# Spaces of non-resultant systems of real bounded multiplicity determined by a toric variety 

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#### Abstract

For each field $\mathbb{F}$ and positive integers $m, n, d$ with $(m, n) \neq(1,1)$, Farb and Wolfson [13 defined the certain affine variety $\operatorname{Poly}_{n}^{d, m}(\mathbb{F})$ as generalizations of spaces first studied by Arnold, Vassiliev, Segal and others. As a natural generalization, for each fan $\Sigma$ and $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right)$ of positive integers, the authors [26] also defined and considered a more general space $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{F})$, where $r$ is the number of one dimensional cones in $\Sigma$. This space can also be regarded as a generalization of the space $\operatorname{Hol}_{D}^{*}\left(S^{2}, X_{\Sigma}\right)$ of based rational curves from the Riemann sphere $S^{2}$ to the toric variety $X_{\Sigma}$ of degree $D$, where $X_{\Sigma}$ denotes the toric variety (over $\mathbb{C}$ ) corresponding to the fan $\Sigma$.

In this paper, we define a space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{F})(\mathbb{F}=\mathbb{R}$ or $\mathbb{C})$ which its real analogue and which can be viewed as a generalization of spaces considered by Arnold, Vassiliev and others in the context of real singularity theory. We prove that homotopy stability holds for this space and compute the stability dimension explicitly.


## 1 Introduction

1.1 Historical survey. For a complex manifold $X$, let $\operatorname{Map}^{*}\left(S^{2}, X\right)=$ $\Omega^{2} X$ (resp. $\left.\operatorname{Hol}^{*}\left(S^{2}, X\right)\right)$ denote the space of all based continuous maps (resp. based holomorphic maps) from the Riemann sphere $S^{2}$ to $X$. The relationship between the topology of the space $\operatorname{Hol}^{*}\left(S^{2}, X\right)$ and that of the

[^0]space $\Omega^{2} X$ has played a significant role in several different areas of geometry and mathematical physics (e.g. [2, 5). In particular there arose the question whether the inclusion $\operatorname{Hol}^{*}\left(S^{2}, X\right) \xrightarrow{\subset} \Omega^{2} X$ is a homotopy equivalence (or homology equivalence) up to a certain dimension, which we will refer to as the stability dimension. Since G. Segal [31] studied this problem for the case $X=\mathbb{C P}^{m}$, a number of mathematicians have investigated various closely related ones (e.g. [1], [15], [17, [22], [23], [27, [28], [29]).

Similar stabilization results appeared in the work of Arnold ([3], [4), and Vassiliev ([32], [33]) in connection with singularity theory. They considered spaces of polynomials without roots of multiplicity greater than a certain natural number. These spaces are examples of "complement of discriminants" in Vassiliev's terminology [32] (cf. [20]).

Inspired by these results, Farb and Wolfson [13] introduced a new family of spaces Poly ${ }_{n}^{d, m}(\mathbb{F})$, which is defined for every field $\mathbb{F}$ and integers $m, n, d \geq 1$ with $(m, n) \neq(1,1)$. The present authors generalised this further in [26], by considering a fan $\Sigma$ (or toric variety) and a field $\mathbb{F}$, and define a space $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{F})$ as follows.

Definition 1.1 ([26]). Let $\mathbb{F}$ be a field with its algebraic closure $\overline{\mathbb{F}}$, and let $\Sigma$ be a fan in $\mathbb{R}^{m}$ such that $\Sigma(1)=\left\{\rho_{1}, \cdots, \rho_{r}\right\}$, where $\Sigma(1)$ denotes the set of all one dimensional cones in $\Sigma$ as in (2.7) 1 Let $X_{\Sigma}$ denote the toric variety over $\mathbb{C}$ associated to the fan $\Sigma$, and let $\mathbb{N}$ denote the set of all positive integers.

For each $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, let $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{F})$ denote the space of all $r$-tuples $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathbb{F}[z]^{r}$ of $\mathbb{F}$-coefficients monic polynomials satisfying the following two conditions (1.1]) and (1.1)):
(1.1]a) $f_{i}(z) \in \mathbb{F}[z]$ is an $\mathbb{F}$-coefficients monic polynomial of the degree $d_{i}$ for each $1 \leq i \leq r$.
(1.1b) For each $\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \in I\left(\mathcal{K}_{\Sigma}\right)$, polynomials $f_{i_{1}}(z), \cdots, f_{i_{s}}(z)$ have no common root $\alpha \in \overline{\mathbb{F}}$ of multiplicity $\geq n$.
Here, $\mathcal{K}_{\Sigma}$ denotes the underlying simplicial complex of the fan $\Sigma$ on the index set $[r]=\{1,2, \cdots, r\}$ defined by (2.8), and $I\left(\mathcal{K}_{\Sigma}\right)$ is the set $I\left(\mathcal{K}_{\Sigma}\right)=\left\{\sigma \subset[r]: \sigma \notin \mathcal{K}_{\Sigma}\right\}$ as in (2.2).

Remark 1.2. (i) By using the classical theory of resultants, one can show that $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{F})$ is an affine variety over $\mathbb{F}$ and that it is the complement of the set of solutions of a system of polynomial equations (called a generalized

[^1]resultant) with integer coefficients. For this reason, we call it the space of non-resultant systems of bounded multiplicity determined by a toric variety.
(ii) Note that
\[

$$
\begin{equation*}
\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{C})=\operatorname{Hol}_{D}^{*}\left(S^{2}, X_{\Sigma}\right) \quad \text { if } n=1 \text { and } \quad \sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m} \tag{1.1}
\end{equation*}
$$

\]

where $\operatorname{Hol}_{D}^{*}\left(S^{2}, X_{\Sigma}\right)$ denotes the space of based rational curves of of degree $D$ on $X$ (i.e. rational maps of degree $D$ from the Riemann surface $S^{2}$ to $X_{\Sigma}$ ) (see [23] for further details). Thus, the space $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{C})$ can be also regarded as a generalization of the space $\operatorname{Hol}_{D}^{*}\left(S^{2}, X_{\Sigma}\right)$.

Now recall the following homotopy stability result.
Theorem 1.3 ([26]). Let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}, n \geq 2$, and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety over $\mathbb{C}$ such that the condition (2.18)* holds.
(i) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the natural map

$$
i_{D}: \operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega_{D}^{2} X_{\Sigma}(n) \simeq \Omega_{0}^{2} X_{\Sigma}(n) \simeq \Omega^{2} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$

is a homotopy equivalence through dimension $d_{\mathrm{poly}}(D ; \Sigma, n)$.
(ii) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, there is a map

$$
j_{D}: \operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega^{2} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$

which is a homotopy equivalence through dimension $d_{\text {poly }}(D ; \Sigma, n)$.
Here, we denote by $\lfloor x\rfloor$ the integer part of a real number $x$. Moreover, let $d_{\min }=\min \left\{d_{1}, \cdots, d_{r}\right\}$ and $r_{\min }(\Sigma)$ denote positive integers given by (2.35), and let $d_{\text {poly }}(D ; \Sigma, n)$ denote the positive integer defined by ${ }^{2}$

$$
\begin{equation*}
d_{\text {poly }}(D ; \Sigma, n)=\left(2 n r_{\min }(\Sigma)-3\right)\left\lfloor d_{\min } / n\right\rfloor-2 \tag{1.2}
\end{equation*}
$$

1.2 Basic definitions. In this paper, we replace the space $\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{F})$ by its real analogue $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{F})$ for $\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$. Its formal definition is below.

Definition 1.4. Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ such that $\Sigma(1)=\left\{\rho_{1}, \cdots, \rho_{r}\right\}$, where $\Sigma(1)$ denotes the set of all one dimensional cones in $\Sigma$ as in Definition 1.1.

For each $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ and $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$, let $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ denote the space of all $r$-tuples $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathbb{K}[z]^{r}$ of $\mathbb{K}$-coefficients monic polynomials satisfying the following two conditions (1.2a) and (1.2b):

[^2](1.2 a$)$ For each $1 \leq i \leq r, f_{i}(z) \in \mathbb{K}[z]$ is an $\mathbb{K}$-coefficients monic polynomial of the degree $d_{i}$.
(1.2b) For each $\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \in I\left(\mathcal{K}_{\Sigma}\right)$, polynomials $f_{i_{1}}(z), \cdots, f_{i_{s}}(z)$ have no common real root $\alpha \in \mathbb{R}$ of multiplicity $\geq n$ (but may have a common root $\alpha \in \mathbb{C} \backslash \mathbb{R}$ of any multiplicity).

Note that the following inclusion holds:

$$
\begin{equation*}
\operatorname{Poly}_{n}^{D, \Sigma}(\mathbb{K}) \subset \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) \quad \text { for } \mathbb{K}=\mathbb{R} \text { or } \mathbb{C} \tag{1.3}
\end{equation*}
$$

Recall that the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ was already investigated for the case $n=1$ in [24], $3^{3}$ and that the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ was already extensively studied in [25] for the the case $\left(X_{\Sigma}, D\right)=\left(\mathbb{C P}^{m-1}, D_{m}(d)\right), 4$ where $D_{m}(d) \in \mathbb{N}^{m}$ denotes the $m$-tuple of positive integers defined by

$$
\begin{equation*}
D_{m}(d)=(d, d, \cdots, d) \quad(m \text {-times }) . \tag{1.4}
\end{equation*}
$$

1.3 The main results. In this paper we will study the homotopy type of the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ for $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$. In particular, we will show that Atiyah-Jones-Segal type homotopy stability holds for the space $Q_{n}^{D, \Sigma}(\mathbb{K})$.

In our result we will need the following two conditions (1.4)* and (1.4) $\dagger^{\dagger} \cdot 5$
(1.4)* $d_{\text {min }} \geq n \geq 1$.
$(1.4)^{\dagger} d_{\text {min }} \geq n \geq 1$ and $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$.
Let $d(D ; \Sigma, n, \mathbb{K})$ denote the positive integer defined by

$$
d(D ; \Sigma, n, \mathbb{K})= \begin{cases}\left(2 n r_{\min }(\Sigma)-2\right)\left\lfloor d_{\min } / n\right\rfloor-2 & \text { if } \mathbb{K}=\mathbb{C}  \tag{1.5}\\ \left(n r_{\min }(\Sigma)-2\right)\left\lfloor d_{\min } / n\right\rfloor-2 & \text { if } \mathbb{K}=\mathbb{R}\end{cases}
$$

Then we can state the main result of this article as follows.
Theorem 1.5 (Theorems 2.14 and 2.15). Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in$ $\mathbb{N}^{r}$, and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety satisfying the condition (2.18)*.

[^3](i) If the condition $(1.4)^{*}$ is satisfied, the map (given by (2.25) and (10.1))
$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$
is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.
(ii) If the condition (1.4) ${ }^{\dagger}$ is satisfied, the map (given by (2.30) and (10.3))
$$
j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)
$$
is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.
Remark 1.6. (i) Recall that a map $g: V \rightarrow W$ is called a homology (resp. homotopy) equivalence through dimension $N$ if the induced homomorphism $g_{*}: H_{k}(V ; \mathbb{Z}) \rightarrow H_{k}(W ; \mathbb{Z})$ (resp. $\left.g_{*}: \pi_{k}(V) \rightarrow \pi_{k}(W)\right)$ is an isomorphism for all $k \leq N$.
(ii) Similarly, when $G$ is a topological group and a map $g: V \rightarrow W$ is a $G$-equivariant map between $G$-spaces $V$ and $W$, the map $g$ is called a $G$ equivariant homology (resp. G-equivariant homology homotopy) equivalence through dimension $N$ if the restriction $g^{H}=g \mid V^{H}: V^{H} \rightarrow W^{H}$ is a homology (resp. homotopy) equivalence through dimension $N$ for any subgroup $H \subset G$. Here, for each $G$-space $X$ and a subgroup $H \subset G$, let $X^{H}$ denote the $H$-fixed subspace of $X$ defined by
\[

$$
\begin{equation*}
X^{H}=\{x \in X: h \cdot x=x \quad \text { for any } h \in H\} . \tag{1.6}
\end{equation*}
$$

\]

1.4 Organization. This paper is organized as follows. In $\$ 2$ we recall the basic definitions and facts which is needed for the statements of the results of this article. After then precise statements of the main results (Theorems 2.14, 2.15, and Corollary 2.16) are stated. In $\S 3$ we recall several basic facts related to polyhedral products and toric varieties. In §4, we summarize the definition of the non-degenerate simplicial resolution, and we construct the Vassiliev spectral sequence. In $\$ 5$ we define the stabilization maps, and in §6, we construct the truncated spectral sequence induced from the spectral sequence obtained in $\$ 4$. By using this truncated spectral sequence, we shall prove the homology stability result (Theorems 6.5, 6.8, and Corollary 6.6). In \$7 we investigate about the connectivity of the space $Q_{n}^{D, \Sigma}(\mathbb{K})$. In particular, we prove that the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})\left(\right.$ resp. $\left.\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})\right)$ is simply connected if the condition (1.4)* (resp. (1.4) ${ }^{\dagger}$ ) is satisfied. In $\$ 8$ we consider the configuration model for the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ and recall the stabilized horizontal scanning map (see Theorem 8.7). In $\$ 9$ we prove the stability result (Theorem 9.2), and in $\S 10$ we give the proofs of the main results (Theorems 2.14, 2.15, and Corollary (2.16) by using it.

## 2 Toric varieties and the main results

In this section we recall several basic definitions and facts related to toric varieties (convex rational polyhedral cones, toric varieties, a fan of toric variety, polyhedral products, homogenous coordinate, rational curves on a toric variety etc). Then by using these definitions and notations we give precise statements of the main results of this paper. From now on, we always assume that $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$. Moreover, if $d_{\text {min }}<n,\left\lfloor d_{\min } / n\right\rfloor=0$ and $d(D ; \Sigma, n, \mathbb{K})=-2<0$. So we also assume that $d_{\min } \geq n \geq 1$.
2.1 Fans, toric varieties and Polyhedral products. A convex rational polyhedral cone in $\mathbb{R}^{m}$ is a subset of $\mathbb{R}^{m}$ of the form

$$
\begin{equation*}
\sigma=\operatorname{Cone}(S)=\operatorname{Cone}\left(\boldsymbol{m}_{1}, \cdots, \boldsymbol{m}_{s}\right)=\left\{\sum_{k=1}^{s} \lambda_{k} \boldsymbol{m}_{k}: \lambda_{k} \geq 0\right\} \tag{2.1}
\end{equation*}
$$

for a finite set $S=\left\{\boldsymbol{m}_{1}, \cdots, \boldsymbol{m}_{s}\right\} \subset \mathbb{Z}^{m}$. The dimension of $\sigma$ is the dimension of the smallest subspace of $\mathbb{R}^{m}$ which contains $\sigma$. A convex rational polyhedral cone $\sigma$ is called strongly convex if $\sigma \cap(-\sigma)=\left\{\mathbf{0}_{m}\right\}$, where we set $\mathbf{0}_{m}=\mathbf{0}=(0,0, \cdots, 0) \in \mathbb{R}^{m}$. A face $\tau$ of a convex rational polyhedral cone $\sigma$ is a subset $\tau \subset \sigma$ of the form $\tau=\sigma \cap\left\{\boldsymbol{x} \in \mathbb{R}^{m}: L(\boldsymbol{x})=0\right\}$ for some linear form $L$ on $\mathbb{R}^{m}$, such that $\sigma \subset\left\{\boldsymbol{x} \in \mathbb{R}^{m}: L(\boldsymbol{x}) \geq 0\right\}$. Note that if $\sigma$ is a strongly convex rational polyhedral cone, so is any of its faces ${ }^{6}$

Definition 2.1. Let $\Sigma$ be a finite collection of strongly convex rational polyhedral cones in $\mathbb{R}^{m}$.
(i) The set $\Sigma$ is called $a$ fan (in $\mathbb{R}^{m}$ ) if the following two conditions hold:
(2.1]) Every face $\tau$ of $\sigma \in \Sigma$ belongs to $\Sigma$.
(2.1b) If $\sigma_{1}, \sigma_{2} \in \Sigma, \sigma_{1} \cap \sigma_{2}$ is a common face of each $\sigma_{k}$ and $\sigma_{1} \cap \sigma_{2} \in \Sigma$.
(ii) An $m$ dimensional irreducible normal variety $X$ (over $\mathbb{C}$ ) is called $a$ toric variety if it has a Zariski open subset $\mathbb{T}_{\mathbb{C}}^{m}=\left(\mathbb{C}^{*}\right)^{m}$ and the action of $\mathbb{T}_{\mathbb{C}}^{m}$ on itself extends to an action of $\mathbb{T}_{\mathbb{C}}^{m}$ on $X$.

The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan $\Sigma$. We denote by $X_{\Sigma}$ the toric variety associated to a fan $\Sigma$ (see [11] for the details).
(iii) Let $K$ be some set of subsets of $[r]$. Then the set $K$ is called an abstract simplicial complex on the index set $[r]$ if the following condition $(\dagger)_{K}$ holds:

[^4]$(\dagger)_{K} \quad \tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$.
Remark 2.2. (i) It is well known that there are no holomorphic maps $\mathbb{C P}^{1}=$ $S^{2} \rightarrow \mathbb{T}_{\mathbb{C}}^{m}$ except the constant maps, and that the fan $\Sigma$ of $\mathbb{T}_{\mathbb{C}}^{m}$ is $\Sigma=\left\{\mathbf{0}_{m}\right\}$. Hence, without loss of generality we always assume that $X_{\Sigma} \neq \mathbb{T}_{\mathbb{C}}^{m}$, and that any fan $\Sigma$ in $\mathbb{R}^{m}$ satisfies the condition $\left\{\mathbf{0}_{m}\right\} \varsubsetneqq \Sigma$.
(ii) In this paper by a simplicial complex $K$ we always mean an abstract simplicial complex, and we always assume that a simplicial complex $K$ contains the empty set $\emptyset$.

Definition 2.3. Let $K$ be a simplicial complex on the index set $[r]=$ $\{1,2, \cdots, r\}$, and let $(X, A)$ be a pairs of based spaces.
(i) Let $I(K)$ denote the collection of subsets $\sigma \subset[r]$ defined by

$$
\begin{equation*}
I(K)=\{\sigma \subset[r]: \sigma \notin K\} . \tag{2.2}
\end{equation*}
$$

(ii) Define the polyhedral product $\mathcal{Z}_{K}(X, A)$ with respect to $K$ by

$$
\begin{align*}
\mathcal{Z}_{K}(X, A) & =\bigcup_{\sigma \in K}(X, A)^{\sigma}, \quad \text { where }  \tag{2.3}\\
(X, A)^{\sigma} & =\left\{\left(x_{1}, \cdots, x_{r}\right) \in X^{r}: x_{k} \in A \text { if } k \notin \sigma\right\} .
\end{align*}
$$

(iii) For each subset $\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \subset[r]$, let $L_{\sigma}\left(\mathbb{K}^{n}\right)$ denote the subspace of $\mathbb{K}^{n r}$ defined by

$$
\begin{equation*}
L_{\sigma}\left(\mathbb{K}^{n}\right)=\left\{\left(\boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{r}\right) \in\left(\mathbb{K}^{n}\right)^{r}=\mathbb{K}^{n r}: \boldsymbol{x}_{i_{1}}=\cdots=\boldsymbol{x}_{i_{s}}=\mathbf{0}_{n}\right\} \tag{2.4}
\end{equation*}
$$

and let $L_{n}^{K}(\mathbb{K})$ denote the subspace of $\mathbb{K}^{n r}$ defined by

$$
\begin{equation*}
L_{n}^{K}(\mathbb{K})=\bigcup_{\sigma \in I(K)} L_{\sigma}\left(\mathbb{K}^{n}\right)=\bigcup_{\sigma \subset[r], \sigma \notin K} L_{\sigma}\left(\mathbb{K}^{n}\right) \tag{2.5}
\end{equation*}
$$

Then it is easy to see that

$$
\begin{equation*}
\mathcal{Z}_{K}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)=\mathbb{K}^{n r} \backslash L_{n}^{K}(\mathbb{K}), \quad \text { where }\left(\mathbb{K}^{n}\right)^{*}=\mathbb{K}^{n} \backslash\left\{\mathbf{0}_{n}\right\} \tag{2.6}
\end{equation*}
$$

2.2 Homogenous coordinates. Next we recall the basic facts about homogenous coordinates on toric varieties.

Definition 2.4. Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ such that $\left\{\mathbf{0}_{m}\right\} \varsubsetneqq \Sigma$, and let

$$
\begin{equation*}
\Sigma(1)=\left\{\rho_{1}, \cdots, \rho_{r}\right\} \tag{2.7}
\end{equation*}
$$

denote the set of all one dimensional cones in $\Sigma$.
(i) For each $1 \leq k \leq r$, we denote by $\boldsymbol{n}_{k} \in \mathbb{Z}^{m}$ the primitive generator of $\rho_{k}$, such that $\rho_{k} \cap \mathbb{Z}^{m}=\mathbb{Z}_{\geq 0} \cdot \boldsymbol{n}_{k}$. Note that $\rho_{k}=\operatorname{Cone}\left(\boldsymbol{n}_{k}\right)$.
(ii) Let $\mathcal{K}_{\Sigma}$ denote the underlying simplicial complex of $\Sigma$ defined by

$$
\begin{equation*}
\mathcal{K}_{\Sigma}=\left\{\left\{i_{1}, \cdots, i_{s}\right\} \subset[r]: \boldsymbol{n}_{i_{1}}, \boldsymbol{n}_{i_{2}}, \cdots, \boldsymbol{n}_{i_{s}} \text { span a cone in } \Sigma\right\} . \tag{2.8}
\end{equation*}
$$

