# CABLES OF THE FIGURE-EIGHT KNOT VIA REAL FRØYSHOV INVARIANTS 

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#### Abstract

We prove that the $(2 n, 1)$-cable of the figure-eight knot is not smoothly slice when $n$ is odd, by using the real Seiberg-Witten Frøyshov invariant of Konno-Miyazawa-Taniguchi. For the computation, we develop an $O(2)$-equivariant version of the lattice homotopy type, originally introduced by Dai-Sasahira-Stoffregen. This enables us to compute the real Seiberg-Witten Floer homotopy type for a certain class of knots. Additionally, we present some computations of Miyazawa's real framed Seiberg-Witten invariant for 2-knots.


## 1. Introduction

Casson and Gordon CG83, Theorem 5.1] proved that a fibered knot in a homology sphere is homotopically ribbon if and only if its closed monodromy extends over a handlebody. Utilizing this characterization, Miyazaki Miy94 constructed a large family of fibered knots and proved that each knot in this family is not ribbon. Within this family, there are two important sets of knots: the first one [Miy94, Example 1] is the set of nontrivial connected sums of iterated torus knots. This set relates to Rudolph's conjecture Rud76, which asserts that the set of algebraic knots is linearly independent in the smooth knot concordance group (see Lit84, HKL12, AT16, Bak16, CKP23 for related results).

The other set Miy94, Example 2] consists of the ( $2 n, 1$ )-cables of fibered negative-amphiciral knots with irreducible Alexander polynomial ${ }^{1}$ These knots are known to be algebraically slice and strongly rationally slice Kaw80, Cha07, KW18, ${ }^{2}$ While these knots attracted considerable attention due to their relation to the slice-ribbon conjecture Fox62, Problem 25], no proof of nonsliceness had been established for them until recently. In $\mathrm{DKM}^{+} 22$, Theorem 1.1] (see also $\mathrm{ACM}^{+} 23$, Theorem 2.1] and KMT23, Corollary 1.20]), Dai, Kang, Mallick, Park, and Stoffregen proved that the simplest case-the ( 2,1 )-cable of the figure-eight knot-is not smoothly slice. In fact, they show that a $(2,1)$-cable of a Floer-thin knot with nonvanishing Arf invariant has infinite order in the smooth concordance group. In this article, we consider ( $2 n, 1$ )-cables in general and obtain the following:

Theorem 1.1. Let $E$ be the figure-eight knot, and let $E_{2 n, 1}$ denote the $(2 n, 1)$-cable of $E$. For each positive odd integer $n$, the knot $E_{2 n, 1}$ does not bound a normally immersed disk in $B^{4}$ with only negative double points. In particular, for each odd integer $n$, the knot $E_{2 n, 1}$ is not smoothly slice.

Here, we say a surface is normally immersed if it is smoothly immersed in a manifold such that the only singularities are transverse double points in the interior of the surface. Recall that the 4-dimensional clasp number $c_{4}(K)$ of a knot $K$ Shi74 is the minimal number of double points in a normally immersed disk in $B^{4}$ bounded by $K$. A refinement $c_{4}^{+}(K)$, considered for example in DS24, JZ20, FP22, Mil22, Liv22, is the minimal number of positive double points in such a normally immersed disk. With this terminology, the main theorem can be compactly stated as $0<c_{4}^{+}\left(E_{2 n, 1}\right)$ for each positive odd integer $n$. Since a smoothly slice knot has vanishing $c_{4}^{+}$, the theorem is a strict improvement over previous results, even for the case $n=1$.

Note that the figure-eight knot $E$ can be transformed into the unknot by changing a negative crossing to a positive one. This implies that $E$ bounds a normally immersed disk with only one negative double point, and $E_{2 n, 1}$ bounds a normally immersed disk with $2 n$ positive double points and $4 n^{2}$ negative double points in $B^{4}$. For the special case $E_{2,1}$, with some extra consideration, one can find two crossing changes, one from positive to negative and one from negative to positive, that turn $E_{2,1}$ into a smoothly slice knot, which in particular implies that $c_{4}^{+}\left(E_{2,1}\right)=1$ (see Remark 3.4). Determining $c_{4}\left(E_{2 n, 1}\right)$ and $c_{4}^{+}\left(E_{2 n, 1}\right)$ in general seems to be an interesting yet challenging problem.

[^0]Our proof shows that for each odd integer $n$, the double-branched cover of $E_{2 n, 1}$ does not bound a 4-manifold $W$ with the following properties:

- $W$ is a smooth spin 4-manifold with a spin stucture $\mathfrak{s}$,
- $\tau: W \rightarrow W$ is a smooth involution such that $\left.\tau\right|_{\partial W}$ is the deck transformation and $\tau^{*} \mathfrak{s} \cong \mathfrak{s}$, and
- $b_{1}(W)=0, b_{2}^{+}(W)-b_{2}^{+}(W / \tau)=0$, and $\sigma(W) \leq 0$.

In particular, the double-branched cover does not bound an equivariant $\mathbb{Z}_{2}$-homology ball; that is, a $\mathbb{Z}_{2}$-homology ball over which the branching involution extends as a smooth involution. From the nonexistence of such a spin 4-manifold filling of the branched cover, we can further conclude that the knot $E_{2 n, 1}$ does not bound a normally immersed disk with only negative double points in any $\mathbb{Z}_{2}$-homology ball.

The topological input to the theorem is the existence of a smooth concordance from the figure-eight knot to the unknot in a twice-punctured $2 \mathbb{C P}^{2}$, denoted by $X$, that represents $(1,3)$ in $H_{2}(X, \partial X ; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, as proved by Aceto, Castro, Miller, Park, and Stipsicz in $\mathrm{ACM}^{+} 23$. Our obstruction applies to all knots that permit such a concordance to a smoothly slice knot, which is the case for $\mathrm{ACM}^{+} 23$, Theorem 2.3] as well.

Theorem 1.2. Let $K$ be a knot, and let $K_{2 n, 1}$ denote the $(2 n, 1)$-cable of $K$. Suppose that $K$ can be transformed into a slice knot by applying full negative twists along two disjoint disks, where one intersects $K$ algebraically once and the other intersects it algebraically three times. Then, for each positive odd integer $n$, the knot $K_{2 n, 1}$ does not bound a normally immersed disk in $B^{4}$ with only negative double points.

There are infinitely many knots that satisfy the assumptions of Theorem 1.2 . In fact, $\mathrm{ACM}^{+} 23$, Remark 2.6] provides an infinite family of strongly negative-amphichiral knots meeting the assumptions. Recall that a knot $K$ is called strongly negative-amphichiral if there is an orientation-reversing involution $\tau: S^{3} \rightarrow S^{3}$ such that $\tau(K)=K$. Since each knot in the family is strongly negative-amphichiral, the $(2 n, 1)$-cables of these knots are algebraically slice and strongly rationally slice Kaw09. In particular, the usual concordance invariants from knot Floer homology OS04 Ras03 (cf. Hom17,HKPS22) and the concordance invariants $s^{\#}, f_{\sigma}, \tau^{\#}, \nu^{\#}, \tau_{I}$ ${ }^{3}$, $\widetilde{s}, \Gamma, r_{s}$ from instanton knot Floer theory KM13, DS19 GLW19 KM21, BS21, DIS ${ }^{+} 22$, and the concordance invariants $\theta^{p}, q_{M}$ from equivariant Seiberg-Witten theory [BH24, Bar22, IT24] vanish. Moreover, it can also be proved that the $s$-invariant Ras10 from Khovanov homology Kho00 vanishes (cf. MMSW23]). Additionally, we note that the $(2,1)$-cable (i.e., when $n=1$ ) is the only case where $\mathrm{DKM}^{+} 22, \mathrm{ACM}^{+} 23, \mathrm{KMT} 23$ can be directly applied.

Our main tools are the real Frøyshov inequalities involving the three concordance invariants

$$
\delta_{R}(K), \underline{\delta}_{R}(K), \text { and } \bar{\delta}_{R}(K) \in \frac{1}{16} \mathbb{Z}
$$

which are called real Frøyshov invariants, introduced by Konno, Miyazawa, and the third author in KMT23b. The invariants are defined as certain Frøyshov type invariants for the fixed point spectrum of an order 2 subgroup $\langle I\rangle$ in $O(2)$, acting on the Manolescu's Seiberg-Witten Floer homotopy type Man03 of the double-branched cover of a knot $K$ :

$$
S W F_{R}(K):=\left(S W F\left(\Sigma_{2}(K), \mathfrak{s}_{0}\right)\right)^{I}
$$

where $\mathfrak{s}_{0}$ is the unique spin structure on the double-branched cover $\Sigma_{2}(K)$. Note that $S W F_{R}(K)$ has a $\mathbb{Z}_{4^{-}}$ symmetry, which comes from the $j$-action in $\operatorname{Pin}(2)$. The invariant $\delta_{R}(K)$ is a $\mathbb{Z}_{2}$-equivariant Frøyshov invariant, $\underline{\delta}_{R}$ which can be seen as an analog of the Heegaard Floer $d$-invariant OS03. The latter two invariants, $\underline{\delta}_{R}(K)$ and $\bar{\delta}_{R}(K)$, are $\mathbb{Z}_{4}$-equivariant Frøyshov invariants similar to $\underline{d}$ and $\bar{d}$ in involutive Heegaard Floer theory HM17. There are several variants of real Seiberg-Witten theory; for examples, see TW09, Nak13, Nak15, Kat22, KMT21, Li22, KMT23b, Miy23 Li23 BH24b.

To prove Theorem 1.2 we shall show that if a knot $K$ satisfies the assumptions of the theorem, then $\underline{\delta}_{R}\left(K_{2 n, 1}\right)<0$ for each odd $n$. To accomplish this, we make use of a smooth concordance from $K_{2 n, 1}$ to the torus knot $T_{2 n, 1-20 n}$ in a twice-punctured $2 \mathbb{C P}^{2}$. This approach simplifies the calculation of $\underline{\delta}_{R}\left(K_{2 n, 1}\right)$ to calculating $\bar{\delta}_{R}\left(T_{2 n, 1-20 n}\right)$. For the computation of $\bar{\delta}_{R}\left(T_{2 n, 1-20 n}\right)$, we develop a theory of the $O(2)$-homotopy type of the Seiberg-Witten Floer spectrum, which we describe below.

We introduce a method to compute both the real and the $O(2)$-equivariant Seiberg-Witten Floer homotopy type for an almost-rational plumbed homology sphere. Our main tool is based on the Pin(2)-equivariant

[^1]lattice homotopy type, developed by Dai, Stoffregen, and Sasahira DSS23. Additionally, we develop an $O(2)-$ equivariant version of the lattice homotopy type. For a given negative-definite plumbing graph $\Gamma$, the associated plumbed 4-manifold is denoted by $W_{\Gamma}$, and its boundary is denoted by $Y_{\Gamma}$. If the plumbing graph $\Gamma$ is almostrational (abbreviated as $A R$, see Ném05, Definition 8.1]), then we say that $Y_{\Gamma}$ is an almost-rational plumbed homology sphere. The following theorem enables us to compute the invariants $\underline{\delta}_{R}, \delta_{R}$, and $\bar{\delta}_{R}$ for all torus knots.

Theorem 1.3. Let $K$ be a knot in $S^{3}$ and $\Sigma_{2}(K)$ be its double-branched cover. Suppose there is an almostrational plumbing graph $\Gamma$ with an involution $\tau$ such that the induced involution on $Y_{\Gamma}$ coincides with the deck transformation of $\Sigma_{2}(K)$. Moreover, assume that $\Gamma$ admits an almost I-invariant path $h^{4} \gamma$ that carries the lattice homology. Then there is an $O(2)$-equivariant map

$$
\mathcal{T}^{O(2)}: \mathcal{H}\left(\gamma, \mathfrak{s}_{0}\right) \rightarrow S W F\left(\Sigma_{2}(K) ; \mathfrak{s}_{0}\right)
$$

which is an $S^{1}$-equivariant homotopy equivalence with respect to a certain $O(2)$-action on $\mathcal{H}\left(\gamma, \mathfrak{s}_{0}\right)$. Here, $\mathfrak{s}_{0}$ denotes the unique self-conjugate $\operatorname{spin}^{c}$ structure on $\Sigma_{2}(K)$.

This can be applied to compute a 2-knot invariant from real Seiberg-Witten theory. In Miy23, Miyazawa defined the numerical invariant

$$
|\operatorname{deg}(S)| \in \mathbb{Z}_{\geq 0}
$$

for a smoothly embedded 2-knot $S$ in $S^{4}$ as the absolute value of the mapping degree of the $\{ \pm 1\}$-framed real Bauer-Furuta invariant. Furthermore, in Miy23, Proposition 4.25, Lemma 4.27, and Proposition 4.30], he provided the following formula:

$$
\begin{equation*}
\left|\operatorname{deg}\left(\tau_{(k, \alpha)}(K)\right)\right|=|\operatorname{deg}(K)| \tag{1}
\end{equation*}
$$

for a determinant one knot $K$ in $S^{3}$, where $\tau_{(k, \alpha)}(K)$ is the $\alpha$-roll $k$-twisted spun 2 -knot of $K$, and $|\operatorname{deg}(K)|$ is the absolute value of signed counting of $\{ \pm 1\}$-framed real Seiberg-Witten solutions on the double-branched cover of $K$ with respect to its unique spin structure. Since Theorem 1.3 enables us to give non-equivariant homotopy type of $S W F_{R}(K)$, combined with (1), we can give a general formula of $\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)$ as follows:

Corollary 1.4. If $K$ is a determinant one knot in $S^{3}$ satisfying the same assumptions as in Theorem 1.3 , then we have that

$$
\left|\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)\right|=|\operatorname{deg}(K)|=\left|\chi\left(\mathcal{H}\left(\gamma, \mathfrak{s}_{0}\right)^{I}\right)\right|=1
$$

for integers $k$ and $\alpha$ such that $\frac{k}{2}+\alpha$ is an odd integer.
We also consider the case when $K$ is a Montesinos knot.
Theorem 1.5. Let $\Gamma$ be a negative-definite almost-rational plumbing graph, and $K_{\Gamma}$ be the corresponding arborescent knot. If $\gamma$ is a path that carries the lattice homology of $(\Gamma, \mathfrak{s})$ for a $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ on the doublebranched covering space $\Sigma_{2}(K)$, then there is an $O(2)$-equivariant map

$$
\mathcal{T}^{O(2)}: \mathcal{H}(\gamma, \mathfrak{s}) \rightarrow S W F\left(\Sigma_{2}(K) ; \mathfrak{s}\right)
$$

which is an $S^{1}$-equivariantly homotopy equivalence, where the $I$-action on $\mathcal{H}(\Gamma, \mathfrak{s})$ is given by the complex conjugation.

