^\pm|^2dxdy - \int\eta[\underline{\rho}N]dx - g'\int\eta^2dx - \sigma'\int\kappa\eta dx$$

$$
\begin{aligned}
&= \sum_{\pm} \underline{\rho}^{\pm} \left(\iint_{\Omega^{\pm}} |u^{\pm}|^2 dxdy \mp \int \eta N^{\pm} dx \right) - g' \int \eta^2 dx - \sigma' \int \frac{|\nabla \eta|^2}{\sqrt{1+|\nabla \eta|^2}} dx \\
&= \sum_{\pm} \underline{\rho}^{\pm} \left(\iint_{\Omega^{\pm}} |u^{\pm}|^2 dxdy \mp \int \eta N^{\pm} dx \right) - 2E_p + \sigma' \int \frac{\left(\sqrt{1+|\nabla \eta|^2}-1\right)^2}{\sqrt{1+|\nabla \eta|^2}} dx,
\end{aligned}
$$

where the last equality follows from

$$
\begin{aligned}
2E_p =& g' \int \eta^2 dx + 2\sigma' \int \left(\sqrt{1+|\nabla \eta|^2}-1\right) dx \\
=& g' \int \eta^2 dx + \sigma' \int \frac{|\nabla \eta|^2}{\sqrt{1+|\nabla \eta|^2}} dx + \sigma' \int \frac{\left(\sqrt{1+|\nabla \eta|^2}-1\right)^2}{\sqrt{1+|\nabla \eta|^2}} dx.
\end{aligned}
$$

To conclude the desired identity (3.1), it suffices to apply (B.4), which makes appear the modified kinetic energy $\tilde{E}_k$,

$$
\begin{aligned}
\iint_{\Omega^{\pm}} |u^{\pm}|^2 dxdy \mp \int \eta N^{\pm} dx =& \iint_{\Omega^{\pm}} |\nabla_{x,y} \phi^{\pm}|^2 dxdy - \iint_{\Omega^{\pm}} \frac{|\nabla \phi^{\pm}|^2 - (\phi^{\pm}_y)^2}{2} dxdy \\
&+ \frac{H^{\pm}}{2} \int_{\Gamma^{\pm}} |\nabla \phi^{\pm}|^2 dx \\
=& \iint_{\Omega^{\pm}} \left(\frac{1}{2} |\nabla \phi^{\pm}|^2 + \frac{3}{2} (\phi^{\pm}_y)^2 \right) dxdy + \frac{H^{\pm}}{2} \int_{\Gamma^{\pm}} |\nabla \phi^{\pm}|^2 dx.
\end{aligned}
$$

To sum up,

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int \eta \psi dx =& \sum_{\pm} \underline{\rho}^{\pm} \left(\iint_{\Omega^{\pm}} \left(\frac{1}{4} |\nabla \phi^{\pm}|^2 + \frac{3}{4} (\phi^{\pm}_y)^2 \right) dxdy + \frac{H^{\pm}}{4} \int_{\Gamma^{\pm}} |\nabla \phi^{\pm}|^2 dx \right) - E_p \\
&+ \sigma' \int \frac{\left(\sqrt{1+|\nabla \eta|^2}-1\right)^2}{2\sqrt{1+|\nabla \eta|^2}} dx \\
=& \tilde{E}_k - E_p + \sum_{\pm} \underline{\rho}^{\pm} \frac{H^{\pm}}{4} \int_{\Gamma^{\pm}} |\nabla \phi^{\pm}|^2 dx + \sigma' \int \frac{\left(\sqrt{1+|\nabla \eta|^2}-1\right)^2}{4\sqrt{1+|\nabla \eta|^2}} dx \\
=& \tilde{E}_k - E_p + R.
\end{aligned}
$$

□

4 Instability of system

In this section, we consider the case where the heavy fluid is placed above the light one, namely $\rho^- \geqslant \rho^+$.

Theorem 4.1. *Let (η, ψ) be a regular solution to (2.17) with $\rho^- > \rho^+$ and $\sigma' \geqslant 0$. If the total energy E is strictly negative, there exists a universal constant $C > 0$, such that*

$$
2|E|t + \int \eta_0 \psi_0 dx \leqslant C \|\nabla \eta\|^2_{L^\infty} \sqrt{1 + \|\nabla \eta\|_{L^\infty}}. \tag{4.1}
$$

To prove this result, we need a trace theorem :

Proposition 4.2. *The following estimate holds true, at least for regular η, ψ :*

$$\|\psi\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim (1+\|\nabla\eta\|_{L^\infty})\sum_{\pm}\pm\int(\underline{\rho}^\pm)^2\psi^\pm G^\pm(\eta)\psi^\pm dx. \tag{4.2}$$

Proof. Due to the definition (2.14), we have

$$\begin{aligned}\|\psi\|^2_{\dot{H}^{\frac{1}{2}}} \leqslant& 2\sum_{\pm}(\underline{\rho}^\pm)^2\|\psi^\pm\|^2_{\dot{H}^{\frac{1}{2}}}\\ \lesssim&(1+\|\nabla\eta\|_{L^\infty})\sum_{\pm}\pm\int(\underline{\rho}^\pm)^2\psi^\pm G^\pm(\eta)\psi^\pm dx,\end{aligned}$$

where the second inequality is the classical trace estimate for Dirichlet-to-Neumann operator:

$$\|\psi^\pm\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim (1+\|\nabla\eta\|_{L^\infty})\int\psi^\pm\left(\pm G^\pm(\eta)\right)\psi^\pm dx \tag{4.3}$$

A prove of this inequality could be found in Proposition B.3 of [3], where the dependence on depth is specified. □

Remark 4.3. *A refined version of* (4.3) *reads*

$$\|\psi^\pm\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim (1+\|\nabla\eta\|_{BMO})\int\psi^\pm\left(\pm G^\pm(\eta)\right)\psi^\pm dx, \tag{4.4}$$

whose proof can be found in [1]. As a result, in the sequel, it is possible to replace the $\|\nabla\eta\|_{L^\infty}$ by $\|\nabla\eta\|_{BMO}$. However, since it gives no further information, we shall only focus on L^∞ norm for simplicity. We also refer to Section A.3 of [38] and [40] for a systematical study of the trace inequality.

Proof of Theorem 4.1. Since the total energy $E<0$, one may deduce from (3.1) that

$$\frac{1}{2}\frac{d}{dt}\int\eta\psi dx \geqslant \tilde{E}_k - E_p = -E+\tilde{E}_k+E_k \geqslant |E|. \tag{4.5}$$

As a consequence,

$$\int\eta\psi dx \geqslant 2|E|t+\int\eta_0\psi_0 dx.$$

On the other hand, the left hand side of the inequality above can be controlled via Proposition 4.2,

$$\begin{aligned}\left|\int\eta\psi dx\right| \leqslant&\|\eta\|_{\dot{H}^{-\frac{1}{2}}}\|\psi\|_{\dot{H}^{\frac{1}{2}}}\\ \lesssim&\|\eta\|_{L^\infty}\sqrt{(1+\|\nabla\eta\|_{L^\infty})\sum_{\pm}\pm\int(\underline{\rho}^\pm)^2\psi^\pm G^\pm(\eta)\psi^\pm dx}\\ \leqslant&\|\eta\|_{L^\infty}\sqrt{(1+\|\nabla\eta\|_{L^\infty})\sum_{\pm}\pm\int\underline{\rho}^\pm\psi^\pm G^\pm(\eta)\psi^\pm dx}\end{aligned}$$

$$=\|\eta\|_{L^\infty}\sqrt{(1+\|\nabla\eta\|_{L^\infty})E_k}.$$

The assumption $\rho^- > \rho^+$ ensures that $g' < 0$, which implies

$$E_k = -|E| - E_p \leqslant -|E| - \frac{g'}{2}\|\eta\|_{L^2}^2 \lesssim -|E| + \|\eta\|_{L^\infty}^2 \leqslant \|\eta\|_{L^\infty}^2.$$

Moreover, the zero mass assumption (2.25) yields

$$\|\eta\|_{L^\infty} \lesssim \|\nabla\eta\|_{L^\infty}.$$

By combining all the inequalities above, one may conclude that

$$2|E|t + \int \eta_0\psi_0 dx \lesssim \|\nabla\eta\|_{L^\infty}^2\sqrt{1+\|\nabla\eta\|_{L^\infty}}.$$

□

Remark 4.4. *The above result gives a growing rate in large time (if the solution exists)*

$$\|\nabla\eta(t)\|_{L^\infty} \geqslant Ct^{\frac{2}{5}}.$$

An induction argument implies that, in the absence of surface tension, $\|\nabla\eta\|_{L^\infty}$ cannot be controlled by $t^{2-\epsilon}$ for all $0 < \epsilon \ll 1$:

$$\limsup_{t>0} \frac{\|\nabla\eta(t)\|_{L^\infty}}{t^{2-\epsilon}} = +\infty.$$

Here we assume that the solution (η, ψ) is regular and global (we do not know whether such solution exists). Assume that the following uniform control holds true

$$\|\nabla\eta(t)\|_{L^\infty} \leqslant C_0 t^{2-\epsilon}.$$

Then a modification of (4.5) gives

$$\frac{1}{2}\frac{d}{dt}\int \eta\psi dx \geqslant \tilde{E}_k - E_p = -\frac{1}{2}E + \tilde{E}_k + \frac{1}{2}E_k - \frac{1}{2}E_p \geqslant \frac{1}{2}|E| - \frac{1}{2}E_p.$$

By repeating the trace estimate, one has

$$|E|t - \int_0^t E_p(\tau)d\tau + \int \eta_0\psi_0 dx \lesssim \sqrt{1+\|\nabla\eta(t)\|_{L^\infty}}\|\eta(t)\|_{L^2}^2.$$

Since the potential energy is composed only by the gravity part, the inequality above can be written as

$$|E|t + |g'|\int_0^t \|\eta(\tau)\|_{L^2}^2 d\tau + \int \eta_0\psi_0 dx \lesssim \sqrt{1+\|\nabla\eta(t)\|_{L^\infty}}\|\eta(t)\|_{L^2}^2 \lesssim t^{1-\frac{\epsilon}{2}}\|\eta(t)\|_{L^2}^2. \tag{4.6}$$