It is easy to see that $\mathcal{K}_{\Sigma}$ is a simplicial complex on the index set $[r]$.
(iii) Let $G_{\Sigma, \mathbb{K}} \subset \mathbb{T}_{\mathbb{K}}^{r}=\left(\mathbb{K}^{*}\right)^{r}$ be the subgroup

$$
\begin{equation*}
G_{\Sigma, \mathbb{K}}=\left\{\left(\mu_{1}, \cdots, \mu_{r}\right) \in \mathbb{T}_{\mathbb{K}}^{r}: \prod_{k=1}^{r}\left(\mu_{k}\right)^{\left\langle n_{k}, m\right\rangle}=1 \text { for all } \boldsymbol{m} \in \mathbb{Z}^{m}\right\} \tag{2.9}
\end{equation*}
$$

where $\langle\boldsymbol{u}, \boldsymbol{v}\rangle=\sum_{k=1}^{m} u_{k} v_{k}$ for $\boldsymbol{u}=\left(u_{1}, \cdots, u_{m}\right)$ and $\boldsymbol{v}=\left(v_{1}, \cdots, v_{m}\right) \in \mathbb{R}^{m}$.
(iv) There is a natural $G_{\Sigma, \mathbb{K}}$-action on $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$ by coordinate-wise multiplication,

$$
\begin{equation*}
\left(\mu_{1}, \cdots, \mu_{r}\right) \cdot\left(\boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{r}\right)=\left(\mu_{1} \boldsymbol{x}_{1}, \cdots, \mu_{r} \boldsymbol{x}_{r}\right) \tag{2.10}
\end{equation*}
$$

for $\left(\left(\mu_{1}, \cdots, \mu_{r}\right),\left(\boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{r}\right)\right) \in G_{\Sigma, \mathbb{K}} \times \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$, where we set

$$
\begin{equation*}
\mu \boldsymbol{x}=\left(\mu x_{1}, \cdots, \mu x_{n}\right) \quad \text { if }(\mu, \boldsymbol{x})=\left(\mu,\left(x_{1}, \cdots, x_{n}\right)\right) \in \mathbb{K}^{*} \times \mathbb{K}^{n} \tag{2.11}
\end{equation*}
$$

(v) Let $X_{\Sigma, \mathbb{K}}(n)$ denote the corresponding orbit space

$$
\begin{gather*}
X_{\Sigma, \mathbb{K}}(n)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) / G_{\Sigma, \mathbb{K}}, \quad \text { where }  \tag{2.12}\\
q_{n, \mathbb{K}}: \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \rightarrow X_{\Sigma, \mathbb{K}}(n)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) / G_{\Sigma, \mathbb{K}} \tag{2.13}
\end{gather*}
$$

denotes the corresponding canonical projection. In particular, we also write

$$
\begin{equation*}
X_{\Sigma}(n)=X_{\Sigma, \mathbb{C}}(n) \text { and } G_{\Sigma}=G_{\Sigma, \mathbb{C}} \quad \text { if } \mathbb{K}=\mathbb{C} \tag{2.14}
\end{equation*}
$$

Theorem 2.5 ( 9 , Theorem 2.1). If the set $\left\{\boldsymbol{n}_{k}\right\}_{k=1}^{r}$ of all primitive generators spans $\mathbb{R}^{m}$ (i.e. $\sum_{k=1}^{r} \mathbb{R} \cdot \boldsymbol{n}_{k}=\mathbb{R}^{m}$ ), there is a natural isomorphism

$$
\begin{equation*}
X_{\Sigma} \cong \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}, \mathbb{C}^{*}\right) / G_{\Sigma, \mathbb{C}}=X_{\Sigma}(1)=X_{\Sigma, \mathbb{C}}(1) \tag{2.15}
\end{equation*}
$$

Hence, we can identify $X_{\Sigma}(n)$ with the toric variety $X_{\Sigma}$ if $n=1$.
Remark 2.6. Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ as in Definition 2.4. Then the fan $\Sigma$ is completely determined by the pair $\left(\mathcal{K}_{\Sigma},\left\{\boldsymbol{n}_{k}\right\}_{k=1}^{r}\right)$ (see [23, Remark 2.3] for the details).

For each $1 \leq i \leq r$, let $F_{i}=\left(f_{1 ; i}, \cdots, f_{n ; i}\right) \in \mathbb{K}\left[z_{0}, \cdots, z_{s}\right]^{n}$ be an $n$-tuple of homogenous polynomials of the same degree $d_{i}$ satisfying the following condition:
(2.16)* For each $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$, the homogenous polynomials $\left\{f_{k ; i}\right\}_{k \in \sigma}$ have no common real root except $\mathbf{0}_{s+1} \in \mathbb{R}^{s+1}$.

In this situation, consider the map

$$
\begin{gather*}
F=\left(F_{1}, \cdots, F_{r}\right): \mathbb{R}^{s+1} \backslash\left\{\mathbf{0}_{s+1}\right\} \rightarrow\left(\mathbb{K}^{n}\right)^{r}=\mathbb{K}^{r n} \text { given by }  \tag{2.16}\\
\left\{\begin{array}{l}
F(\boldsymbol{x})=\left(F_{1}(\boldsymbol{x}), \cdots F_{r}(\boldsymbol{x})\right) \quad \text { for } \boldsymbol{x} \in \mathbb{R}^{m+1} \backslash\left\{\mathbf{0}_{m+1}\right\} \\
F_{i}(\boldsymbol{x})=\left(f_{1 ; i}(\boldsymbol{x}), f_{2 ; i}(\boldsymbol{x}), \cdots, f_{n ; i}(\boldsymbol{x})\right) \quad \text { for } 1 \leq i \leq r .
\end{array}\right.
\end{gather*}
$$

By the assumption (2.16)*, homogenous polynomials $\left\{f_{k ; i}\right\}_{k \in \sigma}$ have no common real root except $\mathbf{0}_{s+1} \in \mathbb{R}^{s+1}$ for each $1 \leq i \leq r$ and $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$. Thus, we see that the image of the map $F$ is contained in $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$, and we may regard the map $F$ as the map

$$
\begin{equation*}
F=\left(F_{1}, \cdots, F_{r}\right): \mathbb{R}^{s+1} \backslash\left\{\mathbf{0}_{s+1}\right\} \rightarrow \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \tag{2.17}
\end{equation*}
$$

The following lemma, whose proof we postpone until the end of §3, plays a crucial role in the proof of the main result of this paper.

Lemma 2.7 (cf. [10], Theorem 3.1; [19], Lemma 2.6). Suppose that the set $\left\{\boldsymbol{n}_{k}\right\}_{k=1}^{r}$ of all primitive generators spans $\mathbb{R}^{m}$. For each $1 \leq i \leq r$ and $\sigma \in$ $I\left(\mathcal{K}_{\Sigma}\right)$, let $F_{i}=\left(f_{1 ; i}, \cdots, f_{n ; i}\right) \in \mathbb{K}\left[z_{0}, \cdots, z_{s}\right]^{n}$ be an $n$-tuple of homogenous polynomials of the same degree $d_{i}$ satisfying the condition (2.16)*.

Then there is a unique map $f: \mathbb{R} \mathrm{P}^{s} \rightarrow X_{\Sigma, \mathbb{K}}(n)$ such that the diagram

is commutative if and only if the condition $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$. holds.
Here, $\gamma_{s}: \mathbb{R}^{s+1} \backslash\left\{\mathbf{0}_{s+1}\right\} \rightarrow \mathbb{R} P^{s}$ denotes the canonical double covering, and the map $F=\left(F_{1}, \cdots, F_{r}\right)$ is given by (2.17).
2.3 Assumptions. From now on, let $\Sigma$ be a fan in $\mathbb{R}^{m}$ as in Definition 2.4, and we always assume that $X_{\Sigma}$ is simply connected and non-singular. Moreover, we shall assume the following condition holds.
(2.18)* There is an $r$-tuple $D_{*}=\left(d_{1}^{*}, \cdots, d_{r}^{*}\right) \in \mathbb{N}^{r}$ such that $\sum_{k=1}^{r} d_{k}^{*} \boldsymbol{n}_{k}=\mathbf{0}_{m}$.

Remark 2.8. (i) It follows from [11, Theorem 12.1.10] that $X_{\Sigma}$ is simply connected if and only if the following condition ( $\dagger \dagger$ ) holds:
( $\dagger \dagger$ ) The set $\left\{\boldsymbol{n}_{k}\right\}_{k=1}^{r}$ of all primitive generators spans $\mathbb{Z}^{m}$ over $\mathbb{Z}$, i.e. $\sum_{k=1}^{r} \mathbb{Z} \cdot \boldsymbol{n}_{k}=\mathbb{Z}^{m}$.

Thus we see that if $X_{\Sigma}$ is simply connected then the set $\left\{\boldsymbol{n}_{k}\right\}_{k=1}^{r}$ of all primitive generators spans $\mathbb{R}^{m}$. In particular, if $X_{\Sigma}$ is a compact smooth toric variety then $X_{\Sigma}$ is simply connected (see Lemma 3.8).
(ii) We make the identification $\mathbb{R P}^{1}=S^{1}=\mathbb{R} \cup \infty$ and choose the points $\infty$ and $[1,1, \cdots, 1]$ as the base points of $\mathbb{R} \mathrm{P}^{1}$ and $X_{\Sigma}$, respectively. Then, by setting $z=\frac{z_{0}}{z_{1}}$, for each $1 \leq k \leq r$, we can view $f_{k}$ as a monic polynomial $f_{k}(z) \in \mathbb{K}[z]$ of degree $d_{k}$ in the real variable $z$.
2.4 Spaces of algebraic maps of real bounded multiplicity. Now we can define the space of algebraic maps as follows.

Definition 2.9. From now on, let $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$ as before.
(i) For a monic polynomial $f(z) \in \mathbb{K}[z]$ of degree $d$, let $F_{n}(f)(z)$ denote the $n$-tuple of monic polynomials of the same degree $d$ defined by

$$
\begin{equation*}
F_{n}(f)(z)=\left(f(z), f(z)+f^{\prime}(z), f(z)+f^{\prime \prime}(z), \cdots, f(z)+f^{(n-1)}(z)\right) . \tag{2.19}
\end{equation*}
$$

Note that a monic polynomial $f(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C}$ of multiplicity $\geq n$ iff $F_{n}(f)(\alpha)=0_{n} \in \mathbb{C}^{n}$.
(ii) For each $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ and a fan $\Sigma$ in $\mathbb{R}^{m}$, let $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ denote the space of $r$-tuples $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathbb{K}[z]^{r}$ of $\mathbb{K}$-coefficients monic polynomials satisfying the conditions (1.2 a$)$ and (1.2b) (as in Definition 1.4).

Remark 2.10. (i) Note that $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is path-connected, and that $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is path-connected if $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$ (which will be explained in Remark 8.4).
(ii) Let $\mathbb{Z}_{2}=\{ \pm 1\}$ denote the multiplicative cyclic group of order 2, and let $X^{\mathbb{Z}_{2}}$ denote the $\mathbb{Z}_{2}$-fixed point set of a $\mathbb{Z}_{2}$-space $X$ as in (1.6). Complex conjugation on $\mathbb{C}$ extends to a $\mathbb{Z}_{2}$-actions on the spaces $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right)$ and $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ such that

$$
\begin{equation*}
\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right)^{\mathbb{Z}_{2}}, \quad \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})=\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})^{\mathbb{Z}_{2}} . \tag{2.20}
\end{equation*}
$$

It is easy to see that complex conjugation on $\mathbb{C}$ also naturally extends to a $\mathbb{Z}_{2}$-action on $X_{\Sigma}(n)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) / G_{\Sigma}$ such that

$$
\begin{equation*}
X_{\Sigma, \mathbb{R}}(n)=X_{\Sigma}(n)^{\mathbb{Z}_{2}} . \tag{2.21}
\end{equation*}
$$

It easily follows from the definition of the above actions that the following diagram is commutative:

$$
\begin{gather*}
\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \xrightarrow{q_{n, \mathbb{R}}} X_{\Sigma, \mathbb{R}}(n)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right) / G_{\Sigma, \mathbb{R}} \\
i_{n}^{X} \downarrow n \tag{2.22}
\end{gather*}
$$

where let $i_{n}^{D}$ and $i_{n}^{X}$ denote he corresponding inclusion maps.
Remark that $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ (resp. $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ ) is simply connected if the condition (1.4)* (resp. (1.4) ${ }^{\dagger}$ ) is satisfied (which will be proved in Corollary 7.3).

Definition 2.11. Suppose that the condition (2.18)* holds, and let $D=$ $\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ be an $r$-tuple of positive integers satisfying the condition

$$
\begin{equation*}
\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m} . \tag{2.23}
\end{equation*}
$$

(i) First, consider the case $\mathbb{K}=\mathbb{C}$. By Lemma 2.7, one can define a map

$$
\begin{align*}
& i_{D, n, \mathbb{C}}: \mathbb{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega X_{\Sigma}(n) \quad \text { by }  \tag{2.24}\\
& i_{D, n, \mathbb{C}}(f)(\alpha)= \begin{cases}{\left[F_{n}\left(f_{1}\right)(\alpha), F_{n}\left(f_{2}\right)(\alpha), \cdots, F_{n}\left(f_{r}\right)(\alpha)\right]} & \text { if } \alpha \in \mathbb{R} \\
{[\boldsymbol{e}, \boldsymbol{e}, \cdots, \boldsymbol{e}]} & \text { if } \alpha=\infty\end{cases}
\end{align*}
$$

for $f=\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ and $\alpha \in \mathbb{R} \cup \infty=S^{1}$, where we set $e=(1,1, \cdots, 1) \in \mathbb{C}^{n}$.

Since the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is simply connected and $\Omega q_{n, \mathbb{C}}$ is a universal covering (by (iv) of Remark 2.10 and (ii) of Corollary 3.10), the map $i_{D, n, \mathbb{C}}$ lifts to the space $\Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$, and there is a based map

$$
\begin{equation*}
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \simeq \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \tag{2.25}
\end{equation*}
$$

such that the following equality holds:

$$
\begin{equation*}
\Omega q_{n, \mathbb{C}} \circ j_{D, n, \mathbb{C}}=i_{D, n, \mathbb{C}} \tag{2.26}
\end{equation*}
$$

(ii) Next, consider the case $\mathbb{K}=\mathbb{R}$.

Recall the $\mathbb{Z}_{2}$-action on the spaces $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ and $X_{\Sigma}$ induced from complex conjugation on $\mathbb{C}$, and remark that the map $i_{D, n, \mathbb{C}}$ is a $\mathbb{Z}_{2}$-equivariant map. Then, by (2.20) and (2.21), we see that

$$
\begin{equation*}
i_{D, n, \mathbb{C}}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})\right) \subset \Omega X_{\Sigma}(n)^{\mathbb{Z}_{2}}=\Omega X_{\Sigma, \mathbb{R}}(n) \tag{2.27}
\end{equation*}
$$

Thus, the restriction $i_{D, n, \mathbb{C}} \mid \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ defines the map

$$
\begin{equation*}
i_{D, n, \mathbb{R}}=i_{D, n, \mathbb{C}} \mid Q_{n}^{D, \Sigma}(\mathbb{R}): \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega X_{\Sigma, \mathbb{R}}(n) \tag{2.28}
\end{equation*}
$$

such that the following diagram is commutative:

$$
\begin{align*}
& \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \xrightarrow{i_{D, n, \mathbb{R}}} \Omega X_{\Sigma, \mathbb{R}}(n) \underset{\simeq}{\stackrel{\Omega q_{n, \mathbb{R}}}{\simeq}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right) \\
& i_{n}^{D} \downarrow \quad \Omega i_{n}^{X} \downarrow \quad \Omega j_{n} \downarrow  \tag{2.29}\\
& \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \xrightarrow{i_{D, n, \mathbb{C}}} \Omega X_{\Sigma}(n) \stackrel{\Omega q_{n, \mathbb{C}}}{\rightleftarrows} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right)
\end{align*}
$$

where the $j_{n}: \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right) \xrightarrow{\subset} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right)$ denotes the inclusion map. Note that $\Omega q_{n, \mathbb{R}}$ is a homotopy equivalence (which will be proved in Corollary 3.10). Thus, there is a based map

$$
\begin{equation*}
j_{D, n, \mathbb{R}}: Q_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right) \simeq \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right) \tag{2.30}
\end{equation*}
$$

which satisfies the following equality:

$$
\begin{equation*}
\Omega q_{n, \mathbb{R}} \circ j_{D, n, \mathbb{R}}=i_{D, n, \mathbb{R}} \quad(\text { up to homotopy }) . \tag{2.31}
\end{equation*}
$$

Remark 2.12. When $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{n}$, by (2.25) and (2.31), we obtain the map

$$
\begin{equation*}
j_{D, n, \mathbb{K}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \simeq \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right) \tag{2.32}
\end{equation*}
$$

where the number $d(\mathbb{K})$ is defined by

$$
d(\mathbb{K})=\operatorname{dim}_{\mathbb{R}} \mathbb{K}= \begin{cases}2 & \text { if } \mathbb{K}=\mathbb{C}  \tag{2.33}\\ 1 & \text { if } \mathbb{K}=\mathbb{R}\end{cases}
$$

2.5 The numbers $r_{\min }(\Sigma)$ and $d(D ; \Sigma, n, \mathbb{K})$. Before stating the main results of this paper, we need to define the positive integers $r_{\text {min }}(\Sigma)$ and $d(D ; \Sigma, n, \mathbb{K})$ (which already appeared in the statements of our results).

Definition 2.13. (i) We say that a set $S=\left\{\boldsymbol{n}_{i_{1}}, \cdots, \boldsymbol{n}_{i_{s}}\right\}$ is a primitive collection if Cone $(S) \notin \Sigma$ and Cone $(T) \in \Sigma$ for any proper subset $T \varsubsetneqq S$.
(ii) For each $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, define the positive integer $d(D, \Sigma, n, \mathbb{K})$ by

$$
d(D ; \Sigma, n, \mathbb{K})= \begin{cases}\left(2 n r_{\min }(\Sigma)-2\right)\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor-2 & \text { if } \mathbb{K}=\mathbb{C}  \tag{2.34}\\ \left(n r_{\text {min }}(\Sigma)-2\right)\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor-2 & \text { if } \mathbb{K}=\mathbb{R}\end{cases}
$$

as in (1.5), where $r_{\min }(\Sigma)$ and $d_{\min }$ are the positive integers given by

$$
\left\{\begin{array}{l}
r_{\min }(\Sigma)=\min \left\{s \in \mathbb{N}:\left\{\boldsymbol{n}_{i_{1}}, \cdots, \boldsymbol{n}_{i_{s}}\right\} \text { is a primitive collection }\right\}  \tag{2.35}\\
d_{\min }=\min \left\{d_{1}, d_{2}, \cdots, d_{r}\right\}
\end{array}\right.
$$

Note that

$$
\begin{equation*}
r_{\min }(\Sigma) \geq 2 . \tag{2.36}
\end{equation*}
$$

2.6 The main results. Note that the space $Q_{n}^{D, \Sigma}(\mathbb{C})$ has already been extensively studied in the case $n=1$ in [24]. The main purpose of this paper is to generalize the results of [24] to the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ for $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$ and for any $n \geq 1$. Indeed, the main results of this article are stated as follows.

Theorem 2.14 (The case $\mathbb{K}=\mathbb{C}$ ). Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety such that the two conditions (2.18)* and (1.4)* holds.
(i) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the map (given by (2.25))

$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$

is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.
(ii) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, there is a map

$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$

which is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{C}) \cdot 7$
Theorem 2.15 (The case $\mathbb{K}=\mathbb{R}$ ). Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety such that the two conditions (2.18) ${ }^{*}$ and $(1.4)^{\dagger}$ hold.
(i) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the map (given by (2.30))

$$
j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)
$$

is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.
(ii) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, there is a map

$$
j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)
$$

which is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.

[^5]Corollary 2.16. Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$. and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety such that the two conditions (2.18)* and (1.4)* hold.
(i) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the map $i_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega X_{\Sigma}(n)$ induces an isomorphism

$$
\left(i_{D, n, \mathbb{C}}\right)_{*}: \pi_{s}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})\right) \xrightarrow{\cong} \pi_{s}\left(\Omega X_{\Sigma}\right) \cong \pi_{s+1}\left(X_{\Sigma}(n)\right)
$$

for any $2 \leq s \leq d(D ; \Sigma, n, \mathbb{C})$.
(ii) If $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, the map $i_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega X_{\Sigma}(n)$ defined by

$$
\begin{equation*}
i_{D, n, \mathbb{C}}:=\Omega q_{n, \mathbb{C}} \circ j_{D, n, \mathbb{C}} \tag{2.37}
\end{equation*}
$$

induces an isomorphism

$$
\left(i_{D, n, \mathbb{C}}\right)_{*}: \pi_{s}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})\right) \xrightarrow{\cong} \pi_{s}\left(\Omega X_{\Sigma}(n)\right) \cong \pi_{s+1}\left(X_{\Sigma}(n)\right)
$$

for any $2 \leq s \leq d(D ; \Sigma, n, \mathbb{C})$.
Consider the $\mathbb{Z}_{2}$-action on the spaces $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ and $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$ induced from the complex conjugation on $\mathbb{C}$, where we identify

$$
\begin{equation*}
D^{2 n}=\left\{\left(x_{1}, \cdots, x_{n}\right) \in \mathbb{C}^{n}: \sum_{k=1}^{n}\left|x_{k}\right|^{2} \leq 1\right\} \tag{2.38}
\end{equation*}
$$

Note that we can regard the space $D^{2 n}$ as a $\mathbb{Z}_{2}$-space whose $\mathbb{Z}_{2}$ action is given by the complex conjugation.

$$
\begin{equation*}
(-1) \cdot\left(x_{1}, \cdots, x_{n}\right)=\left(\overline{x_{1}}, \cdots, \overline{x_{n}}\right) \quad \text { for }\left(x_{1}, \cdots, x_{n}\right) \in D^{2 n} \tag{2.39}
\end{equation*}
$$

Since $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})=\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})^{\mathbb{Z}_{2}}, \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)=\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)^{\mathbb{Z}_{2}}$, and $j_{D, n, \mathbb{R}}=\left(j_{D, n, \mathrm{C}}\right)^{\mathbb{Z}_{2}}$, we also obtain the following result.