Corollary 1.6. Let $k$ and $\alpha$ be integers so that $\frac{k}{2}+\alpha$ is odd. Under the same assumptions as in Theorem 1.5 , suppose that the lattice homology of $(\Gamma, \mathfrak{s})$ is expressed as a graded root $R$. Denote the sets of leaves and angles of $R$ by $L(R)$ and $A(R)^{5}$, respectively, and shift the grading (if necessary) so that all vertices of $R$ lie on even degrees. Additionally, we assume the determinant of $K$ is one. Then we have

$$
\left|\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)\right|=|\operatorname{deg}(K)|=\left|\sum_{v \in L(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}-\sum_{v \in A(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}\right|
$$

[^2]Originally, Miyazawa used the computation of Seiberg-Witten moduli spaces using the analytical result given in MOY97. Alternatively, Corollary 1.6 gives a combinatorial computation using the lattice homotopy type DSS23 for a certain class of twisted roll spun 2-knots.

We also consider an invariant of a 2 -knot or a $\mathbb{R P}^{2}$-knot $S$ in $S^{4}$. For simplicity, we assume the doublebranched cover of $S$ is homology $\overline{\mathbb{C P}}^{2}$ in this paper, in order to consider a canonical spin ${ }^{c}$ structure up to sign, whose first Chern class is a generator of $H_{2}\left(\overline{\mathbb{C P}}^{2} ; \mathbb{Z}\right)$. For the strongest invariant in the real setting for a 2-knot or a $\mathbb{R P}^{2}$-knot $S$ in $S^{4}$, we have the $O(2)$-equivariant Bauer-Furuta invariants ${ }^{6}$

$$
B F_{S}: V^{+} \rightarrow V^{+},
$$

which were introduced in BH 24 b , where $V$ denotes an $O(2)$-representation space and + denotes the one-point compactification. If we consider the $\langle I\rangle \subset O(2)$ fixed point part of $B F_{S}$, we can recover the Miyazawa's degree invariant. We give some structural theorem for $O(2)$-Bauer-Furuta invariant.
Theorem 1.7. For any 2-knot or $\mathbb{R P}^{2}$-knot in $S^{4}$, the $O(2)$-equivariant Bauer-Furuta invariant of it is $O(2)$ stably homotopic to $\pm$ identity up to the coordinate changes of the domain $7^{7}$ if Miyazawa's degree invariant is one.

A similar structural theorem for $S^{1} \times \mathbb{Z}_{p}$-equivariant Bauer-Furuta invariants for 2-knots introduced in BH24 is also proved in IT24, Theorem 1.18], based on a similar technique.

Remark 1.8. As a refinement of Theorem 1.7, one can also observe the following: For a given pair of 2-knots or $\mathbb{R P}^{2}$-knots, suppose that Miyazawa's degree invariants of them are the same, then the corresponding $O(2)$ -Bauer-Furuta invariants are $O(2)$-equivariantly stably homotopic up to sign and coordinate change. Note that we can also define the $O(2)$-stable homotopy class of real Bauer-Furuta invariants even for orientable surfaces in $S^{4}$ by considering their double-branched covers with invariant spin structures with respect to the covering involutions. However, a similar technique proves that if the genus is positive, then the $O(2)$-stable homotopy class of the Bauer-Furuta invariant does not depend on the embeddings.

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## 2. Some topological facts

2.1. Concordance to torus knots. In $\mathrm{ACM}^{+} 23$ (see also $\mathrm{Bal22}$ ), it was observed that the 0 -framed figureeight knot can be transformed into a -10 -framed unknot by performing two full negative twists, as described in Figure 1. This observation provided a new proof that the ( 2,1 )-cable of the figure-eight knot is not smoothly slice in $B^{4}$. Furthermore, it will be crucially used in this article.

The 1-framed red circles in Figure 1 link the 0 -framed figure-eight knot, one linking algebraically once and the other algebraically three times, respectively. This implies that there is a concordance $S$ in

$$
X:=2 \mathbb{C P}^{2} \backslash\left(\stackrel{\circ}{B}^{4} \sqcup \stackrel{\circ}{B}^{4}\right) \cong 2 \mathbb{C P}^{2} \#\left(S^{3} \times I\right)
$$

from the figure-eight knot to the unknot, such that $S$ represents the homology class $(1,3)$ in $H_{2}(X, \partial X ; \mathbb{Z}) \cong$ $H_{2}\left(2 \mathbb{C P}^{2} ; \mathbb{Z}\right)=\mathbb{Z} \oplus \mathbb{Z}$. Due to the framing change, applying a cabling operation along the annulus results in a new concordance $S_{2 n}$ in $X$ from the $(2 n, 1)$-cable of the figure-eight knot to the $(2 n, 1-20 n)$-cable of the unknot, namely the $T_{2 n, 1-20 n}$ torus knot. Moreover, $S_{2 n}$ represents the homology class $(2 n, 6 n)$ in $H_{2}(X, \partial X ; \mathbb{Z})$. For this, we only needed the fact that the figure-eight knot can be converted into a slice knot by introducing full negative twists along two disjoint disks, one intersecting $K$ algebraically once and the other intersecting it algebraically three times. We record this as a proposition:

[^3]Proposition 2.1. Let $K$ be a knot, such as the figure-eight knot, which can be transformed into a slice knot by applying full negative twists along two disjoint disks-one that intersects $K$ algebraically once and another that intersects it algebraically three times. Then, for each positive integer $n$, there is a smooth concordance $S_{2 n}$ in the twice-punctured $2 \mathbb{C P}^{2}$, denoted by $X$, from $K_{2 n, 1}$ to $T_{2 n, 1-20 n}$. Moreover, $S_{2 n}$ represents the homology class $(2 n, 6 n)$ in $H_{2}(X, \partial X ; \mathbb{Z})$.


Figure 1. The 0-framed figure-eight knot becomes the -10 -framed unknot after two full negative twists.
2.2. Topological invariants for torus knots. The signature of a positive torus knot $T_{p, q}$ is computed via the following recursive formulae GLM81, Theorem 5.2]; note that we are using the convention where positive torus knots have negative signature. When $2 q<n$, we have

$$
\sigma\left(T_{n, q}\right)= \begin{cases}\sigma\left(T_{n-2 q, q}\right)-q^{2}+1 & \text { if } q \text { is odd } \\ \sigma\left(T_{n-2 q, q}\right)-q^{2} & \text { if } q \text { is even }\end{cases}
$$

When $q \leq n<2 q$, we have

$$
\sigma\left(T_{n, q}\right)= \begin{cases}-\sigma\left(T_{2 q-n, q}\right)-q^{2}+1 & \text { if } q \text { is odd } \\ -\sigma\left(T_{2 q-n, q}\right)-q^{2}+2 & \text { if } q \text { is even }\end{cases}
$$

Using this formula, we compute the signature of $T_{2 n, 1-20 n}$ as follows.

$$
\begin{aligned}
\sigma\left(T_{2 n, 1-20 n}\right) & =-\sigma\left(T_{20 n-1,2 n}\right) \\
& =-\sigma\left(T_{4 n-1,2 n}\right)+16 n^{2} \\
& =-2+20 n^{2}+\sigma\left(T_{1,2 n}\right)=-2+20 n^{2}
\end{aligned}
$$

Now we compute the Neumann-Siebenmann invariant $\bar{\mu}$ Neu80, Sie80 of the double-branched cover of $S^{3}$ along $T_{2 n, 1-20 n}$, which is the rational Brieskorn sphere $\Sigma(2,2 n, 1-20 n)$ with respect to its unique spin structure. Since $\bar{\mu}$ satisfies $\bar{\mu}(-Y)=-\bar{\mu}(Y)$, we will instead compute $\bar{\mu}(\Sigma(2,2 n, 20 n-1))$.

To compute it, we follow [NR78]. We first represent $\Sigma(2,2 n, 20 n-1)$ as the boundary of a plumbed 4manifold. One can do this by first representing it as a Seifert manifold and then translating each singular fiber as a leg in a star-shaped plumbing graph. To do so, we first write down the circle action on $\Sigma(2,2 n, 20 n-1)$, which is given as:

$$
t \cdot\left(z_{1}, z_{2}, z_{3}\right)=\left(t^{n(20 n-1)} z_{1}, t^{20 n-1} z_{2}, t^{2 n} z_{3}\right)
$$

It is then clear that, when $n>1$, the above action has three singular orbits, with Seifert coefficients given by $(20 n-1,-20 n+11),(20 n-1,-20 n+11)$, and $(n,-1)$, respectively. Note that the $(n,-1)$ orbit becomes nonsingular when $n=1$, resulting in only two singular orbits.

Now, we can draw a plumbing graph $\Gamma$ such that the boundary of the 4 -manifold obtained by plumbing disk bundles corresponding to $\Gamma$ is $\Sigma(2,2 n, 20 n-1)$. Recall that a singular orbit of Seifert coefficient $(p, q)$
contributes to a leg of the form $\left[a_{1}, \ldots, a_{n}\right]$ in the resulting star-shaped plumbing graph, where $a_{1}, \ldots, a_{n}$ satisfy $a_{i} \leq-2$ and are uniquely determined by the continued fraction expansion

$$
\frac{p}{q}=a_{1}-\frac{1}{a_{2}-\frac{1}{\ddots-\frac{1}{a_{n}}}}
$$

of $p / q$. When $n>1$, we obtain a graph with three legs, given by

$$
[\underbrace{-2, \ldots,-2}_{2 n-2},-3, \underbrace{-2, \ldots,-2}_{8}], \quad[\underbrace{-2, \ldots,-2}_{2 n-2},-3, \underbrace{-2, \ldots,-2}_{8}], \quad \text { and } \quad[-n] .
$$

Moreover, the central vertex has a coefficient of -2 . Hence the plumbing graph is given as follows.


On the other hand, when $n=1$, we only have two identical legs, given by

$$
[-3, \underbrace{-2, \ldots,-2}_{8}]
$$

and the central vertex has coefficient -1 . Hence, in this case, the plumbing graph is given as follows.


Given these plumbing graphs, it is now easy to compute $\bar{\mu}$ using the formula

$$
\bar{\mu}(Y)=\frac{\sigma(\Gamma)-w^{2}}{8}
$$

where $\Gamma$ is a plumbing graph for $Y, \sigma(\Gamma)$ is the signature of $\Gamma$, and $w$ is the spherical Wu class. When $n$ is odd and $n>1$, using the plumbing graph from above, we find that $\sigma(\Gamma)=-4 n-16$, and $w$ satisfies $w^{2}=-4 n+2$. For $n=1$, we have $\sigma(\Gamma)=-18$ and $w=0$. Therefore, we find:

$$
\bar{\mu}(\Sigma(2,2 n, 20 n-1))=-\frac{18}{8}
$$

for each positive odd integer $n$.
Remark 2.2. We do a brief sanity check here in the case $n=1$. Since $T_{2,19}$ is a 2 -bridge knot, $\Sigma(2,2,19)$ is a lens space, so we should have

$$
\bar{\mu}(\Sigma(2,2,19))=\frac{\sigma\left(T_{2,19}\right)}{8}
$$

Since $T_{2,19}$ has signature -18 , we see that our computation is correct for $n=1$.

## 3. Review of the $\delta_{R}$ INVARIANT AND THE CASE $n=1$

3.1. Category of spectrums for $S W F_{R}(K)$. In this section, we introduce a category $\mathfrak{C}_{G}$ that contains the real stable equivariant Floer homotopy type $S W F_{R}(K)$ for a knot $K$ in $S^{3}$. For a finite-dimensional vector space $V$, let $V^{+}$be the one-point compactification of $V$. We define the group $G$ to be the cyclic group of order 4 generated by $j \in \operatorname{Pin}(2)$, i.e.,

$$
G=\mathbb{Z}_{4}=\{1, j,-1,-j\} \text { with a subgroup } H=\mathbb{Z}_{2}=\{1,-1\} \subset G
$$

We will use the following representations of $G$ :

- $\tilde{\mathbb{R}}$ : the 1-dimensional real representation space of $G$ defined by the surjection $G \rightarrow \mathbb{Z}_{2}=\{1,-1\}$ with $j \mapsto-1$,
- $\mathbb{C}$ : the complex 1-dimensional representation defined by assigning $j \in G$ to $i$ in $\mathbb{C}$.

As representations for the suspensions, we shall only use subspaces of $\mathcal{V}=\oplus_{\mathbb{N}} \tilde{\mathbb{R}}$ and $\mathcal{W}=\oplus_{\mathbb{N}} \mathbb{C}$. A pointed finite $G$-CW complex $X$ is called a space of type $(G, H)-S W F$, if $X^{H}$ is $G$-homotopy equivalent to $V^{+}$, where $V$ is a finite dimensional subspace of $\mathcal{V}$, and $H$ acts freely on $X \backslash X^{H}$. The dimension $\operatorname{dim} V$ is called the level of $X$.

Now we introduce the category $\mathfrak{C}_{G}$ whose object is the equivalence classes of $(X, m, n)$ up to $G$-stably equivalence, where $X$ is a space of type $(G, H)$-SWF, $m \in \mathbb{Z}$, and $n \in \mathbb{Q}$. We say that $(X, m, n)$ and $\left(X^{\prime}, m^{\prime}, n^{\prime}\right)$ are $G$-stably equivalent if $n-n^{\prime} \in \mathbb{Z}$ and there exist finite dimensional subspaces $V, V^{\prime} \subset \mathcal{V}$ and $W, W^{\prime} \subset \mathcal{W}$ and a pointed $G$-homotopy equivalence

$$
\Sigma^{V} \Sigma^{W} X \rightarrow \Sigma^{V^{\prime}} \Sigma^{W^{\prime}} X^{\prime}
$$

where $\operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{R}} V^{\prime}=m^{\prime}-m$ and $\operatorname{dim}_{\mathbb{C}} W-\operatorname{dim}_{\mathbb{C}} W^{\prime}=n^{\prime}-n$.
Informally, we may think of the triple $(X, m, n)$ as the formal desuspension of $X$ by $V$ and $W$, where $V \subset \mathcal{V}$ and $W \subset \mathcal{W}$ with $\operatorname{dim} V=m$ and $\operatorname{dim} W=n$. So, symbolically one may write

$$
(X, m, n)=\Sigma^{-m \tilde{\mathbb{R}}} \Sigma^{-n \mathbb{C}} X
$$

Let $(X, m, n)$ and $\left(X^{\prime}, m^{\prime}, n^{\prime}\right)$ be triples as above. A $G$-stable map $(X, m, n) \rightarrow\left(X^{\prime}, m^{\prime}, n^{\prime}\right)$ is called a $G$-local map, if $\operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{R}} V^{\prime}=m^{\prime}-m$ and it induces a $G$-homotopy equivalence on the $H$-fixed-point sets. We say that $(X, m, n)$ and $\left(X^{\prime}, m^{\prime}, n^{\prime}\right)$ are $G$-locally equivalent if there exist $G$-local maps $(X, m, n) \rightarrow\left(X^{\prime}, m^{\prime}, n^{\prime}\right)$ and $\left(X^{\prime}, m^{\prime}, n^{\prime}\right) \rightarrow(X, m, n)$. The invariants $\delta_{R} \bar{\delta}_{R}$ and $\underline{\delta}_{R}$ are invariant under $G=\mathbb{Z}_{4}$ local equivalence. By considering the action comes from the inclusion $\mathbb{Z}_{2} \rightarrow \mathbb{Z}_{4}$, we have the corresponding representations:

- the trivial representation $\mathbb{R}$
- the non-trivial real representation $\tilde{\mathbb{R}}$.