Trivially, this estimate gives $t^{\frac{\epsilon}{2}} \lesssim \|\eta(t)\|_{L^2}^2$ for large time. Provided that $\|\eta(t)\|_{L^2}^2$ grows at speed $t^{\frac{N\epsilon}{2}}$ $(N > 0)$, one may deduce from (4.6) that

$$t^{1-\frac{\epsilon}{2}}\|\eta(t)\|_{L^2}^2 \geqslant C\left(\int_0^t \tau^{N\epsilon}d\tau + \int \eta_0\psi_0 dx\right) \geqslant C' t^{\frac{N\epsilon}{2}+1},$$

which implies a faster increase $t^{\frac{(N+1)\epsilon}{2}} \lesssim \|\eta(t)\|_{L^2}^2$. Thus by induction $\|\eta(t)\|_{L^2}^2$ admits any polynomial growth, while the contradiction arises from the estimate :

$$\|\eta(t)\|_{L^2}^2 \lesssim \|\nabla\eta(t)\|_{L^\infty}^2 \lesssim t^{4-2\epsilon}.$$

Remark 4.5. *In the absence of surface tension, we may prove the same result* (4.1) *with* $2|E|$ *replaced by* $|E|$ *for data of strictly positive energy* $E > 0$*. In fact, when* $\sigma = 0$*, the inequality* (4.5) *can be obtained by*

$$\frac{1}{2}\frac{d}{dt}\int \eta\psi dx \geqslant \tilde{E}_k - E_p \geqslant \frac{1}{2}E_k - E_p = \frac{1}{2}|E| - \frac{3}{2}E_p \geqslant \frac{1}{2}|E|.$$

And all the proof remains unchanged. The estimate above is not possible when $\sigma > 0$ *since the potential energy is not necessary negative.*

Remark 4.6. *In the case of* $\rho^+ = \rho^-, orequivalently\ g' = 0$*, we may obtain the KHI under the hypothesis of zero surface tension* $\sigma = 0$*. As in previous remark, it suffices to establish a similar estimate as* (4.5)*. In this case, we have automatically* $E_p = 0$*, which yields*

$$\frac{1}{2}\frac{d}{dt}\int \eta\psi dx = \tilde{E}_k + R \geqslant \frac{1}{2}E_k = \frac{1}{2}E,$$

where, by definition, total energy E *should always be strictly positive unless the solution is trivial.*

5 Non-graph case

In this section, we shall focus on the 2D case $\mathbb{T} \times \mathbb{R}$ and derive the virial identity (3.1) together with an analogue of Theorem 4.1 without assuming that the interface $\Sigma(t)$ is a general curve represented as a graph $\{y = \eta(x)\}$. Instead, $\Sigma(t)$ is parameterized by arc-length parameter s, namely

$$\Sigma(t) = \{\gamma(t,s) = (\alpha(t,s), \beta(t,s)) \in \mathbb{T} \times \mathbb{R} : t \in \mathbb{R},\ \ s \in \mathbb{R}/L(t)\}, \tag{5.1}$$

where $|\gamma_s| \equiv 1$ and $L(t)$ is the length of interface at time t. Remark that, the length L should be at least 2π to separate two fluid domains $\Omega^{\pm}$. Here, one should assume that there exists a uniform-in-time parameterization $\tilde{\gamma}(t,\tilde{s})$ with $t \in \mathbb{R}$, $\tilde{s} \in \mathbb{T} = \mathbb{R}/2\pi$ and define the arc-length parameter s via

$$s = s(t,\tilde{s}) = \int_0^{\tilde{s}} |\tilde{\gamma}_{\tilde{s}}(t,r)|dr, \quad \text{or equivalently } \frac{d\tilde{s}}{ds} = |\tilde{\gamma}_{\tilde{s}}(t,\tilde{s})|.$$

By construction, $s(t,\tilde{s}+2\pi) = s(t,\tilde{s}) + L(t)$, which induces a mapping, still noted by $s(t,\cdot)$, from $\mathbb{R}/2\pi$ to $\mathbb{R}/L(t)$. In the same way, $\tilde{s}(t,\cdot)$ should be understood as a mapping $\mathbb{R}/L(t) \to \mathbb{R}/2\pi$. Note that $\tilde{s}$ is time-independent while s is not, and this fact may lead to some technical difficulties in the calculation below.

We also assume that the interface is non-self-intersecting. Thus, it is natural to add the *chord-arc* condition:

$$c_0|s-s'| \leqslant |\gamma(t,s) - \gamma(t,s')| \leqslant |s-s'|, \quad \forall s,s' \in \mathbb{R}/L(t). \tag{5.2}$$

Here $|s-s'|$ should be understood as the distance in $\mathbb{R}/L(t)$ and $c_0 > 0$ is a universal constant.

In this setting, the normal direction reads $n = (-\beta_s, \alpha_s)$, and the curvature κ is given by

$$\tau_s(s) = -\kappa(s)n(s), \tag{5.3}$$

where $\tau := \gamma_s = (\alpha_s, \beta_s)$ is the unit tangent direction. The kinematic boundary condition (1.7) should be rewritten as

$$\tilde{\gamma}_t \cdot n = u^+ \cdot n = u^- \cdot n, \quad \text{on } \Sigma(t). \tag{5.4}$$

One observes that $\gamma_t(t)$, which is not periodic in s, equals to $\tilde{\gamma}_t$ in normal direction,

$$\partial_t \gamma = \partial_t \left(\tilde{\gamma}(t, \tilde{s}(t,s))\right) = \tilde{\gamma}_t + \tilde{\gamma}_{\tilde{s}} \partial_t \tilde{s}(t,s).$$

Thus, the kinematic boundary condition (5.4) is equivalent to

$$\gamma_t \cdot n = u^+ \cdot n = u^- \cdot n, \quad \text{on } \Sigma(t). \tag{5.5}$$

To sum up, the system of two phase water-wave governed by (2.9) and (1.5) - (1.8) can be generalized as

$$\begin{cases} \rho^\pm \left(\partial_t \phi^\pm + \frac{1}{2}|\nabla_{x,y}\phi^\pm|^2\right) + P^\pm + \rho^\pm g y = 0, & \text{in } \Omega^\pm(t), \\ \gamma_t \cdot n = u^+ \cdot n = u^- \cdot n, & \text{on } \Sigma(t), \\ [P] = \sigma\kappa, & \text{on } \Sigma(t), \\ n_b^\pm \cdot \nabla_{x,y}\phi^\pm = 0, & \text{on } \Gamma^\pm. \end{cases} \tag{5.6}$$

Recall that $n_b^\pm = (0, \mp 1)$ is the unit outward normal vector to the bottom $\Gamma^\pm$. A detailed study of the formulation of water-wave equation for non-graph interface can be found in [9], where general parameterization is considered. Note that the graph case can be recovered by $\tilde{\gamma}(t,\tilde{s}) = (\tilde{s}, \eta(t,\tilde{s}))$ with parameter $\tilde{s} = x$. In the same paper [9], the author also established a Hamiltonian formulation, which is not the same symplectic form as in (2.19). Nevertheless, we are still able to prove a virial identity as (3.1). To begin with, we check the conservation laws for mass and energy.

5.1 Conservation laws

In non-graph case, the kinetic energy is defined as in (2.23),

$$E_k := \frac{1}{2} \sum_\pm \iint_{\Omega^\pm} \underline{\rho}^\pm \left((\phi_x^\pm)^2 + (\phi_y^\pm)^2\right) dxdy. \tag{5.7}$$

To generalize the potential energy, we observe that, in graph case $\Sigma = \{y = \eta\}$, the potential energy defined by (2.24) can be written, by change of variable into arc-length coordinate $(x,y) = \gamma(s) = (\alpha(s), \beta(s))$, as

$$\begin{aligned} E_p =& \frac{g'}{2} \int_0^{2\pi} \frac{y^2}{2} dx + \sigma' \int_0^{2\pi} \left(\sqrt{1 + |\nabla\eta|^2} - 1\right) dx \\ =& \frac{g'}{2} \int_0^L \frac{\beta^2}{2} \alpha_s ds + \sigma'(L - 2\pi), \end{aligned}$$

which is also well-defined for non-graph case. In the sequel, we will define the potential energy as

$$E_p := \frac{g'}{2} \int \beta^2 \alpha_s ds + \sigma' L, \tag{5.8}$$

where the constant $-2\pi\sigma'$ is omitted for simplicity since it has no impact in the conservation law.

Before checking the conversation laws, we begin with a technical formula to be used frequently in this section.

Lemma 5.1. *Let $f = f(t, x, y)$ be a smooth function defined in $\Omega^\pm(t)$, then*

$$\frac{d}{dt}\iint_{\Omega^\pm(t)} f dxdy = \iint_{\Omega^\pm(t)} \partial_t f dxdy \pm \int_{\Sigma(t)} n\cdot\gamma_t f(t,\gamma(t,s))ds. \tag{5.9}$$

Proof. Consider the time-space fluid domain $\tilde{\Omega}^\pm = \{(t,x,y) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R} : (x,y) \in \Omega^\pm(t)\}$ and time-space free boundary $\tilde{\Sigma}^\pm = \{(t,x,y) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R} : (x,y) \in \Sigma(t)\}$, which can be parameterized as $(t,s) \mapsto (t,\gamma(t,s))$. Therefore, the tangent space of $\tilde{\Sigma}$ is spanned by $(1,\gamma_t)$ and $(0,\tau)$, which induce the formula for unit normal vector and surface element

$$\tilde{n} = \frac{(-n\cdot\gamma_t, n)}{\sqrt{1+(n\cdot\gamma_t)^2}}, \quad d\tilde{S} = \sqrt{1+(n\cdot\gamma_t)^2}dsdt.$$

For any test function $\varphi = \varphi(t) \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}}\left(\frac{d}{dt}\iint_{\Omega^\pm(t)} f dxdy\right)\varphi dt &= -\iiint_{\tilde{\Omega}^\pm} f\varphi' dtdxdy\\
&= \iiint_{\tilde{\Omega}^\pm} \partial_t f\varphi dtdxdy - \iiint_{\tilde{\Omega}^\pm} \partial_t(f\varphi)dtdxdy\\
&= \int_{\mathbb{R}}\iint_{\Omega^\pm(t)} \partial_t f dxdy\varphi dt - \iiint_{\tilde{\Omega}^\pm} \nabla_{t,x,y}\cdot(f\varphi,0,0)dtdxdy\\
&= \int_{\mathbb{R}}\iint_{\Omega^\pm(t)} \partial_t f dxdy\varphi dt \mp \iint_{\tilde{\Sigma}} \tilde{n}\cdot(f\varphi,0,0)d\tilde{S}\\
&= \int_{\mathbb{R}}\iint_{\Omega^\pm(t)} \partial_t f dxdy\varphi dt \pm \int_{\mathbb{R}}\int_{\Sigma(t)} n\cdot\gamma_t f ds\varphi dt\\
&= \int_{\mathbb{R}}\left(\iint_{\Omega^\pm(t)} \partial_t f dxdy \pm \int_{\Sigma(t)} n\cdot\gamma_t f ds\right)\varphi dt,
\end{aligned}$$

which gives (5.9). ◻

Now, we are in a position to prove the conservation laws.

