# Techniques for Showing the Decidability of the Boundedness Problem of Language Acceptors * 

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#### Abstract

There are many types of automata and grammar models that have been studied in the literature, and for these models, it is common to determine whether certain problems are decidable. One problem that has been difficult to answer throughout the history of automata and formal language theory is to decide whether a given system $M$ accepts a bounded language (whether there exist words $w_{1}, \ldots, w_{k}$ such that $L(M) \subseteq w_{1} \cdots w_{k}$ ?). Decidability of this problem has gone unanswered for the majority of automata/grammar models in the literature. Boundedness was only known to be decidable for regular and contextfree languages until recently when it was shown to also be decidable for finite-automata and pushdown automata augmented with reversalbounded counters, and for vector addition systems with states. In this paper, we develop new techniques to show that the boundedness problem is decidable for larger classes of one-way nondeterministic automata and grammar models, by reducing the problem to the decidability of boundedness for simpler classes of automata. One technique involves characterizing the models in terms of multi-tape automata. We give new characterizations of finite-turn Turing machines, finite-turn Turing machines augmented with various storage structures (like a pushdown, multiple reversal-bounded counters, partially-blind counters, etc.), and simple matrix grammars. The characterizations are then used to show that the boundedness problem for these models is decidable. Another technique uses the concept of the store language of an automaton. This is used to show that the boundedness problem is decidable for pushdown automata that can "flip" their pushdown a bounded number of times, and boundedness remains decidable even if we augment this device with additional stores.


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## 1 Introduction

There are many well-studied models of automata/grammars that are more powerful than finite automata (denoted by NFA) but less powerful than Turing machines. Perhaps the most well-studied is the one-way nondeterministic pushdown automata (NPDA) which accept the context-free languages. This model is very practical - for example, the non-emptiness problem ("given a machine $M$, is $L(M) \neq \emptyset$ ?"), as well as the infiniteness problem ("given a machine $M$, is $L(M)$ infinite?"), can both be determined in polynomial time for NPDA 18 .

Authors have studied models that are more powerful than NPDA, such as $t$ flip NPDA (resp. finite-flip NPDA), which have the ability to flip their pushdown stack at most $t$ (resp. a finite number of) times [16|17]. Non-emptiness and infiniteness are decidable for this model as well [17] (implied from their semilinear Parikh image). Another more powerful model is simple matrix grammars, which are a class of grammars that generates a family of languages properly between the context-free languages and the matrix languages [20.

Other well-studied models with power between that of finite automata and Turing machines is the one-way nondeterministic reversal-bounded multicounter machines [21 (NCM). This is an NFA with some number of counters, where each counter contains a non-negative integer, and transitions can detect if a counter is non-empty or not. The condition of being $r$-reversal-bounded (resp. reversalbounded) enforces that in each accepting computation, the number of changes between non-decreasing and non-increasing on each counter is at most $r$ (resp. a finite number). It is also possible to combine different types of stores. For example, another class of automata is NPDA augmented by reversal-bounded counters, denoted by NPCM. This device, which is strictly more powerful than either NPDA or NCM, has an NP-complete non-emptiness problem [13.

We will also consider nondeterministic Turing machines with a one-way readonly input tape and a single two-way read/write worktape, denoted by NTM. While all of the problems above are undecidable for NTM, a $t$-turn (resp. finiteturn) NTM are machines with at most $t$ (resp. some number of) changes in direction on the worktape in every accepting computation (called reversal-bounded in [12], but we call it finite-turn here). Again, the non-emptiness and infiniteness problems are decidable for this model.

Another important property beyond emptiness and infiniteness is that of boundedness. A language $L \subseteq \Sigma^{*}$ is bounded if there exist non-empty words $w_{1}, \ldots, w_{k}$ such that $L \subseteq w_{1}^{*} \cdots w_{k}^{*}$. Here, we explore further the important decision problem called the boundedness problem: given machine $M$, is $L(M)$ a bounded language?". In the early years of the study of formal language theory, this property was shown to be decidable for NFA and NPDA by Ginsburg and Spanier using a rather complicated procedure 89. In contrast, if a class of machines with an undecidable emptiness problem accepts languages that are closed under concatenation with the language $\$ \Sigma^{*}$ (where $\$$ is a new symbol and $\Sigma$ is an at least two letter alphabet), then the boundedness problem is also undecidable for the class, because $\$ \Sigma^{*}$ concatenated with anything nonempty is not bounded, and so $L \$ \Sigma^{*}$ is bounded if and only if $L$ is empty. Until
recently, the status of the boundedness problem had been elusive for essentially all other machine/grammar models (besides NPDA) studied in the literature that have a decidable emptiness problem. Finally, Czerwinski, Hofman, and Zetzsche showed that the boundedness problem is decidable for vector addition systems with states [4] (equivalent to one-way partially blind multicounter machines [10], denoted by PBCM, that properly contain NCM). With PBCM, machines can add and subtract from counters but cannot detect whether counters are empty or not except that a machine crashes if a counter goes below zero, and a word is accepted if it hits a final state with all counters being zero. Also, in [3], it was determined that the boundedness problem for NPCM (and NCM) is not only decidable, but also coNP-complete.

Here, we develop techniques for showing that the boundedness problem is decidable. One technique involves creating characterizations in terms of multitape versions of NFA, NCM, NPCM, and PBCM of the following:

1. finite-turn NTM in terms of multi-tape NFA,
2. finite-turn NTM augmented with reversal-bounded counters in terms of multitape NCM,
3. finite-turn NTM augmented with a pushdown and reversal-bounded counters where in each accepting computation, the pushdown can only be changed during one sweep of the Turing tape, in terms of multi-tape NPCM,
4. finite-turn NTM augmented with partially blind counters in terms of multitape PBCM.

These characterizations are then used to show decidability of the boundedness (also emptiness and infiniteness) problem for each of the models. These results are strong as any combination of two 1-turn stores has an undecidable emptiness and thus boundedness problem. In model (3) above, the restriction that the pushdown can only be used during one sweep (between two consecutive turns) of the read/write tape cannot be dropped, as allowing one more sweep would make both emptiness and boundedness undecidable. Note that the model in (3) is more powerful than NPDA and finite-turn NTM, and can even accept nonindexed languages [1]. For model (4), this model is strictly more powerful than the family of PBCM languages. Hence, it is the most powerful model containing non-semilinear languages with a known decidable boundedness problem. Using a similar technique, we show that the boundedness problem is decidable for simple matrix grammars (even when augmented with reversal-bounded counters).

Another technique involves the store language of a machine, which is the set of strings that encode the contents of the internal stores that can appear in any accepting computation. There are some automata models in the literature where the family of store languages for that class can be accepted by a simpler type of automata. We use this to show that the boundedness problem is decidable for finite-flip NPDA. This is also true if augmented by reversal-bounded counters, and by a finite-turn worktape where the flip-pushdown is only used during one sweep of the worktape. Hence, this is the most powerful model properly containing the context-free languages with a known decidable boundedness problem.

All omitted proofs and some definitions are in the Appendix to help reviewers.

## 2 Preliminaries and Notation

We assume knowledge of introductory automata and formal language theory [18], including deterministic and nondeterministic finite automata, context-free grammars, pushdown automata, and Turing machines.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_{0}$ the non-negative integers. Given a set $X$ and $t \in \mathbb{N}$, let $\langle X\rangle^{t}$ be the set of all $t$-tuples over $X$. Given a finite alphabet $\Sigma$, let $\Sigma^{*}$ (resp. $\Sigma^{+}$) be the set of all words (resp. non-empty words) over $\Sigma$. $\Sigma^{*}$ includes the empty word $\lambda$. A language $L$ is any subset of $\Sigma^{*}$, and a $t$-tuple language $L$ is any subset of $\left\langle\Sigma^{*}\right\rangle^{t}$. Given a word $w$, the reverse of $w$, denoted $w^{R}$ is equal to $\lambda$ if $w=\lambda$, and $a_{n} a_{n-1} \cdots a_{1}$ if $w=a_{1} a_{2} \cdots a_{n}, a_{i} \in \Sigma$ for $1 \leq i \leq n$. The length of $w$, denoted by $|w|$, is equal to the number of characters in $w$, and given $a \in \Sigma,|w|_{a}$ is the number of $a$ 's in $w$. Given alphabet $\Sigma=$ $\left\{a_{1}, \ldots, a_{m}\right\}$ and $w \in \Sigma^{*}$, the Parikh image of $w, \psi(w)=\left(|w|_{a_{1}}, \ldots,|w|_{a_{m}}\right)$; and the Parikh image of a language $L \subseteq \Sigma^{*}$ is $\psi(L)=\{\psi(w) \mid w \in L\}$. Although we will not provide the formal definition of a language being semilinear, equivalently, a language is semilinear if and only if it has the same Parikh image as some regular language [10]. Similarly, the Parikh image of $\left(w_{1}, \ldots, w_{t}\right) \in$ $\left\langle\Sigma^{*}\right\rangle^{t}, \psi\left(w_{1}, \ldots, w_{t}\right)=\psi\left(w_{1} \cdots w_{t}\right)$, and for $L \subseteq\left\langle\Sigma^{*}\right\rangle^{t}, \psi(L)=\{\psi(x) \mid x \in L\}$. A class of machines/grammars is said to be effectively semilinear if, given such a machine/grammar, a finite automaton with the same Parikh image can be effectively constructed.