Corollary 2.17. Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, and let $X_{\Sigma}$ be an $m$ dimensional simply connected non-singular toric variety satisfying the two conditions (2.18)* and (1.4) ${ }^{\dagger}$. Then the map

$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)
$$

is a $\mathbb{Z}_{2}$-equivariant homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.

Finally, we easily obtain the following two corollaries.

Corollary 2.18. Let $n \in \mathbb{N}$, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, and let $X_{\Sigma}$ be a simply connected compact non-singular toric variety such that the the condition (2.18)* holds. Let $\Sigma(1)$ denote the set of all one dimensional cones in $\Sigma$, and let $\Sigma_{1}$ be any fan in $\mathbb{R}^{m}$ satisfying the condition

$$
\begin{equation*}
\Sigma(1) \subset \Sigma_{1} \varsubsetneqq \Sigma . \tag{2.40}
\end{equation*}
$$

(i) Then $X_{\Sigma_{1}}$ is a non-compact smooth toric subvariety of $X_{\Sigma}$.
(ii) If the condition (1.4) holds and $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the map

$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\Sigma_{1}}\left(D^{2 n}, S^{2 n-1}\right)
$$

is a homotopy equivalence through the dimension $d\left(D ; \Sigma_{1}, n, \mathbb{C}\right)$.
Moreover, the map $i_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{C}) \rightarrow \Omega X_{\Sigma_{1}}$ induces an isomorphism
for any $2 \leq s \leq d\left(D ; \Sigma_{1}, n, \mathbb{C}\right)$.
(iii) If the condition (1.4)* holds and $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, then there is a map

$$
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_{1}}}\left(D^{2 n}, S^{2 n-1}\right)
$$

which is a homotopy equivalence through dimension $d\left(D ; \Sigma_{1}, n, \mathbb{C}\right)$.
Moreover, the map $i_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{C}) \rightarrow \Omega X_{\Sigma_{1}}$ defined by

$$
\begin{equation*}
i_{D, n, \mathbb{C}}:=\Omega q_{n, \mathbb{C}} \circ j_{D, n, \mathbb{C}} \tag{2.41}
\end{equation*}
$$

induces an isomorphism

$$
\left(i_{D, n, \mathbb{C}}\right)_{*}: \pi_{s}\left(\mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{C})\right) \xrightarrow{\cong} \pi_{s}\left(\Omega X_{\Sigma_{1}}(n)\right) \cong \pi_{s+1}\left(X_{\Sigma_{1}}(n)\right)
$$

for any $2 \leq s \leq d\left(D ; \Sigma_{1}, n, \mathbb{C}\right)$.
(iv) If the condition (1.4) holds and $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$, then the map

$$
j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\Sigma_{1}}\left(D^{n}, S^{n-1}\right)
$$

is a homotopy equivalence through the dimension $d\left(D ; \Sigma_{1}, n, \mathbb{K}\right)$.
(v) If the condition (1.4) ${ }^{\dagger}$ holds and $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{m}$, there is a map

$$
j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma_{1}}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_{1}}}\left(D^{n}, S^{n-1}\right)
$$

which is a homotopy equivalence through dimension $d\left(D ; \Sigma_{1}, n, \mathbb{R}\right)$.

## 3 Basic facts about toric varieties

In this section, we recall some basic definitions and known results.
Definition 3.1 ([7, Definition 6.27, Example 6.39). Let $K$ be a simplicial complex on the index set $[r]$, and let $I(K)=\{\sigma \subset[r]: \sigma \notin K\}$ as in (2.2).
(i) An element $\sigma \in I(K)$ is called a minimal non-face of $K$ if $\tau \in K$ for any proper subset $\tau \varsubsetneqq \sigma$.
(ii) Then we denote by $I_{\min }(K)$ the set of all minimal non-faces of $K$. It is easy to see that the following equality holds.

$$
\begin{equation*}
K=\left\{\sigma \subset[r]: \tau \not \subset \sigma \text { for any } \tau \in I_{\min }(K)\right\} . \tag{3.1}
\end{equation*}
$$

(iii) We denote by $\mathcal{Z}_{K}$ and $D J(K)$ the moment-angle complex of $K$ and the Davis-Januszkiewicz space of $K$ which are defined by

$$
\begin{equation*}
\mathcal{Z}_{K}=\mathcal{Z}_{K}\left(D^{2}, S^{1}\right), \quad D J(K)=\mathcal{Z}_{K}\left(\mathbb{C P}^{\infty}, *\right) \tag{3.2}
\end{equation*}
$$

Remark 3.2. Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ and let $X_{\Sigma}$ be a smooth toric variety such that the condition (2.18)* holds. Then it is easy to see that $\left\{\boldsymbol{n}_{i_{1}}, \boldsymbol{n}_{i_{2}}, \cdots, \boldsymbol{n}_{i_{s}}\right\}$ is primitive if and only if $\sigma=\left\{i_{1}, i_{2}, \cdots, i_{s}\right\} \in I_{\min }\left(\mathcal{K}_{\Sigma}\right)$. Thus, we also obtain the following equality:

$$
\begin{equation*}
r_{\min }(\Sigma)=\min \left\{\operatorname{card}(\sigma): \sigma \in I\left(\mathcal{K}_{\Sigma}\right)\right\} \tag{3.3}
\end{equation*}
$$

Lemma 3.3 ( 7 ; Corollary 6.30, Theorems $6.33,8.9$ ). Let $K$ be a simplicial complex on the index set $[r]$.
(i) The space $\mathcal{Z}_{K}$ is 2 -connected, and there is a fibration sequence

$$
\begin{equation*}
\mathcal{Z}_{K} \longrightarrow D J(K) \xrightarrow{\subset}\left(\mathbb{C P}^{\infty}\right)^{r} \tag{3.4}
\end{equation*}
$$

(ii) There are $\mathbb{T}^{r}$-equivariant deformation retraction

$$
\begin{equation*}
\text { ret : } \mathcal{Z}_{K}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \xrightarrow{\simeq} \mathcal{Z}_{K}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right) . \tag{3.5}
\end{equation*}
$$

where we set $\mathbb{T}^{r}=\left(S^{1}\right)^{r}$.
Lemma 3.4 ([30]). Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ and let $X_{\Sigma}$ be a smooth toric variety such that the condition (2.18)* holds.
(i) There is an isomorphism

$$
\begin{equation*}
G_{\Sigma, \mathbb{K}} \cong \mathbb{T}_{\mathbb{K}}^{r-m}=\left(\mathbb{K}^{*}\right)^{r-m} \tag{3.6}
\end{equation*}
$$

(ii) The group $G_{\Sigma, \mathbb{K}}$ acts on the space $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$ freely as in (2.10) and there is a principal $G_{\Sigma, \mathbb{K}}$-bundle sequence

$$
\begin{equation*}
G_{\Sigma, \mathbb{K}} \longrightarrow \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \xrightarrow{q_{n, \mathbb{K}}} X_{\Sigma, \mathbb{K}} \tag{3.7}
\end{equation*}
$$

(iii) If $\mathbb{K}=\mathbb{R}$, there is a homotopy equivalence $\mathbb{T}_{\mathbb{R}}^{r} \simeq\left(\mathbb{Z}_{2}\right)^{r-m}$ and the map $q_{n, \mathbb{R}}$ is a covering projection with fiber $\left(\mathbb{Z}_{2}\right)^{r-m}$ (up to homotopy).

Proof. First, consider the case $\mathbb{K}=\mathbb{C}$. Then the assertions (i) and (ii) follow from [30, (6.2) page 527; Proposition 6.7].

Next, let $\mathbb{K}=\mathbb{R}$. Since $G_{\Sigma}=G_{\Sigma, \mathbb{C}} \cong\left(\mathbb{C}^{*}\right)^{r-m}$ and $G_{\Sigma, \mathbb{R}}=G_{\Sigma} \cap\left(\mathbb{R}^{*}\right)^{r}$, we have the isomorphism $G_{\Sigma, \mathbb{R}} \cong\left(\mathbb{R}^{*}\right)^{r-m}=\mathbb{T}_{\mathbb{R}}^{r-m}$. Since $G_{\Sigma, \mathbb{C}}$ acts on the space $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right)$ freely, the subgroup $\mathbb{G}_{\Sigma, \mathbb{R}}$ also acts on the space $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right)$ freely and we obtain the $G_{\Sigma, \mathbb{R}^{-}}$principal fibration sequence (3.7) for the case $\mathbb{K}=\mathbb{R}$. This proves (i) and (ii) for the case $\mathbb{K}=\mathbb{R}$. Since $G_{\Sigma, \mathbb{R}} \simeq\left(\mathbb{Z}_{2}\right)^{r-m}, q_{n, \mathbb{R}}$ is a covering projection with fiber $\left(\mathbb{Z}_{2}\right)^{r-m}$, and we obtain (iii).

Definition 3.5 (c.f. [26], (5.26)). Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ and let $X_{\Sigma}$ be a smooth toric variety such that the condition (2.18)* holds.
(i) Let $\mathcal{K}_{\Sigma}(n)$ denote the simplicial complex on the index set $[r] \times[n]$ defined by

$$
\begin{equation*}
\mathcal{K}_{\Sigma}(n)=\left\{\tau \subset[r] \times[n]: \sigma \times[n] \not \subset \tau \text { for any } \sigma \in I\left(\mathcal{K}_{\Sigma}\right)\right\} \tag{3.8}
\end{equation*}
$$

(ii) For each $(i, j) \in[r] \times[n]$, let $\boldsymbol{n}_{i, j} \in \mathbb{Z}^{m n}$ denote the lattice vector defined by

$$
\boldsymbol{n}_{i, j}=\left(\boldsymbol{a}_{1}, \cdots, \boldsymbol{a}_{n}\right), \text { where we set } \boldsymbol{a}_{k}= \begin{cases}\boldsymbol{n}_{i} & (k=j)  \tag{3.9}\\ \mathbf{0}_{m} & (k \neq j)\end{cases}
$$

and define a fan $F_{n}(\Sigma)$ in $\mathbb{R}^{m n}$ by

$$
\begin{equation*}
F_{n}(\Sigma)=\left\{c_{\tau}: \tau \in \mathcal{K}_{\Sigma}(n)\right\}, \tag{3.10}
\end{equation*}
$$

where $c_{\tau}$ denotes the cone in $\mathbb{R}^{m n}$ given by

$$
\begin{equation*}
c_{\tau}=\operatorname{Cone}\left(\left\{\boldsymbol{n}_{i, j}:(i, j) \in \tau\right\}\right)=\left\{\sum_{(i, j) \in \tau} x_{i, j} \boldsymbol{n}_{i, j}: x_{i, j} \geq 0\right\} . \tag{3.11}
\end{equation*}
$$

Lemma 3.6. (i) If $\mathbb{T}^{r}=\left(S^{1}\right)^{r}$, there is a $\mathbb{T}^{r}$-equivariant homeomorphism

$$
\begin{equation*}
\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \cong \mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(D^{2}, S^{1}\right) . \tag{3.12}
\end{equation*}
$$

(ii) If $\mathbb{T}_{\mathbb{C}}^{r}=\left(\mathbb{C}^{*}\right)^{r}$, there is a $\mathbb{T}_{\mathbb{C}}^{r}$-equivariant homeomorphsim

$$
\begin{equation*}
\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \simeq \mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(\mathbb{C}, \mathbb{C}^{*}\right) \tag{3.13}
\end{equation*}
$$

(iii) The space $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$ is 2 -connected.

Proof. (i) Let $J=(n, n, \cdots, n) \in \mathbb{N}^{r}$ and let $\mathcal{K}_{\Sigma}(J)$ denote the simplical complex on the index set $[r] \times[n]$ defined by [6, Definition 2.1]:8 Then it follows from [6, Definition 2.1] that the following equality holds:

$$
\begin{equation*}
I_{\min }\left(\mathcal{K}_{\Sigma}(J)\right)=\left\{\tau \times[n]: \tau \in I_{\min }\left(\mathcal{K}_{\Sigma}\right)\right\} \tag{3.14}
\end{equation*}
$$

Hence, by (3.1) and (3.8), we obtain the following equality:

$$
\mathcal{K}_{\Sigma}(J)=\left\{\sigma \subset[r] \times[n]: \tau \times[n] \not \subset \sigma \text { for any } \tau \in I_{\min }\left(\mathcal{K}_{\Sigma}\right)\right\} .
$$

Thus, we we have $\mathcal{K}_{\Sigma}(J)=\mathcal{K}_{\Sigma}(n)$ by (3.14). Hence, it follows from [6, Theorem 7.5] that there is a $\mathbb{T}^{r}$-equivariant homeomorphism $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \cong$ $\mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(D^{2}, S^{1}\right)$, and the assertion (i) follows.
(ii) It follows from [26, (5.32)] that there is a homeomorphism

$$
\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \cong \mathcal{Z}_{\mathcal{K}_{\mathcal{K}_{\Sigma}(n)}}\left(\mathbb{C}, \mathbb{C}^{*}\right) .
$$

One can easily check that the above homeomorphism is $\mathbb{T}_{\mathbb{C}}^{r}$-equivariant, and the assertion (ii) follows.
(iii) It follows from (i) and (ii) that there is the following homotopy equivalence

$$
\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \simeq \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \cong \mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(\mathbb{C}, \mathbb{C}^{*}\right) \simeq \mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(D^{2}, S^{1}\right) .
$$

Since the moment-angle complex $\mathcal{Z}_{\mathcal{K}_{\Sigma}(n)}\left(D^{2}, S^{1}\right)$ is 2 -connected by [7, Theorem 6.33], the space $\mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$ is also 2-connected.

Definition 3.7 ([11]). Let $\Sigma$ be a fan in $\mathbb{R}^{m}$. Then a cone $\sigma \in \Sigma$ is called smooth if it is generated by a subset of a basis of $\mathbb{Z}^{m}$.

Lemma 3.8 ([1]). Let $X_{\Sigma}$ be a toric variety determined by a fan $\Sigma$ in $\mathbb{R}^{m}$.
(i) $X_{\Sigma}$ is compact if and only if $\mathbb{R}^{m}=\bigcup_{\sigma \in \Sigma} \sigma$.
(ii) $X_{\Sigma}$ is smooth if and only if every cone $\sigma \in \Sigma$ is smooth.

Lemma 3.9. The space $X_{\Sigma}(n)$ is a non-singular toric variety associated to the fan $F_{n}(\Sigma)$. Moreover, there is an isomorphism $X_{\Sigma}(n) \cong X_{F_{n}(\Sigma)}$, and $\mathcal{K}_{\Sigma}(n)$ is the underlying simplicial complex of the fan $F_{n}(\Sigma)$.

[^6]Proof. To see this, consider the toric variety $X(\Sigma)$ determined by the fan $F_{n}(\Sigma)$. By considering the homogenous coordinate representation of $X(\Sigma)$, we easily see that it is isomorphic to $X_{\Sigma}(n)$. Moreover, one can easily show that $X_{\Sigma}(n)$ is non-singular (by using Lemma (3.8). Thus, $X_{\Sigma}(n)$ is a nonsingular toric variety associated to the fan $F_{n}(\Sigma)$. Moreover, by (3.10) we easily see that $\mathcal{K}_{\Sigma}(n)$ is the underlying simplicial complex of $F_{n}(\Sigma)$.

Corollary 3.10. Let $\Sigma$ be a fan in $\mathbb{R}^{m}$ and let $X_{\Sigma}$ be a smooth toric variety such that the condition (2.18)* holds.
(i) The map $\Omega q_{n, \mathbb{C}}: \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \longrightarrow \Omega X_{\Sigma}(n)$ is a universal covering with fiber $\mathbb{Z}^{r-m}$.
(ii) The map $\Omega q_{n, \mathbb{R}}: \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{R}^{n},\left(\mathbb{R}^{n}\right)^{*}\right) \xrightarrow{\simeq} \Omega X_{\Sigma, \mathbb{R}}(n)$ is a homotopy equivalence.
(iii) There is the following fibration sequence (up to homotopy)

$$
\begin{equation*}
\mathbb{T}_{\mathbb{C}}^{m n} \longrightarrow X_{\Sigma}(n) \longrightarrow D J\left(\mathcal{K}_{\Sigma}(n)\right) \tag{3.15}
\end{equation*}
$$

Proof. (i) It follows easily from Lemma 3.4, that the map $\Omega q_{n, \mathbb{C}}$ is a covering projection with fiber $\mathbb{Z}^{r-m}$. Since $\Omega Q_{n}^{D, \Sigma}(\mathbb{C})$ is simply connected (by (i)), $\Omega q_{n, \mathbb{C}}$ is a universal covering with fiber $\mathbb{Z}^{r-m}$.
(ii) The assertion (ii) easily follows from (iii) of Lemma 3.4,
(iii) The assertion (iii) follows from Lemmas 3.6, 3.9 and [23, Proposition 4.4].

Lemma 3.11 ([23]; Lemma 3.4). If the condition (2.18)* is satisfied, the space $X_{\Sigma}$ is simply connected and $\pi_{2}\left(X_{\Sigma}\right)=\mathbb{Z}^{r-m}$.

We end this section with a proof of Lemma 2.7.
Proof of Lemma 2.7. Consider the map $F=\left(F_{1}, \cdots, F_{r}\right)$ is given by (2.17). We let $\mathbb{K}=\mathbb{C}$, as the proof for $\mathbb{K}=\mathbb{R}$ is completely analogous. It suffices to show that $F(\lambda \boldsymbol{x})=F(\boldsymbol{x})$ up to $G_{\Sigma, \mathbb{C}^{-} \text {action for any }(\lambda, \boldsymbol{x}) \in \mathbb{R}^{*} \times\left(\mathbb{R}^{s+1} \backslash\right.}$ $\left.\left\{\mathbf{0}_{s+1}\right\}\right)$ iff $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$.

Since all homogenous polynomials $\left\{f_{k ; i}\right\}_{k=1}^{n}$ have the same degree $d_{i}$, for each $(\lambda, \boldsymbol{x}) \in \mathbb{R}^{*} \times \mathbb{R}^{s+1}$,

$$
\begin{aligned}
F_{i}(\lambda \boldsymbol{x}) & =\left(f_{1 ; i}(\lambda \boldsymbol{x}), \cdots, f_{n ; i}(\lambda \boldsymbol{x})\right)=\left(\lambda^{d_{i}} f_{1 ; i}(\boldsymbol{x}), \cdots, \lambda^{d_{i}} f_{n ; i}(\boldsymbol{x})\right) \\
& =\lambda^{d_{i}}\left(f_{1 ; i}(\boldsymbol{x}), \cdots, f_{n ; i}(\boldsymbol{x})\right)=\lambda^{d_{i}} F_{i}(\boldsymbol{x}) .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
F(\lambda \boldsymbol{x}) & =\left(F_{1}(\lambda \boldsymbol{x}), \cdots, F_{r}(\lambda \boldsymbol{x})\right)=\left(\lambda^{d_{1}} F_{1}(\boldsymbol{x}), \cdots, \lambda^{d_{r}} F_{r}(\boldsymbol{x})\right) \\
& =\left(\lambda^{d_{1}}, \cdots, \lambda^{d_{r}}\right) \cdot\left(F_{1}(\boldsymbol{x}), \cdots, F_{r}(\boldsymbol{x})\right)=\left(\lambda^{d_{1}}, \cdots, \lambda^{d_{r}}\right) \cdot F(\boldsymbol{x}) .
\end{aligned}
$$

Hence, it remains to show that $\left(\lambda^{d_{1}}, \cdots, \lambda^{d_{r}}\right) \in G_{\Sigma, \mathbb{C}}$ for any $\lambda \in \mathbb{R}^{*}$ iff $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$. However, $\left(\lambda^{d_{1}}, \cdots, \lambda^{d_{r}}\right) \in G_{\Sigma, \mathbb{C}}$ for any $\lambda \in \mathbb{R}^{*}$ iff

$$
\prod_{k=1}^{r}\left(\lambda^{d_{k}}\right)^{\left\langle n_{k}, \boldsymbol{m}\right\rangle}=\lambda^{\left\langle\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}, \boldsymbol{m}\right\rangle}=1 \text { for any } \boldsymbol{m} \in \mathbb{Z}^{m} \Leftrightarrow \sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}
$$

and this completes the proof.

## 4 The Vassiliev spectral sequence

4.1 Simplicial resolutions. First, recall the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolution ([21], [27, [28], [32], (33]).

Definition 4.1. (i) For a finite set $\boldsymbol{v}=\left\{v_{1}, \cdots, v_{l}\right\} \subset \mathbb{R}^{N}$, let $\sigma(\boldsymbol{v})$ denote the convex hull spanned by $v$. Let $h: X \rightarrow Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i: X \rightarrow \mathbb{R}^{N}$ be an embedding. Let $\mathcal{X}^{\Delta}$ and $h^{\Delta}: \mathcal{X}^{\Delta} \rightarrow Y$ denote the space and the map defined by

$$
\begin{equation*}
\mathcal{X}^{\Delta}=\left\{(y, u) \in Y \times \mathbb{R}^{N}: u \in \sigma\left(i\left(h^{-1}(y)\right)\right)\right\} \subset Y \times \mathbb{R}^{N}, h^{\Delta}(y, u)=y \tag{4.1}
\end{equation*}
$$

The pair $\left(\mathcal{X}^{\Delta}, h^{\Delta}\right)$ is called the simplicial resolution of $(h, i)$. In particular, it is called $a$ non-degenerate simplicial resolution if for each $y \in Y$ any $k$ points of $i\left(h^{-1}(y)\right)$ span $(k-1)$-dimensional simplex of $\mathbb{R}^{N}$.
(ii) For each $k \geq 0$, let $\mathcal{X}_{k}^{\Delta} \subset \mathcal{X}^{\Delta}$ be the subspace of the union of the ( $k-1$ )-skeletons of the simplices over all the points $y$ in $Y$ given by

$$
\begin{equation*}
\mathcal{X}_{k}^{\Delta}=\left\{(y, u) \in \mathcal{X}^{\Delta}: u \in \sigma(\boldsymbol{v}), \boldsymbol{v}=\left\{v_{1}, \cdots, v_{l}\right\} \subset i\left(h^{-1}(y)\right), l \leq k\right\} . \tag{4.2}
\end{equation*}
$$

We make the identification $X=\mathcal{X}_{1}^{\Delta}$ by identifying $x \in X$ with the pair $(h(x), i(x)) \in \mathcal{X}_{1}^{\Delta}$, and we note that there is an increasing filtration

$$
\begin{equation*}
\emptyset=\mathcal{X}_{0}^{\Delta} \subset X=\mathcal{X}_{1}^{\Delta} \subset \mathcal{X}_{2}^{\Delta} \subset \cdots \subset \mathcal{X}_{k}^{\Delta} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_{k}^{\Delta}=\mathcal{X}^{\Delta} . \tag{4.3}
\end{equation*}
$$

Since the map $h^{\Delta}: \mathcal{X}^{\Delta} \rightarrow Y$ is a proper map, it extends to the map $h_{+}^{\Delta}$ : $\mathcal{X}_{+}^{\Delta} \rightarrow Y_{+}$between the one-point compactifications, where $X_{+}$denotes the one-point compactification of a locally compact space $X$.