With these representations, we also define $\mathfrak{C}_{\mathbb{Z}_{2}}$.
3.2. The real Frøyshov invariants. In this subsection, we review the construction of the real Frøyshov invariants. In KMT23b, the three invariants

$$
\delta_{R}(K), \bar{\delta}_{R}(K), \text { and } \underline{\delta}_{R}(K) \in \frac{1}{16} \mathbb{Z}
$$

are introduced for a knot $K$ in $S^{3}$. In fact, it is defined for any oriented link in $S^{3}$ with non-zero determinant. In the case of the knot, the invariants are independent of the choice of orientations. These invariants are derived from $\mathbb{Z}_{4}$-equivariant stable homotopy type

$$
S W F_{R}(K)
$$

Let $\mathfrak{s}_{0}$ be the unique spin structure on the double-branched cover $\Sigma_{2}(K), \tau: \Sigma_{2}(K) \rightarrow \Sigma_{2}(K)$ be the deck transformation, and $P$ be the principal Spin(3) bundle for $\mathfrak{s}_{0}$. Since the fixed point set is codimension 2, we can take an order 4 lift $\widetilde{\tau}: P \rightarrow P$ of the induced map

$$
\tau_{*}: S O\left(T \Sigma_{2}(K)\right) \rightarrow S O\left(T \Sigma_{2}(K)\right)
$$

where $S O\left(T \Sigma_{2}(K)\right)$ is the orthonormal framed bundle of $\Sigma_{2}(K)$ with respect to a fixed invariant metric $g$ on $\Sigma_{2}(K)$. Then, we have the infinite-dimensional functional

$$
C S D: \mathcal{C}_{K}:=\left(i \operatorname{Ker} d^{*} \subset i \Omega_{\Sigma_{2}(K)}^{1}\right) \oplus \Gamma(\mathbb{S}) \rightarrow \mathbb{R}
$$

called the Chern-Simons Dirac functional, where $\mathbb{S}$ is the spinor bundle with respect to $\mathfrak{s}_{0}$ and $\Gamma(\mathbb{S})$ denotes the set of sections of $\mathbb{S}$. The Seiberg-Witten Floer homotopy type is defined as the Conley index of the finitedimensional approximation of the formal gradient flow of $C S D$. For that purpose, we describe the formal gradient of $C S D$ as the sum $l+c$, where $l$ is a self-adjoint elliptic part and $c$ is a compact map. Then, we decompose $\mathcal{C}_{K}$ into eigenspaces of $l$. Define $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$ to be the direct sums of the eigenspaces of $l$ whose eigenvalues are in $(-\lambda, \lambda]$ and restrict the formal gradient flow $l+c$ to $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$, where $V_{-\lambda}^{\lambda}(K)$ is the eigenspace corresponding to the space of 1-forms and $W_{-\lambda}^{\lambda}(K)$ is the eigenspace corresponding to spinors.

Then by considering the Conley index $(N, L)$ for $\left(V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K), l+p_{-\lambda}^{\lambda} c\right)$ with a certain cutting off, we get the Manolescu's Seiberg-Witten Floer homotopy type

$$
\begin{equation*}
S W F\left(\Sigma_{2}(K), \mathfrak{s}_{0}\right):=\Sigma^{\left.-V_{-\lambda}^{0} \oplus W_{-\lambda}^{0}-n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right)\right) \mathbb{C}} N / L, \tag{2}
\end{equation*}
$$

where $n\left(Y, \mathfrak{s}_{0}, g\right)$ is the quantity given in Man03 and $g$ is a Riemann metric on $\Sigma_{2}(K)$. For the meaning of desuspensions and how to formulate a well-defined homotopy type in a certain category, see Man03. For the latter purpose, we take $g$ as $\mathbb{Z}_{2}$-invariant metric. Since we are working with the spin structure $\mathfrak{s}_{0}$, we have an additional $\operatorname{Pin}(2)$-action on the configuration space $\mathcal{C}_{K}$ which preserves the values of $C S D$. Now, we define an involution on $\mathcal{C}_{K}$

$$
I:=j \circ \widetilde{\tau}
$$

where $j$ is the quaternionic element in $\operatorname{Pin}(2)=S^{1} \cup j \cdot S^{1} \cdot{ }^{8}$
Since $I$ also acts on $S$ anti-complex linearly, the lift $I$ is called a real structure on $\mathfrak{s}_{0}$. Combined with $S^{1}$-action, we can take Conley index so that we have an $O(2)$-action on $S W F\left(\Sigma_{2}(K), \mathfrak{s}_{0}\right)$.

Now, we define

$$
\begin{aligned}
S W F_{R}(K): & =\Sigma^{-\left(V_{-\lambda}^{0} \oplus W_{-\lambda}^{0}\right)^{I}-\frac{1}{2} n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right) \mathbb{C}_{1}} N^{I} / L^{I} \\
& =\left[\left(N^{I} / L^{I}, \operatorname{dim}_{\mathbb{R}}\left(V_{-\lambda}^{0}\right)^{I}, \operatorname{dim}_{\mathbb{C}}\left(W_{-\lambda}^{0}\right)^{I}+n(Y, \mathfrak{t}, g) / 2\right)\right] \in \mathfrak{C}_{G}
\end{aligned}
$$

which we call the real Seiberg-Witten Floer homotopy type for $K$. Here, we take an $O(2)$-invariant index pair $(N, L)$ for the flow $\left(V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K), l+p_{-\lambda}^{\lambda} c\right)$ with a certain cutting off. Since the action of $j$ commutes with $I$, we have a $\mathbb{Z}_{4}$-action on the stable homotopy types $S W F_{R}(K)$. Therefore, we have the following two equivariant cohomologies:

$$
\widetilde{H}_{G}^{*}\left(S W F_{R}(K) ; \mathbb{F}_{2}\right):=\widetilde{H}_{G}^{*+\operatorname{dim}\left(V_{-\lambda}^{0}\right)^{I}+2 \operatorname{dim}_{C}\left(W_{-\lambda}^{0}\right)^{I}+n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right)}\left(N^{I} / L^{I} ; \mathbb{F}_{2}\right)
$$

for $G=\mathbb{Z}_{2}$ or $\mathbb{Z}_{4}$. If we write $H^{*}\left(B \mathbb{Z}_{2}\right) \cong \mathbb{F}_{2}[W]$, we define
$\delta_{R}(K):=\frac{1}{2}\left(\min \left\{m \in \mathbb{Z} \mid x \in H_{\mathbb{Z}_{2}}^{m}\left(N^{I} / L^{I} ; \mathbb{F}_{2}\right), W^{k} x \neq 0, \forall k\right\}-\operatorname{dim}\left(V_{-\lambda}^{0}\right)^{I}-2 \operatorname{dim}_{\mathbb{C}}\left(W_{-\lambda}^{0}\right)^{I}-n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right)\right)$.
Similarly, if we put

$$
\widetilde{H}_{\mathbb{Z}_{4}}^{*}\left(S^{0}\right) \cong \mathbb{Z}_{2}[U, Q] /\left(Q^{2}=0\right)
$$

we can write the definitions of $\underline{\delta}_{R}$ and $\bar{\delta}_{R}$ as

$$
\begin{aligned}
\underline{\delta}_{R}(K):= & \frac{1}{2}\left(\min \left\{m \in \mathbb{Z} \mid x \in H_{\mathbb{Z}_{4}}^{m}\left(N^{I} / L^{I} ; \mathbb{F}_{2}\right), U^{k} x \neq 0, \forall k, m \equiv \operatorname{dim}\left(N^{I} / L^{I}\right)^{\mathbb{Z}_{2}} \bmod 2\right\}\right. \\
& \left.-\operatorname{dim}\left(V_{-\lambda}^{0}\right)^{I}-2 \operatorname{dim}_{\mathbb{C}}\left(W_{-\lambda}^{0}\right)^{I}-n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right)\right) \\
\bar{\delta}_{R}(K):= & \frac{1}{2}\left(\min \left\{m \in \mathbb{Z} \mid x \in H_{\mathbb{Z}_{4}}^{m}\left(N^{I} / L^{I} ; \mathbb{F}_{2}\right), U^{k} x \neq 0, \forall k, m \equiv \operatorname{dim}\left(N^{I} / L^{I}\right)^{\mathbb{Z}_{2}}+1 \bmod 2\right\}\right. \\
& \left.-\operatorname{dim}\left(V_{-\lambda}^{0}\right)^{I}-2 \operatorname{dim}_{\mathbb{C}}\left(W_{-\lambda}^{0}\right)^{I}-n\left(\Sigma_{2}(K), \mathfrak{s}_{0}, g\right)\right)-\frac{1}{2} .
\end{aligned}
$$

3.3. $O(2)$-equivariant cobordism map. In order to calculate the real Seiberg-Witten Floer homotopy type, $S W F_{R}(K)$, we will construct an $O(2)$-equivariant map. This map is obtained as the $O(2)$-equivariant BauerFuruta invariant for the branched covers and the homotopies between them. We review the construction of the $O(2)$-equivariant Bauer-Furuta invariant in this section.

Let $\left(Y_{0}, \mathfrak{t}_{0}\right)$ and $\left(Y_{1}, \mathfrak{t}_{1}\right)$ be $\operatorname{spin}^{c}$ rational homology 3 -spheres with odd involutions $\tau_{i}: Y_{i} \rightarrow Y_{i}$, i.e., an involution $\tau_{i}$ such that

$$
\tau_{i}^{*} \mathfrak{t}_{i} \cong \overline{\mathfrak{t}}_{i}
$$

for each $i=0,1$. A typical situation involves $Y_{0}$ and $Y_{1}$ as the double-branched covers of knots $K$ and $K^{\prime}$, each with unique spin structures $\mathfrak{t}_{0}$ and $\mathfrak{t}_{1}$, respectively. Let $(W, \mathfrak{s})$ be a smooth spin 4-dimensional oriented cobordism from $Y_{0}$ to $Y_{1}$ with $b_{1}(W)=0$. We assume that there is an odd involution $\tau$ on $W$ such that $\left.\tau\right|_{Y_{i}}=\tau_{i}$ for each $i$, i.e., an involution $\tau$ such that

$$
\tau^{*} \mathfrak{s} \cong \overline{\mathfrak{s}}
$$

[^4]and the fixed point set of $\tau$ is of codimension 2. Again, a typical situation is when $W$ is obtained as the double-branched cover along a smoothly embedded surface in a 4-manifold. Let $\mathbb{S}^{ \pm}$be positive and negative spinor bundles on $W$, and let $\mathbb{S}_{i}$ be the spinor bundles on $Y_{i}$ for each $i$. In KMT23b, Section 2], an antilinear lift $I$ on the spinor bundles $\mathbb{S}^{ \pm}, \mathbb{S}_{i}$, and the configuration spaces are constructed. Note that such a choice (of $I)$ corresponds to a choice of splittings of
$$
1 \rightarrow S^{1} \rightarrow G_{\mathfrak{s}} \rightarrow \mathbb{Z}_{2} \rightarrow 1
$$
as it is pointed out in BH24b, Subsection 2.1], where $G_{\mathfrak{s}}$ denotes a certain bundle map of the spinor bundle $\mathbb{S}$ on $W$ which covers $\tau$. We fix a splitting when we consider $O(2)$-equivariant Bauer-Furuta invariant. ${ }^{9}$

In this setting, Konno, Miyazawa, and the third author KMT23b, Section 3.7] (see also BH24b, Section 2.1]) constructed an $I$-equivariant Bauer-Furuta map, which is formally written as

$$
\begin{equation*}
B F_{W, \mathfrak{s}}:\left(\mathbb{C}^{\frac{1}{8}\left(c_{1}(\mathfrak{s})^{2}-\sigma(W)\right)}\right)^{+} \wedge S W F\left(Y_{0}, \mathfrak{t}\right) \rightarrow\left(\mathbb{R}^{b_{2}^{+}(W)}\right)^{+} \wedge S W F\left(Y_{1}, \mathfrak{t}\right) \tag{3}
\end{equation*}
$$

for the 4-manifold $W$ up to stable homotopy, with a certain $I$-action on $\mathbb{C}^{\frac{1}{8}\left(c_{1}(\mathfrak{s})^{2}-\sigma(W)\right)}$ and $\mathbb{R}^{b_{2}^{+}(W)}$.
In this paper, we mainly focus on the case of $b_{2}^{+}(W)=0$ in the construction of a map between $O(2)$-lattice homotopy type and the Seiberg-Witten Floer homotopy type. Note that if we forget the $I$-action, $B F_{W, \mathfrak{s}}$ recovers the usual $S^{1}$-equivariant Bauer-Furuta invariant. Combined it with the $S^{1}$-action, one can see the map $B F_{W, \mathfrak{s}}$ is $O(2)$-equivariant since $I$ and $i \in S^{1}$ anticommute. From the construction combined with Lemma A.1, the $O(2)$-representations that appeared in this setting are the following:

- the trivial 1-dimensional real representation $\mathbb{R}$,
- the non-trivial 1-dimensional real representation $\widetilde{\mathbb{R}}$ obtained via the surjection

$$
O(2) \rightarrow O(1)=\mathbb{Z}_{2}
$$

- the irreducible 2-dimensional representation $\mathbb{C}$, with the natural action of $O(2) \cong S^{1} \rtimes \mathbb{Z}_{2}$, where $S^{1}$ acts as complex multiplication and $\mathbb{Z}_{2}$ acts as complex conjugation.
Therefore, the universe in this setting is written as

$$
\mathcal{U}=\mathbb{R}^{\infty} \oplus \widetilde{\mathbb{R}}^{\infty} \oplus \mathbb{C}^{\infty}
$$

In order to state $B F_{W, \mathfrak{s}}$ is well-defined in a certain $O(2)$-equivariant stable homotopy categeroy, we need to write the dependence of $S W F\left(Y_{0}, \mathfrak{t}\right)$ for the $\mathbb{Z}_{2}$-invariant Riemann metrics. We only need to care about the spectral flows coming from Dirac operators since we are considering rational homology 3 -spheres. However, the $O(2)$-equivariant version of the spectral flows is the same as the usual $S^{1}$-equivariant spectral flows since $O(2)$-actions are actually determined by $S^{1}$-actions from Lemma A.1. Thus, by just using the same formulation as Manolescu did in Man03, Section 6], we can use $n(Y, \mathfrak{t}, g)$ for the data of desuspensions to get well-defined $O(2)$-equivariant Seiberg-Witten Floer homotopy type, i.e., we replace the standard $S^{1}$-representation $\mathbb{C}$ with the standard $O(2)$-representation $\mathbb{C}$. More precisely, we consider the following category:

Also, the corresponding $O(2)$-equivariant stable homotopy category $\mathfrak{C}_{O(2)}$ is given as follows:

- The set of objects consists of tuples $(W, l, m, n)$, where $W$ is a pointed $O(2)$-space, $l, m \in \mathbb{Z}$, and $n \in \mathbb{Q}$.
- For two objects $\left(W_{0}, l_{0}, m_{0}, n_{0}\right),\left(W_{1}, l_{1}, m_{1}, n_{1}\right)$, the set of morphisms is given by

$$
\lim _{p_{0}, p_{1}, q \rightarrow \infty}\left[\Sigma^{p_{0} \mathbb{R} \oplus p_{1} \widetilde{\mathbb{R}} \oplus q \mathbb{C}} W_{0}, \Sigma^{\left(p_{0}+l_{0}-l_{1}\right) \mathbb{R} \oplus\left(p_{1}+m_{0}-m_{1}\right) \widetilde{\mathbb{R}} \oplus\left(q+n_{0}-n_{1}\right) \mathbb{C}} W_{1}\right]_{O(2)}^{0}
$$

if $n_{0}-n_{1} \in \mathbb{Z}$, where $[X, Y]_{O(2)}^{0}$ denotes the set of $O(2)$-equivariant maps up to $O(2)$-equivariant based homotopy.
As in the case of $S^{1}$, we define the (de)suspension by

$$
\Sigma^{V}(W, l, n, m):=\left(\Sigma^{V^{S^{1}}} W, l+2 a, n+2 b, m+c\right)
$$

when $V$ has some trivialization $\mathbb{R}^{a} \oplus \widetilde{\mathbb{R}}^{b} \oplus \mathbb{C}^{c}$. Then, from Lemma A.1, one can see the $O(2)$-homotopy type (2) is well-defined as an object in $\mathfrak{C}_{O(2)}$ and one can see the $O(2)$-equivariant Bauer-Furuta invariant (3) defines

[^5]a morphism in $\mathfrak{C}_{O(2)}$. Consequently, we can define $O(2)$-equivariant Bauer-Furuta invariants of 2-knots or $\mathbb{R} \mathbb{P}^{2}$-knots as introduced earlier.