A $t$-tape NFA over $\Sigma$ is a generalization of an NFA where there are $t$ input tapes and they take $\left(w_{1}, \ldots, w_{t}\right) \in\left\langle\Sigma^{*}\right\rangle^{t}$ as input, and each transition is of the form $q^{\prime} \in \delta(q, a, i)$, where the machine switches from state $q$ to $q^{\prime}$ and reads $a \in \Sigma \cup\{\lambda\}$ from input tape $i$. This allows such a machine to accept a $t$-tuple language. The formal definition appears in the Appendix.

We will augment $t$-tape NFAs with additional stores; e.g. a $t$-tape NPDA is a $t$-tape NFA with an additional pushdown alphabet $\Gamma$, and $\delta$ becomes a partial function with rules of the form $\left(q^{\prime}, \gamma\right) \in \delta(q, a, i, X)$, where $q, q^{\prime}, a, i$ are as with $t$ tape NFAs, $X \in \Gamma$ is the topmost symbol of the pushdown which gets replaced by the word $\gamma \in \Gamma^{*}$. Configurations now include a third component which contains the current pushdown contents, as is standard for pushdown automata 18 . We can similarly define machines with multiple stores by defining the transitions to only read and change one store at a time. Standard one-way single-tape acceptors are a special case with only one input tape. Multi-tape inputs have been studied for NPDA [15], NCM [21, and NPCM [23].

We will use all machine models with a one-way read-only input, including those described in Section 1 . Almost all stores we consider are formally defined in [19], and we omit the formal definitions due to space constraints (see Appendix).

For all multi-tape machine models, they are effectively semilinear if and only if 1-tape machines are effectively semilinear, because given a $t$-tape $M$, there is a 1-tape machine $M^{\prime}$ that reads $a$ from the tape whenever it can read $a$ from any tape of $M$; hence $\psi(L(M))=\psi\left(L\left(M^{\prime}\right)\right)$.

It is known that for any type of nondeterministic machine model with reversalbounded counters, one can equivalently use monotonic counters [22] instead of
reversal-bounded counters. Such machines have an even number $k$ of counters that we identify by $C_{1}, D_{1}, \ldots, C_{k / 2}, D_{k / 2}$ that can only be incremented but not decremented, transitions do not detect the counter status, and acceptance occurs when the machine enters an final state with counters $C_{i}$ and $D_{i}$ having the same value for each $i$. Due to the equivalence, we will use the same notation as above (NPCM, etc.) to mean machines with monotonic counters. Monotonic counters are helpful in this paper because if we simulate an accepting computation of a machine with another machine that applies the same changes but in a different order, then the resulting simulation will still have matching monotonic counters.

## 3 Boundedness Using Multi-Tape Characterizations

### 3.1 Characterizations of Finite-Turn Turing Machines

We first look at finite-turn NTM, and finite-turn NTM with reversal-bounded counters (denoted by finite-turn NTCM). These machines have previously been studied both without counters [12] and with counters [14]. We give characterizations of these machines in terms of multi-tape NFA and multi-tape NCM.

Example 1. Consider $L=\left\{w \# w \$ v \# v\left|w, v \in\{a, b\}^{*},|w|_{a}=|v|_{a},|w|_{b}=|v|_{b}\right\}\right.$. $L$ can be accepted by a 4 -turn NTCM $M$ with four monotonic counters as follows: on input $w_{1} \# w_{2} \$ v_{1} \# v_{2}, M$ reads $w_{1}$ and writes it to the tape while in parallel recording $|w|_{a}$ and $|w|_{b}$ in two monotonic counters $C_{1}$ and $C_{2}$. When it hits \#, it turns and goes to the left end of the tape, and verifies $w_{2}=w_{1}$. When it hits $\$$, it does the same procedure with the read/write tape to the right to verify $v_{1}=v_{2}$, while in parallel putting $\left|v_{1}\right|_{a}$ and $\left|v_{2}\right|_{b}$ on two monotonic counters $D_{1}$ and $D_{2}$. It then accepts if the contents of $C_{1}$ equals $D_{1}$ and $C_{2}$ equals $D_{2}$. Although we do not have a proof, we conjecture this cannot be accepted by an NPCM.

A $t$-turn NTM $M$ is in state normal form if $M$ makes exactly $t$ turns on all inputs accepted, the read/write head always moves to the right or left on every move, and in every accepting computation, it always turns to the left (resp. the right) on the same cell where it writes the current state on the tape. It can be shown (see Appendix) that any $t$-turn NTM can be transformed into another in state normal form that accepts the same language. For such a $t$-turn NTM $M$ in state normal form, define a $t+1$ tuple of symbols $b=\left(b_{1}, \ldots, b_{t+1}\right)$ where each $b_{i}$ is in the worktape alphabet $\Gamma$. Let $\Delta$ be the alphabet of these symbols. We can think of a word in $\Delta^{*}$ as representing a $t+1$ track worktape, where the $i$ th component is the $i$ th track. For $1 \leq i \leq t+1$, define a homomorphism $h_{i}$ from $\Delta^{*}$ to $\Gamma^{*}$ such that $h_{i}\left(\left(b_{1}, \ldots, b_{t+1}\right)\right)=b_{i}$. Given a $t$-turn NTM $M$ in state normal form, define the history language $H(M)$ over $\Delta^{*}$ as follows: $H(M)$ contains all strings $x$ where there is an accepting computation of $M$ such that $h_{i}(x)$ is the string on the worktape after the $i$ th sweep of the worktape and $h_{t+1}(k)$ is the string on the worktape at the end of the computation after it has made the final sweep after the last turn. This means if $t$ is even (it is similar if odd)

$$
h_{1}(x)=q_{0} x_{1} q_{1}, h_{2}(x)=q_{2} x_{2} q_{1}, \ldots, h_{t}(x)=q_{t} x_{t} q_{t-1}, h_{t+1}(x)=q_{t} x_{t+1} q_{t+1}
$$

where $q_{0}$ is the initial state, $M$ writes $q_{0} x_{1} q_{1}$ on the first sweep, etc. until $q_{t} x q_{t+1}$, which is the final worktape contents, and $q_{t+1}$ is a final state.

Let $t \geq 1$. For a $t$-tuple $\left(w_{1}, \ldots, w_{t}\right), w_{i} \in \Sigma^{*}$, let its alternating pattern be:

$$
\left(w_{1}, \ldots, w_{t}\right)^{A}= \begin{cases}w_{1} w_{2}^{R} \cdots w_{t-1} w_{t}^{R} & \text { if } t \text { is even } \\ w_{1} w_{2}^{R} \cdots w_{t-1}^{R} w_{t} & \text { if } t \text { is odd }\end{cases}
$$

If there is a $t \geq 1$ with $L \subseteq\left\langle\Sigma^{*}\right\rangle^{t}$, let $L^{A}=\left\{\left(w_{1}, \ldots, w_{t}\right)^{A} \mid\left(w_{1}, \ldots, w_{t}\right) \in L\right\}$. We now show that every $t$-turn NTM can be "converted" to a $(t+1)$-tape NFA. Starting with $M$ in state normal form, $M^{\prime}$ guesses a $(t+1)$-track string $x \in \Delta^{*}$ letter-by-letter from left-to-right while checking in parallel that the input on tape $i$ would be read by the simulated moves on track $i$ thereby verifying that $x \in H(M)$.
Lemma 2. Let $t \geq 0$, and $M$ be a $t$-turn NTM (resp. $t$-turn NTCM). We can construct a $(t+1)$-tape NFA (resp. $(t+1)$-tape NCM$) M^{\prime}$ such that $L\left(M^{\prime}\right)^{A}=$ $L(M)$.

For the opposite direction, on the first sweep of the worktape, $M^{\prime}$ guesses and writes a guessed sequence of transition labels of $M$, and then sweeps the worktape once for each tape $i$ to make sure the next section of the input word of $M^{\prime}$ would be read by tape $i$ in the simulation.

Lemma 3. Let $t \geq 0$, and let $M$ be a $(t+1)$-tape NFA (resp. $(t+1)$-tape NCM). Then we can construct a t-turn NTM (resp. $t$-turn NTCM) $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$.

From the two lemmas above, we obtain:
Proposition 4. Let $t \geq 0$. There is a ( $t+1$ )-tape NFA (resp. $(t+1)$-tape NCM) $M$ if and only if there is a t-turn NTM (resp. t-turn NTCM) $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$.

Further to the definition of bounded languages, we say $L \subseteq\left\langle\Sigma^{*}\right\rangle^{t}$ is a bounded $t$-tuple language if $L \subseteq B_{1} \times \cdots \times B_{t}$, where each $B_{i}$ is of the form $w_{1}^{*} \cdots w_{n}^{*}$ for some $w_{1}, \ldots, w_{n} \in \Sigma^{+}$. Given $L \subseteq\left\langle\Sigma^{*}\right\rangle^{t}$, let $L^{(i)}=\left\{w_{i} \mid\left(w_{1}, \ldots, w_{t}\right) \in L\right\}$.

Let $\mathcal{M}$ be a class of multi-tape machines consisting of an NFA with zero or more stores. Given a $t$-tape $M \in \mathcal{M}$, for each $i, 1 \leq i \leq t$, let $M_{i}$ be the one tape machine in $\mathcal{M}$ that simulates moves that read from tape $i$ by reading from the input tape, but reads $\lambda$ to simulate a read from other tapes. So, $L\left(M_{i}\right)=L^{(i)}$ for all $i, 1 \leq i \leq t$. The following is easily verified:
Lemma 5. A t-tape $M \in \mathcal{M}$ is a bounded (resp. non-empty, finite) t-tuple language if and only if $L\left(M_{i}\right)$ is a bounded (resp. non-empty, finite) language for each $1 \leq i \leq t$.