Definition 4.2. Let $h: X \rightarrow Y$ be a surjective semi-algebraic map between semi-algebraic spaces, $j: X \rightarrow \mathbb{R}^{N}$ be a semi-algebraic embedding, and let $\left(\mathcal{X}^{\Delta}, h^{\Delta}: \mathcal{X}^{\Delta} \rightarrow Y\right)$ denote the associated non-degenerate simplicial
resolution of $(h, j)$. Then for each positive integer $k \geq 1$, we denote by $h_{k}^{\Delta}: X^{\Delta}(k) \rightarrow Y$ the truncated (after the $k$-th term) simplicial resolution of $Y$ as in [28]. Note that that there is a natural filtration

$$
X_{0}^{\Delta} \subset X_{1}^{\Delta} \subset \cdots \subset X_{l}^{\Delta} \subset X_{l+1}^{\Delta} \subset \cdots \subset X_{k}^{\Delta} \subset X_{k+1}^{\Delta}=X_{k+2}^{\Delta}=\cdots=X^{\Delta}(k),
$$

where $X_{0}^{\Delta}=\emptyset, X_{l}^{\Delta}=\mathcal{X}_{l}^{\Delta}$ if $l \leq k$ and $X_{l}^{\Delta}=X^{\Delta}(k)$ if $l>k$.
4.2 Vassiliev spectral sequences. Next, we shall construct the Vassiliev spectral sequence for computing the homology of the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$.

From now on, we always assume that $\Sigma$ is a fan in $\mathbb{R}^{m}$ such that $X_{\Sigma}$ is simply connected toric variety satisfying the condition (2.18)*. Moreover, let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ will always be a fixed $r$-tuple of positive integers.

Definition 4.3. (i) For each $d \in \mathbb{N}$, let $\mathrm{P}_{d}^{\mathbb{K}} \subset \mathbb{K}[z]$ denote the space of all monic polynomials $f(z)=z^{d}+a_{z}^{d-1}+\cdots+a_{d} \in \mathbb{K}[z]$ of degree $d$. Then for each $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, let $\mathrm{P}_{D}^{\mathbb{K}}$ denote the space of $r$-tuples of monic polynomials defined by

$$
\begin{equation*}
\mathrm{P}_{D}^{\mathbb{K}}=\mathrm{P}_{d_{1}}^{\mathbb{K}} \times \mathrm{P}_{d_{2}}^{\mathbb{K}} \times \cdots \times \mathrm{P}_{d_{r}}^{\mathbb{K}} . \tag{4.4}
\end{equation*}
$$

(ii) For each $\mathrm{f}=\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathrm{P}_{D}^{\mathbb{K}}$, let $F_{(n)}(\mathrm{f})(z)$ denote the $r n$ tuple of monic polynomials defined by

$$
\begin{equation*}
F_{(n)}(\mathrm{f})(z)=\left(F_{n}\left(f_{1}\right)(z), \cdots, F_{n}\left(f_{r}\right)(z)\right) \in \mathbb{K}[z]^{r n}, \tag{4.5}
\end{equation*}
$$

where we denote by $F_{n}\left(f_{i}\right)(z)$ the $n$-tuple of monic polynomials of degree $d_{i}$ given by

$$
\begin{equation*}
F_{n}\left(f_{i}\right)(z)=\left(f_{i}(z), f_{i}(z)+f_{i}^{\prime}(z), f_{i}(z)+f_{i}^{\prime \prime}(z), \cdots, f_{i}(z)+f_{i}^{(n-1)}(z)\right) \tag{4.6}
\end{equation*}
$$

for each $1 \leq i \leq r($ as in (3.10) $)$.
(iii) Let $\Sigma_{D}$ denote the discriminant of $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ in $\mathrm{P}_{D}^{\mathbb{K}}$ given by the complement

$$
\begin{aligned}
\Sigma_{D} & =\mathrm{P}_{D}^{\mathbb{K}} \backslash \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) \\
& =\left\{\mathrm{f}=\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathrm{P}_{D}^{\mathbb{K}}: F_{(n)}(\mathrm{f})(x) \in L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K}) \text { for some } x \in \mathbb{R}\right\},
\end{aligned}
$$

where $L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})$ denotes the set given by $K=\mathcal{K}_{\Sigma}$ in (2.5).
(iv) Let $Z_{D} \subset \Sigma_{D} \times \mathbb{R}$ denote the tautological normalization of $\Sigma_{D}$ consisting of all pairs ( $\mathrm{f}, x)=\left(\left(f_{1}(z), \ldots, f_{r}(z)\right), x\right) \in \Sigma_{D} \times \mathbb{R}$ satisfying the condition $F_{(n)}(\mathrm{f})(x)=\left(F_{n}\left(f_{1}\right)(x), \cdots, F_{n}\left(f_{r}\right)(x)\right) \in L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})$. Projection on the first factor gives a surjective map $\pi_{D}: Z_{D} \rightarrow \Sigma_{D}$.

Remark 4.4. Let $\sigma_{k} \in[r]$ for $k=1,2$. It is easy to see that $L_{\sigma_{1}}\left(\mathbb{K}^{n}\right) \subset$ $L_{\sigma_{2}}\left(\mathbb{K}^{n}\right)$ if $\sigma_{1} \supset \sigma_{2}$. Letting

$$
\operatorname{Pr}(\Sigma)=\left\{\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \subset[r]:\left\{\boldsymbol{n}_{i_{1}}, \cdots, \boldsymbol{n}_{i_{s}}\right\} \text { is a primitive collection }\right\},
$$

we see that $L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})=\bigcup_{\sigma \in \operatorname{Pr}(\Sigma)} L_{\sigma}\left(\mathbb{K}^{n}\right)$, and by using (2.35) we obtain the equality
(4.7) $\quad \operatorname{dim} L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})=n d(\mathbb{K})\left(r-r_{\min }(\Sigma)\right)= \begin{cases}2 n\left(r-r_{\min }(\Sigma)\right) & \text { if } \mathbb{K}=\mathbb{C}, \\ n\left(r-r_{\min }(\Sigma)\right) & \text { if } \mathbb{K}=\mathbb{R} .\end{cases}$

Our goal in this section is to construct, by means of the non-degenerate simplicial resolution of the discriminant, a spectral sequence converging to the homology of $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$.

Definition 4.5. (i) For an $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ of positive integers, let $N(D)$ denote the positive integer given by

$$
\begin{equation*}
N(D)=\sum_{k=1}^{r} d_{k} \tag{4.8}
\end{equation*}
$$

(ii) For each based space $X$, let $F(X, d)$ denote the ordered configuration space of distinct $d$ points in $X$ defined by

$$
\begin{equation*}
F(X, d)=\left\{\left(x_{1}, \cdots, x_{d}\right) \in X^{d}: x_{i} \neq x_{j} \text { if } i \neq j\right\} . \tag{4.9}
\end{equation*}
$$

Note that the symmetric group $S_{d}$ of $d$-letters acts on $F(X, d)$ freely by permuting coordinates. Let $C_{d}(X)$ denote the unordered configuration space of $d$-distinct points in $X$ given by the orbit space

$$
\begin{equation*}
C_{d}(X)=F(X, d) / S_{d} . \tag{4.10}
\end{equation*}
$$

(iii) Let $L_{k ; \Sigma, \mathbb{K}} \subset\left(\mathbb{R} \times L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})\right)^{k}$ denote the subspaces defined by

$$
L_{k ; \Sigma, \mathbb{K}}=\left\{\left(\left(x_{1}, s_{1}\right), \cdots,\left(x_{k}, s_{k}\right)\right) \in\left(\mathbb{R} \times L_{n}^{\mathcal{K}_{\Sigma}}(\mathbb{K})\right)^{k}: x_{i} \neq x_{j} \text { if } i \neq j\right\} .
$$

The symmetric group $S_{k}$ on $k$ letters acts on the space $L_{k ; \Sigma, \mathbb{K}}$ by permuting $k$-elements., and let $C_{k ; \Sigma, \mathbb{K}}$ denote the orbit space defined by

$$
\begin{equation*}
C_{k ; \Sigma, \mathbb{K}}=L_{k ; \Sigma, \mathbb{K}} / S_{k} . \tag{4.11}
\end{equation*}
$$

Note that the space $C_{k ; \Sigma, \mathbb{K}}$ is a cell-complex of dimension (by (4.7))

$$
\operatorname{dim} C_{k ; \Sigma, \mathbb{K}}= \begin{cases}k+2 k n\left(r-r_{\min }(\Sigma)\right) & \text { if } \mathbb{K}=\mathbb{C}  \tag{4.12}\\ k+k n\left(r-r_{\min }(\Sigma)\right) & \text { if } \mathbb{K}=\mathbb{R}\end{cases}
$$

(iv) Let $\left(\mathcal{X}^{D}, \pi_{D}^{\Delta}: \mathcal{X}^{D} \rightarrow \Sigma_{D}\right)$ be the non-degenerate simplicial resolution associated to the surjective map $\pi_{D}: Z_{D} \rightarrow \Sigma_{D}$ with the natural increasing filtration as in Definition 4.1.

$$
\emptyset=\mathcal{X}_{0}^{D} \subset \mathcal{X}_{1}^{D} \subset \mathcal{X}_{2}^{D} \subset \cdots \subset \mathcal{X}^{D}=\bigcup_{k=0}^{\infty} \mathcal{X}_{k}^{D}
$$

By [32, Lemma 1 (page 90)], the map $\pi_{D}^{\Delta}$ extends to a homology equivalence $\pi_{D+}^{\Delta}: \mathcal{X}_{+}^{D} \xrightarrow{\simeq} \Sigma_{D+}$. Since $\mathcal{X}_{k+}^{D} / \mathcal{X}_{k-1+}^{D} \cong\left(\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D}\right)_{+}$, we have a spectral sequence

$$
\begin{equation*}
\left\{E_{t ; D}^{k, s}, d_{t}: E_{t ; D}^{k, s} \rightarrow E_{t ; D}^{k+t, s+1-t}\right\} \Rightarrow H_{c}^{k+s}\left(\Sigma_{D} ; \mathbb{Z}\right) \tag{4.13}
\end{equation*}
$$

where $E_{1: D}^{k, s}=H_{c}^{k+s}\left(\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D} ; \mathbb{Z}\right)$ and $H_{c}^{k}(X ; \mathbb{Z})$ denotes the cohomology group with compact supports given by $H_{c}^{k}(X ; \mathbb{Z})=\tilde{H}^{k}\left(X_{+} ; \mathbb{Z}\right)$.

Since there is a homeomorphism $\mathrm{P}_{D}^{\mathbb{K}} \cong \mathbb{K}^{N(D)} \cong \mathbb{R}^{d(\mathbb{K}) N(D)}$, by Alexander duality there is a natural isomorphism

$$
\begin{equation*}
\tilde{H}_{k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) ; \mathbb{Z}\right) \cong H_{c}^{d(\mathbb{K}) N(D)-k-1}\left(\Sigma_{D} ; \mathbb{Z}\right) \quad \text { for any } k \tag{4.14}
\end{equation*}
$$

By reindexing we obtain a spectral sequence

$$
\begin{equation*}
\left\{E_{k, s}^{t ; D}, \tilde{d}^{t}: E_{k, s}^{t, D} \rightarrow E_{k+t, s+t-1}^{t ; D}\right\} \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) ; \mathbb{Z}\right) \tag{4.15}
\end{equation*}
$$

where $E_{k, s}^{1 ; D}=H_{c}^{d(\mathbb{K}) N(D)+k-s-1}\left(\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D} ; \mathbb{Z}\right)$.
Lemma 4.6. If $d_{\min } \geq n$ and $1 \leq k \leq\left\lfloor\frac{d_{\min }}{n}\right\rfloor$, the space $\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D}$ is homeomorphic to the total space of a real affine bundle $\xi_{D, k, n}$ over $C_{k ; \Sigma, \mathbb{K}}$ with rank $l_{D, k, n}=d(\mathbb{K})(N(D)-n r k)+k-1$.

Proof. Since the proof is completely analogous to that of [26, Lemma 4.9], we omit detail of the proof.

Lemma 4.7. If $d_{\min } \geq n$ and $1 \leq k \leq\left\lfloor\frac{d_{\min }}{n}\right\rfloor$, there is a natural isomorphism

$$
E_{k, s}^{1 ; D} \cong H_{c}^{d(\mathbb{K}) n r k-s}\left(C_{k ; \Sigma, \mathbb{K}} ; \pm \mathbb{Z}\right),
$$

where the twisted coefficients system $\pm \mathbb{Z}$ comes from the Thom isomorphism.
Proof. Suppose that $1 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$. By Lemma 4.6, there is a homeomorphism $\left(\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D}\right)_{+} \cong T\left(\xi_{D, k}\right)$, where $T\left(\xi_{D, k, n}\right)$ denotes the Thom space of $\xi_{D, k, n}$. Since $(d(\mathbb{K}) N(D)+k-s-1)-l_{D, k, n}=d(\mathbb{K}) n r k-s$, the Thom isomorphism gives a natural isomorphism

$$
E_{k, s}^{1 ; d} \cong \tilde{H}^{d(\mathbb{K}) N(D)+k-s-1}\left(T\left(\xi_{d, k, n}\right) ; \mathbb{Z}\right) \cong H_{c}^{d(\mathbb{K}) n r k-s}\left(C_{k ; \Sigma, \mathbb{K}} ; \pm \mathbb{Z}\right),
$$

and the assertion follows.

## 5 Stabilization maps

We will now define two stabilization maps

$$
\left\{\begin{array}{l}
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D, D+a}(\mathbb{C})  \tag{5.1}\\
s_{D, D+a}^{\mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \mathrm{Q}_{n}^{D, D+a}(\mathbb{R})
\end{array} \quad \text { for each } \boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}\right.
$$

Definition 5.1. (i) For an $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, let $U_{D} \subset \mathbb{C}$ denote the subspace defined by

$$
\begin{equation*}
U_{D}=\{w \in \mathbb{C}: \operatorname{Re}(w)<N(D)\}, \tag{5.2}
\end{equation*}
$$

and let $\varphi_{D}: \mathbb{C} \xrightarrow{\cong} U_{D}$ be any homeomorphism (which we now fix) satisfying the following two conditions:

$$
\begin{equation*}
\varphi_{D}(\mathbb{R})=(-\infty, N(D)) \text { and } \varphi_{D}(\bar{\alpha})=\overline{\varphi_{D}(\alpha)} \quad \text { for any } \alpha \in \mathrm{H}_{+}, \tag{5.3}
\end{equation*}
$$

where $\mathrm{H}_{+} \subset \mathbb{C}$ denotes the upper half plane in $\mathbb{C}$ given by

$$
\begin{equation*}
\mathrm{H}_{+}=\{\alpha \in \mathbb{C}: \operatorname{Im} \alpha>0\} . \tag{5.4}
\end{equation*}
$$

(ii) Now let us choose and fix any $r$ points $\left(x_{1}, \cdots, x_{r}\right) \in\left(\mathbb{C} \backslash U_{D}\right)^{r}$ satisfying the condition $x_{i} \neq x_{j}$ if $i \neq j$.

For each monic polynomial $f(z)=\prod_{k=1}^{d}\left(z-\alpha_{k}\right) \in \mathbb{C}[z]$ of degree $d$, let $\varphi_{D}(f)$ denote the monic polynomial of the same degree $d$ given by

$$
\begin{equation*}
\varphi_{D}(f)=\prod_{k=1}^{d}\left(z-\varphi_{D}\left(\alpha_{k}\right)\right) \tag{5.5}
\end{equation*}
$$

(iii) For each $r$-tuple $\boldsymbol{a}=\left(a_{1}, \cdots, a_{r}\right) \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$, define the stabilization map

$$
\begin{gather*}
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{C}) \text { by }  \tag{5.6}\\
s_{D, D+a}(f)=\left(\varphi_{D}\left(f_{1}\right)\left(z-x_{1}\right)^{a_{1}}, \cdots, \varphi_{D}\left(f_{r}\right)\left(z-x_{r}\right)^{a_{r}}\right)
\end{gather*}
$$

for $f=\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$.
Remark 5.2. (i) Note that the definition of the map $s_{D, D+a}$ depends on the choice of the homeomorphism $\varphi_{D}$ and the $r$-tuple $\left(x_{1}, \cdots, x_{r}\right) \in\left(\mathbb{C} \backslash U_{D}\right)^{r}$ of points, but one can show that the homotopy type of it does not depend on these choices.
(ii) Let $\boldsymbol{a}, \boldsymbol{b} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$ be any two $r$-tuples such that $\boldsymbol{a}, \boldsymbol{b} \neq \mathbf{0}_{r}$. Then it is easy to see that the equality

$$
\begin{equation*}
\left(s_{D+\boldsymbol{a}, D+\boldsymbol{a}+\boldsymbol{b}}\right) \circ\left(s_{D, D+\boldsymbol{a}}\right)=s_{D, D+\boldsymbol{a}+\boldsymbol{b}} \quad \text { (up to homotopy) } \tag{5.7}
\end{equation*}
$$

holds. Thus we mostly only consider the stabilization map $s_{D, D+e_{i}}$ for each $1 \leq i \leq r$, where $\boldsymbol{e}_{1}=(1,0, \cdots, 0), \boldsymbol{e}_{2}=(0,1,0, \cdots, 0), \cdots, \boldsymbol{e}_{r}=$ $(0,0, \cdots, 0,1) \in \mathbb{R}^{r}$ denote the standard orthogonal basis of $\mathbb{R}^{r}$.
(iii) From (5.3) it easily follows that

$$
\begin{equation*}
\varphi_{D}(f) \in \mathbb{R}[z] \quad \text { if } f=f(z) \in \mathbb{R}[z] \tag{5.8}
\end{equation*}
$$

Thus, for each $r$-tuple $\boldsymbol{a}=\left(a_{1}, \cdots, a_{r}\right) \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$, one can easily show that the following holds:

$$
\begin{equation*}
s_{D, D+a}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})\right) \subset \mathrm{Q}_{n}^{D+\boldsymbol{a}, \Sigma}(\mathbb{R}) \tag{5.9}
\end{equation*}
$$

Definition 5.3. By (5.9), one can define the stabilization map

$$
\begin{gather*}
s_{D, D+a}^{\mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{R}) \quad \text { by the restriction }  \tag{5.10}\\
s_{D, D+a}^{\mathbb{R}}=s_{D, D+a} \mid \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) .
\end{gather*}
$$

Remark 5.4. By using the definition of (5.6) and (5.8) we easily see that the following equality holds:

$$
\begin{equation*}
s_{D, D+\boldsymbol{a}}^{\mathbb{R}}=\left(s_{D, D+\boldsymbol{a}}\right)^{\mathbb{Z}_{2}} \quad \text { for each } \boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r} . \tag{5.11}
\end{equation*}
$$

## 6 Homology stability

We shall consider the homology stability of the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$.
6.1 The case $\mathbb{K}=\mathbb{C}$. First, consider the case $\mathbb{K}=\mathbb{C}$. Let $1 \leq i \leq r$ and consider the stabilization map

$$
\begin{equation*}
s_{D, D+e_{i}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+e_{i}, \Sigma}(\mathbb{C}) \tag{6.1}
\end{equation*}
$$

It is easy to see that it extends to an open embedding

$$
\begin{equation*}
s_{D, i}: \mathbb{C} \times \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+e_{i}, \Sigma}(\mathbb{C}) \tag{6.2}
\end{equation*}
$$

It also naturally extends to an open embedding $\tilde{s}_{D, i}: \mathrm{P}_{D}^{\mathbb{C}} \rightarrow \mathrm{P}_{D+e_{i}}^{\mathbb{C}}$ and by restriction we obtain an open embedding

$$
\begin{equation*}
\tilde{s}_{D, i}: \mathbb{C} \times \Sigma_{D} \rightarrow \Sigma_{D+e_{i}} . \tag{6.3}
\end{equation*}
$$

Since one-point compactification is contravariant for open embeddings, this map induces a map in the opposite direction

$$
\begin{equation*}
\tilde{s}_{D, i+}:\left(\Sigma_{D+e_{i}}\right)_{+} \rightarrow\left(\mathbb{C} \times \Sigma_{D}\right)_{+}=S^{2} \wedge \Sigma_{D+} \tag{6.4}
\end{equation*}
$$

We obtain the following commutative diagram

$$
\begin{array}{ccc}
\tilde{H}_{k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) ; \mathbb{Z}\right) & \xrightarrow{\left(s_{\left.D, D+e_{i}\right)}\right)} & \tilde{H}_{k}\left(\mathrm{Q}_{n}^{D+e_{i}, \Sigma}(\mathbb{C}) ; \mathbb{Z}\right)  \tag{6.5}\\
A D_{1} \downarrow \cong & A D_{2} \downarrow \cong \\
H_{c}^{2 N(D)-k-1}\left(\Sigma_{D} ; \mathbb{Z}\right) & \xrightarrow{\left(\tilde{s}_{D, i+}\right)^{*}} & H_{c}^{2 N(D)-k+1}\left(\Sigma_{D+e_{i}} ; \mathbb{Z}\right) .
\end{array}
$$