Proposition 5.2. *Let $(\gamma, \phi^\pm)$ be a regular solution to system (5.6). Then the following conservation laws hold true :*

$$\frac{d}{dt}\iint_{\Omega^\pm(t)} dxdy = 0, \qquad \textit{(Mass)} \tag{5.10}$$

$$\frac{d}{dt}E = 0, \qquad \textit{(Energy)} \tag{5.11}$$

where the total energy E is defined by

$$E := E_k + E_p.$$

Remark 5.3. *The total mass defined by $\iint_{\Omega^\pm(t)} dxdy$ equals to infinity in the infinite depth case ($H^\pm = +\infty$). In order to give a proper meaning of the mass, one may replace this quantity by*

$$\iint_{\Omega^\pm_H(t)} dxdy,$$

where $\Omega^+_H(t) := \Omega^+(t) \cap \{y > -H\}$ and $\Omega^-_H(t) := \Omega^-(t) \cap \{y < H\}$, with $H \gg 1$ such that the interface $\Sigma(t)$ does not intersect with the surfaces $\{y = \pm H\}$.

Proof. We only consider the finite depth case $H^{\pm} < +\infty$, while the infinite depth case can be proved by studying $\Omega_H^{\pm}$ defined in previous remark and passing to the limit $H \to +\infty$.

The conservation of mass (5.10) is no more than an application of formula (5.9) with $f = 1$, namely, using (5.5),

$$\begin{aligned}\frac{d}{dt}\iint_{\Omega^{\pm}(t)} dxdy = \pm\int_{\Sigma(t)} n\cdot\gamma_t ds = \pm\int_{\Sigma(t)} n\cdot u^{\pm} ds \\ = \iint_{\Omega^{\pm}} \nabla_{x,y}\cdot u^{\pm} dxdy + \int_{\Gamma^{\pm}} n_b^{\pm}\cdot u^{\pm} dx = 0,\end{aligned}$$

where $n_b^{\pm} = (0, \mp 1)$ is the unit normal vector to the bottom $\Gamma^{\pm}$.

To prove the conservation of energy (5.11), we shall first rewrite the potential energy E_p defined by (5.8) in terms of integrals in $\Omega^{\pm}$ and then apply the formula (5.9) to derive the desired identity.

$$\begin{aligned}E_p =& g'\int_{\Sigma} n\cdot(0,\frac{y^2}{2})ds + \sigma'\int_0^{2\pi}|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} \\ =& \sum_{\pm}\underline{\rho}^{\pm} g\int_{\Sigma}(\pm n)\cdot(0,\frac{y^2}{2})ds + \sigma'\int_0^{2\pi}|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} \\ =& \sum_{\pm}\underline{\rho}^{\pm} g\left(\iint_{\Omega^{\pm}}\nabla_{x,y}\cdot(0,\frac{y^2}{2})dxdy - \int_{\Gamma^{\pm}} n_b^{\pm}\cdot(0,\frac{y^2}{2})dx\right) + \sigma'\int_0^{2\pi}|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} \\ =& \sum_{\pm}\iint_{\Omega^{\pm}}\underline{\rho}^{\pm} gy dxdy + \sigma'\int_0^{2\pi}|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} + C,\end{aligned} \tag{5.12}$$

where $C = \underline{\rho}^{+} g\pi(H^{+})^2 - \underline{\rho}^{-} g\pi(H^{-})^2$ is independent of time and thus has no impact in the calculation of time derivative. With this re-expression of potential energy, one may write the total energy as

$$E = E_k + E_p = \sum_{\pm}\iint_{\Omega^{\pm}(t)}\underline{\rho}^{\pm}\left(\frac{|u^{\pm}|^2}{2} + gy\right)dxdy + \sigma'\int_0^{2\pi}|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} + C.$$

By formula (5.9) and (1.4), its derivative in time reads

$$\begin{aligned}\frac{d}{dt}E =& \sum_{\pm}\left[\iint_{\Omega^{\pm}}\underline{\rho}^{\pm}\partial_t\left(\frac{|u^{\pm}|^2}{2} + gy\right)dxdy \pm \underline{\rho}^{\pm}\int_{\Sigma} n\cdot\gamma_t\left(\frac{|u^{\pm}|^2}{2} + gy\right)ds\right] + \sigma'\int_0^{2\pi}\partial_t|\tilde{\gamma}_{\tilde{s}}|d\tilde{s} \\ =& \sum_{\pm}\left[\underline{\rho}^{\pm}\iint_{\Omega^{\pm}} u^{\pm}\cdot u_t^{\pm} dxdy + \underline{\rho}^{\pm}\int_{\Sigma}(\pm n)\cdot\left(u^{\pm}\left(\frac{|u^{\pm}|^2}{2} + gy\right)\right)ds\right] + \sigma'\int_0^{2\pi}\frac{\tilde{\gamma}_{\tilde{s}}\cdot\partial_t\tilde{\gamma}_{\tilde{s}}}{|\tilde{\gamma}_{\tilde{s}}|}d\tilde{s} \\ =& -\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}} u^{\pm}\cdot\left(u^{\pm}\cdot\nabla_{x,y}u^{\pm} + \frac{\nabla_{x,y}P^{\pm}}{\rho^{\pm}} + g\nabla_{x,y}y\right)dxdy \\ &+ \sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}\nabla_{x,y}\cdot\left(u^{\pm}\left(\frac{|u^{\pm}|^2}{2} + gy\right)\right)dxdy - \sigma'\int_0^{2\pi}\left(\frac{\tilde{\gamma}_{\tilde{s}}}{|\tilde{\gamma}_{\tilde{s}}|}\right)_{\tilde{s}}\cdot\partial_t\tilde{\gamma}d\tilde{s} \\ =& -\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}} u^{\pm}\cdot\frac{\nabla_{x,y}P^{\pm}}{\rho^{\pm}}dxdy - \sigma'\int_0^{2\pi}\left(\frac{\tilde{\gamma}_{\tilde{s}}}{|\tilde{\gamma}_{\tilde{s}}|}\right)_{\tilde{s}}\cdot\partial_t\tilde{\gamma}d\tilde{s} \\ =& -\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\nabla_{x,y}\cdot(u^{\pm}P^{\pm})dxdy - \sigma'\int_0^{2\pi}\tau_{\tilde{s}}\cdot\partial_t\tilde{\gamma}d\tilde{s}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\rho^+ + \rho^-}\sum_{\pm}\int_\Sigma (\pm n)\cdot u^\pm P^\pm ds - \sigma'\int_0^{2\pi} \tau_s \cdot \partial_t\tilde{\gamma}\frac{d\tilde{s}}{|\tilde{\gamma}_{\tilde{s}}|} \\
&= -\frac{1}{\rho^+ + \rho^-}\int_\Sigma n\cdot \gamma_t[P]ds - \sigma'\int_0^{2\pi} \tau_s \cdot \partial_t\tilde{\gamma}\frac{d\tilde{s}}{|\tilde{\gamma}_{\tilde{s}}|}.
\end{aligned}
$$

Since $[P] = \sigma\kappa$, it remains to check that

$$
-\int_\Sigma n\cdot \gamma_t \kappa ds = \int_0^{2\pi} \tau_s \cdot \partial_t\tilde{\gamma}\frac{d\tilde{s}}{|\tilde{\gamma}_{\tilde{s}}|}.
$$

In fact, by change of variable, the right hand side equals to

$$
\int_0^{L(t)} \tau_s \cdot \partial_t\tilde{\gamma} ds = -\int_0^{L(t)} \kappa n\cdot \partial_t\tilde{\gamma} ds = -\int_\Sigma n\cdot\gamma_t\kappa ds,
$$

which completes the proof of (5.11). □

Before entering the next part, let us introduce the following convention

$$
\int_0^{L(t)} \beta\alpha_s ds = 0, \quad \forall t\in\mathbb{R}. \tag{5.13}
$$

Actually, this quantity, by divergence theorem, equals to

$$
\int_\Sigma n\cdot(0,y)ds = \pm\iint_{\Omega^\pm}\nabla_{x,y}\cdot(0,y)dxdy - \int_{\Gamma^\pm} n_b^\pm\cdot(0,y)dx = \pm\left(\iint_{\Omega^\pm} dxdy - 2\pi H^\pm\right),
$$

which is independent of time due to conservation of mass (5.10). Meanwhile, it is clear that

$$
\sum_\pm\left(\iint_{\Omega^\pm} dxdy - 2\pi H^\pm\right) = 0.
$$

These allow us to make the convention (5.13) by plugging a translation in vertical direction into the system (5.6).

5.2 Virial identity

In order to generalize the virial identity (3.1), one observes that, in the graph case, the quantities involved can be written in a more general form so that the graph assumption on interface can be eliminated. More precisely, the modified kinetic energy and potential energy can be defined as (3.2) and (5.8), and

$$
\begin{aligned}
\int \eta\psi dx = \sum_\pm \pm\underline{\rho}^\pm\int\eta\psi^\pm dx &= \sum_\pm \pm\underline{\rho}^\pm\int_{y=\eta} y\phi^\pm dx \\
&= \sum_\pm \underline{\rho}^\pm\left(\iint_{\Omega^\pm}\partial_y(y\phi^\pm)dxdy \pm \int_{\Gamma^\pm} y\phi^\pm dx\right) \\
&= \sum_\pm \underline{\rho}^\pm\left(\iint_{\Omega^\pm}\partial_y(y\phi^\pm)dxdy - H^\pm\int_{\Gamma^\pm} \phi^\pm dx\right).
\end{aligned}
$$