Proof. The proofs for non-emptiness and finiteness are clear.
Assume $L(M)$ is a bounded $t$-tuple language, and therefore there exists $B_{1}, \ldots, B_{t}$, where each $B_{i}$ is of the form $w_{1}^{*} \cdots w_{n}^{*}$ and $L(M) \subseteq B_{1} \times \cdots \times B_{t}$. Thus, for each $i, L\left(M_{i}\right) \subseteq B_{i}$, and is bounded.

Assume each $L\left(M_{i}\right)$ is bounded, and let $B_{i}$ be such that $L\left(M_{(i)}\right) \subseteq B_{i}$ and $B_{i}$ is of the form $w_{1}^{*} \cdots w_{n}^{*}$. Then, $L(M) \subseteq B_{1} \times \cdots \times B_{t}$.

Using this characterization, we can show the following.
Proposition 6. The boundedness, non-emptiness, and infiniteness problems for finite-turn NTM (resp. finite-turn NTCM) are decidable, and they are effectively semilinear.

Proof. From Proposition 4 given a $t$-turn NTM $M^{\prime}$, there is a $t+1$-tape NFA $M$ with $L(M)^{A}=L\left(M^{\prime}\right)$. We will decide if $L(M)^{A}$ is bounded; indeed, we will show $L(M)^{A}$ is bounded if and only if, for each $i, 1 \leq i \leq t+1, L\left(M_{i}\right)$ is bounded.

Assume $L(M)^{A}=L\left(M^{\prime}\right)$ is bounded. Assume $t$ is odd (with the even case being similar). Then $L(M)^{A}=\left\{w_{1} w_{2}^{R} \cdots w_{t} w_{t+1}^{R} \mid\left(w_{1}, \ldots, w_{t+1}\right) \in L(M)\right\}$ is bounded. It is known that given any bounded language $L$, the reverse of $L$ is bounded, the set of subwords of $L$ is bounded, and any subset of $L$ is bounded [9. Hence, for each $i,\left\{w_{i} \mid\left(w_{1}, \ldots, w_{t+1}\right) \in L\left(M^{\prime}\right)\right\}$ is bounded as, for $i$ odd, then this is a subset of the set of subwords of $L(M)^{A}$, and for $i$ even, it is the reverse. This set is $L\left(M_{i}\right)$, and so each $L\left(M_{i}\right)$ is bounded.

Conversely, assume each $L\left(M_{i}\right)$ is bounded for $1 \leq i \leq t+1$, and hence $L\left(M_{i}\right)^{R}$ is also bounded for $i$ even. By Lemma $5, L(M)$ is a bounded $t+1$-tuple language. Since the finite concatenation of bounded languages is bounded [9, $L(M)^{A}$ is bounded.

Hence, $L\left(M^{\prime}\right)$ is bounded if and only if, for each $i, 1 \leq i \leq t+1, L\left(M_{i}\right)$ is bounded. Since these are each regular, we can decide this property.

The proof is the same for finite-turn NTCM using decidability of boundedness for NCM [34].

Semilinearity follows from semilinearity of NFA and NCM 21], and since $\psi(L(M))=\psi\left(L\left(M^{\prime}\right)\right)$.

Although decidability for non-emptiness, finiteness, and effective semilinearity were known for both finite-turn NTM and finite-turn NTCM [14], to our knowledge, decidability of boundedness for both finite-turn NTM and for finite-turn NTCM were not previously known.

### 3.2 Finite-Turn NTM with Pushdown and Counters

In this section, we provide a further generalized model by augmenting finiteturn NTM with not only monotonic counters but also a pushdown where the pushdown can only be used in a restricted manner:

A $t$-turn NTM augmented with monotonic counters and a pushdown is called a $t$-turn NTPCM. Such a machine is called $i$-pd-restricted if, during every accepting computation, the pushdown is only used the $i$ th sweep (either left-to-right or right-to-left) of the worktape; and a machine is pd-restricted if in every accepting computation, the pushdown is only used in a single pass (it can be different passes depending on the computation).

Example 7. Let $D_{1}$ be the language over the alphabet $\left\{a_{1}, b_{1}\right\}$ generated by the context-free grammar with productions $S \rightarrow a_{1} S b_{1} S$ and $S \rightarrow \lambda$. This language is known as the Dyck language over one set of parentheses, and let $L=\left\{x \# x \# x \mid x \in D_{1}\right\}$. $L$ can be accepted by a pd-restricted 4-turn NTPCM machine (even without counters). Indeed, on input $x_{1} \# x_{2} \# x_{3}$, a machine $M$ can use the pushdown to verify $x_{1} \in D_{1}$ while in parallel copying $x_{1}$ to the read/write tape. Since this is the only pass where the pushdown is used, the machine is 1-pd-restricted. Then it can match $x_{1}$ against the input to verify $x_{1}=x_{2}=x_{3}$. It follows from [7] and [25] that $L$ is not even an indexed language, a family that strictly contains the context-free languages [1, and is equal to the family of languages accepted by automata with a "pushdown of pushdowns" 6]. Thus, pdrestricted NTPCM is quite a powerful model, containing all of NPDA, finite-turn NTM, and even some non-indexed languages.

The characterization will use a restriction of multi-tape NPCM as follows. Let $i$ satisfy $1 \leq i \leq t$. A $t$-tape NPCM is $i$-pd-restricted if, for every accepting computation, the pushdown is only used when it reads from a single input tape.

The following is easy to verify. Any transition $\alpha$ that reads $a \in \Sigma \cup\{\lambda\}$ from input tape $j \neq i$ and uses the pushdown can be simulated by first reading $a$ from tape $j$ but not changing another store, then in the next transition, reading and changing the pushdown as in $\alpha$ while reading $\lambda$ from tape $i$.

Lemma 8. Every t-tape NPCM can be converted to an equivalent i-pd-restricted $t$-tape NPCM , for any $1 \leq i \leq t$.

The next proposition follows a proof similar to Lemma 2 for one direction, where the pushdown is only used in one track because it is only used on one sweep of the NTPCM Turing tape; and for the other direction it first uses Lemma 8 and then follows the proof of Lemma 3 where it is verified that the pushdown changes properly according to the guessed transition sequence by simulating the pushdown in the $i$ th sweep of the Turing tape.

Proposition 9. Let $t \geq 0$. There is a $(t+1)$-tape NPCM $M$ if and only if there is an $i$-pd-restricted $t$-turn NTPCM $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$, for any $0 \leq i \leq t$.

For the next proof, we are able to strengthen the result to $M$ being pdrestricted rather than $i$-pd-restricted, seen as follows: Given a pd-restricted machine $M$, we can make $M_{1}, \ldots, M_{t+1}$, where each $M_{i}$ accepts the strings accepted by $M$ for which the pushdown is used in sweep $i$. Thus, $M_{i}$ is $i$-pd-restricted. Furthermore, $L(M)$ is bounded if and only if $L\left(M_{i}\right)$ is bounded for each $i$. The remaining proof is similar to that of Proposition 6 using the fact that emptiness, infiniteness, and boundedness for NPCM are decidable [21]3].

Proposition 10. For every $t \geq 0$, the boundedness, emptiness, and infiniteness problems for pd-restricted $t$-turn NTPCM are decidable, and they are effectively semilinear.

Briefly, the result above can be generalized by replacing the pushdown with other potential types of stores. Consider an NFA augmented with a storage structure $S$ and the specification for updating $S$ and possibly some necessary condition(s) on $S$ for acceptance, in addition to the machine entering a final state. The storage structure $S$ can include multiple storage structures. We do not define such a storage structure formally for simplicity, and the following result can be thought of as a template for other models where decidability of boundedness can be shown. However, definitions such as storage structures [5] and store types [19] work. Examples of $S$ are: pushdown; reversal-bounded counters (or equivalent storage structures such as monotonic counters); partially blind counters; and combinations of the structures, e.g., a pushdown and reversal-bounded counters.

We can examine $S$-restricted $t$-turn NTM augmented by $S$ (denoted NTM $(S)$ ) where in every accepting computation, $S$ can only be changed within a single sweep of the worktape. We can show the following seeing that the pushdown in the proof above can be replaced with other storage types.

Proposition 11. Let $\mathcal{M}$ be a class of NFA with storage structure $S$, whose languages are closed under reversal, where the boundedness (resp. emptiness, infiniteness) problem for $\mathcal{M}$ are decidable. Then for every $t \geq 0$, the boundedness (resp. emptiness, infiniteness) problem for $S$-restricted $t$-turn $\mathrm{NTM}(S)$ are decidable. Furthermore, if $\mathcal{M}$ is augmented with additional reversal-bounded counters (no restrictions on their use) has a decidable boundedness (resp. emptiness, infiniteness) problem, the corresponding problem for $S$-restricted $t$-turn machines with reversal-bounded counters are decidable.

This provides new results for certain general types of automata with decidable properties. For example, checking stack automata provide a worktape that can be written to before the first turn, and then only operate in read-only mode. They have a decidable emptiness and infiniteness problem. If we augment these with a finite-turn worktape where the checking stack could only be used in a single sweep, emptiness and infiniteness are decidable.