Here, $A D_{k}(k=1,2)$ denote the corresponding Alexander duality isomorphisms and $\tilde{s}_{D, i+}^{*}$ denotes the composite of the suspension isomorphism with the homomorphism $\left(\tilde{s}_{D+}\right)^{*}$ given by

$$
\begin{equation*}
H_{c}^{M}\left(\Sigma_{D} ; \mathbb{Z}\right) \xrightarrow{\cong} H_{c}^{M+2}\left(\mathbb{C} \times \Sigma_{D} ; \mathbb{Z}\right) \xrightarrow{\left(\tilde{s}_{D, i}\right)^{*}} H_{c}^{M+2}\left(\Sigma_{D+e_{i}} ; \mathbb{Z}\right), \tag{6.6}
\end{equation*}
$$

where $M=2 N(D)-k-1$.
By the universality of the non-degenerate simplicial resolution [27, the map $\tilde{s}_{D, i}$ also naturally extends to a filtration preserving open embedding

$$
\begin{equation*}
\tilde{s}_{D, i}: \mathbb{C} \times \mathcal{X}^{D} \rightarrow \mathcal{X}^{D+e_{i}} \tag{6.7}
\end{equation*}
$$

between non-degenerate simplicial resolutions. This induces a filtration preserving map

$$
\begin{equation*}
\left(\tilde{s}_{D, i}\right)_{+}: \mathcal{X}_{+}^{D+e_{i}} \rightarrow\left(\mathbb{C} \times \mathcal{X}^{D}\right)_{+}=S^{2} \wedge \mathcal{X}_{+}^{D} \tag{6.8}
\end{equation*}
$$

and we finally obtain the homomorphism of spectral sequences

$$
\begin{align*}
& \left\{\tilde{\theta}_{k, s}^{t}: E_{k, s}^{t ; D} \rightarrow E_{k, s}^{t ; D+a}\right\}, \quad \text { where }  \tag{6.9}\\
& \left\{\begin{array}{ll}
\left\{E_{k, s}^{t ; D}, \tilde{d}^{t}: E_{k, s}^{t ; D} \rightarrow E_{k+t, s+t-1}^{t ; D}\right\} & \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) ; \mathbb{Z}\right), \\
\left\{E_{k, s}^{t, D+e_{i}}, \tilde{d}^{t}: E_{k, s}^{t ; D+e_{i}} \rightarrow E_{k+t, s+t-1}^{t ; D+e_{i}}\right\} & \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D+e_{i}}(\mathbb{C}) ; \mathbb{Z}\right), \\
\begin{cases}E_{k, s}^{1 ; D} & =H_{c}^{2 N(D)+k-1-s}\left(\mathcal{X}_{k}^{D} \backslash \mathcal{X}_{k-1}^{D} ; \mathbb{Z}\right), \\
E_{k, s}^{1 ; D+e_{i}} & =H_{c}^{2 N(D)+k+1-s}\left(\mathcal{X}_{k}^{D+e_{i}} \backslash \mathcal{X}_{k-1}^{D+e_{i}} ; \mathbb{Z}\right) .\end{cases}
\end{array} .\right.
\end{align*}
$$

Lemma 6.1. If $1 \leq i \leq r$ and $0 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor, \tilde{\theta}_{k, s}^{1}: E_{k, s}^{1 ; D} \rightarrow E_{k, s}^{1 ; D+e_{i}}$ is an isomorphism for any $s$.

Proof. Since the proof is completely analogous to that of [26, Lemma 4.13], we omit the detail.

Now we consider the spectral sequences induced by truncated simplicial resolutions.

Definition 6.2. Let $X^{\Delta}$ denote the truncated (after the $\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$-th term) simplicial resolution of $\Sigma_{D}$ with the natural filtration

$$
\emptyset=X_{0}^{\Delta} \subset X_{1}^{\Delta} \subset \cdots \subset X_{\left\lfloor d_{\min } / n\right\rfloor}^{\Delta} \subset X_{\left\lfloor d_{\min } / n\right\rfloor+1}^{\Delta}=X_{\left\lfloor d_{\min } / n\right\rfloor+2}^{\Delta}=\cdots=X^{\Delta}
$$

where $X_{k}^{\Delta}=\mathcal{X}_{k}^{D}$ if $k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$ and $X_{k}^{\Delta}=X^{\Delta}$ if $k \geq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor+1$.
Similarly, let $Y^{\Delta}$ denote the truncated (after the $\left\lfloor\frac{d_{\min }}{n}\right\rfloor$-th term) simplicial resolution of $\Sigma_{D+e_{i}}$ with the natural filtration

$$
\emptyset=Y_{0}^{\Delta} \subset Y_{1}^{\Delta} \subset \cdots \subset Y_{\left\lfloor d_{\min } / n\right\rfloor}^{\Delta} \subset Y_{\left\lfloor d_{\min } / n\right\rfloor+1}^{\Delta}=Y_{\left\lfloor d_{\min } / n\right\rfloor+2}^{\Delta}=\cdots=Y^{\Delta}
$$

where $Y_{k}^{\Delta}=\mathcal{X}_{k}^{D+e_{i}}$ if $k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$ and $Y_{k}^{\Delta}=Y^{\Delta}$ if $k \geq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor+1$.
By [28, §2 and §3], we obtain the following truncated spectral sequences

$$
\begin{cases}\left\{E_{k, s}^{t ; \mathbb{C}}, d^{t}: E_{k, s}^{t ; \mathbb{C}} \rightarrow E_{k+t, s+t-1}^{t ; \mathbb{C}}\right\} & \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) ; \mathbb{Z}\right),  \tag{6.10}\\ \left\{{ }^{\prime} E_{k, s}^{t \cdot \mathbb{C}}, d^{t}:{ }^{\prime} E_{k, s}^{t ; \mathbb{C}} \rightarrow{ }^{\prime} E_{k+t, s+t-1}^{t}\right\} & \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D+e_{i}}(\mathbb{C}) ; \mathbb{Z}\right),\end{cases}
$$

where

$$
\left\{\begin{array}{l}
E_{k, S}^{1 ; \mathbb{C}}=H_{c}^{2 N(D)+k-1-s}\left(X_{k}^{\Delta} \backslash X_{k-1}^{\Delta} ; \mathbb{Z}\right),  \tag{6.11}\\
{ }^{1 ; \mathbb{C}}=H_{c, s}^{2 N(D)+k+1-s}\left(Y_{k}^{\Delta} \backslash Y_{k-1}^{\Delta} ; \mathbb{Z}\right) .
\end{array}\right.
$$

By the naturality of truncated simplicial resolutions, the filtration preserving $\operatorname{map} \tilde{s}_{D, i}: \mathbb{C} \times \mathcal{X}^{D} \rightarrow \mathcal{X}^{D+e_{i}}$ gives rise to a natural filtration preserving map $\tilde{s}_{D, i}^{\prime}: \mathbb{C} \times X^{\Delta} \rightarrow Y^{\Delta}$ which, in a way analogous to (6.9), induces a homomorphism of spectral sequences

$$
\begin{equation*}
\left\{\theta_{k, s}^{t}: E_{k, s}^{t ; \mathbb{C}} \rightarrow{ }^{\prime} E_{k, s}^{t ; \mathbb{C}}\right\} . \tag{6.12}
\end{equation*}
$$

Lemma 6.3. (i) If $k<0$ or $k \geq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor+2, E_{k, s}^{1 ; \mathbb{C}}={ }^{\prime} E_{k, s}^{1 ; \mathbb{C}}=0$ for any $s$.
(ii) $E_{0,0}^{1 ; \mathbb{C}}={ }^{\prime} E_{0,0}^{1 ; \mathbb{C}}=\mathbb{Z}$ and $E_{0, s}^{1 ; \mathbb{C}}={ }^{\prime} E_{0, s}^{1 ; \mathbb{C}}=0$ if $s \neq 0$.
(iii) If $1 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$, there are isomorphisms

$$
E_{k, s}^{1 ; \mathbb{C}} \cong{ }^{\prime} E_{k, s}^{1 ; \mathbb{C}} \cong H_{c}^{2 n r k-s}\left(C_{k ; \Sigma} ; \pm \mathbb{Z}\right)
$$

(iv) If $1 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor, E_{k, s}^{1 ; \mathbb{C}}={ }^{\prime} E_{k, s}^{1 ; \mathbb{C}}=0$ for any $s \leq\left(2 n r_{\min }(\Sigma)-1\right) k-1$.
(v) If $k=\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor+1, E_{k, s}^{1 ; \mathbb{C}}={ }^{\prime} E_{k, s}^{1 ; \mathbb{C}}=0$ for any $s \leq\left(2 n r_{\min }(\Sigma)-1\right)\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor-1$.

Proof. Let us write $r_{\min }=r_{\min }(\Sigma)$ and $d_{\min }^{\prime}=\left\lfloor\frac{d_{\min }}{n}\right\rfloor$. Since the proofs of both cases are identical, it suffices to prove the assertions for $E_{k, s}^{1 ; C}$.
(i), (ii), (iii): Since $X_{k}^{\Delta}=X^{\Delta}$ for any $k \geq d_{\text {min }}^{\prime}+2$, the assertions (i) and (ii) are clearly true. Since $X_{k}^{\Delta}=\mathcal{X}_{k}^{D}$ for any $k \leq d_{\text {min }}^{\prime}$, the assertion (iii) easily follows from Lemma 4.7.
(iv) Suppose that $1 \leq k \leq d_{\min }^{\prime}$. By using the equality (4.12),

$$
2 n r k-s>\operatorname{dim} C_{k ; \Sigma} \Leftrightarrow s \leq\left(2 n r_{\min }-1\right) k-1 .
$$

Thus, the assertion (iv) follows from the isomorphism given by (iii).
(v) By Lemma [28, Lemma 2.1], we see that

$$
\begin{aligned}
\operatorname{dim}\left(X_{d_{\min }^{\prime}+1}^{\Delta} \backslash X_{d_{\min }^{\prime}}^{\Delta}\right) & =\operatorname{dim}\left(\mathcal{X}_{d_{\min }^{\prime}}^{D} \backslash \mathcal{X}_{d_{\min }^{\prime}-1}^{D}\right)+1=l_{D, d_{\min }^{\prime}, n}+\operatorname{dim} C_{d_{\min }^{\prime} ; \Sigma}+1 \\
& =2 N(D)+2 d_{\min }^{\prime}-2 n r_{\min } d_{\min }^{\prime} .
\end{aligned}
$$

Since $E_{d_{\min }^{\prime}+1, s}^{1 ; \mathbb{C}}=H_{c}^{2 N(D)+d_{\min }^{\prime}-s}\left(X_{d_{\min }^{\prime}+1}^{\Delta} \backslash X_{d_{\min }^{\prime}}^{\Delta} ; \mathbb{Z}\right)$ (by (6.11)) and

$$
\begin{aligned}
2 N(D)+d_{\min }^{\prime}-s & >\operatorname{dim}\left(X_{d_{\min }^{\prime}+1}^{\Delta} \backslash X_{d_{\min }^{\prime}}^{\Delta}\right)=2 N(D)+2 d_{\min }^{\prime}-2 n r_{\min } d_{\min }^{\prime} \\
& \Leftrightarrow s<\left(2 n r_{\min }-1\right) d_{\min }^{\prime} \Leftrightarrow s \leqq\left(2 n r_{\min }-1\right) d_{\min }^{\prime}-1,
\end{aligned}
$$

we see that $E_{d_{\min }^{\prime}+1, s}^{1 ; \mathrm{C}}=0$ for any $s \leq\left(2 n r_{\min }-1\right) d_{\min }^{\prime}-1$.
Lemma 6.4. If $0 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor, \theta_{k, s}^{1}: E_{k, s}^{1 ; \mathbb{C}} \xlongequal{\cong}{ }^{\prime} E_{k, s}^{1 ; \mathbb{C}}$ is an isomorphism for any $s$.

Proof. Since $\left(X_{k}^{\Delta}, Y_{k}^{\Delta}\right)=\left(\mathcal{X}_{k}^{D}, \mathcal{X}_{k}^{D+e_{i}}\right)$ for $k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$, the assertion follows from Lemma 6.1

Theorem 6.5. For each $1 \leq i \leq r$, the stabilization map

$$
s_{D, D+e_{i}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+e_{i}, \Sigma}(\mathbb{C})
$$

is a homology equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.
Proof. We write $r_{\min }=r_{\min }(\Sigma)$ and $d_{\min }^{\prime}=\left\lfloor\frac{d_{\min }}{n}\right\rfloor$ as in the proof of Lemma 6.3. Without loss of generality, we may assume that $d_{\text {min }} \geq n \geq 1$.

Let us consider the homomorphism $\theta_{k, s}^{t}: E_{k, s}^{t ; \mathbb{C}} \rightarrow{ }^{\prime} E_{k, s}^{t ; \mathbb{C}}$ of truncated spectral sequences given in (6.12). By using the commutative diagram (6.5) and the comparison theorem for spectral sequences, we see that it suffices to prove that the positive integer $d(D ; \Sigma, n, \mathbb{C})$ has the following property:
$(\dagger) \theta_{k, s}^{\infty}$ is an isomorphism for all $(k, s)$ such that $s-k \leq d(D ; \Sigma, n, \mathbb{C})$.

By Lemma 6.3, we can easily see that:
$(\dagger)_{1}$ if $k<0$ or $k \geq d_{\text {min }}^{\prime}+1, \theta_{k, s}^{\infty}$ is an isomorphism for all $(k, s)$ such that $s-k \leq d(D ; \Sigma, n, \mathbb{C})$.
Next, assume that $0 \leq k \leq d_{\min }^{\prime}$, and investigate the condition that $\theta_{k, s}^{\infty}$ is an isomorphism. Note that the groups $E_{k_{1}, s_{1}}^{1 ; \mathbb{C}}$ and ${ }^{\prime} E_{k_{1}, s_{1}}^{1 ; \mathbb{C}}$ are not known for $(u, v) \in \mathcal{S}_{1}=\left\{\left(d_{\min }^{\prime}+1, s\right) \in \mathbb{Z}^{2}: s \geq\left(2 n r_{\text {min }}-1\right) d_{\text {min }}^{\prime}\right\}$. By considering the differentials $d^{1}$ 's of $E_{k, s}^{1 ; \mathbb{C}}$ and ' $E_{k, s}^{1 ; \mathrm{C}}$, and applying Lemmar.6.4, we see that $\theta_{k, s}^{2}$ is an isomorphism if $(k, s) \notin \mathcal{S}_{1} \cup \mathcal{S}_{2}$, where
$\mathcal{S}_{2}=\left\{(u, v) \in \mathbb{Z}^{2}:(u+1, v) \in \mathcal{S}_{1}\right\}=\left\{\left(d_{\text {min }}^{\prime}, v\right) \in \mathbb{Z}^{2}: v \geq\left(2 n r_{\text {min }}-1\right) d_{\text {min }}^{\prime}\right\}$. A similar argument shows that $\theta_{k, s}^{3}$ is an isomorphism if $(k, s) \notin \bigcup_{t=1}^{3} \mathcal{S}_{t}$, where $\mathcal{S}_{3}=\left\{(u, v) \in \mathbb{Z}^{2}:(u+2, v+1) \in \mathcal{S}_{1} \cup \mathcal{S}_{2}\right\}$. Continuing in the same fashion, considering the differentials $d^{t}$ 's on $E_{k, s}^{t ; \mathbb{C}}$ and ${ }^{\prime} E_{k, s}^{t ; \mathbb{C}}$ and applying the inductive hypothesis, we see that $\theta_{k, s}^{\infty}$ is an isomorphism if $(k, s) \notin \mathcal{S}:=$ $\bigcup_{t \geq 1} \mathcal{S}_{t}=\bigcup_{t \geq 1} A_{t}$, where $A_{t}$ denotes the set

$$
A_{t}=\left\{\begin{array}{l|l}
(u, v) \in \mathbb{Z}^{2} & \begin{array}{l}
\text { There are positive integers } l_{1}, \cdots, l_{t} \text { such that, } \\
1 \leq l_{1}<l_{2}<\cdots<l_{t}, u+\sum_{j=1}^{t} l_{j}=d_{\min }^{\prime}+1 \\
v+\sum_{j=1}^{t}\left(l_{j}-1\right) \geq\left(2 n r_{\min }-1\right) d_{\min }^{\prime}
\end{array}
\end{array}\right\}
$$

Note that if this set was empty for every $t$, then, of course, the conclusion of Theorem 6.5 would hold in all dimensions (this is known to be false in general). If $A_{t} \neq \emptyset$, it is easy to see that

$$
\begin{aligned}
a(t) & =\min \left\{s-k:(k, s) \in A_{t}\right\}=\left(2 n r_{\min }-1\right) d_{\min }^{\prime}-\left(d_{\min }^{\prime}+1\right)+t \\
& =\left(2 n r_{\min }-2\right) d_{\min }^{\prime}+t-1=d(D ; \Sigma, n, \mathbb{C})+t+1
\end{aligned}
$$

Hence, we obtain that $\min \left\{a(t): t \geq 1, A_{t} \neq \emptyset\right\}=d(D ; \Sigma, n, \mathbb{C})+2$. Since $\theta_{k, s}^{\infty}$ is an isomorphism for any $(k, s) \notin \bigcup_{t \geq 1} A_{t}$ for each $0 \leq k \leq d_{\min }^{\prime}$, we have the following:
$(\dagger)_{2}$ If $0 \leq k \leq d_{\text {min }}^{\prime}, \theta_{k, s}^{\infty}$ is an isomorphism for any $(k, s)$ such that $s-k \leq$ $d(D ; \Sigma, n, \mathbb{C})+1$.
Then, by $(\dagger)_{1}$ and $(\dagger)_{2}$, we know that $\theta_{k, s}^{\infty ; \mathbb{C}}: E_{k, s}^{\infty ; \mathbb{C}} \xlongequal{\cong}{ }^{\prime} E_{k, s}^{\infty ; \mathbb{C}}$ is an isomorphism for any $(k, s)$ if $s-k \leq d(D ; \Sigma, n, \mathbb{C})$. Hence, by $(\dagger)$ we have the desired assertion and this completes the proof of Theorem 6.5.
Corollary 6.6. For each $\boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$, the stabilization map

$$
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{C})
$$

is a homology equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.
Proof. The assertion easily follows from (5.7) and Theorem 6.5.
6.2 The case $\mathbb{K}=\mathbb{R}$. Next, we shall consider the case $\mathbb{K}=\mathbb{R}$. By using exactly the same approach as in Lemmas 4.6, 4.7, 6.1, 6.3, 6.4, Theorem 6.5, and Corollary 6.6, we can obtain the following result.

Lemma 6.7. There is the following truncated spectral sequence

$$
\begin{equation*}
\left\{E_{k, s}^{t ; \mathbb{R}}, d^{t}: E_{k, s}^{t ; \mathbb{R}} \rightarrow E_{k+t, s+t-1}^{t ; \mathbb{R}}\right\} \Rightarrow H_{s-k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) ; \mathbb{Z}\right) \tag{6.13}
\end{equation*}
$$

satisfying the following conditions:
(i) If $k<0$ or $k \geq\left\lfloor\frac{d_{\min }}{n}\right\rfloor+2, E_{k, s}^{1 ; \mathbb{R}}=0$ for any $s$.
(ii) $E_{0,0}^{1 ; \mathbb{R}}=\mathbb{Z}$ and $E_{0, s}^{1 ; \mathbb{R}}=0$ if $s \neq 0$.
(iii) If $1 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$, there is a natural isomorphism

$$
E_{k, s}^{1 ; \mathbb{R}} \cong H_{c}^{n r k-s}\left(C_{k ; \Sigma, \mathbb{R}} ; \pm \mathbb{Z}\right)
$$

(iv) If $1 \leq k \leq\left\lfloor\frac{d_{\min }}{n}\right\rfloor, E_{k, s}^{1 ; \mathbb{R}}=0$ for any $s \leq\left(n r_{\min }(\Sigma)-1\right) k-1$.
(v) If $k=\left\lfloor\frac{d_{\min }}{n}\right\rfloor+1, E_{k, s}^{1 ; \mathbb{R}}=0$ for any $s \leq\left(n r_{\min }(\Sigma)-1\right)\left\lfloor\frac{d_{\min }}{n}\right\rfloor-1$.

Theorem 6.8. For each $\boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$, the stabilization map

$$
s_{D, D+a}^{\mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{R})
$$

is a homology equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$, where $d(D ; \Sigma, n, \mathbb{R})$ denotes the integer given by (2.34).

Proof. This assertion can be proved by using the spectral sequence (6.13) in exactly the same way as in the case of $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$, so we omit the details.

Corollary 6.9. For each $\boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$, the stabilization map

$$
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{C})
$$

is a $\mathbb{Z}_{2}$-equivariant homology equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.
Proof. Since $d(D ; \Sigma, n, \mathbb{R})<d(D ; \Sigma, n, \mathbb{C})$, the assertion follows from (5.11), Corollary 6.6 and Theorem 6.8.

## 7 Connectivity

Lemma 7.1. (i) The space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is simply connected.
(ii) The space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is simply connected if $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$.