This inspires us to study the quantity

$$\sum_{\pm} \underline{\rho}^{\pm} \left(\iint_{\Omega^{\pm}} \partial_y (y\phi^{\pm}) dxdy - H^{\pm} \int_{\Gamma^{\pm}} \phi^{\pm} dx \right), \tag{5.14}$$

which can be expressed in the non-graph case as
(5.15)

$$\begin{aligned}
&\sum_{\pm} \underline{\rho}^{\pm} \left(\iint_{\Omega^{\pm}} \nabla_{x,y} \cdot (0, y\phi^{\pm}) dxdy - H^{\pm} \int_{\Gamma^{\pm}} \phi^{\pm} dx \right) \\
&= \sum_{\pm} \underline{\rho}^{\pm} \left(\int_{\Sigma} (\pm n) \cdot (0, y\phi^{\pm}) ds + \int_{\Gamma^{\pm}} n_b^{\pm} \cdot (0, y\phi^{\pm}) dx - H^{\pm} \int_{\Gamma^{\pm}} \phi^{\pm} dx \right) \\
&= \sum_{\pm} \underline{\rho}^{\pm} \int_{\Sigma} (\pm n) \cdot (0, y\phi^{\pm}) ds = \sum_{\pm} \pm \underline{\rho}^{\pm} \int_0^{L(t)} \alpha_s \beta \phi^{\pm}(t, \gamma(t,s)) ds = \int_0^{L(t)} \alpha_s \beta \psi ds,
\end{aligned}$$

where $\psi = \psi(t,s)$ is defined as in (2.14),

$$\psi(t,s) = \sum_{\pm} \pm \underline{\rho}^{\pm} \phi^{\pm}(t, \gamma(t,s)). \tag{5.16}$$

Theorem 5.4. *Let $(\gamma, \phi^{\pm})$ be a regular solution to (5.6). Then we have*

$$\frac{1}{2} \frac{d}{dt} \int_0^{L(t)} \alpha_s \beta \psi ds = \tilde{E}_k - E_p + R, \tag{5.17}$$

where $\tilde{E}_k$ is the modified kinetic energy defined by (3.3),

$$\tilde{E}_k = \sum_{\pm} \iint_{\Omega^{\pm}} \underline{\rho}^{\pm} \left(\frac{1}{4} |\nabla \phi^{\pm}|^2 + \frac{3}{4} (\phi_y^{\pm})^2 \right) dxdy,$$

and the remainder R equals to

$$R = \sum_{\pm} \frac{\underline{\rho}^{\pm} H^{\pm}}{4} \int_{\Gamma^{\pm}} |\nabla_{x,y} \phi^{\pm}|^2 dx + \sigma' \int_0^L (|\alpha_s|^2 + \frac{1}{2} |\beta_s|^2) ds > 0. \tag{5.18}$$

Proof. As before, we only prove the finite depth case, while the other cases can be treated by passing to the limit $H^{\pm} \to +\infty$.

By (5.15), it is enough to study instead the time derivative of (5.14). For the second part of (5.14), we may use Bernoulli's equation (2.9) :

$$\begin{aligned}
&\frac{d}{dt} \sum_{\pm} \underline{\rho}^{\pm} H^{\pm} \int_{\Gamma^{\pm}} \phi^{\pm} dx \\
&= -\sum_{\pm} \underline{\rho}^{\pm} H^{\pm} \int_{\Gamma^{\pm}} \left(\frac{|\nabla_{x,y} \phi^{\pm}|^2}{2} + \frac{P^{\pm}}{\rho^{\pm}} + gy \right) dx \\
&= -\sum_{\pm} \underline{\rho}^{\pm} H^{\pm} \int_{\Gamma^{\pm}} \frac{|\nabla_{x,y} \phi^{\pm}|^2}{2} dx - \sum_{\pm} \underline{\rho}^{\pm} \int_{\Gamma^{\pm}} n_b^{\pm} \cdot (0, \frac{yP^{\pm}}{\rho^{\pm}}) dx + \sum_{\pm} \pm 2\pi g \underline{\rho}^{\pm} (H^{\pm})^2 \\
&= -\sum_{\pm} \underline{\rho}^{\pm} H^{\pm} \int_{\Gamma^{\pm}} \frac{|\nabla_{x,y} \phi^{\pm}|^2}{2} dx - \sum_{\pm} \underline{\rho}^{\pm} \iint_{\Omega^{\pm}} \nabla_{x,y} \cdot (0, \frac{yP^{\pm}}{\rho^{\pm}}) dxdy
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\pm}\pm\underline{\rho}^{\pm}\int_{\Sigma}n\cdot(0,\frac{yP^{\pm}}{\rho^{\pm}})ds+\sum_{\pm}\pm 2\pi g\underline{\rho}^{\pm}(H^{\pm})^2\\
=&-\sum_{\pm}\underline{\rho}^{\pm}H^{\pm}\int_{\Gamma^{\pm}}\frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}dx-\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\partial_y(yP^{\pm})dxdy+\sigma'\int_{\Sigma}\alpha_s\beta\kappa ds\\
&+\sum_{\pm}\pm 2\pi g\underline{\rho}^{\pm}(H^{\pm})^2,
\end{aligned}
$$

where the last equality follows from boundary condition (1.8). Besides, we have

$$
\int_{\Sigma}\alpha_s\beta\kappa ds=-\int_{\Sigma}n\cdot(0,\beta)\tau_s\cdot nds=-\int_{\Sigma}(0,\beta)\cdot\tau_s ds=-\int_0^{L(t)}\beta\beta_{ss}ds=\int_0^{L(t)}\beta_s^2 ds.
$$

The first part of (5.14) can be calculated via formula (5.9),

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}\partial_y(y\phi^{\pm})dxdy\\
=&\sum_{\pm}\underline{\rho}^{\pm}\left(\iint_{\Omega^{\pm}}\partial_y(y\phi_t^{\pm})dxdy\pm\int_{\Sigma}n\cdot\gamma_t\partial_y(y\phi^{\pm})ds\right)\\
=&\sum_{\pm}\underline{\rho}^{\pm}\left[-\iint_{\Omega^{\pm}}\partial_y\left(y\frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}\right)dxdy-\iint_{\Omega^{\pm}}\partial_y\left(gy^2\right)dxdy\pm\int_{\Sigma}n\cdot u^{\pm}\partial_y(y\phi^{\pm})ds\right]\\
&-\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\partial_y(yP^{\pm})dxdy\\
=&\sum_{\pm}\underline{\rho}^{\pm}\Big[-\iint_{\Omega^{\pm}}\partial_y\left(y\frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}\right)dxdy+\iint_{\Omega^{\pm}}\nabla_{x,y}\cdot\left(u^{\pm}\partial_y(y\phi^{\pm})\right)ds\\
&-\int_{\Gamma^{\pm}}n_b^{\pm}\cdot u^{\pm}\partial_y(y\phi^{\pm})ds\Big]-\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\partial_y(yP^{\pm})dxdy-2\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}gydxdy\\
=&\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}\left(\frac{1}{2}(\phi_x^{\pm})^2+\frac{3}{2}(\phi_y^{\pm})^2\right)dxdy-\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\partial_y(yP^{\pm})dxdy\\
&-2\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}gydxdy,
\end{aligned}
$$

where we use the boundary condition (1.5) and the identity

$$
-\partial_y\left(y\frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}\right)+\nabla_{x,y}\cdot\left(u^{\pm}\partial_y(y\phi^{\pm})\right)=\frac{1}{2}(\phi_x^{\pm})^2+\frac{3}{2}(\phi_y^{\pm})^2,
$$

which can be computed directly. Due to the reformulation (5.12) of potential energy E_p, one may deduce

$$
\begin{aligned}
\frac{d}{dt}\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}\partial_y(y\phi^{\pm})dxdy=&2\tilde{E}_k-\frac{1}{\rho^{+}+\rho^{-}}\sum_{\pm}\iint_{\Omega^{\pm}}\partial_y(yP^{\pm})dxdy-2E_p+2\sigma' L\\
&+\sum_{\pm}\pm\underline{\rho}^{\pm}2g\pi(H^{\pm})^2.
\end{aligned}
$$

By combining all the identities above, we conclude that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^{L(t)}\alpha_s\beta\psi ds =&\frac{d}{dt}\sum_{\pm}\underline{\rho}^{\pm}\iint_{\Omega^{\pm}}\partial_y(y\phi^{\pm})dxdy-\frac{d}{dt}\sum_{\pm}\underline{\rho}^{\pm}H^{\pm}\int_{\Gamma^{\pm}}\phi^{\pm}dx\\
=&2\tilde{E}_k-2E_p+2\sigma' L+\sum_{\pm}\underline{\rho}^{\pm}H^{\pm}\int_{\Gamma^{\pm}}\frac{|\nabla_{x,y}\phi^{\pm}|^2}{2}dx-\sigma'\int_0^L\beta_s^2ds\\
=&2\tilde{E}_k-2E_p+\sum_{\pm}\frac{\underline{\rho}^{\pm}H^{\pm}}{2}\int_{\Gamma^{\pm}}|\nabla_{x,y}\phi^{\pm}|^2dx+\sigma'\int_0^L(2|\alpha_s|^2+|\beta_s|^2)ds\\
=&2(\tilde{E}_k-E_p+R).
\end{aligned}
$$

□

5.3 Trace estimate

As in Section 4, we shall apply some trace estimates to obtain an upper bound of

$$\int_0^L\alpha_s\beta\psi ds,$$

which leads to instability results. By definition (5.16), we may split ψ into $\psi^{\pm}:=\phi^{\pm}(t,\gamma(t,s))$ and study each trace separately. Thus, in this part, we shall return to one-phase case, i.e. $\rho^-=0$ and $\rho^+=1$. Since trace estimates are stationary properties, the dependence in time t and the superscript $\pm$ will be omitted for simplicity.

Theorem 5.5. *Let $\epsilon>0$ be defined as*

$$\epsilon:=\min\{\frac{c_0}{N_0\|\kappa\|_{L^\infty}},\ d(\Sigma,\Gamma^{\pm}),\frac{1}{N_0}\},\tag{5.19}$$

where c_0 is the chord-arc constant defined in (5.2) and $N_0\gg 1$ is a universal constant. Then for all $F\in H^1(\Omega)$ whose trace $f=F|_{\Sigma}$ has zero mean

$$\int_{\Sigma}fds=0,$$

we have

$$\|f\|_{L^2(\Sigma)}\lesssim\frac{L}{\sqrt{\epsilon}}\|\nabla_{x,y}F\|_{L^2(\Omega)},\tag{5.20}$$

where L is the length of free boundary.