As with other results in this paper, a $t$-turn NTM (combined with other stores) can be replaced with a $t$-turn checking stack. The restriction on NTPCM to be $S$-restricted in Proposition 10 is needed, as the next proposition shows. Let DCSA be deterministic checking stack automata. The first point uses undecidability of non-emptiness of the intersection of two 1-turn deterministic pushdown automata [2], the second problem uses undecidability of the halting problem for Turing machines [18], the third point uses the undecidability of the halting problem for 2-counter machines [24], and the fourth point uses the third point.

Proposition 12. The emptiness (boundedness, infiniteness) problems are undecidable for the following models:

1. 1-turn NTM (or DCSA) with a 1-turn pushdown.
2. 2-turn DCSA with a 1-turn pushdown, even when the pushdown is used only during the checking stack reading phase (i.e., after turn 1).
3. 1-turn NTM with an unrestricted counter.
4. 1-turn deterministic pushdown automata with an unrestricted counter.

### 3.3 Finite-Turn NTM with Partially Blind Counters

Partially blind counter machines are multicounter machines (PBCM) where the counters can be incremented or decremented but not tested for zero, however the machine crashes if any of the counters becomes negative, and acceptance occurs when the machine enters an accepting state with all the counters being zero. The emptiness, finiteness, [10] and boundedness problems [3] have been shown decidable for vector addition systems with states, which are equivalent to partially blind multicounter machines [10]. The family of languages accepted by these machines is a recursive family, does not contain all context-free languages (as in the example below), but contains non-semilinear languages 10 (unlike all the other models considered so far in this paper).

Here, we look at $t$-turn NTM augmented with partially blind counters, called $t$-turn NTPBCM. It is pointed out in [10] that $L=\left\{w \# w^{R} \mid w \in\{a, b\}^{*}\right\}$ is not accepted by any PBCM. However, it is easily accepted by a NTPBCM (or even a 2-turn NTM), which are therefore strictly more powerful.

We could augment a NTPBCM (or PBCM, $t$-tape PBCM) with monotonic counters, but it is straightforward to see that each pair of monotonic counters can be simulated by a pair of partially-blind counters, and we therefore do not consider these machines with additional reversal-bounded counters.

The results in Section 3.1 concerned finite-turn NTM, optionally augmented with monotonic counters. We will see next that these results hold if "monotonic counters" is replaced by "partially blind counters". However, monotonic counters are easy to handle, as we can permute the order that counter changes are applied in an accepting computation and the resulting computation does not change the counter values. But this is not so for partially blind counters as changing the orders can cause counters to go below zero, which is not allowed. But we can modify the proof as follows:

Proposition 13. Let $t \geq 0$. There is a $(t+1)$-tape PBCM $M$ if and only if there is a t-turn NTPBCM $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$.

Proof. One half of the proof follows an identical construction to that in the proof of Lemma 3 where all counters are simulated on the first sweep while guessing the transition sequence and the order of counter changes is the same.

For the reverse direction, the construction is a modification to that of Lemma 2. We describe the construction of $M$ from $M^{\prime}$. Note that $M^{\prime}$ makes $s=(t+1)$ left-to-right and right-to-left sweeps on its worktape. If $M^{\prime}$ has $k$ partially blind counters $C_{1}, \ldots, C_{k}, M$ will have $s k$ partially blind counters called $C_{1, j}, \ldots, C_{k, j}$ for $1 \leq j \leq s$. The simulation of all sweeps of the computation of $M^{\prime}$ on its worktape are done in parallel. The counters in $C_{1, j}, \ldots, C_{k, j}$ are used to simulate the counters of $M$ in sweep $j$. For odd $j$, the simulation is faithful, but for even $j$, the simulation is backwards. The counters in $C_{i, 1}$ are initially zero, as are $C_{i, s}$ if $s$ is even. For all $j$ even, $C_{i, j}$ and $C_{i, j+1}$ are set nondeterministically to be the same guessed values. The simulation of the computation of $M^{\prime}$ on the even tracks of the worktape (using counters $C_{i, j}, j$ even) is done in reverse and in parallel with the simulation of the odd tracks (using counters in $C_{i, j} j$ odd). When the


Fig. 1: In (a), it shows the counter values of counter $i$ is an original accepting computation of $M^{\prime}$, where $i_{j}$ is the value at turn $j$ of the Turing tape. In (b), we see the modified computation of $M$ with each counter simulating in parallel.
simulation reaches the end of the worktape and $M^{\prime}$ enters an accepting state, for all odd $j, j<s$, the counters in $C_{i, j}$ and $C_{i, j+1}$ are decremented simultaneously a nondeterministically guessed number of times to verify that they are the same (and if $j=s$ is is not changed thereby verifying that it is zero). Then $M$ enters a final state. Figure 1 demonstrates an example. This technique allows counters to be adjusted in a different order.

From Proposition 13 and decidability of boundedness for PBCM, we obtain:
Proposition 14. The boundedness, emptiness, and finitenesss problems for finiteturn NTPBCM are decidable.

We believe that this is a new result for all three decision problems, and in particular, decidability of boundedness is quite powerful.

### 3.4 Simple Matrix Grammars

A matrix grammar has a finite set of matrix rules of the form $\left[A_{1} \rightarrow w_{1}, \ldots, A_{k} \rightarrow\right.$ $w_{k}$ ], where each $A_{i} \rightarrow w_{i}$ is a context-free production. In the derivation, at each step a matrix is chosen nondeterministically, whereby the context-free rules of the matrix must be applied in order to the sentential form to produce the next sentential form. An $n$-simple matrix grammar ( $n$-SMG) (from [20]), a restricted form of a matrix grammar, is a tuple $G=\left(V_{1}, \ldots, V_{n}, \Sigma, P, S\right)$, where $V_{1}, \ldots, V_{n}$ are disjoint sets of nonterminals, $\Sigma$ is the terminal alphabet, $S$ is a start nonterminal not in $\left(V_{1} \cup \cdots \cup V_{n}\right)$, and $P$ is a finite set of rules of the form:

1. $S \rightarrow A_{1} \cdots A_{n}$, where each $A_{i} \in V_{i}$,
2. $\left[X_{1} \rightarrow w_{1}, \ldots, X_{n} \rightarrow w_{n}\right]$, where $X_{i} \in V_{i}$ and $w_{i} \in\left(V_{i} \cup \Sigma\right)^{*}$, and the number of nonterminals in $w_{i}$ is equal to the number of nonterminals in $w_{j}$ for all $i \neq j$.

The derivation relation enforces that in each rule of type 2., always the leftmost nonterminal of $V_{i}$ in the sentential form is rewritten (precise definition of the
derivation relation is in [20]. The language $L(G)$ consists of all strings $w \in \Sigma^{*}$ that can be derived starting from $S$ and applying the rules in such a leftmost derivation. Note that a $1-\mathrm{SMG}$ is just a context-free grammar. It is known 20 that without either the restriction of the number of nonterminals being the same, or not requiring leftmost derivation, grammars can generate more languages.

The following result follows from [20] and [15].
Proposition 15. L is generated by an n-SMG $G$ if and only if there is an n-tape NPDA $M$ accepting $L^{\prime} \subseteq\left\langle\Sigma^{*}\right\rangle^{n}$ such that $L=\left\{x_{1} \cdots x_{n} \mid\left(x_{1}, \ldots, x_{n}\right) \in L^{\prime}\right\}$.

We can generalize the definition of a simple matrix grammar by augmenting it with monotonic counters. Then, in every matrix rule, each context-free production includes $2 k$ counter increments, and for a string to be generated, the counter values in counter $i$ and $i+1$ have to be equal, for $i$ odd. Proposition 15 can be generalized to include counters.
Proposition 16. $L$ is generated by an n-SMG with $2 k$ monotonic counters $G$ if and only if there a an n-tape NPDA with $2 k$ monotonic counters (which is equivalent to an n-tape NPCM) $M$ accepting $L^{\prime} \subseteq\left\langle\Sigma^{*}\right\rangle^{n}$ such that $L=$ $\left\{x_{1} \cdots x_{n} \mid\left(x_{1}, \ldots, x_{n}\right) \in L^{\prime}\right\}$.

Using a proof similar to the decidability problems shown using other multitape characterizations in this paper, we obtain:
Proposition 17. The emptiness, infiniteness, and boundedness problems for simple matrix grammars (resp. with monotonic counters) are decidable, and they are effectively semilinear.

## 4 Store Languages for the Boundedness Problem

To summarize, so far we have determined several new classes of machines for which the boundedness problem is decidable. One of the largest is finite-turn NTM with reversal-bounded counters and a pushdown where, in each accepting computation, the pushdown can only be used within a single sweep of the Turing worktape. In this section, we determine one more class that is even more general than this one. The algorithm provides an entirely different technique than multitape characterizations that we have used thus far.

We focus on finite-flip NPDA [17. A $t$-flip (resp. finite-flip) NPCM augments a $t$-flip NPDA with $k$-reversal-bounded counters. With this model, configurations are of the form $\left(q, w, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right)$ where $q$ is the current state, $w$ is the remaining input, $Z_{0} \gamma$ is the current pushdown contents, and $i_{j}$ is the current contents of counter $j$.