Proof. Note that an element of $\pi_{1}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})\right)$ can be represented by an $r$ tuple $\left(\eta_{1}, \cdots, \eta_{r}\right)$ of strings of $r$-different colors where each $\eta_{k}(1 \leq k \leq r)$ has total multiplicity $d_{k}$, as in the case of strings representing elements of the classical braid group $\mathrm{Br}_{d}=\pi_{1}\left(C_{d}(\mathbb{C})\right)$ [18]. However, in our case an $r$-tuple $\left(\eta_{1}, \cdots, \eta_{r}\right)$ of strings of $r$-different colors can move continuously representing the same element of the fundamental group, 9 as long as the following situation $(*)_{\sigma}$ does not occur for each $\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \in I\left(\mathcal{K}_{\Sigma}\right)$ :
$(*)_{\sigma}$ The strings $\left\{\eta_{i}\right\}_{i \in \sigma}$ of $s$-different colors with multiplicity $\geq n$ pass through a single point of the real line $\mathbb{R}$.
(i) In the case $\mathbb{K}=\mathbb{C}$, we can continuously deform the strings $\left(\eta_{1}, \cdots, \eta_{r}\right)$ and, if necessary, make them pass through one another in $\mathbb{C} \backslash \mathbb{R}$, so that any collection of strings can be continuously deformed to a trivial one. Thus $\pi_{1}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})\right)$ is trivial.
(ii) Let $\mathbb{K}=\mathbb{R}$. If $n \geq 2$, a similar argument as above shows that the fundamental group must be trivial, since any string of multiplicity $\geq n$ can be split into stings of multiplicity less than $n$ (by the continuous deformation). Thus, the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is path-connected and simply connected if $n \geq 2$.

Next, consider the case $n=1$ with $r_{\text {min }}(\Sigma) \geq 3$. Then the space $\mathrm{Q}_{1}^{D, \Sigma}(\mathbb{R})$ is path-connected and it is simply connected. To see this, let $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$ and $\{i, j\} \subset \sigma$. Since $\operatorname{card}(\sigma) \geq r_{\min }(\Sigma) \geq 3$ (by (3.3)), there is some number $k \in \sigma$ such that $k \notin\{i, j\}$. But this means that the $i$-th braid and the $j$-th braid can pass through one another on the real line, as long as they both don't pass through the $k$-th braid at the same time. By using this fact, we see that any collection of strings can be continuously deformed to a trivial one. Thus, $\mathrm{Q}_{1}^{D, \Sigma}(\mathbb{R})$ is path connected and that $\pi_{1}\left(\mathrm{Q}_{1}^{D, \Sigma}(\mathbb{R})\right)$ is trivial. Since $n \geq 2$ or $n=1$ with $r_{\min }(\Sigma) \geq 3 \Leftrightarrow\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$, the assertion (ii) follows.

Remark 7.2. The space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is not path-connected if $\left(n, r_{\text {min }}(\Sigma)\right)=$ $(1,2)$. But its each path-component is simply connected.

To see this, suppose that $\left(n, r_{\min }(\Sigma)\right)=(1,2)$. Since $r_{\min }(\Sigma)=2$, there has to exist $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$ such that $\sigma=\{i, j\}$ (by (3.3)). Since $n=1$, this

[^7]means that particles on the real line corresponding to the $i$-th and the $j$-th polynomial cannot cross one another on the real line (i.e. the $i$-th and the $j$-th polynomials cannot have common real roots). Thus, $\mathrm{Q}_{1}^{D, \Sigma}(\mathbb{R})$ is not path-connected. However, since there are no restrictions on the movement of roots (particles) within a connected component, each path-component is simply connected.

Corollary 7.3. (i) If the condition (1.4)* holds, $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is simply connected. (ii) If the condition $(1.4)^{\dagger}$ holds, $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is simply connected.

Proof. The assertions follow from Lemma 7.1 .
Lemma 7.4. (i) If $k<0$, or $k \geq\left\lfloor\frac{d_{\min }}{n}\right\rfloor+2$, or $k=0$ and $s \neq 0, E_{k, s}^{1 ; \mathbb{K}}=0$.
(ii) If $1 \leq k \leq\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$ and $s-k \leq\left(d(\mathbb{K}) n r_{\min }(\Sigma)-2\right) k-1, E_{k, s}^{1 ; \mathbb{K}}=0$.
(iii) If $k=\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor+1$ and $s-k \leq\left(d(\mathbb{K}) n r_{\text {min }}(\Sigma)-2\right)\left\lfloor d_{\text {min }} / n\right\rfloor-2, E_{k, s}^{1 ; \mathbb{K}}=0$.

Proof. The assertions follow from Lemmas 6.3 and 6.13.
Lemma 7.5. (i) If $\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor \geq 2$,

$$
\tilde{H}_{i}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) ; \mathbb{Z}\right)=0 \quad \text { for any } i \leq d(\mathbb{K}) n r_{\min }(\Sigma)-3 .
$$

(ii) If $\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor=1$,

$$
\tilde{H}_{i}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) ; \mathbb{Z}\right)=0 \quad \text { for any } i \leq d(\mathbb{K}) n r_{\min }(\Sigma)-4
$$

Proof. Let us write $d_{\text {min }}^{\prime}=\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor$. Consider the spectral sequences (6.10) and (6.13). Define the integer $a(k)$ by

$$
a(k)=\left(d(\mathbb{K}) n r_{\min }(\Sigma)-2\right) n_{0}(k)-\epsilon(k) \quad \text { for each } 1 \leq k \leq d_{\min }^{\prime}+1,
$$

where $n_{0}(k)$ and $\epsilon(k)$ denote the integers given by

$$
\left(n_{0}(k), \epsilon(k)\right)= \begin{cases}(k, 1) & \text { if } 1 \leq k \leq d_{\min }^{\prime} \\ \left(d_{\min }^{\prime}, 2\right) & \text { if } k=d_{\min }^{\prime}+1 .\end{cases}
$$

Then, by Lemma 7.4, we see that $E_{k, s}^{1 ; \mathbb{K}}=0$ for any $(k, s) \neq(0,0)$ if $s-k \leq$ $m_{0}=\min \left\{a(k): 1 \leq k \leq d_{\text {min }}^{\prime}+1\right\}$ is satisfied. Hence, $\tilde{H}_{k}\left(\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) ; \mathbb{Z}\right)=0$ for any $k \leq m_{0}$. However, we see that

$$
\begin{aligned}
m_{0} & =\min \left\{a(k): 1 \leq k \leq d_{\min }^{\prime}+1\right\}=\min \left\{a(1), a\left(d_{\min }^{\prime}+1\right)\right\} \\
& = \begin{cases}d(\mathbb{K}) n r_{\min }(\Sigma)-3 & \text { if } d_{\min }^{\prime} \geq 2, \\
d(\mathbb{K}) n r_{\min }(\Sigma)-4 & \text { if } d_{\min }^{\prime}=1 .\end{cases}
\end{aligned}
$$

Hence, we obtain the assertions (i) and (ii).

Corollary 7.6. (i) If $n \geq 2$ and $\left\lfloor\frac{d_{\min }}{n}\right\rfloor \geq 2, \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is $\left(2 n r_{\min }(\Sigma)-3\right)$ connected.
(ii) If $n \geq 2$ and $\left\lfloor\frac{d_{\text {min }}}{n}\right\rfloor=1, Q_{n}^{D, \Sigma}(\mathbb{C})$ is $\left(2 n r_{\min }(\Sigma)-4\right)$-connected.
(iii) If $n=1$ and $d_{\min } \geq 2, Q_{n}^{D, \Sigma}(\mathbb{C})$ is $\left(2 r_{\min }(\Sigma)-3\right)$-connected.
(iv) Let $n=d_{\text {min }}=1$. Then $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is $\left(2 r_{\min }(\Sigma)-4\right)$-connected if $r_{\min }(\Sigma) \geq 3$, and it is simply connected if $r_{\min }(\Sigma)=2$.

Proof. Since $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is simply connected (by Lemma 7.1), the assertions follow from the Hurewicz Theorem and Lemma 7.5

Corollary 7.7. (i) If $n \geq 2$ and $\left\lfloor\frac{d_{\min }}{n}\right\rfloor \geq 2, \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is $\left(n r_{\text {min }}(\Sigma)-3\right)$ connected.
(ii) Let $n \geq 2$ and $\left\lfloor\frac{d_{\min }}{n}\right\rfloor=1$. Then $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is $\left(n r_{\min }(\Sigma)-4\right)$-connected if $n r_{\min }(\Sigma) \geq 5$, and it is simply connected if $n=r_{\min }(\Sigma)=2$.
(iii) Let $n=1, d_{\min } \geq 2$. Then $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is $\left(r_{\min }(\Sigma)-3\right)$-connected if $r_{\min }(\Sigma) \geq 4$, and it is simply connected if $r_{\min }(\Sigma)=3$.
(iv) Let $n=d_{\text {min }}=1$. Then $\mathbb{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is $\left(r_{\min }(\Sigma)-4\right)$-connected if $r_{\min }(\Sigma) \geq 5$, and it is simply connected if $r_{\min }(\Sigma)=3$ or 4 .

Proof. If $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$, the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is simply connected (by Lemma 7.1). Thus the assertions easily follow from he Hurewicz Theorem and Lemma 7.5

Corollary 7.8. Let $\boldsymbol{a} \neq \mathbf{0}_{r} \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$.
(i) If the condition $(1.4)^{*}$ holds, the stabilization map

$$
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{C})
$$

is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.
(ii) If the condition $(1.4)^{\dagger}$ holds, the stabilization map

$$
s_{D, D+a}^{\mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{R})
$$

is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.
(iii) If the condition $(1.4)^{\dagger}$ holds, the stabilization map

$$
s_{D, D+a}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{D+a, \Sigma}(\mathbb{C})
$$

is a $\mathbb{Z}_{2}$-equivariant homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.
Proof. The assertions (i) and (ii) follow from Theorem 6.8, Corollaries 6.6 and 7.3. Since $d(D ; \Sigma, n, \mathbb{R})<d(D ; \Sigma, n, \mathbb{C})$ and $\left(s_{D, D+a}\right)^{\mathbb{Z}_{2}}=s_{D, D+a}^{\mathbb{R}}$, the assertion (iii) follows from (i) and (ii).

## 8 Scanning maps

In this section we study about the configuration space model of $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ and the corresponding scanning map.
8.1 Configuration space models. First, consider about the configuration space model of $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$.

Definition 8.1. For a positive integer $d \geq 1$ and a based space $X$, let $\operatorname{SP}^{d}(X)$ denote the $d$-th symmetric product of $X$ defined as the orbit space

$$
\begin{equation*}
\operatorname{SP}^{d}(X)=X^{d} / S_{d}, \tag{8.1}
\end{equation*}
$$

where the symmetric group $S_{d}$ of $d$ letters acts on the $d$-fold product $X^{d}$ in the natural manner.

Remark 8.2. (i) Note that an element $\eta \in \operatorname{SP}^{d}(X)$ may be identified with a formal linear combination

$$
\begin{equation*}
\eta=\sum_{k=1}^{s} n_{k} x_{k} \tag{8.2}
\end{equation*}
$$

where $\left\{x_{k}\right\}_{k=1}^{s} \in C_{s}(X)$ and $\left\{n_{k}\right\}_{k=1}^{s} \subset \mathbb{N}$ with $\sum_{k=1}^{s} n_{k}=d$. In this situation we shall refer to $\eta$ as configuration (or divisor) of points, the point $x_{k} \in X$ having a multiplicity $n_{k}$.
(ii) For example, when $X=\mathbb{C}$, we have the natural homeomorphism

$$
\begin{equation*}
\psi_{d}: \mathrm{P}_{d}^{\mathbb{C}} \xrightarrow{\cong} \mathrm{SP}^{d}(\mathbb{C}) \tag{8.3}
\end{equation*}
$$

given by using the above identification

$$
\begin{equation*}
\psi_{d}(f(z))=\sum_{k=1}^{s} d_{k} \alpha_{k} \quad \text { for } f(z)=\prod_{k=1}^{s}\left(z-\alpha_{k}\right)^{d_{k}} \in \mathrm{P}_{d}^{\mathbb{C}} \tag{8.4}
\end{equation*}
$$

Definition 8.3. (i) For a subspace $A \subset X$, let $\mathrm{SP}^{d}(X, A)$ denote the quotient space

$$
\begin{equation*}
\operatorname{SP}^{d}(X, A)=\operatorname{SP}^{d}(X) / \sim \tag{8.5}
\end{equation*}
$$

where the equivalence relation $\sim$ is defined by

$$
\begin{equation*}
\xi \sim \eta \Leftrightarrow \xi \cap(X \backslash A)=\eta \cap(X \backslash A) \quad \text { for } \xi, \eta \in \mathrm{SP}^{d}(X) \tag{8.6}
\end{equation*}
$$

Thus, the points of $A$ are ignored. When $A \neq \emptyset$, by adding a point in $A$ we have the natural inclusion $\mathrm{SP}^{d}(X, A) \subset \operatorname{SP}^{d+1}(X, A)$. Thus, when $A \neq \emptyset$, one can define the space $\operatorname{SP}^{\infty}(X, A)$ by the union

$$
\begin{equation*}
\operatorname{SP}^{\infty}(X, A)=\bigcup_{d \geq 0} \operatorname{SP}^{d}(X, A) \tag{8.7}
\end{equation*}
$$

where we set $\mathrm{SP}^{0}(X, A)=\{\emptyset\}$ and $\emptyset$ denotes the empty configuration.
(ii) From now on, we always assume that $X \subset \mathbb{C}$. For each $r$-tuple $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, let $\mathrm{SP}^{D}(X)=\prod_{i=1}^{r} \mathrm{SP}^{d_{i}}(X)$, and define the space $\mathcal{Q}_{D, n}^{\Sigma}(X)$ by

$$
\begin{equation*}
\mathcal{Q}_{D, n}^{\Sigma}(X)=\left\{\left(\xi_{1}, \cdots, \xi_{r}\right) \in \mathrm{SP}^{D}(X): \text { the condition }(*)_{n}^{\Sigma} \text { holds }\right\} \tag{8.8}
\end{equation*}
$$

where the condition $(*)_{n}^{\Sigma}$ is given by
$(*)_{n}^{\Sigma}$ : The configuration $\left(\bigcap_{k \in \sigma} \xi_{k}\right) \cap \mathbb{R}$ contains no point $x \in X$ of multiplicity $\geq n$ for any $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$.
(iii) When $A \subset X$ is a subspace, define an equivalence relation " $\sim$ " on the space $\mathcal{Q}_{D, n}^{\Sigma}(X)$ by

$$
\left(\xi_{1}, \cdots, \xi_{r}\right) \sim\left(\eta_{1}, \cdots, \eta_{r}\right) \quad \text { if } \quad \xi_{i} \cap(X \backslash A)=\eta_{i} \cap(X \backslash A)
$$

for each $1 \leq j \leq r$. Let $\mathcal{Q}_{D, n}^{\Sigma}(X, A)$ be the quotient space defined by

$$
\begin{equation*}
\mathcal{Q}_{D, n}^{\Sigma}(X, A)=\mathcal{Q}_{D, n}^{\Sigma}(X) / \sim \tag{8.9}
\end{equation*}
$$

When $A \neq \emptyset$, by adding points in $A$ we have natural inclusion

$$
\begin{equation*}
\mathcal{Q}_{D, n}^{\Sigma}(X, A) \subset \mathcal{Q}_{D+e_{i}, n}^{\Sigma}(X, A) \quad \text { for each } 1 \leq i \leq r, \tag{8.10}
\end{equation*}
$$

where $D+\boldsymbol{e}_{i}=\left(d_{1}, \cdots, d_{i-1}, d_{i}+1, d_{i+1}, \cdots, d_{r}\right)$.
Thus, when $A \neq \emptyset$, one can define the space $\mathcal{Q}_{n}^{\Sigma}(X, A)$ by the union

$$
\begin{equation*}
\mathcal{Q}_{n}^{\Sigma}(X, A)=\bigcup_{D \in \mathbb{N}^{r}} \mathcal{Q}_{D, n}^{\Sigma}(X, A), \tag{8.11}
\end{equation*}
$$

where the empty configuration $(\emptyset, \cdots, \emptyset)$ is the base-point of $\mathcal{Q}_{n}^{\Sigma}(X, A)$.
Remark 8.4. (i) Let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$. Then by using the identification (8.3) we easily obtain the homeomorphism

$$
\begin{align*}
\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) & \stackrel{\Psi_{D}}{\cong} \quad \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})  \tag{8.12}\\
\left(f_{1}(z), \cdots, f_{r}(z)\right) & \xrightarrow{\cong}\left(\psi_{d_{1}}\left(f_{1}(z)\right), \cdots, \psi_{d_{r}}\left(f_{r}(z)\right)\right)
\end{align*}
$$

(ii) Now let $\varphi_{D}: \mathbb{C} \xrightarrow{\cong} U_{D}$ and $\boldsymbol{x}_{D}=\left(x_{D, 1}, \cdots, x_{D, r}\right) \in F\left(\mathbb{C} \backslash \overline{U_{D}}, r\right)$ be the homeomorphism and the point for defining the stabilization map $s_{D}$ given in Definition 5.1. Then define the map

$$
\begin{align*}
& s_{D}^{\Sigma}: \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C}) \rightarrow \mathcal{Q}_{D+e, n}^{\Sigma}(\mathbb{C}) \quad \text { by }  \tag{8.13}\\
& s_{D}^{\Sigma}\left(\xi_{1}, \cdots, \xi_{r}\right)=\left(\varphi_{D}\left(\xi_{1}\right)+x_{D, 1}, \cdots, \varphi_{D}\left(\xi_{r}\right)+x_{D, r}\right)
\end{align*}
$$

for $\left(\xi_{1}, \cdots, \xi_{r}\right) \in \mathcal{Q}_{D, n}^{\Sigma}$, where we write $\varphi_{D}(\xi)=\sum_{k=1}^{s} n_{k} \varphi_{D}\left(x_{k}\right)$ if $\xi=$ $\sum_{k=1}^{s} n_{k} x_{k} \in \mathrm{SP}^{d}(\mathbb{C})$ and $\left(n_{k}, x_{k}\right) \in \mathbb{N} \times \mathbb{C}$ with $\sum_{k=1}^{s} n_{k}=d$.

Then by using the above homeomorphism (8.12), we have the following commutative diagram

$$
\begin{array}{ll}
\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \xrightarrow{s_{D, D+e}} & \mathrm{Q}_{n}^{D+e, \Sigma}(\mathbb{C}) \\
\Psi_{D} \downarrow \cong & \Psi_{D+e} \downarrow \cong  \tag{8.14}\\
\mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C}) \xrightarrow{s_{D}^{\Sigma}} & \mathcal{Q}_{D+e, n}^{\Sigma}(\mathbb{C})
\end{array}
$$

(iv) Note that $\mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})$ is path-connected. Indeed, for any two points $\xi_{0}, \xi_{1} \in \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})$, one can construct a path $\omega:[0,1] \rightarrow \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})$ such that $\omega(i)=\xi_{i}$ for $i \in\{0,1\}$ by using the string representation used in [16, $\S$ Appendix]. Thus the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ is also path-connected. By choosing the path $\omega \mathbb{Z}_{2}$-equivariant way, one can show that $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is also pathconnected if $n \geq 2$ or if $n=1$ and $r_{\text {min }}(\Sigma) \geq 3 \Leftrightarrow\left(n, r_{\text {min }}(\Sigma)\right) \neq(1,2)$ (see also the proof of Lemma 7.1 and Remark (7.2).
Definition 8.5. Define the stabilized space $Q_{D+\infty}^{\Sigma}(\mathbb{C})$ by the colimit

$$
\begin{equation*}
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})=\lim _{k \rightarrow \infty} \mathrm{Q}_{n}^{D+k e, \Sigma}(\mathbb{C}) \tag{8.15}
\end{equation*}
$$

where the colimit is taken from the family of stabilization maps

$$
\begin{equation*}
\left\{s_{D+k e, D+(k+1) e}: \mathrm{Q}_{n}^{D+k e, \Sigma}(\mathbb{C}) \rightarrow \mathrm{Q}_{n}^{\Sigma, D+(k+1) e}(\mathbb{C})\right\}_{k \geq 0} \tag{8.16}
\end{equation*}
$$

8.2 Scanning maps. Now we are ready to define the scanning map. From now on, we identify $\mathbb{C}=\mathbb{R}^{2}$ in a usual way.

Definition 8.6. For a rectangle $X$ in $\mathbb{C}=\mathbb{R}^{2}$, let $\sigma X$ denote the union of the sides of $X$ which are parallel to the $y$-axis, and for a subspace $Z \subset \mathbb{C}=\mathbb{R}^{2}$, let $\bar{Z}$ be the closure of $Z$. From now on, let $I$ denote the interval $I=[-1,1]$ and let $0<\epsilon<\frac{1}{1000000}$ be any fixed real number.

For each $x \in \mathbb{R}$, let $V(x)$ be the set defined by

$$
\begin{align*}
V(x) & =\{w \in \mathbb{C}:|\operatorname{Re}(w)-x|<\epsilon,|\operatorname{Im}(w)|<1\}  \tag{8.17}\\
& =(x-\epsilon, x+\epsilon) \times(-1,1)
\end{align*}
$$

and let's identify $I \times I=I^{2}$ with the closed unit rectangle $\{t+s \sqrt{-1} \in \mathbb{C}$ : $-1 \leq t, s \leq 1\}$ in $\mathbb{C}$.