Remark 5.6. *In the demonstration below, we will see the equivalence between this problem and the usual trace inequality on the finite strip $\mathbb{T}\times[-\epsilon,0]$. The main difficulty for non-periodic case $\mathbb{R}\times[-\epsilon,0]$ is that the trace estimate holds only in homogeneous Sobolev spaces [51, 40] and we cannot recover L^2 norm due to the low frequency part of f, which is zero in periodic case.*

Proof. The proof of this theorem is based on flattening by *normal geodesic coordinate* defined by

$$
\begin{array}{rccc}
\Phi: & \mathbb{R}/L\times[-\epsilon,\epsilon] & \to & \mathbb{T}\times\mathbb{R}\\
 & (s,l) & \mapsto & \gamma(s)+l\cdot n(s)
\end{array}\tag{5.21}
$$

It is clear that, for ϵ small enough, Φ is diffeomorphic (a proof of this is given in Appendix D) and its Jacobian $j(s,l) = 1 + l\kappa(s) \in [\frac{1}{2}, \frac{3}{2}]$, since

$$|l\kappa(s)| \leqslant \epsilon \|\kappa\|_{L^\infty} \leqslant \frac{c_0}{N_0} \leqslant \frac{1}{N_0} \ll 1.$$

Remark that $\{l < 0\} \subset \Omega$, $\{l = 0\} = \Sigma$, and $\{l > 0\} \subset \bar{\Omega}^c$. In the sequel, we shall use the restriction

$$-\epsilon \leqslant l \leqslant 0.$$

To begin with, we truncate F inside $\{-\epsilon < l \leqslant 0\}$ using

$$G(s,l) := F(s,l)\chi\left(\frac{l}{\epsilon}\right),$$

where $\chi \in C_c^\infty(\mathbb{R})$ is supported inside $]-1,1[$ and equals to 1 near zero. One may observe that G has the same trace as F and

$$\begin{aligned}\|\partial_l G\|_{L^2}^2 &= \iint \left| n \cdot \nabla_{x,y} F \chi\left(\frac{l}{\epsilon}\right) + \frac{1}{\epsilon} F \chi'\left(\frac{l}{\epsilon}\right)\right|^2 dsdl \\ &\lesssim \iint |\nabla_{x,y} F|^2 \, dxdy + \frac{1}{\epsilon^2}\int_{-\epsilon}^0 \int_0^L |F|^2 \, dsdl.\end{aligned}$$

Since G is supported in $\{-\epsilon < l \leqslant 0\}$, we have trivially,

$$f(s) = \int_{-\epsilon}^0 \partial_l G(s,l) dl,$$

thus

$$|f(s)|^2 \leqslant \epsilon \int_{-\epsilon}^0 |\partial_l G|^2 dl.$$

By integrating on both sides, one obtains

$$\|f\|_{L^2(\Sigma)}^2 \leqslant \epsilon \iint |\partial_l G|^2 dsdl \lesssim \epsilon \iint |\nabla_{x,y} F|^2 \, dxdy + \frac{1}{\epsilon}\int_{-\epsilon}^0 \int_0^L |F|^2 \, dsdl.$$

In order to conclude (5.20), it suffices to apply Poincaré's inequality for the last term on right hand side. The difficulty is that the mean of F in $[0,L] \times [-\epsilon, 0]$ is non zero in general. Let us denote the mean of F as

$$m = \frac{1}{\epsilon L}\int_{-\epsilon}^0 \int_0^L F dsdl.$$

By repeating the argument above, due to the zero mean condition for F, one may deduce that

$$\begin{aligned} m = \frac{1}{\epsilon L}\int_{-\epsilon}^0 \int_0^L F(s,l) dsdl =& \frac{1}{\epsilon L}\int_{-\epsilon}^0 \int_0^L \left(F(s,0) + \int_0^l \partial_l F(s,r) dr\right) dsdl \\ =& \frac{1}{\epsilon L}\int_{-\epsilon}^0 \int_0^L \int_0^l \partial_l F(s,r) dr dsdl,\end{aligned}$$

which implies the desired control of m :

$$
\begin{aligned}
|m| \leqslant & \frac{\sqrt{\epsilon}}{\epsilon L} \int_{-\epsilon}^{0} \int_{0}^{L} \sqrt{\int_{-\epsilon}^{\epsilon} |\partial_l F(s,r)|^2 dr ds dl} \\
\leqslant & \frac{\sqrt{\epsilon}\sqrt{\epsilon L}}{\epsilon L} \sqrt{\int_{-\epsilon}^{0} \int_{0}^{L} \int_{-\epsilon}^{0} |\partial_l F(s,r)|^2 dr ds dl} \\
= & \sqrt{\frac{\epsilon}{L}} \|\partial_l F\|_{L^2([0,L]\times[-\epsilon,0])} \\
= & \sqrt{\frac{\epsilon}{L}} \|n \cdot \nabla_{x,y} F\|_{L^2([0,L]\times[-\epsilon,0])} \lesssim \sqrt{\frac{\epsilon}{L}} \|\nabla_{x,y} F\|_{L^2(\Omega)}
\end{aligned}
$$

Now we are able to apply Poincaré's inequality for $F - m$ on $[0, L] \times [-\epsilon, 0]$:

$$
\begin{aligned}
\int_{-\epsilon}^{0} \int_{0}^{L} |F|^2 \, ds dl \leqslant & 2 \int_{-\epsilon}^{0} \int_{0}^{L} |F - m|^2 \, ds dl + 2\epsilon L m^2 \\
\lesssim & \max\{L, \epsilon\}^2 \int_{-\epsilon}^{\epsilon} \int_{0}^{L} |\nabla_{s,l} F|^2 \, ds dl + \epsilon^2 \|\nabla_{x,y} F\|_{L^2(\Omega)}^2 \lesssim L^2 \|\nabla_{x,y} F\|_{L^2(\Omega)}^2,
\end{aligned}
$$

where the last inequality follows from the fact that $L \geqslant 2\pi$ and $\epsilon \ll 1$ when $N_0 \gg 1$. $\square$

Corollary 5.7. *Let $\gamma = (\alpha, \beta)$ and $\phi^{\pm}$ be regular solution to the system (5.6) and ψ be defined by (5.16). Then we have*

$$
\left| \int_0^L \alpha_s \beta \psi ds \right| \lesssim \frac{L}{\sqrt{\epsilon}} \|\alpha_s \beta\|_{L^2} \sqrt{E_k}, \tag{5.22}
$$

where L is the length of interface and $0 < \epsilon \ll 1$ is defined in (5.19).

To prove this corollary, as explained before, it suffices to study the case of one phase problem and prove

$$
\left| \int_0^L \alpha_s \beta \psi ds \right| \lesssim \frac{L}{\sqrt{\epsilon}} \|\alpha_s \beta\|_{L^2} \|\nabla_{x,y} \phi\|_{L^2(\Omega)}. \tag{5.23}
$$

Let us introduce the normalized Dirichlet-to-Neumann operator :

$$
\mathcal{N}\psi := n \cdot \nabla_{x,y} \phi|_{\Sigma}. \tag{5.24}
$$

Remark that this definition is equivalent to (2.11), up to multiple of positive function. Thus, this normalized operator is linear, strictly positive, self-adjoint (w.r.t L^2 scalar product) on H^1 modulo constants. Moreover, since Σ (diffeomorphic to torus $\mathbb{T}$) is compact, $\mathcal{N}$ admits discrete spectrum. Further spectral properties of this operator and its comparison with $-\Delta_\Sigma$ could be found in [17] and [16]. As a consequence, one may define $\mathcal{N}^{\pm\frac{1}{2}}$ via spectral decomposition, at least in the space of regular functions on Σ with zero mean.

Besides, by definition (5.24), it can be verified directly that

$$
\|\mathcal{N}^{\frac{1}{2}}(\psi - m(\psi))\|_{L^2(\Sigma)}^2 = \int_\Sigma \psi \mathcal{N} \psi ds = \int_\Sigma \phi n \cdot \nabla_{x,y} \phi ds = \iint_\Omega |\nabla_{x,y} \phi|^2 dx dy, \tag{5.25}
$$

where $m(\psi)$ is the mean of ψ on Σ and the first equality follows from the fact that $\mathcal{N}$ maps constants to zero. Now we are ready to prove (5.23) and thus (5.22).

Proof of Corollary 5.7. Thanks to the convention (5.13), one may write the left hand side of (5.23) as

$$\left|\int_0^L \alpha_s\beta(\psi - m(\psi))ds\right| = \left|\int_0^L \mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)\mathcal{N}^{\frac{1}{2}}(\psi - m(\psi))ds\right|,$$

which is bounded by

$$\|\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)\|_{L^2(\Sigma)}\|\mathcal{N}^{\frac{1}{2}}(\psi - m(\psi))\|_{L^2(\Sigma)} = \|\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)\|_{L^2(\Sigma)}\|\nabla_{x,y}\phi\|_{L^2(\Omega)}.$$

The definition of $\mathcal{N}^{-\frac{1}{2}}$ ensures that $\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)$ has also zero mean and this allows us to apply Theorem 5.5 with F equal to the harmonic extension of $\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)$:

$$\|\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)\|_{L^2(\Sigma)} \lesssim \frac{L}{\sqrt{\epsilon}}\|\nabla_{x,y}F\|_{L^2(\Omega)} = \frac{L}{\sqrt{\epsilon}}\|\mathcal{N}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}(\alpha_s\beta)\|_{L^2(\Sigma)} = \frac{L}{\sqrt{\epsilon}}\|\alpha_s\beta\|_{L^2(\Sigma)},$$

where we repeat the argument (5.25) to obtain the first equality and use the fact that $\mathcal{N}^{\frac{1}{2}}\mathcal{N}^{-\frac{1}{2}}$ equals to identity modulo constants to conclude the proof. □

5.4 Instability

In this section, we shall establish an analogue of Theorem 4.1 in terms of length L and maximal curvature $\|\kappa\|_{L^\infty}$. In fact, with Theorem 3.1 replaced by Theorem 5.4 and Proposition 4.2 replaced by Theorem 5.5, we can apply exactly the same argument as in Theorem 4.1 to get the following result

Theorem 5.8. *Let $(\gamma, \phi^\pm)$ be a regular solution to (5.6). In the case with zero surface tension $\sigma = 0$ and nonzero total energy $E \neq 0$, or with nonzero surface tension $\sigma > 0$ and negative total energy $E < 0$, there exists a constant C, such that*

$$C\frac{L(t)}{\sqrt{\epsilon}}\|\alpha_s(t)\beta(t)\|_{L^2}\sqrt{E_k(t)} \geqslant \int_0^L \alpha_s\beta\psi ds\Big|_{t=0} + |E|t, \tag{5.26}$$

where L is the length of interface and $0 < \epsilon \ll 1$ is defined in (5.19), which we recall here

$$\epsilon = \min\{\frac{c_0}{N_0\|\kappa\|_{L^\infty}},\ d(\Sigma, \Gamma^\pm), \frac{1}{N_0}\},\quad N_0 \gg 1.$$

Proof. By integrating both sides of (5.17) and applying the inequality (5.22), we have

$$C\frac{L}{\sqrt{\epsilon}}\|\alpha_s\beta\|_{L^2}\sqrt{E_k} \geqslant \int_0^L \alpha_s\beta\psi ds\Big|_{t=0} + 2\int_0^t (\tilde{E}_k - E_p + R)d\tau.$$

It remains to give a lower bound of $\tilde{E}_k - E_p + R$ with $R > 0$.