In [19|23], the authors study the concept of a store language of a machine $M$ for arbitrary types of automata, which is essentially a language description of all the store contents that can appear in any accepting computation of the machine. So, for a $t$-flip NPCM $M=\left(Q, \Sigma, \delta, q_{0}, F\right)$, the store language of $M$,

$$
\begin{aligned}
S(M)=\left\{q Z_{0} \gamma c_{1}^{i_{1}} \cdots c_{k}^{i_{k}} \mid\right. & \left(q_{0}, w, Z_{0}, 0, \ldots, 0\right) \vdash_{M}^{*}\left(q, w^{\prime}, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \vdash_{M}^{*} \\
& \left(q_{f}, \lambda, Z_{0} \gamma^{\prime}, i_{1}^{\prime}, \ldots, i_{k}^{\prime}\right), \\
& \left.q_{f} \in F, w, w^{\prime} \in \Sigma^{*}, \gamma^{\prime} \in \Gamma^{*}, i_{1}^{\prime}, \ldots, i_{k}^{\prime} \geq 0\right\},
\end{aligned}
$$

where $c_{1}, \ldots, c_{k}$ are new special symbols associated with the counters. In [23], the authors showed that the store language of every $t$-flip NPDA (resp. $t$-flip NPCM) is in fact a regular (resp. NCM) language. Therefore, the pushdown can be essentially eliminated. This is a generalization of the important result that the store language of any NPDA is regular [11.

The next proof uses an inductive procedure (informally described without counters) where we know 0-flip NPDAs (equal to the context-free languages) have a boundedness problem. And inductively, if we have an $r+1$-flip NPDA, we can create two machines, an $r$-flip machine that accepts the parts of the inputs of $M$ read during the first $r$ flips that eventually leads to acceptance, and a 0-flip machine that accepts the parts of the inputs of $M$ from which, with no flips, it will eventually accept. These two languages use the store languages, which can be accepted by finite automata. The purpose of the store languages should be noted. Simply using the fact that $r+1$-flip NPDA are closed under gsm mappings, it is immediately evident that both of these languages can be accepted by $r+1$-flip NPDA (just using closure properties). But, by using the store language, it is possible to accept the first with only an $r$-flip NPDA and the second with a 0 -flip NPDA. This is needed to make the induction work, so that essentially we can decide boundedness up to any given $r$.

Proposition 18. The boundedness, emptiness, and infiniteness problems are decidable for finite-flip NPDA (resp. finite-flip NPCM).

Lastly, we consider machines with a finite-flip pushdown, reversal-bounded counters, and a finite-turn worktape. Such a machine is pd-restricted if, in every accepting computation, the finite-flip pushdown can only be used in one left-toright sweep or right-to-left sweep of the worktape. Finally, by Proposition 11.

Corollary 19. The class of pd-restricted finite-flip NPCM augmented with a finite-turn worktape has a decidable boundedness, emptiness, and infiniteness problem.

## 5 Conclusions

In this paper, we study powerful one-way nondeterministic machine models, and find new models where the boundedness, emptiness, and infiniteness problems are decidable. The largest of these are finite-turn Turing machines augmented by partially blind counters, and finite-turn Turing machines augmented by a pushdown that can be flipped a finite number of times, and reversal-bounded counters, where the pushdown can only be used in one sweep of the Turing worktape. It also shows two new techniques to show these problems are decidable.

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## Appendix

## Definitions

For $t \geq 1$, a one-way $t$-tape nondeterministic finite automaton ( $t$-tape NFA) is a tuple $M=\left(Q, \Sigma, \delta, q_{0}, F\right)$ where $Q$ is a finite set of states, $\Sigma$ is the finite input alphabet, $q_{0} \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta$ is a partial function from $Q \times(\Sigma \cup\{\lambda\}) \times\{i \mid 1 \leq i \leq t\}$ (for 1-tape machines, we unambiguously leave off the last component) to finite subsets of $Q$. We usually denote an element $q^{\prime} \in \delta(q, a, i)$ by $\delta(q, a, i) \rightarrow q^{\prime}$. A configuration of $M$ is a tuple $\left(q,\left(w_{1}, \ldots, w_{t}\right)\right)$ where $q \in Q$ is the current state, and $\left(w_{1}, \ldots, w_{t}\right), w_{1}, \ldots, w_{t} \in$ $\Sigma^{*}$ is the remainder of the $t$-tape input. Two configurations change as follows:

$$
\left(q,\left(w_{1}, \ldots, w_{i-1}, a w_{i}, w_{i+1}, \ldots, w_{t}\right)\right) \vdash\left(q^{\prime},\left(w_{1}, \ldots, w_{t}\right)\right),
$$

if there is a transition $\delta(q, a, i) \rightarrow q^{\prime}$. We let $\vdash^{*}$ be the reflexive and transitive closure of $\vdash$. An accepting computation on $\left(w_{1}, \ldots, w_{t}\right) \in\left\langle\Sigma^{*}\right\rangle^{t}$ is a sequence

$$
\begin{equation*}
\left(q_{0},\left(w_{1}, \ldots, w_{t}\right)\right) \vdash \cdots \vdash\left(q_{n},(\lambda, \ldots, \lambda)\right), \tag{1}
\end{equation*}
$$

where $q_{n} \in F$. The language accepted by $M, L(M) \subseteq\left\langle\Sigma^{*}\right\rangle^{t}$ is the set of all $\left(w_{1}, \ldots, w_{t}\right)$ for which there is an accepting computation.

A $t$-tape $k$-counter machine is a tuple $M=\left(Q, \Sigma, \delta, q_{0}, F\right)$, where $Q, \Sigma, q_{0}, F$ are just like $t$-tape NFA, and $\delta$ has transitions $\delta(q, a, i, s, j) \rightarrow\left(q^{\prime}, x\right)$ where $q, q^{\prime} \in Q, a \in \Sigma \cup\{\lambda\}, 1 \leq i \leq t, s \in\{0,1\}, x \in\{-1,0,1\}, 1 \leq j \leq k$. A configuration is a tuple $\left(q,\left(w_{1}, \ldots, w_{t}\right),\left(z_{1}, \ldots, z_{k}\right)\right)$ where $q \in Q$ is the current state, $\left(w_{1}, \ldots, w_{t}\right)$ is the remaining contents of the input tapes, and $\left(z_{1}, \ldots, z_{k}\right)$ are the contents of the counters, where $z_{i} \in \mathbb{N}_{0}$, for each $i$. Configurations change by

$$
\begin{aligned}
& \left(q,\left(w_{1}, \ldots, w_{i-1}, a w_{i}, w_{i+1}, \ldots, w_{t}\right),\left(z_{1}, \ldots, z_{k}\right)\right) \vdash^{\alpha} \\
& \quad\left(q^{\prime},\left(w_{1}, \ldots, w_{t}\right),\left(z_{1}, \ldots, z_{j-1}, z_{j}+x, z_{j+1}, \ldots, z_{k}\right)\right)
\end{aligned}
$$

if $\alpha$ is $\delta(q, a, i, s, j) \rightarrow\left(q^{\prime}, x\right), s$ is 0 if $z_{j}=0$, and $s$ is 1 if $z_{j}$ is positive. As such, $s$ is known as the counter status as it is used to check if a counter is empty or not. A $k$-counter machine $M$ is $r$-reversal-bounded (resp. reversal-bounded) if in each accepting computation, the number of changes between non-decreasing and non-increasing (or vice versa) on each counter is at most $r$ (resp. a finite number). A $k$-counter machine is partially-blind if $\delta(q, a, i, 0, j)=\delta(q, a, i, 1, j)$ for each $q \in Q, a \in \Sigma \cup\{\lambda\}, 1 \leq i \leq t, 1 \leq j \leq k$. For this reason, typically the counter status component is left off the transitions.

We also examine one-way machines with a two-way (Turing) read/write worktape denoted by NTM. These machines (for this model, we only use 1-tape inputs) have the same components as 1-tape NFA, but also have a worktape alphabet $\Gamma$, (including a fixed blank character $\sqcup$ ), and $\delta$ is from $Q \times(\Sigma \cup\{\lambda\}) \times \Gamma$ to subsets of $Q \times \Gamma \times\{\mathrm{L}, \mathrm{S}, \mathrm{R}\}$. Each transition $\delta(q, a, y) \rightarrow\left(q^{\prime}, z, x\right)$ consists of, the current state $q \in Q$, the state to switch to $q^{\prime} \in Q$, the input $a \in \Sigma \cup\{\lambda\}$, the
symbol currently being scanned on the worktape $y \in \Gamma$, the symbol to replace it with $z \in \Gamma$, and the direction $x$ (left, stay, or right) moved by the read/write head.

An NTM (resp. NPDA) $M$ is l-turn if, in every accepting computation, the worktape makes at most $l$ changes in direction, between moving towards the right and moving towards the left, and vice versa. A machine is finite-turn if it is $l$-turn for some $l$.

Let $M$ be a $t$-turn NTM (resp. $t$-turn NTCM), where $t \geq 0$. We say $M$ is in normal form if: $M$ makes exactly $t$ turns on all inputs accepted; the read/write worktape head always moves left or right at every step that uses the worktape (no stay transitions); on every accepting computation, there is a worktape cell $d$, and $M$ only turns left on cell $d$ and right on cell 1 (the cell it starts on); the worktape never moves left of cell 1 or right of cell $d$; and $M$ accepts only in cell 1 or $d$.

Normal Form Lemma. Let $t \geq 0$. Given a $t$-turn NTM (resp. $t$-turn NTCM) $M$, we can construct a $t$-turn NTM (resp. $t$-turn NTCM) $M^{\prime}$ in normal form such that $L\left(M^{\prime}\right)=L(M)$.
Proof. First, we assume without loss of generality that $M$ starts by moving towards the right (if it does not, then another machine can be built which uses the worktape in the opposite direction).