For each $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$, define the horizontal scanning map

$$
\begin{equation*}
s c_{D}: \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \partial I \times I\right)=\Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right) \tag{8.18}
\end{equation*}
$$

as follows. For each $r$-tuple $\alpha=\left(\xi_{1}, \cdots, \xi_{r}\right) \in \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})$ of configurations, let $s c_{D}(\alpha): \mathbb{R} \rightarrow \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \partial I \times I\right)=\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)$ denote the map given by

$$
\mathbb{R} \ni x \mapsto\left(\xi_{1} \cap \bar{V}(x), \cdots, \xi_{r} \cap \bar{V}(x)\right) \in \mathcal{Q}_{n}^{\Sigma}(\bar{V}(x), \sigma \bar{V}(x)) \cong \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)
$$

where we use the canonical identification $(\bar{V}(x), \sigma \bar{V}(x)) \cong\left(I^{2}, \sigma I^{2}\right)$.
Since $\lim _{x \rightarrow \pm \infty} s c_{D}(\alpha)(x)=(\emptyset, \cdots, \emptyset)$, by setting $s c_{D}(\alpha)(\infty)=(\emptyset, \cdots, \emptyset)$ we obtain a based map $s c_{D}(\alpha) \in \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)$, where we identify $S^{1}=$ $\mathbb{R} \cup \infty$ and we choose the empty configuration $(\emptyset, \cdots, \emptyset)$ as the base-point of $\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)$. One can show that the following diagram is homotopy commutative:

$$
\begin{gather*}
\mathcal{Q}_{D+k e, n}^{\Sigma}(\mathbb{C}) \xrightarrow{s c_{D+k e}} \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right) \\
\|  \tag{8.19}\\
\mathcal{Q}_{D+(k+1) e, n}^{\Sigma}(\mathbb{C}) \xrightarrow{s c_{D+(k+1) e}} \downarrow \\
\| \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)
\end{gather*}
$$

By using the above diagram and by identifying $Q_{n}^{D+k e, \Sigma}(\mathbb{C})$ with $\mathcal{Q}_{D+k e, n}^{\Sigma}(\mathbb{C})$, we finally obtain the stable horizontal scanning map

$$
\begin{equation*}
S^{H}=\lim _{k \rightarrow \infty} s c_{D+k e}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right), \tag{8.20}
\end{equation*}
$$

where $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})$ is defined in (8.15).
Theorem 8.7 ([31], (cf. [14], [25])). The stable horizontal scanning map

$$
S^{H}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \xrightarrow{\simeq} \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)
$$

is a homotopy equivalence.
Proof. The proof is analogous to the one given in [31, Prop. 3.2, Lemma 3.4] and [14, Prop. 2]. However, as it appears to be probably most difficult and least familiar part of the article [31], we gave its precise proof in [25, Theorem 5.6] (see also [25, Remark 5.8]).

Definition 8.8. (i) Define the stabilized space $\mathrm{Q}_{D+\infty}^{\Sigma}(\mathbb{R})$ by the colimit

$$
\begin{equation*}
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})=\lim _{k \rightarrow \infty} \mathrm{Q}_{n}^{D+k e, \Sigma}(\mathbb{R}) \tag{8.21}
\end{equation*}
$$

where the colimit is taken from the family of stabilization maps

$$
\begin{equation*}
\left\{s_{D+k e, D+(k+1) e}^{\mathbb{R}}: \mathrm{Q}_{n}^{D+k e, \Sigma}(\mathbb{R}) \rightarrow \mathrm{Q}_{n}^{\Sigma, D+(k+1) e}(\mathbb{R})\right\}_{k \geq 0} \tag{8.22}
\end{equation*}
$$

(ii) Recall the $\mathbb{Z}_{2}$-action on the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ induced from the complex conjugation on $\mathbb{C}$. Then by using (5.11), one can easily see the following:

$$
\begin{equation*}
\mathrm{Q}_{n}^{D+\infty}(\mathbb{R})=\left(\mathrm{Q}_{n}^{D+\infty}(\mathbb{C})\right)^{\mathbb{Z}_{2}} \tag{8.23}
\end{equation*}
$$

Moreover, since $s_{D, k}^{\mathbb{R}}=\left(s_{D, k}\right)^{\mathbb{Z}_{2}}$ as in Remark 5.4, one can define the horizontal scanning map

$$
\begin{equation*}
S^{\mathbb{Z}_{2}}=\lim _{k \rightarrow \infty}\left(s c_{D+k e}\right)^{\mathbb{Z}_{2}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)^{\mathbb{Z}_{2}} \tag{8.24}
\end{equation*}
$$

in a complete similar way as (8.20).
Since $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})=\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})^{\mathbb{Z}_{2}} \subset \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$, one can identify the space $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})$ with the subspace of $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})$. By this identification, we can identify

$$
\begin{equation*}
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})=\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})^{\mathbb{Z}_{2}} \quad \text { and } \quad S^{\mathbb{Z}_{2}}=S^{H} \mid \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})=\left(S^{H}\right)^{\mathbb{Z}_{2}} \tag{8.25}
\end{equation*}
$$

Theorem 8.9 ([31], (cf. [14], [25)). The stable horizontal scanning map

$$
\left(S^{H}\right)^{\mathbb{Z}_{2}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R}) \xrightarrow{\simeq} \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)^{\mathbb{Z}_{2}}
$$

is a homotopy equivalence if $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$.
Proof. Note that $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})$ is simply connected if $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$. The proof is completely analogous to that of Theorem 8.7.

Corollary 8.10. The stable horizontal scanning map

$$
S^{H}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \xrightarrow{\simeq} \Omega \mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)
$$

is a $\mathbb{Z}_{2}$-equivariant homotopy equivalence if $\left(n, r_{\min }(\Sigma)\right) \neq(1,2)$.
Proof. The assertion follows from Theorems 8.7 and 8.9 .

## 9 The stable result

In this section we prove the stable theorem (Theorem 9.2).
Definition 9.1. From now on, let $\boldsymbol{e}=\left(a_{1}, \cdots, a_{r}\right) \in \mathbb{N}^{r}$ be any fixed an $r$-tuple of positive integers such that $\sum_{k=1}^{r} a_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$.

Let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ be an $r$-tuple of positive integers such that $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$. Then it is easy to see that the following two diagram is homotopy commutative for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ :

$$
\begin{gathered}
\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K}) \quad \xrightarrow{j_{D, n, \mathbb{K}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \simeq \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right) \\
\| \\
s_{D, D+e}^{s^{\mathbb{K}}} \downarrow \\
\mathrm{Q}_{n}^{D+e, \Sigma}(\mathbb{K}) \xrightarrow{j_{D+e, n, \mathbb{K}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \simeq \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right)
\end{gathered}
$$

where we set

$$
\begin{equation*}
s_{D, D+e}^{\mathbb{K}}=s_{D, D+e} \quad \text { if } \mathbb{K}=\mathbb{C} . \tag{9.1}
\end{equation*}
$$

Hence, for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, we obtain the stabilized map

$$
\begin{equation*}
j_{D+\infty, n, \mathbb{K}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{K}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right), \tag{9.2}
\end{equation*}
$$

where we set

$$
\begin{equation*}
j_{D+\infty, n, \mathbb{K}}=\lim _{t \rightarrow \infty} j_{D+t e, D+(t+1) e, \mathbb{K}} . \tag{9.3}
\end{equation*}
$$

The main purpose of this section is to prove the following result.
Theorem 9.2. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and let $D=\left(d_{1}, \cdots, d_{r}\right) \in \mathbb{N}^{r}$ be an $r$-tuple of positive integers such that $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{m}$. Then the stabilized map

$$
j_{D+\infty, n, \mathbb{K}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{K}) \xrightarrow{\simeq} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{d(\mathbb{K}) n}, S^{d(\mathbb{K}) n-1}\right)
$$

is a homotopy equivalence.
Before proving Theorem 9.2 we need the following definition and lemma.
Definition 9.3. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ as before. Now we identify $\mathbb{C}=\mathbb{R}^{2}$ in a usual way and let us write $U=\{w \in \mathbb{C}:|\operatorname{Re}(w)|<1,|\operatorname{Im}(w)|<1\}=$ $(-1,1) \times(-1,1)$ and $I=[-1,1]$.
(i) For an open set $X \subset \mathbb{C}$, let $F_{n}^{\mathbb{K}}(X)$ denote the space of $r$-tuples $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathbb{K}[z]^{r}$ of (not necessarily monic) polynomials satisfying the following condition $(*)_{n, \mathbb{R}}$ :
$(*)_{n, \mathbb{R}}$ For any $\sigma=\left\{i_{1}, \cdots, i_{s}\right\} \in I\left(\mathcal{K}_{\Sigma}\right)$, the polynomials $f_{i_{1}}(z), \cdots, f_{i_{s}}(z)$ have no common real roots of multiplicity $\geq n$ in $X$ (i.e. no common roots of multiplicity $\geq n$ in $X$ ).
(ii) Let $e v_{0, \mathbb{K}}: F_{n}^{\mathbb{K}}(U) \rightarrow \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$ denote the map given by evaluation at 0, i.e.

$$
\begin{equation*}
e v_{0, \mathbb{K}}\left(f_{1}(z), \cdots, f_{r}(z)\right)=\left(F_{n}\left(f_{1}\right)(0), \cdots, F_{n}\left(f_{r}\right)(0)\right) \tag{9.4}
\end{equation*}
$$

for $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in F_{n}^{\mathbb{K}}(U)$, where $F_{n}\left(f_{i}\right)(z)$ denotes the $n$-tuple of monic polynomials of the same degree $d_{i}$ given by (4.6).
(iii) Let $\tilde{F}_{n}^{\mathbb{K}}(U) \subset F_{n}^{\mathbb{K}}(U)$ denote the subspace of all $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in$ $F_{n}^{\Sigma, \mathbb{K}}(U)$ such that no $f_{i}(z)$ is identically zero.

Let $e v_{\mathbb{K}}: \tilde{F}_{n}^{\mathbb{K}}(U) \rightarrow \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$ denote the map given by the restriction

$$
\begin{equation*}
e v_{\mathbb{K}}=e v_{0, \mathbb{K}} \mid \tilde{F}_{n}^{\mathbb{K}}(U) \tag{9.5}
\end{equation*}
$$

It is easy to see that the following two equality holds:

$$
\begin{equation*}
e v_{\mathbb{R}}=\left(e v_{\mathbb{C}}\right)^{\mathbb{Z}_{2}} \tag{9.6}
\end{equation*}
$$

(iv) Note that the group $\mathbb{T}_{\mathbb{K}}^{r}=\left(\mathbb{K}^{*}\right)^{r}$ acts freely on the space $\tilde{F}_{n}^{\mathbb{K}}(U)$ in a natural way, and let

$$
\begin{equation*}
p_{\mathbb{K}}: \tilde{F}_{n}^{\mathbb{K}}(U) \rightarrow \tilde{F}_{n}^{\mathbb{K}}(U) / \mathbb{T}_{\mathbb{K}}^{r} \tag{9.7}
\end{equation*}
$$

denote the natural projection, where $\tilde{F}_{n}^{\mathbb{K}}(U) / \mathbb{T}_{\mathbb{K}}^{r}$ denotes the corresponding orbit space.

Lemma 9.4. Let $X_{\Sigma}$ be a simply connected non-singular toric variety such that the condition (2.18)* is satisfied.
(i) If the condition (1.4)* is satisfied, the space $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})$ is simply connected. Similarly, if the condition $(1.4)^{\dagger}$ is satisfied, the space $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})$ is simply connected.
(ii) The map ev $v_{\mathbb{K}}: \tilde{F}_{n}^{\mathbb{K}}(U) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$ is a homotopy equivalence.

Proof. (i) The assertion easily follows from Corollary 7.3,
(ii) For each $\boldsymbol{b}=\left(b_{0}, b_{1}, \cdots, b_{n-1}\right) \in \mathbb{K}^{n}$, let $f_{\boldsymbol{b}}(z) \in \mathbb{K}[z]$ denote the polynomial of degree $\leq n$ defined by

$$
\begin{equation*}
f_{b}(z)=b_{0}+\sum_{k=1}^{n-1} \frac{b_{k}-b_{0}}{k!} z^{k} . \tag{9.8}
\end{equation*}
$$

Let $i_{0}: \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right) \rightarrow F_{n}^{\mathbb{K}}(U)$ be the inclusion map given by

$$
\begin{equation*}
i_{0}\left(\boldsymbol{b}_{1}, \cdots, \boldsymbol{b}_{r}\right)=\left(f_{b_{1}}(z), \cdots, f_{\boldsymbol{b}_{r}}(z)\right) \tag{9.9}
\end{equation*}
$$

for $\left(\boldsymbol{b}_{1}, \cdots, \boldsymbol{b}_{r}\right) \in \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)$. Since the degree of each polynomial $f_{b_{1}}(z)$ has at most $n-1$, it has no root of multiplicity $\geq n$. Thus, the map $i_{0}$ is well-defined, and clearly the equality $e v_{0} \circ i_{0}=$ id holds.

Let $f: F_{n}^{\mathbb{K}}(U) \times[0,1] \rightarrow F_{n}^{\mathbb{K}}(U)$ be the homotopy given by

$$
f\left(\left(f_{1}, \cdots, f_{t}\right), t\right)=\left(f_{1, t}(z), \cdots, f_{r, t}(z)\right),
$$

where $f_{i, t}(z)=f_{i}(t z)$. This gives a homotopy between the map $i_{0} \circ e v_{0, \mathbb{K}}$ and the identity map, and this proves that the map

$$
e v_{0, \mathbb{K}}: F_{n}^{\mathbb{K}}(U) \xrightarrow{\simeq} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{K}^{n},\left(\mathbb{K}^{n}\right)^{*}\right)
$$

is a deformation retraction. Since $F_{n}^{\mathbb{K}}(U)$ is an infinite dimensional manifold and $\tilde{F}_{n}^{\mathbb{K}}(U)$ is a closed submanifold of $F_{n}^{\mathbb{K}}(U)$ of infinite codimension, it follows from [12, Theorem 2] that the inclusion

$$
\begin{equation*}
i_{n}^{\Sigma, \mathbb{K}}: \tilde{F}_{n}^{\mathbb{K}}(U) \xrightarrow{\simeq} F_{n}^{\mathbb{K}}(U) \tag{9.10}
\end{equation*}
$$

is a homotopy equivalence. Hence the restriction $e v_{\mathbb{K}}=e v_{0, \mathbb{K}} \circ i_{n}^{\Sigma, \mathbb{K}}$ is also a homotopy equivalence.

Definition 9.5. Note that $(\bar{U}, \sigma \bar{U})=\left(I^{2}, \sigma I^{2}\right)=(I \times I, \partial I \times I)$. Let

$$
\left\{\begin{array}{l}
w_{n}^{\mathbb{C}}: \tilde{F}_{n}^{\mathbb{C}}(U) \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})=\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)  \tag{9.11}\\
w_{n}^{\mathbb{R}}: \tilde{F}_{n}^{\mathbb{R}}(U) \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}=\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)^{\mathbb{Z}_{2}}
\end{array}\right.
$$

denote the natural maps which assigns to an $r$-tuple $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in$ $\tilde{F}_{n}^{\mathbb{K}}(U)(\mathbb{K}=\mathbb{C}$ or $\mathbb{R})$ the $r$-tuple of their configurations represented by their real roots which lie in $\bar{U}=I^{2}$. These maps clearly induce the maps

$$
\left\{\begin{array}{l}
v_{n}^{\mathbb{C}}: \tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r} \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})=\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)  \tag{9.12}\\
v_{n}^{\mathbb{R}}: \tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r} \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{Z_{2}}=\mathcal{Q}_{n}^{\Sigma}\left(I^{2}, \sigma I^{2}\right)^{\mathbb{Z}_{2}}
\end{array}\right.
$$

such that the following diagram is commutative:


Lemma 9.6. Any fiber of the map $w_{n}^{\mathbb{K}}$ is homotopy equivalent to the space $\mathbb{T}_{\mathbb{K}}^{r}$.
Proof. Any fiber of the map $w_{n}^{\mathbb{K}}$ is homeomorphic to the space $f i b(r)$ consisting of all $r$-tuples $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in \mathbb{K}[z]^{r}$ of $\mathbb{K}$-coefficients polynomials such that each polynomial $f_{i}(z)$ has no root in $U$. It suffices to show that there is a homotopy equivalence

$$
\begin{equation*}
f i b(r) \simeq \mathbb{T}_{\mathbb{K}}^{r} . \tag{9.13}
\end{equation*}
$$

First define the inclusion map $j_{0}: \mathbb{T}_{\mathbb{K}}^{r} \rightarrow f i b(r)$ by $j_{0}(\boldsymbol{x})=\left(x_{1}, \cdots, x_{r}\right)$ for $\boldsymbol{x}=\left(x_{1}, \cdots, x_{r}\right) \in \mathbb{T}_{\mathbb{K}}^{r}$. Next, let $f=\left(f_{1}(z), \cdots, f_{r}(z)\right) \in f i b(r)$ be any element. Since $0 \in U,\left(f_{1}(0), \cdots, f_{r}(0)\right) \in \mathbb{T}_{\mathbb{K}}^{r}$. Hence, one can define the evaluation map $\epsilon_{0}: f i b(r) \rightarrow \mathbb{T}_{\mathbb{K}}^{r}$ by $\epsilon_{0}(\mathrm{f})=\left(f_{1}(0), \cdots, f_{r}(0)\right)$ for $\mathrm{f}=$ $\left(f_{1}(z), \cdots, f_{r}(z)\right) \in f i b(r)$. It is easy to see that $\epsilon_{0} \circ j_{0}=\mathrm{id}_{\mathbb{T}_{\mathbb{R}}^{r}}$.

Now consider the map $j_{0} \circ \epsilon_{0}$. Note that if a polynomial $g(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C} \backslash U$ and $0<t \leq 1$, the polynomial $g(t z)$ has a root $\alpha / t \in \mathbb{C} \backslash U$. Thus, one can define the homotopy $F: f i b(r) \times[0,1] \rightarrow f i b(r)$ by $F(\mathrm{f}, t)=$ $\left(f_{1}(t z), \cdots, f_{r}(t z)\right)$ for $(\mathrm{f}, t)=\left(\left(f_{1}(z), \cdots, f_{r}(z)\right), t\right) \in \operatorname{fib}(r) \times[0,1]$. It is easy to see that the map $F$ gives a homotopy between the maps $j_{0} \circ \epsilon_{0}$ and $\mathrm{id}_{f i b(r)}$. Hence, we see that the map $\epsilon_{0}: f i b(r) \xrightarrow{\simeq} \mathbb{T}_{\mathbb{K}}^{r}$ is a desired homotopy equivalence.
Lemma 9.7. The map $w_{n}^{\mathbb{C}}: \tilde{F}_{n}^{\mathbb{C}}(U) \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})$ is a quasifibration with fiber $\mathbb{T}_{\mathbb{C}}^{r}$. Similarly, the map $w_{n}^{\mathbb{R}}: \tilde{F}_{n}^{\mathbb{R}}(U) \rightarrow \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}$ is a quasifibration with fiber $\mathbb{T}_{\mathbb{R}}^{r}$.

Proof. Since the proof is completely analogous, we give the proof only for the map $w_{n}^{\mathbb{C}}$. The assertion may be proved by using the well-known Dold-Thom criterion. Recall that the base space $B=\mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})$ consists of $r$-tuple of divisors (or configurations) $\left(\xi_{1}, \cdots, \xi_{r}\right)$ satisfying the condition
$(\dagger)_{\Sigma}$ The configuration $\left(\cap_{k \in \sigma} \xi_{k}\right) \cap \mathbb{R} \cap(\bar{U} \backslash \sigma \bar{U})$ does not contains no points of multiplicity $\geq n$ for any $\sigma \in I\left(\mathcal{K}_{\Sigma}\right)$.

For each $r$-tuple $\left(d_{1}, \cdots, d_{r}\right) \in\left(\mathbb{Z}_{\geq 0}\right)^{r}$ of non-negative integers, we denote by $B_{\leq d_{1}, \cdots, \leq d_{r}}$ the subspace of $B$ consisting of all $r$-tuples $\left(\xi_{1}, \cdots, \xi_{r}\right) \in B$ satisfying the condition

$$
\begin{equation*}
\operatorname{deg}\left(\xi_{k} \cap \mathbb{R} \cap(\bar{U} \backslash \sigma \bar{U})\right) \leq d_{k} \quad \text { for each } 1 \leq k \leq r \tag{9.14}
\end{equation*}
$$

We filter the base space $B$ by an increasing family of subspaces $\left\{B_{\leq d_{1}, \cdots, \leq d_{r}}\right\}$. It suffices to prove that each restriction

$$
\begin{equation*}
w_{n} \mid w_{n}^{-1}\left(B_{\leq d_{1}, \cdots, \leq d_{r}}\right): w_{n}^{-1}\left(B_{\leq d_{1}, \cdots, \leq d_{r}}\right) \rightarrow B_{\leq d_{1}, \cdots, \leq d_{r}} \tag{9.15}
\end{equation*}
$$

is a quasifibration. Its proof is essentially completely analogous to that of [25, Lemma 5.13] (cf. [31, Lemmas 3.3, 3.4]). The difference is only the condition which we treated. In the case of [25, Lemma 5.13], we consider the $m$-tuple $\left(\xi_{1}, \cdots, \xi_{m}\right)$ of configurations which satisfies the condition $(\dagger)_{1}$, where
$(\dagger)_{1}$ The configuration $\left(\cap_{k=1}^{m} \xi_{k}\right) \cap \mathbb{R} \cap(\bar{U} \backslash \sigma \bar{U})$ does not contains no points of multiplicity $\geq n$.

On the other hand, in our case, we need to consider $r$-tuple $\left(\xi_{1}, \cdots, \xi_{r}\right)$ of configurations satisfying the condition $(\dagger)_{\Sigma}$. If we replace by the condition $(\dagger)_{\Sigma}$ in the proof of [25, Lemma 5.13], we can prove that each restriction (9.15) is a quasifibration by the completely identical similar way. So we omit the detail.

Corollary 9.8. The map $v_{n}^{\mathbb{C}}: \tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r} \xrightarrow{\simeq} \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})$ is a homotopy equivalence. Similarly, the map $v_{n}^{\mathbb{R}}: \tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r} \xrightarrow{\simeq} \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}$ is also a homotopy equivalence.

Proof. Since the proof is completely analogous, we give the proof only for the $\operatorname{map} v_{n}^{\mathbb{C}}$. Let $F_{n}$ denote the homotopy fiber of the map $w_{n}^{\mathbb{C}}$. It follows from [8, Lemma 2.1] that there is the following homotopy commutative diagram

where all above vertical and horizontal sequences are fibration sequences. By this diagram, we easily see that $F_{n}$ is contractible. Thus, $v_{n}^{\mathbb{C}}$ is a homotopy equivalence.