If $\sigma = 0$ and $E > 0$, by observing

$$\tilde{E}_k \geqslant \frac{1}{2}E_k,$$

we have

$$\tilde{E}_k - E_p + R \geqslant \frac{1}{2}E_k - E_p = \frac{1}{2}E - \frac{3}{2}E_p \geqslant \frac{1}{2}E,$$

where the last estimate follows from the assumption $g' \leqslant 0$ and $\sigma = 0$.

If $E < 0$, the estimate can be achieved by

$$\tilde{E}_k - E_p + R = \tilde{E}_k - E + E_k + R \geqslant -E = |E|.$$

□

Remark 5.9. *(5.26) can be considered as an alternative of (4.1) with* $\sqrt{1+\|\nabla\eta\|_{L^\infty}}$ *replaced by* $L/\sqrt{\epsilon}$. *In fact, in graph case*

$$\|\alpha_s\beta\|_{L^2}^2 \leqslant \int_0^L \beta^2|\alpha_s|^2 ds \leqslant \int_0^L \beta^2\alpha_s ds = \|\eta\|_{L^2}^2, \quad \int_0^L \alpha_s\beta\psi ds = \int \eta\psi dx, \tag{5.27}$$

where we use $0 < \alpha_s \leqslant |\gamma_s| = 1$.

Remark 5.10. *Before the presence of singularity* $\|\nabla\eta\|_{L^\infty} = +\infty$, *i.e. in the graph case, if one assumes that length* L *and curvature* κ *are both bounded, then in the case without surface tension* $\sigma = 0$, *a similar proof of (5.26) gives*

$$\int \eta_0\psi_0 dx + \frac{|E|}{2}t + \frac{g'}{2}\int_0^t \|\eta\|_{L^2}^2 \leqslant C\frac{L}{\sqrt{\epsilon}}\|\eta\|_{L^2}\sqrt{E - \frac{g'}{2}\|\eta\|_{L^2}^2}. \tag{5.28}$$

In fact, (5.27) and (5.22) gives that

$$\left|\int \eta\psi dx\right| \leqslant C\frac{L}{\sqrt{\epsilon}}\|\eta\|_{L^2}\sqrt{E_k} = C\frac{L}{\sqrt{\epsilon}}\|\eta\|_{L^2}\sqrt{E - \frac{g'}{2}\|\eta\|_{L^2}^2}.$$

Again, by integrating (5.17) and using the estimate

$$2\left(\tilde{E}_k - E_p + R\right) \geqslant \left(\tilde{E}_k - E_p + R\right) + \frac{1}{2}|E| \geqslant -E_p + \frac{1}{2}|E| = -\frac{g'}{2}\|\eta\|_{L^2}^2 + |E|,$$

we can conclude the refined inequality (5.28).

Proposition 5.11. *When the interface is a graph* $\Sigma = \{y = \eta(t,x)\}$, *with* $\sigma = 0$, $g' < 0$, *and*

$$\int \eta_0\psi_0 dx > 0,$$

if the following estimate holds for some $T > 0$

$$M := \sup_{t\in[0,T]} \frac{L(t)}{\sqrt{\epsilon(t)}} \leqslant +\infty, \tag{5.29}$$

we have, for all $t \in [0,T]$

$$\frac{|E|}{2} + \frac{|g'|}{2}\|\eta(t)\|_{L^2}^2 \geqslant \frac{1}{C'}\exp\left(\frac{t}{C'}\right)\int \eta_0\psi_0 dx, \tag{5.30}$$

where C' *is a constant depending on* E, g', *and* M.

Proof. Let us denote

$$F(t) := \frac{|E|}{2} - \frac{g'}{2}\|\eta(t)\|_{L^2}^2 \geqslant \frac{|g'|}{2}\|\eta(t)\|_{L^2}^2.$$

The inequality (5.28) implies that, for all $t \in [0,T]$,

$$G(t) := \int \eta_0\psi_0 dx + \int_0^t F(\tau)d\tau \leqslant \sqrt{2}CM\|\eta(t)\|_{L^2}\sqrt{F(t)} \leqslant \frac{2}{\sqrt{|g'|}}CMF(t).$$

With $C' = \frac{2}{\sqrt{|g'|}}CM$, one observes that G satisfies

$$G'(t) = F(t) \geqslant \frac{1}{C'}G(t), \quad G(0) = \int \eta_0\psi_0 dx.$$

By solving this problem, on obtains (5.30). ∎

This exponential growth, however, can exist only in short time, since the trivial observation

$$\|\eta\|_{L^2} \leqslant \sqrt{2\pi}L \leqslant \sqrt{2\pi}\frac{L}{\sqrt{\epsilon}} \leqslant \sqrt{2\pi}M$$

will lead to the contradiction in large time.

To end this section, we attempt to replace left hand side of (5.26) by a function of only L or $\|\kappa\|_{L^\infty}$. In general, there is few relation between L and $\|\kappa\|_{L^\infty}$. One can easily construct interfaces with large length and small curvature, or with limited length but huge curvature. In finite depth case $H^\pm < +\infty$, however, the length L can be controlled by ϵ, thus by curvature.

Lemma 5.12. *In finite depth $H^\pm < +\infty$, we have*

$$L\epsilon \leqslant 2\pi(H^+ + H^-). \tag{5.31}$$

Proof. To prove this inequality, we shall use the change of variable Φ defined in (5.21),

$$\Phi(s,l) = \gamma(s) + l \cdot n(s).$$

Let us denote by $\mathcal{O}_\epsilon$ the image of $[0,L]\times[-\epsilon,\epsilon]$ by Φ, which is a subdomain of $\Omega^+\cup\Sigma\cup\Omega^-$. Therefore,

$$2\pi(H^+ + H^-) = \mathrm{Vol}(\Omega^+ \cup \Sigma \cup \Omega^-) \geqslant \mathrm{Vol}(\mathcal{O}_\epsilon) = \iint j(s,l)dsdl \geqslant L\epsilon,$$

where j is the Jacobian of Φ and due to our definition of ϵ, $j = l\kappa(s) \in [\frac{1}{2}, \frac{3}{2}]$ due to (5.21). $\square$

By combining Lemma 5.12 with the trivial estimates

$$\|\alpha_s\beta\|_{L^2}^2 \leqslant \int_0^L \beta^2|\alpha_s|ds \lesssim L^3, \quad E_k \leqslant |E| + \frac{|g'|}{2}\int_0^L \beta^2\alpha_s ds \lesssim L^3,$$

from (5.26) and (5.31), we have in large time (if solution exists),

$$\epsilon \leqslant Ct^{-\frac{2}{9}},$$

for some constant C depending on total energy E, depth $H^\pm$, and initial data. Once such estimate holds, by definition (5.19), we have, at least for large time,

$$\epsilon = \frac{c_0}{N_0\|\kappa\|_{L^\infty}},$$

which yields

$$\frac{c_0}{CN_0}t^{\frac{2}{9}} \leqslant \|\kappa\|_{L^\infty}.$$

This means, for large time, the curvature will tend to infinity, no matter whether the singularity $\|\nabla\eta\|_{L^\infty} = +\infty$ occurs.

A Local well-posedness and functional spaces

The objective of this section is to review the local well-posedness of RTI and KHI problem in the space of analytic functions, which is proved in [53, 52], and to explain why our choice of regularity (1.11) is reasonable. For simplicity, we focus on the problem in $\mathbb{R}^2$ without bottom, while the results are similar in $3D$ and with bottom. More details on this topic can be found in the survey [4] and the references therein.

To begin with, we introduce another formulation, known as *vortex sheet*, of the system (1.4) with kinematic boundary condition (1.7) and zero surface tension $\sigma = 0$. Consider the following vector field defined in the whole space $\mathbb{R}^2$,

$$v := u^+ \mathbb{1}_{\Omega^+} + u^- \mathbb{1}_{\Omega^-}.$$

In the sense of distribution, one may check easily that

$$\langle \operatorname{curl}_{x,y} v, f \rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} = \int_{\mathbb{R}} \omega(x) f(x, \eta(x)) dx,$$

where the interface Σ is represented by $(x, \eta(x))$, and $\omega(x)$ equals to

$$\omega(x) = \big(u^+(x, \eta(x)) - u^-(x, \eta(x))\big) \cdot (1, \eta_x(x)), \tag{A.1}$$

where $(1, \eta_x(x))$ is the tangent direction. That is to say, the rotation of unified velocity field v is a measure concentrated on interface Σ with density ω. Additionally, via Biot-Savart law, one may recover v:

$$v(t, x, y) = \mathrm{P.\,V.} \int_{\mathbb{R}} \nabla_{x,y} G(x, y; x', \eta(s')) \times \omega(x') dx'.$$

Here G is the Green function of Laplacian in $\mathbb{R}^2$, namely

$$G(x, y; x', y') = -\frac{1}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2}.$$

This formula gives a proper definition of v at interface Σ:

$$v(x, \eta(x)) := -\frac{1}{2\pi} \mathrm{P.\,V.} \int_{\mathbb{R}} \frac{(-(\eta(x) - \eta(x')), x - x)}{(x - x')^2 + (\eta(x) - \eta(s'))^2} \omega(x') dx'. \tag{A.2}$$

Note that for flat interface, the singular integral on the right hand side is the Fourier multiplier $|D_x|$, and for η smooth enough, this operator is also well-defined. We refer to [52] for more details.

The next step is to derive equations for the new variables (η, ω). The motion of interface η is governed by the kinematic boundary condition (1.7),

$$\eta_t - v^2 + \eta_x v^1 = 0, \tag{A.3}$$

where $v = (v^1, v^2)$ represents its trace on Σ, defined in (A.2) via singular integral.