Suppose $Y_{0}$ and $Y_{1}$ are the double-branched covers along knots $K$ and $K^{\prime}$, each with unique spin structures $\mathfrak{t}_{0}$ and $\mathfrak{t}_{1}$, respectively. Assume $W$ is obtained as the double-branched cover along a surface cobordism $S$ from $K$ to $K^{\prime}$ properly and smoothly embedded in a 4 -dimensional cobordism from $S^{3}$ to $S^{3}$. If we consider the $I$-invariant part of $B F_{W, \mathfrak{s}}$, we obtain a cobordism map in real Seiberg-Witten theory:

$$
B F_{S, \mathfrak{s}}:=B F_{W, \mathfrak{s}}^{I}
$$

which is used to prove Frøyshov type inequalities in KMT23b. Taking the fixed point part can be understood as the functor

$$
\mathfrak{C}_{O(2)} \rightarrow \mathfrak{C}_{\mathbb{Z}_{2}}
$$

obtained by taking $\langle I\rangle$-fixed point part of the spectrums and stable morphisms.
3.4. The case $n=1$ : a toy model. We now offer an alternative proof of the main theorem for the case of $n=1$, previously established using Heegaard Floer theory in $\mathrm{DKM}^{+} 22$ and minimal genus functions in $\mathrm{ACM}^{+} 23$. This proof serves as a useful toy model for the case of general odd $n>1$.

Let $K$ be a knot in $S^{3}$, and let $\Sigma_{2}(K)$ denote its double-branched cover. Recall from KMT23b, Proposition 1.10] that when $\Sigma_{2}(K)$ is a lens space, we have

$$
\delta_{R}(K)=\underline{\delta}_{R}(K)=\bar{\delta}_{R}(K)=-\frac{\sigma(K)}{16}
$$

where $\sigma(K)$ is the signature of $K$. Given that the torus knot $T_{2,-19}$ is a two-bridge knot, and thus its doublebranched cover is the lens space $\Sigma(2,2,-19)=L(19,1)$, we deduce

$$
\begin{equation*}
\bar{\delta}_{R}\left(T_{2,-19}\right)=-\frac{\sigma\left(T_{2,-19}\right)}{16}=-\frac{9}{8} . \tag{4}
\end{equation*}
$$

Now, we invoke the following theorem.
Theorem 3.1 (KMT23b, Theorem 1.6]). Let $K$ and $K^{\prime}$ be knots in $S^{3}$, let $X$ be an oriented, smooth, compact, connected 4-manifold cobordism from $S^{3}$ to $S^{3}$ with $H_{1}(X ; \mathbb{Z})=0$, and let $S$ be a connected surface cobordism that is smoothly embedded in $X$ from $K$ to $K^{\prime}$, such that the homology class $[S] / 2$ in $H_{2}(X, \partial X ; \mathbb{Z})$ reduces to $w_{2}(X)$. Let $\Sigma_{2}(S)$ be the double-branched cover of $X$ branched along $S$ and $\sigma\left(\Sigma_{2}(S)\right)$ be its signature.

If $b_{2}^{+}\left(\Sigma_{2}(S)\right)-b_{2}^{+}(X)=1$, then we have

$$
\underline{\delta}_{R}(K)-\frac{1}{16} \sigma\left(\Sigma_{2}(S)\right) \leq \bar{\delta}_{R}\left(K^{\prime}\right)
$$

If $b_{2}^{+}\left(\Sigma_{2}(S)\right)-b_{2}^{+}(X)=0$, then the following stronger inequality holds:

$$
\underline{\delta}_{R}(K)-\frac{1}{16} \sigma\left(\Sigma_{2}(S)\right) \leq \underline{\delta}_{R}\left(K^{\prime}\right)
$$

The latter part is a stronger conclusion since we have $\underline{\delta}_{R}(K) \leq \delta_{R}(K) \leq \bar{\delta}_{R}(K)$ for each knot $K$.
Remark 3.2. The following can be computed using the Mayer-Vietoris sequence and the $G$-signature theorem (see KMT23b, Lemma 4.5]). Suppose that $S$ is an annulus; then, we have

$$
\begin{aligned}
b_{2}^{+}\left(\Sigma_{2}(S)\right)-b_{2}^{+}(X) & =b_{2}^{+}(X)-\frac{1}{4}[S]^{2}-\frac{1}{2} \sigma(K)+\frac{1}{2} \sigma\left(K^{\prime}\right) \\
\sigma\left(\Sigma_{2}(S)\right) & =2 \sigma(X)-\frac{1}{2}[S]^{2}-\sigma(K)+\sigma\left(K^{\prime}\right)
\end{aligned}
$$

We will use these to compute the quantities $b_{2}^{+}\left(\Sigma_{2}(S)\right)$ and $\sigma\left(\Sigma_{2}(S)\right)$.
We have the following immediate corollary of Theorem 3.1.
Corollary 3.3. Let $K$ be a knot with vanishing signature. Suppose $K$ bounds a normally immersed disk in $B^{4}$ with only negative double points. Then, we have $0 \leq \underline{\delta}_{R}(K)$.

Proof. If $K$ bounds a normally immersed disk in $B^{4}$ with $m$ negative double points, then there is a smooth concordance $S$ in twice-punctured $m \mathbb{C P}^{2}$, denoted by $X$, from the unknot to $K$. Moreover, $S$ represents $[S]=(2,2, \ldots, 2)$ in $H_{2}(X, \partial X ; \mathbb{Z})$. Moreover, by Remark 3.2, we have that $b_{2}^{+}\left(\Sigma_{2}(S)\right)-b_{2}^{+}(X)=0$ and $\sigma\left(\Sigma_{2}(S)\right)=0$. Then the conclusion follows from Theorem 3.1

Let $E$ be the figure-eight knot. Consider the smooth concordance $S$, as described in Proposition 2.1, from $E_{2,1}$ to $T_{2,-19}$ in a twice-punctured $2 \mathbb{C P}^{2}$, which is denoted by $X$. This concordance has the homology class $(2 n, 6 n)$. To check that the assumptions of Theorem 3.1 are satisfied for $S$, we calculate:

$$
\begin{aligned}
b_{2}^{+}\left(\Sigma_{2}(S)\right)-b_{2}^{+}(X) & =b_{2}^{+}(X)-\frac{1}{4}[S]^{2}+\frac{1}{2} \sigma\left(T_{2,-19}\right) \\
& =2-\frac{1}{4}\left(2^{2}+6^{2}\right)+\frac{1}{2}(18) \\
& =2-10+9 \\
& =1
\end{aligned}
$$

Hence the assumptions are satisfied, and thus we get

$$
\underline{\delta}_{R}\left(E_{2,1}\right)-\frac{1}{16}\left(2 \sigma(X)-\frac{1}{2}[S]^{2}+\sigma\left(T_{2,-19}\right)\right) \leq \bar{\delta}_{R}\left(T_{2,-19}\right)
$$

Since we have

$$
\begin{aligned}
-\frac{1}{16}\left(2 \sigma(X)-\frac{1}{2}[S]^{2}+\sigma\left(T_{2,-19}\right)\right) & =-\frac{1}{16}\left(2 \cdot 2-\frac{1}{2}\left(2^{2}+6^{2}\right)+18\right) \\
& =-\frac{1}{8}
\end{aligned}
$$

use (4) to conclude that

$$
\underline{\delta}_{R}\left(E_{2,1}\right) \leq-1
$$

Thus, by applying Corollary 3.3 , we conclude that $E_{2,1}$ does not bound a normally immersed disk in $B^{4}$ with only negative double points. In particular, it is not smoothly slice.
Remark 3.4. Consider the unique minimal genus Seifert surface $S$ for the figure-eight knot $E$. It consists of two bands, one with a full positive twist and the other with a full negative twist. Take two parallel copies of $S$ and denote them by $S^{+}$and $S^{-}$. Connecting them with a half-twisted band yields a Seifert surface $S^{\prime}$ for $E_{2,1}$. Perform a crossing change on $E_{2,1}$ that corresponds to undoing the full positive twist on $S^{+}$and a crossing change that corresponds to undoing the full negative twist on $S^{-}$. These crossing changes produce a new knot $R$ and a Seifert surface $S^{\prime \prime}$ derived from $S^{\prime}$ for $R$. Moreover, on $S^{\prime \prime}$, we have a two-component unlink $U_{1} \cup U_{2}$ such that the Seifert form restricted to the homology classes of the unlink vanishes (i.e., it forms a derivative link for $R$ ), which in particular implies that $R$ is a ribbon knot. In fact, one can check that $R$ is the ribbon knot 12 n 268 . Therefore, we conclude that $c_{4}^{+}\left(E_{2,1}\right)=1$.
3.5. Two technical lemmas. Before ending this section, we shall show the following lemmas, which will be used later. We say that a spectrum $X$ is a $\mathbb{Z}_{2}$-homology sphere if $\tilde{H}^{*}\left(X ; \mathbb{Z}_{2}\right) \cong \pi_{*}\left(X \wedge H \mathbb{Z}_{2}\right)$ is 1-dimensional over $\mathbb{Z}_{2}$.

Lemma 3.5. If $S W F_{R}(K)$ is a $\mathbb{Z}_{2}$-homology sphere, then we have

$$
\delta_{R}(K)=\underline{\delta}_{R}(K)=\bar{\delta}_{R}(K)
$$

Proof. Since Seiberg-Witten spectra are finite, we may assume for simplicity that it is actually a finite CWcomplex by stabilizing it many times. Then, since $S W F^{I}(K)$ is a $\mathbb{Z}_{2}$-homology sphere, then it is also a $\mathbb{Z}_{2}$-cohomology sphere. Consider the Serre spectral sequence

$$
E_{2}=\tilde{H}^{*}\left(S W F_{R}(K) ; \mathbb{Z}_{2}\right) \otimes_{\mathbb{Z}_{2}} H^{*}\left(B \mathbb{Z}_{4} ; \mathbb{Z}_{2}\right) \Rightarrow \tilde{H}_{\mathbb{Z}_{4}}^{*}\left(S W F_{R}(K) ; \mathbb{Z}_{2}\right)=E_{\infty}
$$

We already know that the $E_{2}$ page is free of rank 1 over $H^{*}\left(B \mathbb{Z}_{4} ; \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2}[U, Q] /\left(Q^{2}\right)$. On the other hand, it follows from discussions in KMT23b, Section 3] that the $E_{\infty}$ page, after localizing by formally inverting $U$,
is free of rank 1 over $\mathbb{Z}_{2}\left[U, U^{-1}, Q\right]$. Therefore we see that the spectral sequence collapses at the $E_{2}$ page, and hence we get

$$
\delta_{R}(K)=\underline{\delta}_{R}(K)=\bar{\delta}_{R}(K)
$$

as desired.

Lemma 3.6. Let $G$ be a finite 2-group and $X, Y$ be finite $G$ - $C W$-complexes. Suppose that there exists a homotopy equivalence $f: X \rightarrow Y$ which is $G$-equivariant; note that $f$ might not be a $G$-equivariant homotopy equivalence. Then the restriction of $f$ to $G$-fixed point loci, i.e.,

$$
f^{G}: X^{G} \rightarrow Y^{G}
$$

induces an isomorphism between $\mathbb{Z}_{2}$-coefficient singular homology.
Proof. Consider the commutative diagram

where $(-)^{h G}$ denotes the homotopy fixed point, i.e.,

$$
Z^{h G}=[E G, Z]^{G}
$$

and $(-)_{2}^{\wedge}$ denotes the Bousfield-Kan 2-adic completion. Note that for any $G$-space $Z$, we have a canonically defined 2 -adic completion map

$$
Z \rightarrow Z_{2}^{\wedge}
$$

and the (2-completed) comparison map

$$
\left(Z^{G}\right)_{2}^{\wedge} \rightarrow\left(Z_{2}^{\wedge}\right)^{h G}
$$

But 2-adic completion maps are mod 2 homotopy equivalences. Furthermore, for finite $G$-complexes, the 2-completed comparison map is a weak homotopy equivalence, due to the Sullivan conjecture DMN89, Car91, Lan92. By the mod $p$ Whitehead theorem Sch81, this is equivalent to saying that all horizontal maps in the diagram above, as well as $\left(f_{2}^{\wedge}\right)^{h G}$, induce isomorphisms between $\mathbb{Z}_{2}$-coefficient homology. Therefore $f^{G}$ also induces an isomorphism between $\mathbb{Z}_{2}$-coefficient homology.

Remark 3.7. Since completion is a stable operation, by replacing fixed points with geometric fixed points, we can easily see that Lemma 3.6 also applies to the case when $X$ and $Y$ are finite $G$-spectra.