Next, we introduce three new worktape symbols: $\triangleright$, $\#$, and $\triangleleft$ (plus a marker that can be added to any letter). $M^{\prime}$ starts by writing $\triangleright$ on the first cell. Throughout, if $M$ has counters, then transitions that use them are simulated verbatim. Then $M^{\prime}$ simulates $M$ before the first turn, whereby at each step, $M^{\prime}$ can either simulate a transition verbatim that moves right. It can simulate a sequence of stay transitions before moving right as follows: if the symbol on the worktape cell is $x$, it guesses the final contents of the cell before eventually moving right, $y$, and replaces $x$ with $y$, but by immediately moving right. It then simulates the sequence of stay transitions appropriately from $x$ to $y$ but using the state to store the current simulated symbol and by moving right on \# on each step, ultimately verifying that the simulated sequence of stay transitions ends with $y$. Also, at each step, instead of simulating a transition of $M$, it can instead nondeterministically write any number of $\#$ symbols and move right on the store (reading $\lambda$ on the input). At some point, $M^{\prime}$ writes a $\triangleleft$ on the worktape (this is the guessed rightmost cell $d$ to be visited during the entire computation). Next, it continues the simulation but only towards the left by only simulating transitions that move left verbatim (only on a non-\# symbol from the worktape), or that stay on the worktape in a similar fashion as above starting on $x$, guessing the final contents $y$, replacing $x$ with $y$, then simulating the sequence appropriately using the state, by moving left on $\#$ at each step. It can also skip over arbitrarily many \# symbols on $\lambda$ input. If it simulates a turn transition, it instead marks the current cell, moves left to $\triangleright$ and back to the marked cell, where it unmarks it and continues the simulation. If $t \geq 2$, it again switches direction and continues this same simulation towards the right in a similar fashion, and so on. $M^{\prime}$ remembers how many turns have occurred in
the finite control to make sure it turns exactly $t$ times. Lastly, if $M^{\prime}$ hits a final state of $M$, it remembers this in the finite control and continues making a full $t+1$ sweeps of the worktape whence it enters a final state of $M^{\prime}$.

Altogether, $M^{\prime}$ can shuffle in arbitrarily many \# symbols into the worktape, it writes $\triangleright$ on the first cell, $\triangleleft$ on some cell $d$ (nondeterministically guessed so that $M$ would not move to the right of that cell on any turn), it only turns at those two designated cells, and always turns at those two cells, there are no stay transitions, and it accepts only on either $\triangleright$ or $\triangleleft$. It is evident that $L\left(M^{\prime}\right)=L(M)$ and $M^{\prime}$ is in normal form.

Also, given such a machine $M$ in normal form, we can have the machine write the current state in the first and last cell ( 1 and $d$ ) every time it reaches them. We call this state normal form.

Lemma 2. Let $t \geq 0$, and let $M$ be a t-turn NTM (resp. $t$-turn NTCM). We can construct a $(t+1)$-tape NFA (resp. $(t+1)$-tape NCM$) M^{\prime}$ such that $L\left(M^{\prime}\right)^{A}=$ $L(M)$.

Proof. We will only describe the case when $k$ is odd, with the even case being similar.

Assume without loss of generality by the previous lemma that $M$ is in state normal form. We construct $M^{\prime}$ which accepts input ( $w_{1}, \ldots, w_{t+1}$ ) if and only if $w_{1} w_{2}^{R} \cdots w_{t} w_{t+1}^{R} \in L(M)$, as follows:

On input $\left(w_{1}, \ldots, w_{t+1}\right), M^{\prime}$ guesses a $(t+1)$-track string $x \in \Delta^{*}$ letter-byletter (here, $x$ does not need to be stored as it is guessed one letter at a time from left-to-right), and simulates the computation of $M$ on the $t+1$ input tapes by making sure that the computation of $M$ is "compatible" with the guessed string $x$ while checking that $x \in H(M)$. To do this, $M^{\prime}$ verifies that on input $w_{1}, M$ could read $w_{1}$ before the first turn and finish its first sweep with $h_{1}(x)$ on its tape, on input $x_{2}^{R}$ and starting with state and tape contents of $h_{1}(x), M$ could finish its second sweep with $h_{2}(x)$ on its tape, etc. Furthermore, on the guessed $x$, it will be possible to verify in parallel that, for each $i, 1 \leq i \leq t$, $h_{i}(x)$ produced $h_{i+1}(x)$ while reading $w_{i}$ if $i$ is odd, or $w_{i}^{R}$ if $i$ is even. Also, it is possible to do so from left-to-right when guessing $x$, even when $i$ is even. We will describe the even case which is slightly more complicated. The simulation using the even tracks are done in reverse by "flipping the directions". It only needs to simulate transitions that move left. To simulate $\delta(q, a, y) \rightarrow(p, z, \mathrm{~L})$ on track $i, M^{\prime}$ switches the simulation of track $i$ from $p$ to $q$ while reading $a$ from tape $i$, and verifying that the $(t+1)$-track string $x$ has $y$ in the current letter of track $i-1$ (or $\sqcup$ if $i=1$ ), and $z$ in the current letter of track $i$. If $M$ has monotonic counters, then $M^{\prime}$ simulates them verbatim, as applying transitions in a different order preserves their total.

Lemma 3. Let $t \geq 0$, and let $M$ be a $(t+1)$-tape NFA (resp. $(t+1)$-tape NCM). Then we can construct a t-turn NTM (resp. $t$-turn NTCM) $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$.

Proof. We will describe the case when $t$ is odd, with the even case being similar. First consider the case without counters.

Each transition of the $(t+1)$-tape NFA is of the form $\delta(q, a, i) \rightarrow p$, where $q, p$ are states, $a \in \Sigma \cup\{\lambda\}$ and $1 \leq i \leq t+1$. Let $T$ be a set of labels in bijective correspondence with the transitions of $M$.

On input $w=w_{1} w_{2}^{R} \cdots w_{t} w_{t+1}^{R}$ (note that the $w_{i}$ 's need not have the same lengths and can even be the empty word since some transitions are on $\lambda$ input), $M^{\prime}$ operates as follows:

1. $M^{\prime}$ uses the worktape to guess a sequence of transition labels of $M$. So $M^{\prime}$ writes $\alpha_{0} \alpha_{1} \cdots \alpha_{n}\left(\right.$ each $\left.\alpha_{i} \in T\right)$, where it verifies that $\alpha_{0}$ is a transition from an initial state of $M, \alpha_{n}$ is a transition into a final state, and the ending state of $\alpha_{i}$ is the starting state of $\alpha_{i+1}$ for all $i, 0 \leq i<n$. In parallel, $M^{\prime}$ reads input $w_{1}$ and verifies that the letters of $\Sigma \cup\{\lambda\}$ on input tape 1 that are read by the transition sequence $\alpha_{0} \cdots \alpha_{n}$ are $w_{1}$.
2. $M^{\prime}$ turns on the worktape and reads input $w_{2}^{R}$ and makes sure the letters read by $\alpha_{n} \cdots \alpha_{0}$ on tape 2 are $w_{2}^{R}$.
3. $M$ turns on the worktape and reads $w_{3}$ and makes sure the letters read by $\alpha_{0} \cdots \alpha_{n}$ on tape 3 are $w_{3}$.
$\vdots$
until tape $t+1$.
Because the transition sequence $\alpha_{0} \cdots \alpha_{n}$ is fixed after step $1, M^{\prime}$ can verify that $\left(w_{1}, \ldots, w_{t+1}\right)$ could be read by $\alpha_{0} \cdots \alpha_{n}$ by making a turn on the store after reading each $w_{i}$.

If $M$ has counters, then the guessed sequence of transition labels remains the same, and all counter changes can be applied when guessing it.

Proposition 9. Let $t \geq 0$. There is a $(t+1)$-tape NPCM $M$ if and only if there is an $i$-restricted $t$-turn NTPCM $M^{\prime}$ such that $L\left(M^{\prime}\right)=L(M)^{A}$, for any $0 \leq i \leq t$.

Proof. The construction from $i$-restricted $t$-turn NTPCM to ( $i$-restricted) $(t+1)$ tape NPCM is similar to the proof of Lemma 2 noting that since the pushdown in the $t$-turn NTPCM is only used between two consecutive turns (or the start or end of the computation) of the worktape head, the simulation of the pushdown is only done on a single track of the $(t+1)$-track guessed string when reading that one input tape. (Note that the simulation of the pushdown is done in reverse if $i$ is odd).

For the reverse construction, by Lemma 8, given a $(t+1)$-tape NPCM, we can convert to an equivalent $i$-restricted $(t+1)$-tape NPCM. To convert that to an $i$-restricted $t$-turn NTPCM $(0 \leq i \leq t)$ is similar to the one in the proof of Lemma 3 where it only verified that the pushdown changes properly according to the guessed transition sequence by simulating the pushdown only between the $i$ th and $i+1$ st turns of the Turing tape (or the start or the end).

Proposition 12. The emptiness (boundedness, infiniteness) problems are undecidable for the following models:

1. 1-turn NTM (or DCSA) with a 1-turn pushdown.
2. 2-turn DCSA with a 1-turn pushdown, even when the pushdown is used only during the checking stack reading phase (i.e., after turn 1).
3. 1-turn NTM with an unrestricted counter.
4. 1-turn deterministic pushdown automata with an unrestricted counter.