In order to obtain an equation for ω, we observe that v is actually a solution to incompressible Euler equation in the sense of distribution,

$$\begin{cases} \partial_t \rho + \nabla_{x,y}(\rho v) = 0, & \text{in } \mathbb{R}^2, \\ \partial_t(\rho v) + \operatorname{div}_{x,y}(\rho v \otimes v) + \nabla_{x,y} P + \rho(0, y) = 0, & \text{in } \mathbb{R}^2, \\ \operatorname{div}_{x,y} v = 0, & \text{in } \mathbb{R}^2, \end{cases}$$

where

$$\mathrm{div}_{x,y}(\rho v \otimes v)^j = \nabla_{x,y}(\rho v^j v), \quad j = 1, 2,$$

and the unified quantities ρ, P are defined as

$$\rho := \rho^+ \mathbb{1}_{\Omega^+} + \rho^- \mathbb{1}_{\Omega^-}, \quad P := P^+ \mathbb{1}_{\Omega^+} + P^- \mathbb{1}_{\Omega^-}.$$

Thus, the rotation of v satisfies the vorticity equation in the sense of distribution,

$$\partial_t \,\mathrm{curl}_{x,y}(\rho v) + \mathrm{div}_{x,y}\,\mathrm{curl}_{x,y}(\rho v \otimes v) + \mathrm{curl}_{x,y}\left(\rho(0, y)\right) = 0.$$

By applying the kinematic boundary condition and the formula (A.1), the equation above is equivalent to
(A.4)

$$\partial_t \left(\frac{\omega}{2} + A(v^1 + \eta_x v^2)\right) + \partial_x \left(v^1 \left(\frac{\omega}{2} + A(v^1 + \eta_x v^2)\right) + A\left(\frac{\omega^2}{8(1+\eta_x^2)} - \frac{|v|^2}{2} + gy\right)\right) = 0,$$

where $A = (\rho^+ - \rho^-)/(\rho^+ + \rho^-)$ is the Atwood number and $v = (v^1, v^2)$ should be understood as its trace on interface, which can be calculated from ω, η by (A.2). This step requires some careful calculus due to the singularity near the interface. We refer to [52] for more details and to [4] for the general case.

Now we introduce the functional space we are going to use.

Definition A.1. *Let $r > 0$ and S_r be the complex strip of width r,*

$$S_r := \{x + iw \in \mathbb{C} : x \in \mathbb{R}, \ |w| < r\}.$$

The space B_r^1 is the collection of analytic functions f defined in S_r such that

$$\|f\|_{B_r^1} := \sup_{S_r} |f| + \sup_{x \neq x', |w| < r} \frac{|f(x + iw) - f(x' + iw)|}{|x - x'|^\iota} = \sup_{|w|<r} \|f(\cdot + iw)\|_{C^\iota(\mathbb{R})} < \infty,$$

where $\iota \in]0, 1[$ is a fixed index.

The space B_r^2 is a subspace of B_r^1, with norm

$$\|f\|_{B_r^2} := \|f\|_{B_r^1} + \sup_{|w|<r} \|f(\cdot + iw)\|_{L^2(\mathbb{R})}.$$

It is easy to check that B_r^1 and B_r^2 are Banach spaces. In these spaces of analytic functions, we are able to prove the local well-posedness by a Cauchy-Kowalewski-type theorem.

Theorem A.2 (Theorem 5.1 of [52])**.** *Let $r_0 > 0$ and $\kappa \in]0, 1[$. There exists constants $k_0 > 0$ and $a > 0$, such that for all initial data $(\eta_0, \omega_0) \in B_{r_0}^1 \times B_{r_0}^2$, with $\partial_x \eta_0 \in B_{r_0}^2$ and smallness conditions*

$$\|\operatorname{Im} \partial_x \eta_0\|_{L^\infty(S_{r_0})} < \kappa, \quad |A| \|\partial_x \eta_0\|_{B_{r_0}^2} < k_0,$$

there exists a unique solution to (A.3) and (A.4), such that for all $|t| < a r_0$,

$$(\eta(t), \eta_x(t), \omega(t)) \in B^1_{r_0 - \frac{|t|}{a}} \times B^2_{r_0 - \frac{|t|}{a}} \times B^2_{r_0 - \frac{|t|}{a}}.$$

We refer to Theorem 5.2 of [52] for the $3D$ case and Theorem 6.1, 6.2 of the same paper for finite depth case. As mentioned in Section 1, in this paper, we shall only study the solutions obtained in the theorem above with $r_0 \gg 1$ in order that the solutions are analytic in strip of width $r_0 - \frac{|t|}{a}$ and the lifespan $\frac{r_0}{a}$ is large enough. To end this section, we shall illustrate the relation between B_r^2 and usual Sobolev spaces.

Proposition A.3. *Let $r, r' > 0$. We assume that $r = r'$ in dimension $d = 1$, $r > r'$ in higher dimensions. Then the space B_r^2 is embedded continuously in the Gevrey space $\mathcal{H}_{r'}$, defined as the collection of functions on $\mathbb{R}^d$ with*

$$\|f\|_{\mathcal{H}_{r'}}^2 := \int_{\mathbb{R}^d} \left| \hat{f}(\xi) e^{r'|\xi|} \right|^2 d\xi < +\infty.$$

This is actually a consequence of the L^2 norm in the definition of B_r^2. A proof can be found in [2], Theorem A.1.

Recall that Sobolev space H^m is a subspace of $\mathcal{H}_{r'}$ for all $m \in \mathbb{R}$ and $r' > 0$. Thus, it is reasonable to assume that all the functions studied in this paper belong to H^m with $m \gg 1$, which is the convention we made in (1.11).

B Useful identities

In the section, we collect some identities used in this paper. As before, all the functions are assumed to be regular.

Proposition B.1. *Let $\psi^\pm$ be defined by (2.14) and $B^\pm$, $V^\pm$ be defined by (2.7) and (2.8) respectively. Then the following identities hold for regular (η, ψ)*

$$G^\pm(\eta)\psi^\pm = B^\pm - V^\pm \cdot \nabla\eta, \tag{B.1}$$

$$\nabla\psi^\pm = V^\pm + \nabla\eta B^\pm, \tag{B.2}$$

Proposition B.2. *Let ψ and $G(\eta)$ be defined by (2.14) and (2.16) respectively. Then for all regular solutions, we have*

$$\int \psi G(\eta)\psi dx = \sum_\pm \iint_{\Omega^\pm} \underline{\rho}^\pm |u^\pm|^2 dxdy. \tag{B.3}$$

Proof. An application of divergence theorem gives

$$\begin{aligned}
\int \psi G(\eta)\psi dx &= \sum_\pm \pm \int \underline{\rho}^\pm \psi^\pm G(\eta)\psi dx = \sum_\pm \pm \int \underline{\rho}^\pm \psi^\pm G^\pm(\eta)\psi^\pm dx \\
&= \sum_\pm \pm \int \underline{\rho}^\pm \left(\phi^\pm n \cdot \nabla_{x,y}\phi^\pm \right)|_{y=\eta(x)\mp 0} \sqrt{1 + |\nabla\eta|^2} dx \\
&= \sum_\pm \pm \int_\Sigma \underline{\rho}^\pm \phi^\pm n \cdot \nabla_{x,y}\phi^\pm dS \\
&= \sum_\pm \iint_{\Omega^\pm} \underline{\rho}^\pm \operatorname{div}_{x,y}\left(\phi^\pm \nabla_{x,y}\phi^\pm \right) dxdy \\
&= \sum_\pm \iint_{\Omega^\pm} \underline{\rho}^\pm |\nabla_{x,y}\phi^\pm|^2 dxdy = \sum_\pm \iint_{\Omega^\pm} \underline{\rho}^\pm |u^\pm|^2 dxdy.
\end{aligned}$$

$\square$

Proposition B.3. *For regular η and $\phi^\pm$, we have*

$$\pm\int \eta N^\pm = \iint_{\Omega^\pm} \frac{|\nabla\phi^\pm|^2 - (\phi_y^\pm)^2}{2} dxdy - \frac{H^\pm}{2}\int_{\Gamma^\pm} |\nabla\phi^\pm|^2 dx, \tag{B.4}$$

where $N^\pm$ is defined in (2.6).

Proof. First recall that we use the notation ∇, Δ for the derivatives in horizontal variable x, namely $\nabla = \nabla_x$, $\Delta = \Delta_x$.

The proof of $+$ part can be found in [3]. With the same argument we can proof the analogue for $-$ part :

$$\begin{aligned}
0 = \int \nabla \cdot \int_\eta^{H^-} y\phi_y^- \nabla\phi^- dydx &= -\int \nabla\eta \cdot V^- B^- \eta dx + \iint_{\Omega^-} y\nabla\phi_y^- \cdot \nabla\phi^- dxdy \\
&\quad + \iint_{\Omega^-} y\phi_y^- \Delta\phi^- dxdy \\
&= -\int \nabla\eta \cdot V^- B^- \eta dx + \iint_{\Omega^-} y\left(\nabla\phi_y^- \cdot \nabla\phi^- - \phi_y^-\phi_{yy}^-\right) dxdy \\
&= -\int \nabla\eta \cdot V^- B^- \eta dx + \int\int_\eta^{H^-} y\partial_y \frac{|\nabla\phi^-|^2 - (\phi_y^-)^2}{2} dxdy \\
&= -\int \nabla\eta \cdot V^- B^- \eta dx - \iint_{\Omega^-} \frac{|\nabla\phi^-|^2 - (\phi_y^-)^2}{2} dxdy \\
&\quad - \int \eta \frac{|V^-|^2 - (B^-)^2}{2} dx + \frac{H^-}{2}\int_{\Gamma^\pm} |\nabla\phi^\pm|^2 dx \\
&= -\int \eta N^- dx - \iint_{\Omega^-} \frac{|\nabla\phi^-|^2 - (\phi_y^-)^2}{2} dxdy \\
&\quad + \frac{H^-}{2}\int_{\Gamma^\pm} |\nabla\phi^\pm|^2 dx,
\end{aligned}$$

which gives (B.4). ◻

C Hamiltonian structure of the system

This purpose of this section is the proof of the Hamiltonian system (2.19) under the hypothesis that all the functions concerned are regular. To begin with, we prove the following variational formula related to Dirichlet-to-Neumann operator under some acceptable constraints, while a complete proof in Sobolev spaces could be found in Section 3.3 of [38].

Lemma C.1. *For all regular $(\eta, \psi^\pm)$ and regular h, if h is non-negative (or non-positive), we have*

(C.1)
$$d\left(G^\pm(\eta)\psi^\pm\right)h := \lim_{\epsilon\to 0+} \frac{1}{\epsilon}\left(G^\pm(\eta+\epsilon h) - G^\pm(\eta)\right)\psi^\pm = -G^\pm(\eta)\left(B^\pm h\right) - \operatorname{div}\left(V^\pm h\right),$$

where $B^\pm$, $V^\pm$ are defined in (2.7) *and* (2.8) *respectively. As a consequence, one has*

$$\frac{\delta}{\delta\eta}\int f G^\pm(\eta)\psi^\pm dx = -B^\pm G^\pm(\eta) f + \nabla f \cdot V^\pm. \tag{C.2}$$

Proof. We focus on the $-$ part and non-negative h, as the other cases can be treated in the same way. In what follows, we denote by Ω_ϵ^- the domain

$$\Omega_\epsilon^- := \{\eta(x) + \epsilon h(x) < y < H^-\} \supset \Omega^-,$$

and by ϕ_ϵ^- the harmonic extension of ψ^- in Ω_ϵ^-. Moreover, all the quantities involving Ω_ϵ^- will be written with subscript ϵ.