## 4. Lattice homotopy type and the proof of Theorem 1.2

Our strategy utilizes the work of Dai, Sasahira, and Stoffregen DSS23 on the lattice homotopy computation of Floer homotopy type. For background materials regarding lattice homology, we refer the readers to OS03b, Ném05, Ném08 for the general theory, and DSS23 for the modernized constructions. We will mainly follow [DSS23] for notations in lattice homology. In this section, we shall construct $O(2)$-equivariant maps between the $O(2)$-lattice homotopy type and the $O(2)$-equivariant Seiberg-Witten Floer homotopy type in two different situations, which can be regarded as morphisms in $\mathcal{C}_{O(2)}$. Note that these stable equivariant morphisms are assumed to be based maps, but from tD87, Chapter II, Lemma (4.15)], there is no difference between based $O(2)$-equivariant maps and $O(2)$-equivariant maps between $O(2)$ - CW complexes $X$ and $Y$ if $\pi_{1}\left(X^{O(2)}\right)=$ $\pi_{1}\left(Y^{O(2)}\right)=0$. In our situation, both the $O(2)$-lattice homotopy type and the $O(2)$-Seiberg-Witten Floer homotopy type satisfy such a condition, we will not care about the based points in the construction.
4.1. Computation sequences in lattice homology. In this subsection, we will review the construction of computation sequences in lattice homology, following [Ném08], as a detailed understanding of it is crucial in understanding the construction of $j$-action on the Dai-Sasahira-Stoffregen lattice homotopy type. We will then modify it a little bit to constuct $I(=j \tau)$-lattice homotopy type in the later subsection.

Given an almost rational, negative-definite plumbing graph $\Gamma$, let $W_{\Gamma}$ denote the corresponding 4-manifold, $K$ its canonical divisor, and $k=K+2 l^{\prime}$ a characteristic element; here, $l^{\prime}$ represents an element of $H^{2}\left(W_{\Gamma} ; \mathbb{Z}\right)$, which is considered via the intersection form, as a subgroup of $H_{2}\left(W_{\Gamma} ; \mathbb{Q}\right)$. Denote the vertices of $\Gamma$ by $b_{1}, \ldots, b_{n}$ and consider the cone

$$
S_{\mathbb{Q}}=\left\{x \in H_{*}\left(W_{\Gamma} ; \mathbb{Q}\right) \mid\left(x, b_{i}\right) \leq 0 \text { for all } j\right\}
$$

It follows from the negative definiteness of $\Gamma$ that every element $x \in S_{\mathbb{Q}}$ satisfy $x \geq 0$, where $\geq$ denotes the partial ordering on $H_{*}\left(W_{\Gamma} ; \mathbb{Q}\right)$ defined by inequalities on coefficients of each vertex $b_{j}$ of $\Gamma$.

We find a minimal representative of $[k] \in \operatorname{Spin}^{c}\left(Y_{\Gamma}\right)$ as follows: Consider the intersection

$$
\left(l^{\prime}+H_{2}\left(W_{\Gamma} ; \mathbb{Z}\right)\right) \cap S_{\mathbb{Q}} .
$$

With respect to the partial ordering on $H_{2}\left(W_{\Gamma} ; \mathbb{Z}\right)$, this subset admits a unique minimal element $l_{[k]}^{\prime}$ Ném05, Lemma 5.4]. Thus we take the corresponding distinguished representative $k_{r}$ of $[k]$ as follows:

$$
k_{r}=K+2 l_{[k]}^{\prime}
$$

Now fix a vertex $b_{o}$ among the vertices of $\Gamma$. Then, for each integer $i \geq 0$, we construct a sequence of cycles $x(i) \in H_{2}\left(W_{\Gamma} ; \mathbb{Z}\right)$ as the minimal element satisfying the following conditions:

- the coefficient of $x(i)$ for the vertex $b_{o}$ is $i$;
- $\left(x(i)+l_{[k]}^{\prime}, b_{j}\right) \leq 0$ for any vertex $b_{j} \neq b_{o}$.

It follows from Ném05, Lemma 7.6] that $x(i)$ is uniquely defined and satisfies $x(i) \geq 0$. Furthermore, every leaf of the graded root $R_{\Gamma}$ induced by $\Gamma$ contains at least one $x(i)$ Ném05, Lemma 9.2].

We then construct a computation sequence between $x(i)$ and $x(i+1)$ as follows. Set $x_{0}=x(i)$ and $x_{1}=x(i)+$ $b_{o}$. Assuming that $x_{1}, \ldots, x_{l}$ are already constructed, we inductively define $x_{l+1}$ as follows. If $\left(x_{l}+l_{[k]}^{\prime}, b_{j}\right) \leq 0$ for all vertices $b_{j} \neq b_{o}$, then we stop, as $x_{l}=x(i+1)$ is satisfied by Ném05, Lemma 7.7]. Otherwise, we take $x_{l+1}=x_{l}+b_{j(l)}$, where $b_{j(l)}$ is a vertex of $\Gamma$ which is not $b_{o}$ and satisfies $\left(x_{l}+l_{[k]}^{\prime}, b_{j(l)}\right)>0$.

Now we amalgamate computation sequence between $x(i)$ and $x(i+1)$ for each $i \geq 0$ to obtain an infinite sequence of cycles. We can truncate this sequence after sufficiently many terms to get a finite sequence. This sequence is the computation sequence for the lattice homology of $M=\partial W_{\Gamma}$; more precisely, this sequence carries the lattice homology of $M$ in the sense of DSS23. Theorem 4.9].

Remark 4.1. It is possible to make sense of computation sequences between $x(i)$ and $x(i+s)$ for positive integers $s$, by going from $x(i)$ to $x(i)+s b_{o}$ by adding one $b_{o}$ at a time and then applying the same algorithm to go from $x(i)+s b_{o}$ to $x(i+s)$. If there exists an increasing sequence $0 \leq i_{1}<i_{2}<\cdots<i_{m}$ such that each leaf of the graded root $R_{\Gamma}$ contains at least one of the cycles $x\left(i_{1}\right), \ldots, x\left(i_{m}\right)$, one can generate computations sequences between $x\left(i_{s}\right)$ and $x\left(i_{s+1}\right)$ and then merge them to obtain a sequence which also carries the lattice homology of $M$. This observation will be used in the next subsection.
4.2. Review of $S^{1}$-lattice homotopy type. In this subsection, we review the construction of the Pin(2)lattice homotopy type. We will closely follow the arguments of DSS23.

We start by defining the weight function $w$ as follows. Given a $\operatorname{spin}^{c}$-structure $[k] \in \operatorname{Spin}^{c}(M)$ and its element $k \in[k]$, we define its weight as

$$
w(k)=\frac{1}{4}\left(k^{2}+|\Gamma|\right) .
$$

Also, given a pair of elements $k, k^{\prime} \in[k]$ which differ by $b_{j}$ for some $j$, we consider the pair as an "edge" $e_{k, k^{\prime}}$ and define its weight as

$$
w\left(e_{k, k^{\prime}}\right)=\min \left(w(k), w\left(k^{\prime}\right)\right)
$$

Then, given a sequence $\gamma=\left(x_{1}, \ldots, x_{m}\right)$ such that for each $i, x_{i}$ and $x_{i+1}$ differ by $b_{j}$ for some $j$, we consider a CW-complex $\mathcal{F}(\gamma, h)$ for very big positive even integers $h$ as

$$
\begin{equation*}
\mathcal{F}(\gamma, h)=\left(( \bigsqcup _ { i = 1 , \ldots , m } ( \mathbb { C } ^ { \frac { w ( x _ { i } ) + h } { 2 } } ) ^ { + } ) \cup \left(\bigsqcup _ { i = 1 , \ldots , m - 1 } \left(\mathbb{C}^{\left.\left.\left.\frac{w\left(e_{\left.x_{i}, x_{i+1}\right)+h}^{2}\right.}{2}\right)^{+} \wedge[0,1]\right)\right) / \sim, ~ ; ~, ~}\right.\right.\right. \tag{5}
\end{equation*}
$$

where we identify all basepoints, and furthermore, the points $x \sim(x, 0)$ for $x \in\left(\mathbb{C}^{\frac{w\left(e_{x_{i}, x_{i+1}}\right)+h}{2}}\right)^{+}$, considered as a point in $\left(\mathbb{C}^{\frac{w\left(x_{i}\right)+h}{2}}\right)^{+}$. We also similarly identify $y \sim(y, 1)$ for $y \in\left(\mathbb{C}^{\frac{w\left(e_{\left.x_{i}, x_{i+1}\right)}\right)+h}{2}}\right)^{+}$, considered as a point in $\left(\mathbb{C} \frac{w\left(x_{i+1}\right)+h}{2}\right)^{+}$. Then we define the path homotopy type of $\gamma$ as the formal de-suspension

$$
\mathcal{H}(\gamma,[k])=\Sigma^{-\frac{h}{2} \mathbb{C}} \mathcal{F}(\gamma, h),
$$

where this formal desuspension $\Sigma^{-\frac{h}{2}}$ is taken in a certain $S^{1}$-equivariant stalbe homotopy category. In our situation, we take it in $\mathfrak{C}_{O(2)}$. As the convenient notations, we abbreviate

$$
\mathbb{S}\left(x_{i}\right)=\left(\mathbb{C}^{\frac{w\left(x_{i}\right)+h}{2}}\right)^{+} \text {and } \mathbb{E}\left(e_{x_{i}, x_{i+1}}\right)=\left(\mathbb{C}^{\frac{w\left(e_{\left.x_{i}, x_{i+1}\right)}\right)+h}{2}}\right)^{+} \wedge[0,1] \subset \Sigma^{\frac{h}{2} \mathbb{C}} \mathcal{H}(\gamma,[k])
$$

This spectrum is naturally endowed with an $S^{1}$-action as follows: $S^{1}$ acts by complex multiplication on $\mathbb{C}$ and trivially on $[0,1]$. If $\gamma$ is a sequence which carries the lattice homology of $M$, the homotopy type of $\mathcal{H}(\gamma,[k])$ depends only on the plumbing graph $\Gamma$, and is defined as the $S^{1}$-lattice homotopy type of $\Gamma$.

To upgrade the symmetry group from $S^{1}$ to $\operatorname{Pin}(2)$, under the assumption that $[k]$ is self-conjugate, we have to choose the computation sequence carefully. We say that a computation sequence $\gamma$ is almost J-invariant if it can be written as an amalgamation of three interior-disjoint paths

$$
\gamma=\gamma_{0} \cup \gamma_{\Theta} \cup J \gamma_{0}
$$

where $J$ acts by the negation map, i.e., $k \mapsto-k$. This negation map has a unique invariant lattice cube $\square_{J}$ (see [DSS23, Section 6.1]) for more details); the condition here is that $\gamma_{\Theta}$ should be entirely contained in $\square_{J}$.

To construct an almost $J$-invariant computation sequence which carries the lattice homology of $M$, we proceed as follows. We know from DM19, Theorem 1.1] that $J$ acts on the leaves of the graded root $R_{\Gamma}$ by reflection; it has at most one invariant leaf. If an invariant leaf exists, it is the component containing the $J$-invariant cube $\square_{J}$. Choose a set $S$ of leaves of $R_{\Gamma}$ so that $R \cap J R=\emptyset$ and $R \cup J R$ is the set of all non-invariant leaves of $R_{\Gamma}$.

Since the Wu class of $W_{\Gamma}$ is a linear combination of a subset of nodes of $\Gamma$ which do not contain any pairs of adjacent nodes, we can choose a base node $b_{o}$ so that every vertex of the cube $\square_{J}$ has zero coefficient for $b_{o}$, which implies that $x(0)$, and no other $x(i)$, is contained in $\square_{J}$. For each leaf $C \in S$, choose an integer $i_{C} \geq 0$ such that $x\left(i_{C}\right) \in R_{\Gamma}$, following Ném05, Lemma 9.2]. Consider the set

$$
I=\{0\} \cup\left\{i_{C} \mid C \in S\right\}
$$

and write it as $I=\left\{i_{1}, \ldots, i_{s}\right\}, 0=i_{1}<\cdots<i_{s}$. Then one can take computation sequences between $x\left(i_{t}\right)$ and $x\left(i_{t+1}\right)$ for each $t=1, \ldots, s-1$ and amalgamate them to form a path $\gamma_{0}$. Then, by construction, $\gamma \cap J \gamma=\emptyset$. Then we can choose a path $\gamma_{\Theta}$ inside $\square_{J}$ which connects $x(0)$ and $J x(0)$ and take the amalgamation

$$
\gamma=\gamma_{0} \cup \gamma_{\Theta} \cup J \gamma_{0}
$$

Here, $\gamma_{\Theta}$ is not really a "path". It consists of two points, which are a pair of opposite vertices in the invariant lattice cube $\square_{J}$. Then $\gamma_{0}$ is a path which starts from $\mathfrak{s}$. Its orbit under the $J$ action, which is conjugation, is $J \gamma_{0}$, and this path ends at $\mathfrak{s}^{\prime}=\overline{\mathfrak{s}}$. Such paths are called almost $J$-invariant paths, and they are central in the construction of $\operatorname{Pin}(2)$-lattice homotopy type.

We will slightly modify the construction of Pin(2)-lattice homotopy type so that we can represent the action of $I$, where $I$ denotes the element $j \tau$ in the total symmetry group

$$
O(2)=U(1) \rtimes \mathbb{Z}_{2}
$$

where the order 2 subgroup $I$ generates $\mathbb{Z}_{2}$ corresponds to $j \circ \tau$.
4.3. Almost $I$-invariant path. Given an almost rational negative-definite plumbing graph $\Gamma$, the associated plumbed 4-manifold $W_{\Gamma}$, and an orientation-preserving involution $\tau$ on $W_{\Gamma}$, an almost I-invariant path is a sequence of $\operatorname{spin}^{c}$ structures on $W_{\Gamma}$

$$
\gamma=\left\{\mathfrak{s}_{-n}, \ldots, \mathfrak{s}_{-1}, \mathfrak{s}_{1}, \ldots, \mathfrak{s}_{n}\right\}
$$

such that the following conditions are satisfied:

- $\mathfrak{s}_{-i}=\tau^{*} \overline{\mathfrak{s}}_{i}$, for each $i \in\{1, \ldots, n\}$;
- $\mathfrak{s}_{i+1}-\mathfrak{s}_{i}=P D[S]$, for each $i \in\{1, \ldots, n\}$ and a sphere $S$ which represents a vertex of $\Gamma$;
- $\mathfrak{s}_{-i}-\mathfrak{s}_{-i-1}=P D[S]$, for each $i \in\{1, \ldots, n\}$ and a sphere $S$ which represents a vertex of $\Gamma$;
- $\mathfrak{s}_{1}-\mathfrak{s}_{-1}=P D[S]$ for a smoothly embedded sphere $S$ with $[S]^{2}<0$, where $\tau$ fixes $S$ setwise and acts on $S$ by an orientation-preserving involution (which fixes two points).
Recall that, given a $\operatorname{spin}^{c}$-structure $\mathfrak{s}$ on $Y_{\Gamma}$, a path of $\operatorname{spin}^{c}$-structures on $W_{\Gamma}$ is said to carry the lattice homology of $(\Gamma, \mathfrak{s})$ if the obvious inclusion map

$$
\mathcal{H}(\gamma, \mathfrak{s}) \hookrightarrow \mathcal{H}(\Gamma, \mathfrak{s})
$$

is a chain homotopy equivalence on $S^{1}$-equivariant Borel chain complexes. We will mainly consider almost $I$-invariant paths which carry the lattice homology of $(\Gamma, \mathfrak{s})$ in the proof of Theorem 1.3 .
4.4. Construction of $O(2)$-equivariant maps. We will construct an $O(2)$-equivariant map

$$
\mathcal{T}^{O(2)}: \mathcal{H}(\gamma, \mathfrak{s}) \rightarrow S W F\left(Y_{\Gamma}, \mathfrak{s}\right)
$$

which is an $S^{1}$-equivariantly homotopy equivalence for a given almost $I$-invariant path that carries the lattice homology.