Proof. It was pointed out in the introduction that undecidability of emptiness implies undecidability of boundedness. Similarly, given $M, L(M) \Sigma^{*}$ is infinite if and only if $L(M) \neq \emptyset$, and hence undecidability of emptiness implies undecidability of infiniteness.

The first item above follows from the undecidability of emptiness of the intersection of languages accepted by 1-turn DPDA (deterministic NPDA) 2].

For the second item, we will use the undecidability of the halting problem for single-tape DTM on an initially blank tape. Let $Z$ be a single-tape DTM. Define the following language:
$L=\left\{I_{1} \# I_{3} \cdots \# I_{2 k-1} \$ I_{2 k}^{R} \# \cdots \# I_{4}^{R} \# I_{2}^{R} \mid I_{1} \Rightarrow \cdots \Rightarrow I_{2 k-1} \Rightarrow I_{2 k}\right.$ is a halting computation of $Z\}$.

Construct a 2 -turn DCSA with a 1-turn pushdown $M$ as follows when given an input of the form $w=I_{1} \# I_{3} \cdots \# I_{2 k-1} \$ I_{2 k}^{R} \# \cdots \# I_{4}^{R} \# I_{2}^{R}$ :

1. $M^{\prime}$ writes $I_{1} \# I_{3} \cdots \# I_{2 k-1}$ on the checking stack.
2. $M^{\prime}$ turns on the checking stack, and while reading input $I_{2 k}^{R} \# \cdots \# I_{4}^{R} \# I_{2}^{R}$ does the following in parallel:

- It checks that $I_{1} \Rightarrow I_{2}, I_{3} \Rightarrow I_{4}, \ldots, I_{2 k-1} \Rightarrow I_{2 k}$.
- It pushes $I_{2 k}^{R} \# \cdots \# I_{4}^{R} \# I_{2}^{R}$ on the pushdown.

3. $M^{\prime}$ then makes a second turn on the checking stack and checks (by popping the pushdown and scanning the checking stack) that $I_{2} \Rightarrow I_{3}, \ldots, I_{2 k-2} \Rightarrow$ $I_{2 k-1}$.
$M$ makes only 2 turns on the checking stack and 1-turn on the pushdown, and $L(M)$ is empty if and only if $Z$ does not halt.

For the third point, we show that it is undecidable whether a machine of this type accepts $\lambda$. The proof of the undecidability of the halting problem for 2-counter machine with counters $C_{1}$ and $C_{2}$ (with no input tape) in 24] shows that the counters operate in phases. A phase begins with one counter, say $C_{1}$, having value $d_{i}$ and the other counter, $C_{2}$, having value 0 . During the phase, $C_{1}$ decreases while $C_{2}$ increases. The phase ends with $C_{1}$ having value 0 and $C_{2}$ having value $e_{i}$. Then in the next phase the modes of the counters are interchanged: $C_{2}$ decreases to zero while $C_{1}$ increases to $d_{i+1}$. At the start, $d_{1}=1$. Thus, a halting computation of $M$ (if it halts) will be of the form:

$$
\left(q_{1}, d_{1}, 0\right) \Rightarrow^{*}\left(q_{2}, 0, e_{1}\right) \Rightarrow^{*}\left(q_{3}, d_{2}, 0\right) \Rightarrow^{*}\left(q_{4}, 0, e_{2}\right) \Rightarrow^{*} \cdots \Rightarrow^{*}\left(q_{2 k}, 0, e_{k}\right)
$$

where $q_{1}, \ldots, q_{2 k}$ are states and $d_{1}, e_{1}, d_{2}, e_{2}, \ldots, d_{2 k}, e_{2 k}$ are positive integers with $d_{1}=1$ and the shown configurations are the ends of the phases. Note that the second component of the configuration refers to the value of $C_{1}$, while the third component refers to the value of $C_{2}$. We assume that if $M$ halts, it halts with zero in counter $C_{1}$.

We construct a 1-turn NTM $M^{\prime}$ with an unrestricted counter $D$ to simulate the 2-counter machine $M$ on $\lambda$ input as follows:

1. $M^{\prime}$ writes $z=a^{d_{k}} \# b^{d_{k}} \# \cdots \# a^{d_{3}} \# b^{d_{3}} \# a^{d_{2}} \# b^{d_{2}} \# a^{d_{1}} \# b^{d_{1}}$ on its read/write tape, where $d_{1}=1$ and $k, d_{k}, \ldots, d_{3}, d_{2}$ are nondeterministically chosen positive integers. Clearly, $M^{\prime}$ can do this without reversing on the read/write tape with the help of $D$; When $M^{\prime}$ writes $a^{d_{i}}$, it simultaneously increments $D$ by $d_{i}$, and then it decrements $D$ to zero while writing $\# b^{d_{i}}$.
2. $M^{\prime}$ then reverses its read/write head and simulates $M$. In the simulation, the 1-turn read/write tape will keep track of the changes in counter $C_{1}$ and $D$ will simulate $C_{2}$. The simulation is done as follows: Suppose counter $C_{1}$ has value $d_{i}$ represented by $a^{d_{i}}$ and the read/write head is on \# to the left of $b^{d_{i}}$, and counter $C_{2}$ is zero. $M^{\prime}$ moves the read/write head left to \# simulating $M$ and incrementing $D$ to $e_{i}$. This simulates the phase where $C_{1}$ which has value $d_{i}$ is decremented to zero while $C_{2}$ is incremented to $e_{i}$. In the next phase, $M^{\prime}$ simulating $M$ decrements $D$ to zero while moving the read/write head left of $b^{d_{i+1}}$ to the next \# checking that $d_{i+1}$ is valid. At this point, $D$ is zero, and counter $C_{1}$ has value represented by $a^{d_{i+1}}$. The process is then repeated.

Clearly, $M^{\prime}$ accepts $\lambda$ if and only if $M$ halts, which is undecidable.
Given a 1-turn NPDA $M$ with an unrestricted counter operating on $\lambda$ input, let $T=\left\{t_{1}, \ldots, t_{k}\right\}$ be its set of transitions. Construct a 1-turn DPDA $M^{\prime}$ with an unrestricted counter with inputs in $T^{+}$which operates as follows when given input $w=a_{1} \cdots a_{n}$ in $T^{+}: M^{\prime}$ checks that the transition $a_{1}$ is applicable to the initial condition, i.e., the state is $q_{0}$, the stack symbol $Z_{0}$, and counter 0 . Then $M^{\prime}$ tries to simulate $M$ guided by the transitions in $w . M^{\prime}$ accepts $w$ if and only if the sequence of transitions $w$ leads to $M$ to accept $\lambda$. It follows that $L\left(M^{\prime}\right)$ is not empty if and only if $M$ accepts $\lambda$. The result follows from the third point of this proposition.

Proposition 16. $L$ is generated by an n-SMG with $2 k$ monotonic counters $G$ if and only if there $a$ an n-tape NPDA with $2 k$ monotonic counters (which is equivalent to an n-tape NPCM) $M$ accepting $L^{\prime} \subseteq\left\langle\Sigma^{*}\right\rangle^{n}$ such that $L=$ $\left\{x_{1} \cdots x_{n} \mid \quad\left(x_{1}, \ldots, x_{n}\right) \in L^{\prime}\right\}$.
Proof. Suppose $L^{\prime}$ is accepted by an $n$-tape NPDA $M$ with input alphabet $\Sigma$ and $2 k$ monotonic counters $C_{1}, D_{1}, \ldots, C_{k}, D_{k}$. Let $a_{1}, b_{1}, \ldots, a_{k}, b_{k}$ be new symbols. We construct an $n$-tape NPDA $M^{\prime}$ with input alphabet $\Delta=\Sigma \cup$ $\left\{a_{1}, b_{1}, \ldots, a_{k}, b_{k}\right\} . M^{\prime}$ when given an input $w \in \Delta^{*}$, simulates $M$ on $w$ but in a move, instead of incrementing $C_{r}\left(D_{r}\right)$ by a non-negative integer $i_{r}\left(j_{r}\right), M^{\prime}$ reads $a_{r}^{i_{r}}\left(b_{r}^{j_{r}}\right)$ on input tape 1. By Proposition 15 , since $M^{\prime}$ is a $n$-tape NPDA, we can construct an $n$-SMG $G^{\prime}$ generating $L\left(M^{\prime}\right)$. Next, we construct an $n$-SMG with $2 k$ monotonic counters $G$ from $G^{\prime}$ as follows: If in a rule $R^{\prime}$ of $G^{\prime}$, the symbol $a_{r}\left(b_{r}\right)$ appears $i_{r}\left(j_{r}\right)$ times, we create a rule $R$ of $G$ by deleting these symbols and adding $i_{r}\left(j_{r}\right)$ as increment to counter $C_{r}\left(D_{r}\right)$. Clearly, $L(G)=\left\{x_{1} \cdots x_{n} \mid\right.$ $\left.\left(x_{1}, \ldots, x_{n}\right) \in L\right\}$.

The converse is proved by reversing the construction above: Given an $n$-SMG with $2 k$ monotonic counters $G$, we construct an $n$-SMG $G^{\prime}$ with $2 k$ new terminal symbols to simulate the increments $i_{r}\left(j_{r}\right)$ to counter $C_{r}\left(D_{r}\right)$ by generating $a_{r}^{i_{r}}$ $\left(b_{r}^{j_{r}}\right)$ on the first components of the rules. From $G^{\prime}$ we then construct an $n$ tape NPDA $M^{\prime}$. Finally from $M$, we construct a $n$-tape NPDA with monotonic counters $M$.