Now we can give the proof of Theorem 9.2 ,
Proof of Theorem 9.2. First, we shall prove the assertion for case $\mathbb{K}=\mathbb{C}$. It follows from Lemma 9.4 and Lemma 3.6 that two spaces $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})$ and $\Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$ are simply connected. Thus, it suffices to prove that the map $j_{D+\infty, n, \mathbb{C}}$ induces an isomorphism

$$
\left(j_{D+\infty, n, \mathbb{C}}\right)_{*}: \pi_{k}\left(\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})\right) \xrightarrow{\cong} \pi_{k}\left(\Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)\right) \quad \text { for any } k \geq 2 .
$$

Let us identify $\mathbb{C}=\mathbb{R}^{2}$ and let $U=(-1,1) \times(-1,1)$ as before. Define the scanning map scan : $\tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) \rightarrow \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{C}}(U)\right)$ by

$$
\begin{equation*}
\operatorname{scan}\left(f_{1}(z), \cdots, f_{r}(z)\right)(w)=\left(f_{1}(z+w), \cdots, f_{r}(z+w)\right) \tag{9.17}
\end{equation*}
$$

for $\left.\left(f_{1}(z), \cdots, f_{r}(z)\right), w\right) \in \tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) \times \mathbb{R}$, and consider the diagram

$$
\begin{gathered}
\tilde{F}_{n}^{\mathbb{C}}(U) \xrightarrow{\text { ev }} \underset{p_{\mathbb{C}}}{\simeq} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \\
\quad \downarrow \\
\tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r} \xrightarrow{\simeq} \mathcal{v}_{n}^{\mathbb{C}} \\
\mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})
\end{gathered}
$$

This induces the commutative diagram below

$$
\begin{array}{ccc}
\tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) & \xrightarrow{\text { scan }} \quad \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{C}}(U)\right) \xrightarrow{\left(e v_{\mathrm{C}}\right)_{\#}} \operatorname{Map}\left(\mathbb{R}, \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)\right) \\
p_{\mathbb{C}} \downarrow & \left(p_{\mathrm{C}) \#} \downarrow\right. \\
\tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) / \mathbb{T}_{\mathbb{C}}^{r} \xrightarrow{\text { scan }} \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r}\right) \xrightarrow{\left(v_{n}^{\mathbb{C}}\right) \#} & \operatorname{Map}\left(\mathbb{R}, \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})\right)
\end{array}
$$

Observe that $\operatorname{Map}(\mathbb{R}, \cdot)$ can be replaced by $\operatorname{Map}^{*}\left(S^{1}, \cdot\right)$ by extending from $\mathbb{R}$ to $S^{1}=\mathbb{R} \cup \infty$ (as base-point preserving maps). Thus by setting

$$
\left\{\begin{array}{l}
\widehat{j_{D, n, \mathbb{C}}}: \mathbb{Q}_{n}^{D, \Sigma}(\mathbb{C}) \xrightarrow{C} \tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) \xrightarrow{\text { scan }} \operatorname{Map}^{*}\left(S^{1}, \tilde{F}_{n}^{\mathbb{C}}(U)\right)=\Omega \tilde{F}_{n}^{\mathbb{C}}(U) \\
\widehat{j_{D, n, \mathbb{C}}^{\prime}}: E_{D, n}^{\Sigma, \mathbb{R}}(\mathbb{C}) \xrightarrow{C} \tilde{F}_{n}^{\mathbb{C}}(\mathbb{C}) \xrightarrow{\text { scan }} \operatorname{Map}^{*}\left(S^{1}, \tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r}\right)=\Omega\left(\tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r}\right)
\end{array}\right.
$$

we obtain the following commutative diagram

$$
\begin{align*}
& \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \xrightarrow{\widehat{j_{D, n, \mathrm{C}}}} \quad \Omega \tilde{F}_{n}^{\mathbb{C}}(U) \quad \xrightarrow[\simeq]{\Omega e v_{\mathbb{C}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \\
& \cong \downarrow \quad \Omega_{p_{\mathrm{C}}} \downarrow  \tag{9.18}\\
& \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C}) \xrightarrow{\widehat{j_{D, n, \mathrm{C}}^{\prime}}} \Omega\left(\tilde{F}_{n}^{\mathbb{C}}(U) / \mathbb{T}_{\mathbb{C}}^{r}\right) \xrightarrow[\simeq]{\Omega v_{n}^{\mathbb{C}}} \quad \Omega \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})
\end{align*}
$$

If we identify $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C})$ with the colimit $\lim _{t \rightarrow \infty} \mathcal{Q}_{D+t e, n}^{\Sigma}(\mathbb{C})$, by replacing $D$ by $D+t \boldsymbol{e}(t \in \mathbb{N})$ and letting $t \rightarrow \infty$, we obtain the following homotopy commutative diagram:

$$
\begin{align*}
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \xrightarrow{j_{D+\infty, n, \mathbb{C}}} \quad \Omega \tilde{F}_{n}^{\mathbb{C}}(U) & \xrightarrow[\simeq]{\Omega e v_{\mathbb{C}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \\
\| & \Omega p_{\mathbb{C}} \tag{9.19}
\end{align*}
$$

where we set $\widehat{j_{D+\infty, n, \mathbb{C}}}=\lim _{t \rightarrow \infty} \widehat{j_{D+t e, n, \mathbb{C}}}$ and $\widehat{j_{D+\infty, n, \mathbb{C}}^{\prime}}=\lim _{t \rightarrow \infty} \widehat{j_{D+t e, n, \mathbb{C}}^{\prime}}$.
Since $\left(\Omega e v_{\mathbb{C}}\right) \circ \widehat{j_{D+t e, n, \mathbb{C}}}=j_{D+t e, n, \mathbb{C}}$ and $\left(\Omega v_{n}^{\mathbb{C}}\right) \circ \widehat{j_{D+t e, n, \mathbb{C}}^{\prime}}=s c_{D+t e}$ (by identifying $\mathrm{Q}_{n}^{D+t e, \Sigma}(\mathbb{C})$ with the space $\left.E_{D+t e, n}^{\Sigma}(\mathbb{C})\right)$, we also obtain the following two equalities:

$$
\begin{equation*}
j_{D+\infty, n, \mathbb{C}}=\left(\Omega e v_{\mathbb{C}}\right) \circ \widehat{j_{D+\infty, n, \mathbb{C}}}, \quad S^{H}=\left(\Omega v_{n}^{\mathbb{C}}\right) \circ \widehat{j_{D+\infty, n, \mathbb{C}}^{\prime}} . \tag{9.20}
\end{equation*}
$$

Since the map $e v_{\mathbb{C}}$ is a homotopy equivalence, it suffices to prove that the map
$(\dagger \dagger) \mathrm{c}$

$$
\widehat{j_{D+\infty, n, \mathbb{C}}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \longrightarrow \Omega \tilde{F}_{n}^{\mathbb{C}}(U)
$$

induces an isomorphism on the homotopy group $\pi_{k}()$ for any $k \geq 2$.
Since $S^{H}=\left(\Omega v_{n}^{\mathbb{C}}\right) \circ j_{D+\infty, n, \mathbb{C}}^{\prime}$ and $\Omega v_{n}^{\mathbb{C}}$ are homotopy equivalences (by Theorem 8.7 and Corollary 9.8 ), the map $j_{D+\infty, n, \mathrm{C}}^{\prime}$ is a homotopy equivalence. Since $p_{\mathbb{C}}$ is a fibration with fiber $\mathbb{T}_{\mathbb{C}}^{r}$, the map $\Omega p_{\mathbb{C}}$ induces an isomorphism on the homotopy group $\pi_{k}()$ for any $k \geq 2$. Hence, by using the equality $\left(\Omega p_{\mathbb{C}}\right) \circ \widehat{j_{D+\infty, n, \mathbb{C}}}=\widehat{j_{D+\infty, n, \mathbb{C}}^{\prime}}$ (up to homotopy equivalence), we see that the map $\widehat{j_{D+\infty, n, \mathbb{C}}}$ is induces an isomorphism on the homotopy group $\pi_{k}()$ for any $k \geq 2$. This completes the proof for the case $\mathbb{K}=\mathbb{C}$.

Next, consider the case $\mathbb{K}=\mathbb{R}$. Note that the proof is almost identical to the case $\mathbb{K}=\mathbb{C}$. However, since $\Omega p_{\mathbb{R}}$ is a homotopy equivalence, the proof is easier than that of the case $\mathbb{K}=\mathbb{C}$.

Define the scanning map sca: $\tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) \rightarrow \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{R}}(U)\right)$ by

$$
\begin{equation*}
\operatorname{sca}\left(f_{1}(z), \cdots, f_{r}(z)\right)(w)=\left(f_{1}(z+w), \cdots, f_{r}(z+w)\right) \tag{9.21}
\end{equation*}
$$

for $\left.\left(f_{1}(z), \cdots, f_{r}(z)\right), w\right) \in \tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) \times \mathbb{R}$. Now consider the diagram

$$
\begin{gathered}
\tilde{F}_{n}^{\mathbb{R}}(U) \xrightarrow{\simeq} \xrightarrow{\sim v_{\mathbb{R}}} \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right) \\
p_{\mathbb{R}} \downarrow \\
\tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r} \xrightarrow{\simeq}{ }^{v_{n}^{\mathbb{R}}} \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}
\end{gathered}
$$

This induces the commutative diagram below

$$
\begin{aligned}
& \tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) \xrightarrow{s c a} \quad \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{R}}(U)\right) \xrightarrow[\simeq]{\left(e v_{\mathbb{R}}\right) \#} \operatorname{Map}\left(\mathbb{R}, \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)\right) \\
& p_{\mathbb{R}} \downarrow \quad\left(p_{\mathbb{R}}\right) \neq \downarrow \\
& \tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) / \mathbb{T}_{\mathbb{R}}^{r} \xrightarrow{s c a} \operatorname{Map}\left(\mathbb{R}, \tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r}\right) \xrightarrow[\simeq]{\left(v_{n}^{\mathbb{R}}\right) \#} \quad \operatorname{Map}\left(\mathbb{R}, \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}\right)
\end{aligned}
$$

Observe that $\operatorname{Map}(\mathbb{R}, \cdot)$ can be replaced by $\operatorname{Map}^{*}\left(S^{1}, \cdot\right)$ by extending from $\mathbb{R}$ to $S^{1}=\mathbb{R} \cup \infty$ (as base-point preserving maps). Thus by setting

$$
\left\{\begin{array}{l}
\widehat{j_{D, n, \mathbb{R}}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \xrightarrow{C} \tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) \xrightarrow{s c a} \Omega \tilde{F}_{n}^{\mathbb{R}}(U) \\
\widehat{j_{D, n, \mathbb{R}}^{\prime}}: \mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_{2}} \xrightarrow{C} \tilde{F}_{n}^{\mathbb{R}}(\mathbb{C}) \xrightarrow{s c a} \Omega\left(\tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r}\right)
\end{array}\right.
$$

we obtain the following commutative diagram

$$
\begin{array}{lll}
\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) & \xrightarrow{\widehat{j_{D, n, \mathbb{R}}}} \quad \Omega \tilde{F}_{n}^{\mathbb{R}}(U) & \xrightarrow{\Omega e v_{\mathbb{R}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right) \\
\cong & \Omega p_{\mathbb{R}} \downarrow \simeq  \tag{9.22}\\
& \\
\mathcal{Q}_{D, n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_{2}} \xrightarrow{\widehat{j_{D, n, \mathbb{R}}^{\prime}}} \Omega\left(\tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r}\right) \xrightarrow{\Omega} \Omega \mathcal{Q}_{n}^{\mathbb{R}} \\
\simeq & \Omega, \sigma \bar{U})^{\mathbb{Z}_{2}}
\end{array}
$$

If we identify $\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R})$ with the colimit $\lim _{t \rightarrow \infty} \mathcal{Q}_{D+t e, n}^{\Sigma}(\mathbb{C})^{\mathbb{Z}_{2}}$, by replacing $D$ by $D+t \boldsymbol{e}(t \in \mathbb{N})$ and letting $t \rightarrow \infty$, we obtain the following homotopy commutative diagram:

$$
\begin{array}{ll}
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R}) \xrightarrow{\widehat{j_{D+\infty, n, \mathbb{R}}}} \quad \Omega \tilde{F}_{n}^{\mathbb{R}}(U) \quad \xrightarrow{\Omega e v_{\mathbb{R}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right) \\
\| & \Omega p_{\mathbb{R}} \downarrow \simeq  \tag{9.23}\\
\mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{R}) \xrightarrow{\widehat{j_{D+\infty, n, \mathbb{R}}^{\prime}}} \Omega\left(\tilde{F}_{n}^{\mathbb{R}}(U) / \mathbb{T}_{\mathbb{R}}^{r}\right) \xrightarrow{\Omega v_{n}^{\mathbb{R}}} \Omega \mathcal{Q}_{n}^{\Sigma}(\bar{U}, \sigma \bar{U})^{\mathbb{Z}_{2}}
\end{array}
$$

where we set $\widehat{j_{D+\infty, n, \mathbb{R}}}=\lim _{t \rightarrow \infty} \widehat{j_{D+t e, n, \mathbb{R}}}$ and $\widehat{j_{D+\infty, n, \mathbb{R}}^{\prime}}=\lim _{t \rightarrow \infty} \widehat{j_{D+t e, n, \mathbb{R}}^{\prime}}$.
Since $\left(\Omega e v_{\mathbb{R}}\right) \circ \widehat{j_{D+t e, n, \mathbb{R}}}=j_{D+t e, n, \mathbb{R}}$ and $\left(\Omega v_{n}^{\mathbb{R}}\right) \circ \widehat{j_{D+t e, n, \mathbb{R}}^{\prime}}=\left(s c_{D+t e}\right)^{\mathbb{Z}_{2}}$, we also obtain the following two equalities:

$$
\begin{equation*}
j_{D+\infty, n, \mathbb{R}}=\left(\Omega e v_{\mathbb{R}}\right) \circ \widehat{j_{D+\infty, n, \mathbb{R}}}, \quad\left(S^{H}\right)^{\mathbb{Z}_{2}}=\left(\Omega v_{n}^{\mathbb{R}}\right) \circ \widehat{j_{D+\infty, n, \mathbb{R}}^{\prime}} . \tag{9.24}
\end{equation*}
$$

Since the map $e v_{\mathbb{R}}$ is a homotopy equivalence, it suffices to prove that the map

$$
(\dagger \dagger)_{\mathbb{R}} \quad \widehat{j_{D+\infty, n, \mathbb{R}}}: \mathrm{Q}_{n}^{D+\infty, \Sigma}(\mathbb{C}) \longrightarrow \Omega \tilde{F}_{n}^{\mathbb{R}}(U)
$$

is a homotopy equivalence.
Since $\left(S^{H}\right)^{\mathbb{Z}_{2}}=\left(\Omega v_{n}^{\mathbb{R}}\right) \circ j_{D+\infty, n, \mathbb{R}}^{\prime}$ and $\Omega v_{n}^{\mathbb{R}}$ are homotopy equivalences (by Theorem 8.9 and Corollary 9.8 , the map $j_{D+\infty, n, \mathbb{C}}^{\prime}$ is a homotopy equivalence. On the other hand, since $p_{\mathbb{R}}$ is a covering projection with fiber $\left(\mathbb{Z}_{2}\right)^{r}$, the map $\Omega p_{\mathbb{R}}$ is a homotopy equivalence. Hence, by using the diagram (9.23), we see that the map $\widehat{j_{D+\infty, n, \mathbb{R}}}$ is a homotopy equivalence. This completes the proof of Theorem 9.2.

## 10 Proofs of the main results

Now we give the proofs of the main results (Theorems 2.14, 2.15, and Corollary 2.16).

Proofs of Theorem 2.14, (i) Suppose that $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{n}$. Then the assertion (i) easily follows from Corollary 7.8 and Theorem 9.2.
(ii) Next assume that $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k} \neq \mathbf{0}_{n}$. Recall from (2.18)* that there is an $r$-tuple $D_{*}=\left(d_{1}^{*}, \cdots, d_{r}^{*}\right) \in \mathbb{N}^{r}$ such that $\sum_{k=1}^{r} d_{k}^{*} \boldsymbol{n}_{k}=\mathbf{0}_{n}$. If we choose a sufficiently large integer $m_{0} \in \mathbb{N}$, then the condition $d_{k}<m_{0} d_{k}^{*}$ holds for each $1 \leq k \leq r$. Then consider the map $j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right)$ defined by

$$
\begin{equation*}
j_{D, n, \mathbb{C}}=j_{D_{0}, n, \mathbb{C}} \circ s_{D, D_{0}}, \tag{10.1}
\end{equation*}
$$

where $D_{0}=m_{0} D_{*}=\left(m_{0} d_{1}^{*}, m_{0} d_{2}^{*}, \cdots, m_{0} d_{r}^{*}\right)$ and $j_{D, n, \mathbb{C}}$ is given by the composite of the following maps

$$
\begin{equation*}
j_{D, n, \mathbb{C}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C}) \xrightarrow{s_{D, D_{0}}} \mathrm{Q}_{n}^{D_{0}, \Sigma}(\mathbb{C}) \xrightarrow{j_{D_{0}, n, \mathbb{C}}} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) . \tag{10.2}
\end{equation*}
$$

Since the maps $s_{D, D_{0}}$ and $j_{D_{0}, \mathbb{C}}$ are homotopy equivalences through dimensions $d(D ; \Sigma, n, \mathbb{C})$ and $d\left(D_{0} ; \Sigma, n, \mathbb{C}\right)$, respectively (by Corollary 7.8 and Theorem 2.14), by using $d(D ; \Sigma, n, \mathbb{C}) \leq d\left(D_{0} ; \Sigma, n, \mathbb{C}\right)$ the map $j_{D, n, \mathbb{C}}$ is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{C})$.

Proof of Theorem 2.15, (i) Suppose that $\sum_{k=1}^{r} d_{k} \boldsymbol{n}_{k}=\mathbf{0}_{n}$. Then the assertion (i) easily follows from Corollary 7.8 and Theorem 9.2 .
(ii) This is proved completely analogous way as that of (ii) of Theorem 2.14. Indeed, under the same assumption as (ii) of the proof of Theorem 2.14, we define the map $j_{D, n, \mathbb{R}}: \mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{n}, S^{n-1}\right)$ by

$$
\begin{equation*}
j_{D, n, \mathbb{R}}=j_{D_{0}, n, \mathbb{R}} \circ s_{D, D_{0}}^{\mathbb{R}} . \tag{10.3}
\end{equation*}
$$

Since $d(D ; \Sigma, n, \mathbb{R}) \leq d\left(D_{0} ; \Sigma, n, \mathbb{R}\right)$, it is easy to see that this map is a homotopy equivalence through dimension $d(D ; \Sigma, n, \mathbb{R})$.

Proof of Corollary 2.16. Consider the map of composite

$$
\Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(D^{2 n}, S^{2 n-1}\right) \xrightarrow{\simeq} \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma}}\left(\mathbb{C}^{n},\left(\mathbb{C}^{n}\right)^{*}\right) \xrightarrow{\Omega q_{n, \mathrm{C}}} \Omega X_{\Sigma}(n) .
$$

Since $\Omega q_{n, \mathbb{C}}$ is a universal covering (by Corollary 3.10), the assertions easily follow from Theorem 2.14.

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[^1]:    ${ }^{1}$ Precise definitions and a description of the notation related to toric varieties and their fans will be given in $\S 2$

[^2]:    ${ }^{2}$ Note that the spaces $X_{\Sigma}(n)$ and $\mathcal{Z}_{K}(X, A)$ are the orbit space and the polyhedral product of a pair $(X, A)$ given by (2.12) and Definition 2.3, respectively.

[^3]:    ${ }^{3}$ It is written as $\operatorname{Pol}_{D}^{*}\left(S^{1}, X_{\Sigma}\right)=\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})$ if $n=1$ in [24].
    ${ }^{4}$ It is written as $\mathrm{Q}_{n}^{d, m}(\mathbb{K})=\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{K})$ in [25] for $\left(X_{\Sigma}, D\right)=\left(\mathbb{C P}^{m-1}, D_{m}(d)\right)$.
    ${ }^{5}$ If the condition $\left(\frac{1.4)^{*}}{}\left(\right.\right.$ resp. $\left.(1.4)^{\dagger}\right)$ is satisfied, the space $\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{C})\left(\right.$ resp. $\left.\mathrm{Q}_{n}^{D, \Sigma}(\mathbb{R})\right)$ is simply connected (see Corollary 7.3). Moreover, if the condition (1.44* or (1.44 ${ }^{\dagger}$ is satisfied, the condition $\left\lfloor d_{\text {min }} / n\right\rfloor \geq 1$ holds. Thus, $d(D ; \Sigma, n, \mathbb{K}) \geq 1$ and the main results (Theorem 1.5) Corollary (2.17) are not vacuous. Note that the condition (1.4)* holds if the condition $(1.4)^{\dagger}$ is satisfied.

[^4]:    ${ }^{6}$ When $S$ is the emptyset $\emptyset$, we set $\operatorname{Cone}(\emptyset)=\left\{\mathbf{0}_{m}\right\}$ and we may also regard it as one of strongly convex rational polyhedral cones in $\mathbb{R}^{m}$.

[^5]:    ${ }^{7}$ This map has to be constructed in a slightly different way from the one in (i) but we shall use the same notation for both.

[^6]:    ${ }^{8}$ More precisely, if we set $(K, m)=\left(\mathcal{K}_{\Sigma}, r\right)$ and $J=\left(j_{1}, j_{2}, \cdots, j_{r}\right)=(n, n, \cdots, n)$ ( $r$-times) in the notation of [6, Definition 2.1], we get the simplicial complex $K(J)$ on the index set $[r] \times[n]$.

[^7]:    ${ }^{9}$ Let $f(z) \in \mathbb{R}[z]$ be a real coefficient polynomial and $\alpha \in \mathbb{C} \backslash \mathbb{R}$ be a complex root of $f(z)$ of multiplicity $n_{\alpha}$. Then $f(z)$ has the root $\bar{\alpha}$ of the same multiplicity $n_{\alpha}$. Thus, for the case $\mathbb{K}=\mathbb{R}$, each string $\eta_{k}$ moves symmetrically along the real axis $\mathbb{R}$.