Due to Lemma A.1, we are allowed to choose the following universe for $O(2)$-equivariant Seiberg-Witten theory:

$$
\mathcal{U}=\mathbb{R}^{\infty} \oplus \tilde{\mathbb{R}}^{\infty} \oplus \mathbb{C}^{\infty}
$$

Note that this universe induces the following universe when we restrict to $S^{1}$-equivariance:

$$
\mathcal{U}_{S^{1}}=\mathbb{R}^{\infty} \oplus \mathbb{C}^{\infty}
$$

4.5. $O(2)$-actions on the lattice homotopy type. We suppose our AR-graph $\Gamma$ has a symmetry $\tau$. Let us have an almost $I$-equivariant path. In this setting, we define a class of $O(2)$-actions on the path homotopy type:

$$
\Sigma^{\frac{h}{2} \mathbb{C}} \mathcal{H}(\gamma,[k])=\left(\bigcup_{1 \leq i \leq m} \mathbb{S}\left(\mathfrak{s}_{i}\right)\right) \cup\left(\bigcup_{1 \leq i \leq m-1} \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right)\right)
$$

We will identify the spheres $\mathbb{S}\left(\mathfrak{s}_{i}\right)$ and the "edges" $\mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right)$ with finite-dimensional approximation of the domains of Seiberg-Witten map for $X_{\Gamma}$ in stable homotopy category. From such identifications, we will define $O(2)$-actions. For the subgroup $S^{1} \subset O(2)$, we define the $S^{1}$-action by just multiplication by $S^{1}$ as complex numbers. We only need to define the action of $I \subset O(2)$ on $\mathcal{H}(\gamma,[k])$. Since we have $\tau^{*} \mathfrak{s}_{i} \cong \overline{\mathfrak{s}}_{-i}$ from the definition of almost $I$-invariant paths, we have anti-complex linear bundle map

$$
I: S_{\mathfrak{s}_{i}}^{+} \rightarrow S_{\overline{\mathfrak{s}}_{-i}}^{+}
$$

such that $I^{2}=\mathrm{Id}$. Therefore, $I$ induces some action

$$
I: \mathbb{S}\left(\mathfrak{s}_{i}\right) \rightarrow \mathbb{S}\left(\mathfrak{s}_{-i}\right)
$$

which is the complex conjugation. For the edges $\mathbb{E}_{e_{\mathfrak{s}_{i}, s_{i+1}}}$, we have an induced action

$$
I: \mathbb{E}_{e_{\mathfrak{s}_{i}, \mathfrak{s}_{i+1}}} \rightarrow \mathbb{E}_{e_{\mathfrak{s}_{-i}, \mathfrak{s}_{-i-1}}}
$$

again which can be identified with the complex conjugation $\wedge \mathrm{Id}$. For the central edge $\mathbb{E}\left(e_{\mathfrak{s}_{-1}, \mathfrak{s}_{1}}\right)$, we define

$$
I:\left(\mathbb{C}^{w\left(\mathfrak{s}_{-1}\right) / 2}\right)^{+} \wedge[0,1] \rightarrow\left(\mathbb{C}^{w\left(\mathfrak{s}_{-1}\right) / 2}\right)^{+} \wedge[0,1]
$$

such that $I$ acts on $[0,1]$ by reflection and $I$ on $\left(\mathbb{C}^{w\left(\mathfrak{s}_{-1}\right) / 2}\right)^{+}$is the complex conjugation. Since all $O(2)$ actions are compatible, we have a well-defined $O(2)$-action on $\mathcal{H}(\gamma,[k])$. With this action, we can regard $\Sigma^{\frac{h}{2}} \mathbb{C}_{\mathcal{H}}(\gamma,[k])$ and therefore $\mathcal{H}(\gamma,[k])$ as objects in $\mathfrak{C}_{O(2)}$.
4.6. Proof of Theorem 1.3. Now, we provide the construction of $\mathcal{T}^{O(2)}$ here, which gives the proof of Theorem 1.3. Note that we have a decomposition of the lattice homotopy type

$$
\Gamma:=\mathcal{H}(\gamma,[k])=\Gamma_{0} \cup \Gamma_{\theta}
$$

where

$$
\Sigma^{\frac{h}{2}} \mathbb{C}_{\theta}=\mathbb{S}\left(\mathfrak{s}_{-1}\right) \cup \mathbb{E}\left(\mathfrak{s}_{-1}, \mathfrak{s}_{1}\right) \cup \mathbb{S}\left(\mathfrak{s}_{1}\right)
$$

and $\Gamma_{0}$ is the other part which has a free $I$-action. For each vertex $\mathfrak{s}_{i}$ of an almost $I$-equivariant path $\gamma$, we associate the corresponding Bauer-Furuta invariant

$$
B F_{W_{\Gamma}, \mathfrak{s}_{i}}: \mathbb{S}\left(\mathfrak{s}_{i}\right) \rightarrow \Sigma^{\frac{h}{2} \mathbb{C}} S W F(Y, \mathfrak{s})
$$

with stabilizations by $\mathbb{R}, \widetilde{\mathbb{R}}$ and $\mathbb{C}$, which is $O(2)$-equivariant, which can be regarded as a morphism in $\mathfrak{C}_{O(2)}$. Note that $\mathbb{R}$ and $\widetilde{\mathbb{R}}$ do not appear in the domain of the $B F_{W_{\Gamma}, \mathfrak{s}_{i}}$ since $W_{\Gamma}$ is negative-definite.

Let $\mathfrak{s}_{i}$ and $\mathfrak{s}_{i+1}$ be two successive vertices in $\gamma$, so that $\mathfrak{s}_{i+1}=\mathfrak{s}_{i}+2 v^{*}$ for some vertex $v$ of $\Gamma$ such that the 2 -handle core of $v$. For the edges with $i<0$, we use adjunction relation and obtain a homotopy

$$
B F_{W_{\Gamma}, e_{\mathfrak{s}_{i}, s_{i-1}}}: \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right) \rightarrow \Sigma^{\frac{h}{2} \mathbb{C}} S W F(Y)
$$

For $i>0$, we obtain a map by

$$
B F_{W_{\Gamma}, e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}} \circ I: \mathbb{E}\left(e_{\mathfrak{s}_{-i}, \mathfrak{s}_{-i+1}}\right) \rightarrow \Sigma^{\frac{h}{2} \mathbb{C}} S W F(Y)
$$

This defines $O(2)$-equivariant map

$$
\mathcal{T}_{0}: \Gamma_{0} \rightarrow S W F(Y)
$$

For the unique edge of $\Gamma_{\theta}$, we first only have an $S^{1}$-equivariant map

$$
\mathcal{T}: \Gamma_{\theta}=\mathbb{E}_{e_{\mathfrak{s}_{-1}, s_{1}}} \rightarrow S W F(Y)
$$

again from adjunction relation. It defines an $S^{1}$-euqivariant map

$$
\mathcal{T}: \mathcal{H}(\gamma) \rightarrow S W F(Y)
$$

which is $S^{1}$-equivariantly homotopy equivalence.
In order to give an $O(2)$-equivariant map

$$
\mathcal{T}^{O(2)}: \mathcal{H}(\gamma) \rightarrow S W F(Y)
$$

we need to modify this construction.
The main strategy is almost the same as the Pin(2) case in DSS23. Since the argument is almost the same as that given in DSS23, we just write a flow of the proof and which part is different. First, we define

$$
\Theta:=\operatorname{Cone}\left(\mathcal{T}_{0}: \Gamma_{0} \rightarrow S W F(Y)\right)
$$

and regard it as an $O(2)$-space. Note that we have the following diagram:


Since $\mathcal{T}$ is $S^{1}$-equivariantly homotopic, the $O(2)$-space $\Theta$ is $S^{1}$-equivariantly homotopic to $\Sigma^{\widetilde{\mathbb{R}}} S^{0}$. We will prove $\mathcal{T}$ admits an $O(2)$-equivariant lift. If there is an $O(2)$-equivariant homotopy equivalence

$$
\Theta \rightarrow \Sigma^{\widetilde{\mathbb{R}}} S^{0}
$$

which satisfies the commutativity

up to $O(2)$-homotopy, then from the identifications

$$
\Sigma^{\mathbb{R}} \Gamma=\operatorname{Cone}\left(\Sigma^{\widetilde{\mathbb{R}}} S^{0} \rightarrow \Sigma^{\mathbb{R}} \Gamma_{0}\right) \quad \text { and } \quad \Sigma^{\mathbb{R}} S W F(Y)=\operatorname{Cone}\left(\Theta \rightarrow \Sigma^{\mathbb{R}} \Gamma_{0}\right)
$$

we see $\mathcal{T}$ has an $O(2)$-equivariant lift. More precisely, we consider the following steps:
Step 1 : First we prove there is $\mathbb{Z}_{2}=\langle I\rangle$ equivariant homotopy equivalence:

$$
\begin{equation*}
\Theta^{S^{1}} \cong{ }_{I} \Sigma^{\widetilde{\mathbb{R}}} S^{0} \tag{6}
\end{equation*}
$$

This statement is corresponding to DSS23, Lemma 6.5]. Since several techniques DSS23, (Ho-3), (6.6) in the $O(2)$-setting] to see (6) in the Pin(2)-setting can also work for $O(2)$, we have the desired result.
Step 2 : Next, we prove that, for sufficiently large $p$ and $q$, there exists an $O(2)$-map

$$
M: \Theta \rightarrow \Sigma^{q \widetilde{\mathbb{R}} \oplus p \mathbb{C}} S W F(Y)
$$

which induces homotopy equivalence in $O(2)$-fixed point spectra. This statement is an analog of DSS23, Lemma 6.6]. Here we use the following facts:

- The vanishing result

$$
\left[\Sigma^{r \mathbb{R}} \Gamma_{0}, \Sigma^{r \mathbb{R} \oplus q \widetilde{\mathbb{R}} \oplus p \mathbb{C}} S^{0}\right]_{O(2)}=0
$$

for sufficiently large $p, q$ and $r$. It follows from Ada84, Proposition 4.2]. Since we are working in $\mathfrak{C}_{O(2)}$, we omit $\Sigma^{r \mathbb{R}}$.

- For sufficiently large $p$ and $q$, there is an $O(2)$-equivariant map

$$
N: S W F(Y) \rightarrow \Sigma^{r \mathbb{R} \oplus q \widetilde{\mathbb{R}} \oplus p \mathbb{C}} S^{0}
$$

for some $r$. (We actually can take $r$ as 0 .) This comes from the $O(2)$ Bauer-Furuta invariant for the double-branched cover of an oriented surface $S$ (with high genus) in $D^{4}$ bounded by $K$ with respect to its unique spin structure. Note that $r$ is zero in this situation since $r=b_{2}^{+}\left(\Sigma_{2}(S) / \mathbb{Z}_{2}\right)=$ $b_{2}^{+}\left(D^{4}\right)=0$. Since it is spin structure, the Bauer-Furuta invariant has a $\operatorname{Pin}(2) \times_{\mathbb{Z}_{2}} \mathbb{Z}_{4}$-symmetry (as the maximal symmetry, see Mon22), but we just forget by the homomorphism $S^{1} \rtimes \mathbb{Z}_{2} \cong$ $O(2) \rightarrow \operatorname{Pin}(2) \times_{\mathbb{Z}_{2}} \mathbb{Z}_{4}$, defined by

$$
(u, 0) \mapsto(u, 0) \quad \text { and } \quad(u, 1) \mapsto(j u, j)
$$

where $j$ (resp. 1) denotes the generator of $\mathbb{Z}_{4}$ (resp. $\mathbb{Z}_{2}$ ).
Step 3 : We reduce the numbers $p$ and $q$ so that we have

$$
M^{\prime}: \Theta \rightarrow \Sigma^{\widetilde{\mathbb{R}}} S W F(Y)
$$

and $M^{\prime}$ induces homotopy equivalence for $S^{1}$ - and $O(2)$ - fixed point parts which satisfies the commutativity

up to $O(2)$-homotopy. This is again an analog of DSS23, Lemma 6.7]. Since $O(2)$-analogs of DSS23, (6.8), (ho-03), Theorem 6.4] are still true, their argument still works in our setting.
4.7. Almost $I$-equivariant path for even torus knots. Given a torus knot $K=T_{p, q}$ where $p, q>0$ and $p$ is even, its double-branched cover $\Sigma_{2}(K)=\Sigma(2, p, q)$ admits a symmetric plumbing graph. In particular, it admits a negative-definite almost rational star-shaped plumbing graph $\Gamma$, having three legs, where two of them are identical and the deck transformation $\tau$ acts on $\Gamma$ by swapping those two identical legs and leaving the other leg fixed, as shown in AKS20. It follows that, for the unique spin structure $[k]$ of $\Sigma(2, p, q)$, the distinguished characteristic element $k_{r}$ representing it is $\tau$-invariant.

We then follow the procedure described in the previous subsection to construct an almost $J$-invariant computation sequence $\gamma_{J}$ which carries the lattice homology of $\Sigma(2, p, q)$. Recall that such a sequence is obtained by connecting the cycles $x\left(i_{t}\right)$, and each of those cycles are defined as the minimal cycle whose coefficient at the base vertex $b_{o}$ is $i_{t}$. It is easy to see that we can choose $b_{o}$ to be lying on the invariant leg of $\Gamma$, in which
case it is clear that $x\left(i_{t}\right)$ should also be $\tau$-invariant. Therefore, the action of $\tau$ is trivial on the graded root $R_{\Gamma}$, and given a decomposition

$$
\gamma_{J}=\gamma_{0} \cup \gamma_{\Theta} \cup J \gamma_{0}
$$

the modified path

$$
\gamma_{I}=\gamma_{0} \cup \gamma_{\Theta} \cup J \tau \gamma_{0}
$$

also carries the lattice homology of $\Sigma(2, p, q)$. Hence $\Gamma$ admits an almost $I$-equivariant path carrying the lattice homology for $(\Sigma(2, p, q),[0])$, where $[0]$ denotes the unique spin structure on $\Sigma(2, p, q)$.