Proposition 17. The emptiness, infiniteness, and boundedness problems for simple matrix grammars (resp. with monotonic counters) are decidable, and they are effectively semilinear.

Proof. Given an $n$-SMG with $2 k$ monotonic counters, we construct an $n$-tape NPDA $M$ with monotonic counters (which is equivalent to a $n$-tape NPCM) such that $L(G)=\left\{x_{1} \cdots x_{n} \mid\left(x_{1}, \ldots, x_{n}\right) \in L(M)\right\}$. Let $L_{i}=\left\{x_{i} \mid\left(x_{1}, \ldots, x_{n}\right) \in\right.$ $L(M)\}$. We construct for each $i$, an NPCM $M_{i}$ accepting $L\left(M_{i}\right)$. Then $L(G)$ is non-empty (resp. finite, bounded) if and only if each $L\left(M_{i}\right)$ is non-empty (resp. finite, bounded). The result follows since these problems are decidable for NPCM. Semilinearity is also clear.

Proposition 18. The boundedness, emptiness, and infiniteness problems are decidable for finite-flip NPDA (resp. finite-flip NPCM).

Proof. It suffices to prove it with counters. We will prove by induction on $t \geq 0$, that every $t$-flip NPCM has a decidable boundedness problem. The base case when $t=0$ is true because every 0 -flip NPCM is in fact a normal NPCM which has a decidable boundedness problem [3].

Let $r \geq 0$, assume that every $r$-flip NPCM has a decidable boundedness problem, and let $M=\left(Q, \Sigma, \delta, q_{0}, F\right)$ be a $(r+1)$-flip NPCM. Assume without loss of generality that every flip transition of $M$ is on $\lambda$, and that every flip transition that can be used for the $i$ th flip is from states in $P_{i}$ to $P_{i}^{\prime}$ where these states are not used for any other transitions that do not involve the $i$ th flip. Also assume that each state implies the topmost symbol of the stack (i.e. each transition guesses the topmost stack symbol, and then in the next step verifies that it guessed correctly). Lastly, assume without loss of generality that in every accepting computation, $M$ makes exactly $r+1$ flips. Then $L(M)$ equals

$$
\begin{aligned}
\{w v \mid & \left(q_{0}, w, Z_{0}, 0, \ldots, 0\right) \vdash^{*}\left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \text { with } r \text { flips, } q_{1} \in P_{r+1}, \\
& \left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \vdash\left(q_{2}, \lambda, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \text { with one flip, } q_{2} \in P_{r+1}^{\prime} \\
& \text { and } \left.\left(q_{2}, v, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \vdash^{*}\left(q_{3}, \lambda, \gamma^{\prime}, i_{1}^{\prime}, \ldots, i_{k}^{\prime}\right) \text { with no flips, } q_{3} \in F\right\}
\end{aligned}
$$

Let $X$ be the set of all pairs $(w, v)$ in $L(M)$ above.
Consider $S_{1}=S(M) \cap P_{r+1} \Gamma^{*} c_{1}^{*} \cdots c_{k}^{*}$ and $S_{2}=S(M) \cap P_{r+1}^{\prime} \Gamma^{*} c_{1}^{*} \cdots c_{k}^{*}$. Because the store language of every finite-flip NPCM is an NCM language 23] and NCM is closed under intersection with regular languages, $S_{1}$ an $S_{2}$ are in NCM. Because also NCM is closed under reversal, we can build an NCM $M_{1}$ that accepts the reversal of $S_{1}$, and $M_{2}$ can be built that accepts $S_{2}$ (not the reversal). Let $k_{1}, k_{2}$ be the number of counters in $M_{1}$ and $M_{2}$ respectively.

Let $L_{1}$ equal to

$$
\begin{aligned}
= & \{w \mid \exists v,(w, v) \in X\} \\
= & \left\{w \mid \exists q_{1}, q_{2} \in Q, q_{3} \in F, \gamma, \gamma^{\prime} \in \Gamma^{*}, v \in \Sigma^{*}, i_{j}, i_{j}^{\prime} \geq 0 \text { for } 1 \leq j \leq k\right. \text { such that } \\
& \left(q_{0}, w, Z_{0}, 0, \ldots, 0\right) \vdash^{*}\left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \text { with } r \text { flips, } q_{1} \in P_{r+1}, \\
& \left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \vdash\left(q_{2}, \lambda, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \text { with one flip, } q_{2} \in P_{r+1}^{\prime} \\
& \left.\quad \text { and }\left(q_{2}, v, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \vdash^{*}\left(q_{3}, \lambda, Z_{0} \gamma^{\prime}, i_{1}^{\prime}, \ldots, i_{k}^{\prime}\right) \text { with no flips }\right\} \\
= & \{w \mid \\
& \left(q_{0}, w, Z_{0}, 0, \ldots, 0\right) \vdash^{*}\left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \text { with } r \text { flips, } \\
& \text { and } \left.q_{1} Z_{0} \gamma c_{1}^{i_{1}} \cdots c_{k}^{i_{k}} \in S_{1}\right\} .
\end{aligned}
$$

This is a $r$-flip language because a $r$-flip NPCM with $k+k_{1}$ counters can be built that simulates $M$ until an arbitrarily guessed state $q \in Q_{1}$ (using the first $k$ counters), where, if it has $Z_{0} \gamma$ on the pushdown and $i_{1}, \ldots, i_{k}$ in the counters, it simulates $M_{1}$ (on the other $k_{1}$ counters) on $\left(q Z_{0} \gamma c_{1}^{i_{1}} \cdots c_{k}^{i_{k}}\right)^{R}$ by reducing each counter to zero from counter $k$ to counter 1, then popping from the pushdown until it is empty.

Let $L_{2}$ equal to

$$
\begin{aligned}
= & \{v \mid \exists w,(w, v) \in X\} \\
= & \left\{v \mid \exists q_{1}, q_{2} \in Q, q_{3} \in F, \gamma, \gamma^{\prime} \in \Gamma^{*}, w \in \Sigma^{*}, i_{j}, i_{j}^{\prime} \geq 0 \text { for } 1 \leq j \leq k\right. \text { such that } \\
& \left(q_{0}, w, Z_{0}, 0, \ldots, 0\right) \vdash^{*}\left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \text { with } r \text { flips, } q_{1} \in P_{r+1}, \\
& \left(q_{1}, \lambda, Z_{0} \gamma, i_{1}, \ldots, i_{k}\right) \vdash\left(q_{2}, \lambda, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \text { with one flip, } q_{2} \in P_{r+1}^{\prime} \\
& \left.\quad \text { and }\left(q_{2}, v, Z_{0} \gamma^{R}, i_{1}, \ldots, i_{k}\right) \vdash^{*}\left(q_{3}, \lambda, Z_{0} \gamma^{\prime}, i_{1}^{\prime}, \ldots, i_{k}^{\prime}\right) \text { with no flips }\right\} \\
= & \{v \mid \\
& \left(q_{2}, v, Z_{0} \gamma, i_{1} \ldots, i_{k}\right) \vdash^{*}\left(q_{3}, \lambda, Z_{0} \gamma^{\prime}, i_{1}, \ldots, i_{k}^{\prime}\right) \text { with } 0 \text { flips, } q_{3} \in F, \\
& \left.i_{1}^{\prime}, \ldots, i_{k}^{\prime} \geq 0, \text { and } q_{2} Z_{0} \gamma c_{1}^{i_{1}} \cdots \gamma c_{k}^{i_{k}} \in S_{2}\right\} .
\end{aligned}
$$

This is a NPCM language (no flips) as a machine with $k+k_{2}$ counters can be built which simulates $M_{2}$ by guessing and checking that $q_{0} Z_{0} \gamma c_{1}^{i_{1}} \cdots c_{k}^{i_{k}} \in S_{2}$, while pushing $Z_{0} \gamma$ onto the pushdown and adding $i_{j}$ to each counter $j$. Then it uses the other $k$ counters to simulate $M$ from $q$ without any flips.

Claim. $L$ is bounded (resp. non-empty, finite) if and only if $L_{1}$ and $L_{2}$ are both bounded (resp. non-empty, finite).

Proof. Assume $L$ is bounded. Then there exists $w_{1}, \ldots, w_{n}$ such that $L \subseteq$ $w_{1}^{*} \cdots w_{n}^{*}$. But $L_{1}$ and $L_{2}$ are both subsets of subwords of $L$ and so they are bounded (the set of subwords of a bounded language is bounded, and any subset of a bounded language is bounded [9]).

Assume $L_{1}$ and $L_{2}$ are bounded. Hence, there exists $w_{1}, \ldots, w_{n}, v_{1}, \ldots, v_{m}$ such that $L_{1} \subseteq w_{1}^{*} \cdots w_{n}^{*}$ and $L_{2} \subseteq v_{1}^{*} \cdots v_{m}^{*}$. It is immediate that $L \subseteq L_{1} L_{2}$ because $L=\{x y \mid(x, y) \in X\}$. Hence, $L \subseteq w_{1}^{*} \cdots w_{n}^{*} v_{1}^{*} \cdots v_{m}^{*}$.

As we have an algorithm that checks if a (0-flip) NPCM is bounded [3], the proof above creates an algorithm to check if a 1-flip NPDA is bounded. This provides an inductive algorithm that works up to an arbitrary number of flips.


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