The fact that $\Omega_\epsilon^- \supset \Omega^-$ enables us to write by Taylor expansion that

$$\begin{aligned}\psi^-(x) = \phi_\epsilon^-(x, \eta(x) + \epsilon h(x)) =& \phi_\epsilon^-(x, \eta(x)) + \int_{\eta(x)}^{\eta(x)+\epsilon h(x)} \partial_y \phi_\epsilon^-(x, y) dy \\ =& \phi_\epsilon^-(x, \eta(x)) + \epsilon h(x) B_\epsilon^-(x) + O(\epsilon^2).\end{aligned}$$

This indicates that ϕ_ϵ^- is the harmonic extension in Ω^- of $\psi^- - \epsilon h B_\epsilon^- + O(\epsilon^2)$, which implies

$$G^-(\eta)\left(\psi^- - \epsilon h B_\epsilon^-\right) + O(\epsilon^2) = \partial_y \phi_\epsilon^-(x, \eta) - \nabla\eta \cdot \nabla\phi_\epsilon^-(x, \eta).$$

By definition, we have

$$\begin{aligned}G^-(\eta + \epsilon h)\psi^- =& \partial_y \phi_\epsilon^-(x, \eta + \epsilon h) - (\nabla\eta + \epsilon\nabla h) \cdot \nabla\phi_\epsilon^-(x, \eta + \epsilon h) \\ =& \partial_y \phi_\epsilon^-(x, \eta) + \int_\eta^{\eta+\epsilon h} \partial_y^2 \phi_\epsilon^- dy - \nabla\eta \cdot \nabla\phi_\epsilon^-(x, \eta) - \nabla\eta \cdot \int_\eta^{\eta+\epsilon h} \partial_y \nabla\phi_\epsilon^- dy \\ & - \epsilon\nabla h \cdot \nabla\phi_\epsilon^-(x, \eta + \epsilon h) \\ =& G^-(\eta)\left(\psi^- - \epsilon h B_\epsilon^-\right) + O(\epsilon^2) - \int_\eta^{\eta+\epsilon h} \Delta\phi_\epsilon^- dy - \nabla\eta \cdot \nabla\phi_\epsilon^-(\eta + \epsilon h) \\ & + \nabla\eta \cdot \nabla\phi_\epsilon^-(\eta) - \epsilon\nabla h \cdot \nabla\phi_\epsilon^-(x, \eta + \epsilon h) \\ =& G^-(\eta)\psi^- - \epsilon G^-(\eta)\left(h B_\epsilon^-\right) - \nabla \cdot \left(\int_\eta^{\eta+\epsilon h} \nabla\phi_\epsilon^- dy\right) + O(\epsilon^2).\end{aligned}$$

This implies that, as $\epsilon \to 0 + 0$,

$$\frac{1}{\epsilon}\left(G^-(\eta + \epsilon h)\psi^- - G^-(\eta)\psi^-\right) \to -G^-(\eta)\left(h B^-\right) - \operatorname{div}(V^- h).$$

$\square$

We are now ready to deduce the Hamiltonian system (2.19). The identity (B.3) allows us to rewrite the Hamiltonian $\mathcal{H}$, defined by (2.20) as

$$\mathcal{H} = \frac{1}{2}\int \psi G(\eta)\psi dx + \frac{g'}{2}\int \eta^2 dx + \sigma' \int \left(\sqrt{1 + |\nabla\eta|^2} - 1\right) dx. \tag{C.3}$$

Due to the fact that $G(\eta)$ is self-adjoint, we have immediately

$$\frac{\delta\mathcal{H}}{\delta\psi} = G(\eta)\psi = \partial_t \eta.$$

In order to calculate the variation in η, one may apply (C.2) to conclude that

$$\frac{\delta\mathcal{H}}{\delta\eta} = \frac{1}{2}\sum_\pm \pm\underline{\rho}^\pm \frac{\delta}{\delta\eta}\int \psi^\pm G^\pm(\eta)\psi^\pm dx + g'\eta + \sigma\kappa$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{\pm}\pm\underline{\rho}^{\pm}\left(-B^{\pm}G^{\pm}(\eta)\psi^{\pm}+\nabla\psi^{\pm}\cdot V^{\pm}\right)+g'\eta+\sigma\kappa\\
=&\frac{1}{2}\sum_{\pm}\pm\underline{\rho}^{\pm}\left(2B^{\pm}V^{\pm}\cdot\nabla\eta+|V^{\pm}|^2-(B^{\pm})^2\right)+g'\eta+\sigma\kappa\\
=&[\underline{\rho}N]+g'\eta+\sigma\kappa=-\partial_t\psi,
\end{aligned}
$$

where the third equality follows from identities (B.1) and (B.2).

D Normal geodesic coordinate

In this section, we shall prove that the mapping Φ, known as *normal geodesic coordinate*, defined in (5.21),

$$
\begin{array}{rccc}
\Phi: & \mathbb{R}/L\times]-\epsilon,\epsilon[& \to & \mathbb{T}\times\mathbb{R}\\
 & (s,l) & \mapsto & \gamma(s)+l\cdot n(s)
\end{array}
$$

is a diffeomorphism to its image and $\Omega^{\pm}$, Σ correspond to $\{\pm l<0\}$, $\{l=0\}$ respectively. We also recall the definition (5.19) of ϵ :

$$
\epsilon:=\min\{\frac{c_0}{N_0\|\kappa\|_{L^\infty}},\ d(\Sigma,\Gamma^{\pm}),\frac{1}{N_0}\},
$$

where $N_0\gg 1$ is a large constant to be determined later. It is clear that the Jacobian of Φ reads

$$
j(s,l)=1+l\kappa(s)\sim 1,
$$

if N_0 is large enough. Thus, Φ is at least a local diffeomorphism with the desired correspondence. In the sequel, we shall check that this property is global. In this section, the identification that $[0,2\pi]\times\mathbb{R}$ is a subdomain of $\mathbb{C}$ will also be used, which enables us to define a real function θ such that

$$
\gamma_s(s)=(\alpha_s(s),\beta_s(s))=e^{i\theta(s)}.
$$

Then, by definition, the curvature reads

$$
\kappa=-\tau_s\cdot n=-\gamma_{ss}\cdot n=-\operatorname{Im}(\gamma_{ss}\overline{i\gamma_s})=-\theta'. \tag{D.1}
$$

In fact, once one manages to prove the injectivity of Φ, the diffeomorphism follows from the local diffeomorphism and the correspondence is no more than a consequence of the connectivity of fluid domain $\Omega^{\pm}$.

If Φ is not injective, then there exist distinct points $(s,l),(s',l')\in[0,L]\times[-\epsilon,\epsilon]$, such that

$$
\gamma(s)+l\cdot n(s)=\gamma(s')+l'\cdot n(s'). \tag{D.2}
$$

Without loss of generality, we assume $s<s'$, since $s=s'$ implies $l=l'$ by definition of Φ.

Reduction to local injectivity. We first reduce the problem to the local injectivity of Φ.

Lemma D.1. *When $N_0 \geqslant 2$,*

$$|s - s'| \leqslant \frac{2}{c_0}|l - l'| \leqslant \frac{4}{N_0\|\kappa\|_{L^\infty}}. \tag{D.3}$$

Proof. In fact, the assumption (D.2) can be written as

$$\gamma(s') - \gamma(s) = l \cdot n(s) - l' \cdot n(s') = (l - l')n(s) + l' \int_{s'}^{s} \kappa(r)\tau(r)dr.$$

the length of left hand side can be bounded from below by

$$c_0|s - s'| \leqslant |\gamma(s') - \gamma(s)|,$$

while the length of right hand side is controlled by

$$|l - l'| + \left| l' \int_{s'}^{s} \kappa(r)\tau(r)dr \right| \leqslant |l - l'| + |s - s'|\epsilon\|\kappa\|_{L^\infty} \leqslant |l - l'| + \frac{c_0}{N_0}|s - s'|.$$

By combining the estimates above, we have

$$c_0|s - s'| \leqslant |l - l'| + \frac{c_0}{N_0}|s - s'|,$$

which gives (D.3) providing that $N_0 \geqslant 2$. □

Local injectivity. In previous paragraph, we have seen that

$$|s - s'| \leqslant \frac{4}{N_0\|\kappa\|_{L^\infty}},$$

which implies that $\tau(s)$ is close to $\tau(s')$, namely

Lemma D.2. *Under the assumption above, when $N_0 \gg 1$,*

$$\tau(s) \cdot \tau(s') \geqslant \frac{1}{2}.$$

Proof. By identifying τ as complex number, we have

$$\tau(s) \cdot \tau(s') = \cos(\theta(s) - \theta(s')),$$

while

$$|\theta(s) - \theta(s')| \leqslant |s - s'| \max_{[s,s']} |\theta'| = |s - s'| \max_{[s,s']} |\kappa| \leqslant \frac{4}{N_0\|\kappa\|_{L^\infty}}\|\kappa\|_{L^\infty} = \frac{4}{N_0}.$$

By choosing $\cos(4/N_0) \geqslant 1/2$, we could conclude the desired result. □

Now we are able to deduce a contradiction to (D.2). Let us consider the real-valued function :

$$\begin{array}{rccc} f: & [-\epsilon,\epsilon] & \to & \mathbb{R} \\ & r & \mapsto & (\gamma(s')+rn(s')-\gamma(s))\cdot\tau(s) \end{array}.$$

Once (D.2) holds true, f will admit a zero $r=l'$, while

$$\begin{aligned} f(r) &= (\gamma(s')+rn(s')-\gamma(s)-rn(s))\cdot\tau(s) \\ &= \left(\int_s^{s'}\tau(x)dx + r\int_s^{s'} n_s(x)dx\right)\cdot\tau(s) \\ &= \left(\int_s^{s'}\tau(x)dx + r\int_s^{s'} \kappa(x)\tau(x)dx\right)\cdot\tau(s) \\ &= \int_s^{s'}(1+r\kappa(x))\,\tau(x)\cdot\tau(s)dx. \end{aligned}$$

From previous lemma, $\tau(x)\cdot\tau(s)\geqslant 1/2$, and the definition of ϵ ensures that

$$1+r\kappa(x)\geqslant 1-\epsilon\|\kappa\|_{L^\infty}\geqslant 1-\frac{c_0}{N_0}\geqslant 1/2,$$

for large enough N_0. Therefore, f has lower bound $|s-s'|/4>0$, which is a contradiction.

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