Remark 4.2. It follows directly, at this stage, from the $\tau$-invariance of cycles $x(i)$ for each $i \geq 0$ that the action of $\tau$ on the Heegaard Floer chain complex $C F^{-}\left(Y_{\Gamma}\right)$ of $Y_{\Gamma}=\partial W_{\Gamma}$ is homotopic to the identity. This is stronger than the observations made in AKS20 regarding the deck transformation action on $\Sigma_{2}\left(T_{p, q}\right) \cong \Sigma(2, p, q)$, and thus might be of independent interest.
4.8. The real Frøyshov invariants of $T_{2 n, 1-20 n}$. From the observations we made in the previous subsection, we can prove the following theorem regarding real Frøyshov invariants of even torus knots.

Theorem 4.3. If $K=T_{p, q}$ be a torus knot, where $p, q>0$ and $p$ is even, then we have

$$
\delta_{R}(K)=\underline{\delta}_{R}(K)=\bar{\delta}_{R}(K)=-\frac{1}{2} \bar{\mu}\left(\Sigma_{2}(K)\right),
$$

where $\bar{\mu}$ is the Neumann-Siebenmann invariant for the unique spin structure.
Proof. It follows from the construction in the previous section that the $I$-invariant locus of the $O(2)$-equivariant lattice homotopy type of the double-branched cover $\Sigma_{2}(K)$ of $K$ is the fixed locus of the "central sphere" under the complex conjugation action and thus given by $\left[\left(S^{0}, 0, \bar{\mu}\left(\Sigma_{2}(K)\right)\right)\right]$ as a $\mathbb{Z}_{2}$-homotopy type. More precisely, we see

$$
\mathcal{H}(\gamma,[k])^{I}=\left(\Gamma_{0} \cup \Gamma_{\theta}\right)^{I}=\Gamma_{\theta}^{I}=\mathbb{E}\left(\mathfrak{s}_{-1}, \mathfrak{s}_{+1}\right)^{I}=\left(\left(\mathbb{C}^{\frac{1}{8}\left(c_{1}^{2}\left(\mathfrak{s}_{-1}\right)-\sigma\left(W_{\Gamma}\right)\right)}\right)^{+}\right)^{I} \wedge\left\{\frac{1}{2}\right\}
$$

Note that the $\operatorname{spin}^{c}$ structure corresponding to the definition of $\bar{\mu}$ invariant is $\mathfrak{s}_{1}$ in the previous section. Thus,

$$
\mathcal{H}(\gamma,[k])^{I}=\left(\left(\mathbb{C}^{-\bar{\mu}\left(\Sigma_{2}(K)\right)}\right)^{+}\right)^{I}=\left(\mathbb{R}^{-\bar{\mu}\left(\Sigma_{2}(K)\right)}\right)^{+}=\left[\left(S^{0}, 0, \bar{\mu}\left(\Sigma_{2}(K)\right)\right)\right] \in \mathfrak{C}_{\mathbb{Z}_{2}}
$$

Since both lattice homotopy types and Seiberg-Witten homotopy types are finite $\mathbb{Z}_{2}$-spectra-with $\mathbb{Z}_{2}$ acting by $I$ on the former and by $j \tau$ on the latter-it follows from Theorem 1.3 and Lemma 3.6 that $S W F^{I}(K)$ is a $\mathbb{Z}_{2}$-homology sphere of dimension $-\bar{\mu}\left(\Sigma_{2}(K)\right)$. Therefore we deduce from Lemma 3.5 that

$$
\delta_{R}(K)=\underline{\delta}_{R}(K)=\bar{\delta}_{R}(K)=-\frac{1}{2} \bar{\mu}\left(\Sigma_{2}(K)\right)
$$

as desired.
Using Theorem4.3 we can compute the $\bar{\delta}_{R}$ invariant for the torus knot $T_{2 n, 1-20 n}$.
Corollary 4.4. Let $n \geq 1$ be an odd integer. Then we have

$$
\bar{\delta}_{R}\left(T_{2 n, 1-20 n}\right)=-\frac{9}{8}
$$

Proof. Using Theorem 4.3 and computations from Subsection 2.2, we see that

$$
\underline{\delta}_{R}\left(T_{2 n, 20 n-1}\right)=-\frac{1}{2} \bar{\mu}\left(\Sigma_{2}(K)\right)=\frac{9}{8} .
$$

By KMT23b, Lemma 3.28], we deduce that

$$
\bar{\delta}_{R}\left(T_{2 n, 1-20 n}\right)=-\underline{\delta}_{R}\left(T_{2 n, 20 n-1}\right)=-\frac{9}{8} .
$$

4.9. Proof of Theorem 1.2, We can now prove the main theorem, using our computations of real Frøyshov invariants of $T_{2 n, 1-20 n}$.

Proof of Theorem 1.2. In order to make use of Theorem 3.1, we have to check that its assumptions are satisfied. Consider the smooth concordance, as described in Proposition 2.1, $S_{n}$ from $E_{2 n, 1}$ to $T_{2 n, 1-20 n}$ in a twicepunctured $2 \mathbb{C P}^{2}$, which is denoted by $X$. This concordance has the homology class $(2 n, 6 n)$. We calculate:

$$
\begin{aligned}
b_{2}^{+}\left(\Sigma_{2}\left(S_{n}\right)\right)-b_{2}^{+}(X) & =b_{2}^{+}(X)-\frac{1}{4}\left[S_{n}\right]^{2}+\frac{1}{2} \sigma\left(T_{2 n, 1-20 n}\right) \\
& =2-\frac{1}{4}\left((2 n)^{2}+(6 n)^{2}\right)+\frac{1}{2}\left(20 n^{2}-2\right) \\
& =2-10 n^{2}+\left(10 n^{2}-1\right) \\
& =1
\end{aligned}
$$

Hence the assumptions are satisfied, and as before we get

$$
\underline{\delta}_{R}\left(E_{2 n, 1}\right)-\frac{1}{16}\left(2 \sigma(X)-\frac{1}{2}\left[S_{n}\right]^{2}+\sigma\left(T_{2 n, 1-20 n}\right)\right) \leq \bar{\delta}_{R}\left(T_{2 n, 1-20 n}\right)
$$

Using

$$
\begin{aligned}
-\frac{1}{16}\left(2 \sigma(X)-\frac{1}{2}\left[S_{n}\right]^{2}+\sigma\left(T_{2 n, 1-20 n}\right)\right) & =-\frac{1}{16}\left(2 \cdot 2-\frac{1}{2}\left((2 n)^{2}+(6 n)^{2}\right)+\left(20 n^{2}-2\right)\right) \\
& =-\frac{1}{8}
\end{aligned}
$$

and Corollary 4.4. we conclude that

$$
\underline{\delta}_{R}\left(E_{2 n, 1}\right) \leq-1 .
$$

The proof is complete by applying Corollary 3.3 .
Remark 4.5. As it is observed in KMT23b, Proposition 4.9], the map sends a knot concordance class to the $\left(G=\mathbb{Z}_{4}, H=\mathbb{Z}_{2}\right)$ local equivalence class of the real Floer homotopy type giving a homomorphism:

$$
[K] \mapsto\left[S W F_{R}(K)\right]_{\text {loc }}: \mathcal{C} \rightarrow \mathcal{L} \mathcal{E}_{G}
$$

We have observed that for any torus knot $T_{p, q}$, we have $\left[S W F_{R}\left(T_{p, q}\right)\right]_{\text {loc }}$ is equal to some sphere spectrum. It is also true for any two-bridge knot. In other words, all torus knots are sent to the subgroup in $\mathcal{L} \mathcal{E}_{G}$ generated by sphere spectrums. Note that the invariants $\delta_{R}, \underline{\delta}_{R}, \bar{\delta}_{R}$ factor through the group homomorphism $S W F_{R}: \mathcal{C} \rightarrow \mathcal{L} \mathcal{E}_{G}$. Moreover, one can use the fact that $E_{2,1}$ bounds nullhomologous disks in both $\mathbb{C P}^{2}$ and $\overline{\mathbb{C P}}^{2}$, and apply KMT23b, Theorem 3.23] to conclude that $\delta_{R}\left(E_{2 n, 1}\right)=0$. Then we have $\delta_{R}\left(E_{2 n, 1}\right)>$ $\underline{\delta}_{R}\left(E_{2 n, 1}\right)$ and $S W F_{\text {loc }}^{R}\left(E_{2 n, 1}\right)$ is not equal to some sphere spectrum in the group $\mathcal{L} \mathcal{E}_{G}$. It is already observed in KMT23b, Example 1.11] that certain Montesinos knots also satisfy this property.
4.10. Montesinos cases. Let $\Gamma$ be a negative-definite AR-graph whose corresponding boundary involution is the complex conjugation on the Brieskorn sphere $\Sigma\left(a_{1}, \ldots, a_{n}\right)$, which can be seen as the double-branched covering space along Montesinos knots. Then, we see all paths are strict $I$-invariant paths in the sense that all spin $^{c}$ structures contained in any path are $I$-invariant; in fact, every spin ${ }^{c}$ structure on $W_{\Gamma}$ is $I$-invariant, as observed first in AKS20. Hence, unlike the case of almost $I$-invariant paths, we can just take any path $\gamma$ which carries the lattice homology.

We will construct an $O(2)$-equivariant map

$$
\mathcal{T}: \mathcal{H}\left(\gamma, \mathfrak{s}_{0}\right) \rightarrow S W F\left(\Sigma_{2}(K)\right)
$$

In this setting about Montesinos knots, we define a class of $O(2)$-actions on the path homotopy type:

$$
\mathcal{H}(\gamma,[k])=\bigcup_{1 \leq i \leq m} \mathbb{S}\left(\mathfrak{s}_{i}\right) \cup \bigcup_{1 \leq i \leq m-1} \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right)
$$

Define the action of $I$

$$
I: \mathbb{S}\left(\mathfrak{s}_{i}\right) \rightarrow \mathbb{S}\left(\mathfrak{s}_{i}\right) \quad \text { and } \quad I: \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right) \rightarrow \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right)
$$

by the complex conjugations. This gives a well-defined $O(2)$-action on $\mathcal{H}(\gamma,[k])$.

As in the case of torus knots, for each vertex $\mathfrak{s}_{i}$ of an almost $I$-equivariant path $\gamma$, we associate the corresponding Bauer-Furuta invariant

$$
B F_{W_{\Gamma}, \mathfrak{s}_{i}}: \mathbb{S}\left(\mathfrak{s}_{i}\right) \rightarrow \Sigma^{\frac{h}{2} \mathbb{C}} S W F(Y, \mathfrak{s})
$$

with stabilizations by $\mathbb{R}, \widetilde{\mathbb{R}}$ and $\mathbb{C}$, which is $O(2)$-equivariant. For the maps corresponding to edges, we use the $O(2)$-adjunction relation stated in the next section.
4.10.1. $O(2)$-adjunction. The following is the $O(2)$-adjunction relation, which can be regarded as an $O(2)$ equivariant version of DSS23, Proposition 3.15]:

Proposition 4.6. Let $(X, \mathfrak{s})$ be a $\operatorname{spin}^{c}$ cobordism from $\left(Y_{0}, s_{0}\right)$ to $\left(Y_{1}, \mathfrak{s}_{1}\right)$ with $b_{1}(X)=b_{1}\left(Y_{i}\right)=0$. Suppose there is a smooth involution $\tau$ on $X$ such that

$$
\tau^{*} \mathfrak{s} \cong \overline{\mathfrak{s}}
$$

Suppose that we have an embedded sphere $S$ in $X$ with $S \cdot S<0$ and with $\tau(S)=S$ so that $\left.\tau\right|_{S}: S \rightarrow S$ is the complex conjugation on $\mathbb{C} P^{1}$. Let $L$ be the complex line bundle on $X$ with $c_{1}(K)=P D(S)$. Set

$$
\mathfrak{s}^{\prime}=\mathfrak{s} \otimes L \quad \text { and } \quad n:=\frac{\left\langle c_{1}(\mathfrak{s}),[S]\right\rangle+[S]^{2}}{2} .
$$

We write the $O(2)$-equivariant Bauer-Furuta invariants of $\mathfrak{s}$ and $\mathfrak{s}^{\prime}$ as maps

$$
\begin{aligned}
& B F_{X, \mathfrak{s}}:\left(\mathbb{C}^{\frac{c_{1}\left(s^{2}-\sigma(X)\right.}{8}}\right)^{+} \wedge S W F\left(Y_{0}\right) \rightarrow S W F\left(Y_{1}\right) \\
& B F_{X, \mathfrak{s}^{\prime}}:\left(\mathbb{C}^{\frac{c_{1}\left(\mathfrak{s}^{\prime}\right)^{2}-\sigma(X)}{8}}\right)^{+} \wedge S W F\left(Y_{0}\right) \rightarrow S W F\left(Y_{1}\right)
\end{aligned}
$$

Then, $B F_{X, \mathfrak{s}}$ and $U^{n} B F_{X, \mathfrak{s}^{\prime}}$ are $O(2)$-stably homotopic up to certain coordinate changes if $n>0$, and the same statement holds for $U^{-n} B F_{X, \mathfrak{s}}$ and $B F_{X, \mathfrak{s}^{\prime}}$ if $n<0$. Here $U$ denotes the stable homotopy class of a map

$$
X \rightarrow \Sigma^{\mathbb{C}} X
$$

obtained as $x \mapsto(0, x)$.
Remark 4.7. The meaning of "up to certain coordinate changes" in Proposition 4.6 is the following: if we need, after precomposing an odd permutation

$$
\left(z_{1}, z_{2}, z_{3}, \ldots, z_{n}\right) \mapsto\left(z_{2}, z_{1}, z_{3}, \ldots, z_{n}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}
$$

the maps $B F_{X, \mathfrak{s}}$ and $U^{m} B F_{X, \mathfrak{s}^{\prime}}$ are $O(2)$-equivariantly stably homotopic.
The proof is similar to that given in the proof of DSS23, Proposition 3.15]. The only difference is; we need to analyze the Bauer-Furuta invariants for $I$-fixed point parts in our $O(2)$-setting.
Proof of Proposition 4.6. We first decompose $X$ into

$$
X=\nu(S) \cup(X \backslash \operatorname{int} \nu(S))
$$

$\tau$ equivariantly, where $\nu(S)$ is the disk normal bundle of $S$ identified with a tubular neighborhood of $S$. Then, the equivariant version of the gluing theorem implies

$$
\begin{equation*}
B F_{X, \mathfrak{s}}=B F_{X \backslash \operatorname{int} \nu(S),\left.\mathfrak{s}\right|_{X \backslash \operatorname{int} \nu(S)} \circ B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}} . . . . ~} \tag{7}
\end{equation*}
$$

For such a gluing theorem, the proof of the original gluing argument in Man07] and [KLS23] works without any change. (See also Miy23, Theorem 2.12])

Since the involution $\tau$ preserves the standard positive scalar curvature metric on the lens space $\partial \nu(S)$, one can regard $B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}$ as an $O(2)$-equivariant map

$$
B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}: V^{+} \rightarrow W^{+}
$$

for some $O(2)$-representation spaces. Since $\mathfrak{s}^{\prime}$ and $\mathfrak{s}$ are the same on $X \backslash \operatorname{int} \nu(S)$ and we have $\sqrt[7]{7}$, it is sufficient to give an $O(2)$ homotopy between

$$
B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}: V^{+} \rightarrow W^{+} \quad \text { and } \quad U^{m} \circ B F_{\nu(S),\left.\mathfrak{s}^{\prime}\right|_{\nu(S)}}: V^{+} \rightarrow W^{+}
$$

which are maps between spheres when $m \geq 0$. The case $m<0$ follows from completely the same argument.

We will prove these maps $B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}$ and $U^{m} B F_{\nu(S),\left.\mathfrak{s}^{\prime}\right|_{\nu(S)}}$ are $O(2)$-stably homotopic to $O(2)$-equivariant maps obtained from the inclusions

$$
\iota: \mathbb{C}^{n} \hookrightarrow \mathbb{C}^{n+l}
$$

when $l>0$ and

$$
\pm \mathrm{Id}: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}
$$

when $l=0$ up to certain coordinate changes. Here we are using $\nu(S)$ is negative-definite.
We shall use the equivariant version of Hopf's classification result stated in Theorem A. 2 to make an $O(2)-$ homotopy between $B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}$ and $B F_{\nu(S),\left.\mathfrak{s}^{\prime}\right|_{\nu(S)}}$. We have two cases:

- The case of $\operatorname{dim} V^{I}<\operatorname{dim} W^{I}$ and
- The case of $\operatorname{dim} V^{I}=\operatorname{dim} W^{I}$.

In the first case, we only need to see

$$
\operatorname{deg} B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}^{S^{1}}=\operatorname{deg} B F_{\nu(S),\left.\mathfrak{s}^{\prime}\right|_{\nu(S)}}^{S^{1}}
$$

This is obvious since $S^{1}$-invariant part of the Bauer-Furuta invariant does not depend on the choices of spin ${ }^{c}$ structures. In the second case, we will prove

$$
\operatorname{deg} B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}^{I}= \pm 1
$$

which is a non-trivial computation. Note that, in the second case, the corresponding spin ${ }^{c}$ structure on $\nu(S)$ satisfies

$$
c_{1}^{2}(\mathfrak{s})-\sigma(\nu(S))=4 d\left(-\partial \nu(S)=-L(d, 1),\left.\mathfrak{s}\right|_{-L(d, 1)}\right)
$$

for some $d$, which is equivalent to having a sharp Frøyshov inequality. Here, we use the orientation convention that $L(p, q)$ is obtained by $p / q$-surgery on the unknot. In this case, the $O(2)$-Bauer-Furuta invariant can be written as

$$
B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}:\left(\mathbb{C}^{\frac{c_{1}^{2}(\mathfrak{s})-\sigma(\nu(S))}{8}}\right)^{+} \rightarrow\left(\mathbb{C}^{\frac{d\left(L(d, 1),\left.\mathfrak{s}\right|_{L(d, 1)}\right)}{2}}\right)^{+}
$$

In order to compute the degrees, we use the following key lemma:
Lemma 4.8. Let $d$ be a negative integer and $O(d)$ denote the total space of the disk bundle over $S^{2}$ with Euler number $d$. Define $\tau: O(d) \rightarrow O(d)$ as the complex conjugation on the base and fiber directions. Let $\mathfrak{s}$ be $a \operatorname{spin}^{c}$ structure on $O(d)$ such that $\tau^{*} \mathfrak{s} \cong \overline{\mathfrak{s}}$ and

$$
c_{1}^{2}(\mathfrak{s})-\sigma(O(d))=-4 d\left(O(d)=L(d, 1),\left.\mathfrak{s}\right|_{L(d, 1)}\right) .
$$

Then, there is an equivariant embedding

$$
O(d) \hookrightarrow \#_{-d} \overline{\mathbb{C P}}^{2}
$$

extending the $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ where the action on $\#_{-d} \overline{\mathbb{C P}}^{2}$ is the connected sum of the complex conjugations and $c_{1}(\mathfrak{s})=( \pm 1, \ldots, \pm 1)$. Here, the $\pm$ signs need not be synchronized.

Proof. Let $U$ be the unknot, so that attaching a 2-handle along $U$ to $B^{4}$ with framing $d$ yields $O(d)$. Choose a strong inversion of $U$ and denote its rotation axis as $\ell$. Let $m_{1}, \ldots, m_{-d-1}$ be 0 -framed parallel copies of the meridian of the unknot $U$ so that the rotation along $\ell$ gives a strong inversion of each $m_{i}$.

We then attach $(-1)$-framed 2 -handles to $O(d)$ along each $m_{i}$, and then cap it off with a 4 -handle. We denote the resulting closed 4 -manifold by $W_{d}$. Since we are attaching 2 -handles to each component of a strongly invertible link, we observe that the action of $\tau$ extends smoothly to $W_{d}$. Furthermore, by performing equivariant blowdowns, we see that

$$
W_{d} \cong \#_{-d} \overline{\mathbb{C P}}^{2}
$$

where the diffeomorphism is $\tau$-equivariant.
To prove the statement about extensions of $\operatorname{spin}^{c}$ structures, we recall that $O(d)$, considered as a cobordism from $L(-d, 1)$ to $S^{3}$, is negative-definite and thus it folows from OS03, Section 9] that the Heegaard Floer cobordism map

$$
F_{O(d), \mathfrak{s}}^{-}: H F^{-}\left(L(-d, 1),\left.\mathfrak{s}\right|_{L(-d, 1)}\right) \rightarrow{H F^{-}}^{-}\left(S^{3}\right)=\mathbb{F}_{2}[U]
$$

becomes a homotopy equivalence after localizing by $U^{-1}$. It is easy to see, using the degree shift formula for Heegaard Floer homology OS03, Section 2], that its degree shift is exactly

$$
\operatorname{deg} F_{O(d), \mathfrak{s}}^{-}=\frac{c_{1}(\mathfrak{s})^{2}+1}{4}=\frac{c_{1}(\mathfrak{s})^{2}-\sigma(O(d))}{4}
$$

which is equal to $-d\left(L(d, 1),\left.\mathfrak{s}\right|_{L(-d, 1)}\right)$ by the assumption. Since $d\left(S^{3}\right)=0$ and $L(-d, 1)$ is an L-space, we deduce that $F_{O(d), \mathfrak{s}}$ is an isomorphism. Hence the hat-flavored cobordism map

$$
\hat{F}_{O(d), \mathfrak{s}}: \widehat{H F}\left(L(-d, 1),\left.\mathfrak{s}\right|_{L(-d, 1)}\right) \rightarrow \widehat{H F}\left(S^{3}\right)
$$

is an isomorphism.
It is easy to see via explicit holomorphic triangle counts on Heegaard triple diagrams that for any $n>1$ and
 (given by attaching a ( -1 )-framed 2 -handle to a meridian of an $\left(n-1\right.$ )-surgered unknot) admits a spin ${ }^{c}$ structure $\tilde{\mathfrak{s}}_{0}$, extending $\mathfrak{s}_{0}$, such that the hat-flavored cobordism map

$$
\hat{F}_{W_{n-1,1}, \tilde{\mathfrak{s}}_{0}}: \widehat{H F}\left(L(n-1,1),\left.\tilde{\mathfrak{s}}_{0}\right|_{L(n-1,1)}\right) \rightarrow \widehat{H F}\left(L(n, 1), \mathfrak{s}_{0}\right)
$$

is an isomorphism. By an induction on $-d$, this implies that there exists a spin ${ }^{c}$ structure $\mathfrak{s}_{d}$ on the cobordism

$$
W_{d} \backslash\left(O(d) \sqcup \stackrel{\circ}{B}^{4}\right) \cong X_{-d-1} \cup_{L(-d-1,1)} X_{-d-2} \cup_{L(-d-2,1)} \cdots \cup_{L(2,1)} X_{1}
$$

such that the hat-flavored cobordism map

$$
\hat{F}_{W_{d} \backslash O(d), \mathfrak{s}_{d}}: \widehat{H F}\left(S^{3}\right) \rightarrow \widehat{H F}\left(L(-d, 1),\left.\mathfrak{s}\right|_{L(-d, 1)}\right)
$$

is an isomorphism. By composing this with $\hat{F}_{O(d), \mathfrak{s}}$, we see that the cobordism map

$$
\hat{F}_{W_{d}, \tilde{\mathfrak{s}}_{d}}: \widehat{H F}\left(S^{3}\right) \rightarrow \widehat{H F}\left(S^{3}\right)
$$

is an isomorphism, where $\tilde{\mathfrak{s}}_{d}=\mathfrak{s} \cup \mathfrak{s}_{d}$ is the $\operatorname{spin}^{c}$ structure on $W_{d}$. However, since $W_{d} \cong \#_{-d} \overline{\mathbb{C P}}^{2}$ and thus $\operatorname{spin}^{c}$ structure on $W_{d}$ are classified by their $c_{1}$, if we write $c_{1}\left(\tilde{\mathfrak{s}}_{d}\right)$ as

$$
c_{1}\left(\tilde{\mathfrak{s}}_{d}\right)=\left(\lambda_{1}, \ldots, \lambda_{-d}\right),
$$

where we are choosing the generators of $H^{2}\left(\#_{-d} \overline{\mathbb{C P}}^{2} ; \mathbb{Z}\right)$ to be our choice of basis for $H^{2}\left(W_{d} ; \mathbb{Z}\right)$. Then we have that

$$
\hat{F}_{W_{d}, \tilde{\mathfrak{s}}_{d}}=\hat{F}_{\overline{\mathbb{C P}}^{2}, \mathfrak{s}_{\lambda_{1}}} \circ \cdots \circ \hat{F}_{{\overline{\mathbb{C P}^{2}}{ }^{2}, \mathfrak{s}_{-d}} .}
$$

where $\mathfrak{s}_{\lambda_{i}}$ denotes the unique $\operatorname{spin}^{c}$ structure on $\mathbb{C P}^{2}$ whose $c_{1}$ is $\lambda_{i}$. However, it is easy to check via explicitly counting holomorphic triangles that the map

$$
\hat{F}_{\overline{\mathbb{C P}}^{2}, \mathfrak{s}_{\lambda_{i}}}: \widehat{H F}\left(S^{3}\right) \rightarrow \widehat{H F}\left(S^{3}\right)
$$

is an isomorphism if $\lambda_{i}$ generates $H^{2}\left(\overline{\mathbb{C P}}^{2} ; \mathbb{Z}\right)$ and zero if not. Therefore we deduce that

$$
c_{1}\left(\tilde{\mathfrak{s}}_{d}\right)=( \pm 1, \ldots, \pm 1)
$$

Since $\tilde{\mathfrak{s}}_{d}$ is an extension of the given $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ on $O(d)$, the lemma follows.
Using Lemma 4.8, we have an equivariant embedding $f: \nu(S) \rightarrow \#_{n} \overline{\mathbb{C P}}^{2}$ for some $n>0$. Again, from $O(2)$-equivariant gluing formula of the Bauer-Furuta invariants, we have

$$
B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}} \circ B F_{\#_{n} \overline{\mathrm{CP}}^{2} \backslash \operatorname{int} f(\nu(S)),\left.\mathfrak{s}_{0}\right|_{\#_{n} \overline{\mathrm{CP}}^{2} \backslash \operatorname{int} f(\nu(S))}=B F_{\#_{n} \overline{\mathrm{CP}}^{2}, \mathfrak{s}_{0}} \text { }}
$$

up to $O(2)$-equivariant stable homotopy. Here $\mathfrak{s}_{0}$ denotes the $\operatorname{spin}^{c}$ structure on $\#_{n} \overline{\mathbb{C P}}^{2}$ such that $c_{1}\left(\mathfrak{s}_{0}\right)=$ $( \pm 1, \ldots, \pm 1)$. By an equivariant version of the connected sum formula of $O(2)$-equivariant Bauer-Furuta invariant (which is a special case of the previous gluing result appeared in the proof of Proposition 4.6), we see

$$
\operatorname{deg} B F_{\#_{n} \overline{\mathrm{CP}}^{2}, \mathfrak{s}_{0}}=\left(\operatorname{deg} B F_{\overline{\mathrm{CP}}^{2},\left.\mathfrak{s}_{0}\right|_{\overline{\mathrm{CP}}^{2}}}\right)^{n}
$$

Let $\tau_{\overline{\mathrm{CP}}^{2}}$ denote the complex conjugation. Then this preserves the standard positive scalar curvature metric on $\overline{\mathbb{C P}}^{2}$. So, one can see

$$
\operatorname{deg} B F_{\overline{\mathbb{C P}}^{2},\left.\mathfrak{s}_{0}\right|_{\overline{\mathrm{CP}}^{2}} ^{I}}^{I}= \pm 1
$$

which is stated in Miy23, Theorem 1.9, the third item].
Thus, we have

Thus, one can see

$$
\operatorname{deg}\left(B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}^{I}\right)= \pm 1
$$

Note that the base change

$$
\left(z_{1}, z_{2}, z_{3}, \ldots, z_{n}\right) \mapsto\left(z_{2}, z_{1}, z_{3}, \ldots, z_{n}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}
$$

changes the sign of the mapping degree $B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}^{I}$. Thus, if necessary, after composing it, one can confirm that

$$
\operatorname{deg}\left(B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}^{I}\right)=1
$$

Therefore, up to sign, from Theorem A.2, we see $B F_{\nu(S),\left.\mathfrak{s}\right|_{\nu(S)}}$ and $U^{m} B F_{\nu(S),\left.\mathfrak{s}^{\prime}\right|_{\nu(S)}}$ are $O(2)$-stably homotopic. This completes the proof.

Proof of Theorem 1.5. For a fixed strictly $I$-invariant path for the Montesinos cases and for each vertex $\mathfrak{s}_{i}$, we associate the corresponding Bauer-Furuta invariant

$$
B F_{W_{\Gamma}, \mathfrak{s}_{i}}: \mathbb{S}\left(\mathfrak{s}_{i}\right) \rightarrow \Sigma^{\frac{h}{2}} \mathbb{C} S W F(Y, \mathfrak{s})
$$

with stabilizations by $\mathbb{R}, \widetilde{\mathbb{R}}$ and $\mathbb{C}$, which is $O(2)$-equivariant.
Note that $\mathfrak{s}_{i}-\mathfrak{s}_{i-1}$ can be represented by $P D(S)$, where $S$ is the connected sum of certain 2-handle cores having negative self-intersections. Moreover, from the construction of involution on the graph 4-manifold, we see the 2-handle cores are preserved by the involution and it reverses an orientation of each 2-handle core. Therefore, we can apply Proposition 4.6 to $\mathfrak{s}_{i}$ and $\mathfrak{s}_{i-1}$ to obtain an $O(2)$-equivariant homotopy $H$ after composing a base change if necessary. This gives an $O(2)$-equivariant map

$$
H: \mathbb{E}\left(e_{\mathfrak{s}_{i}, \mathfrak{s}_{i-1}}\right) \rightarrow \Sigma^{\frac{h}{2} \mathbb{C}} S W F(Y)
$$

which gives a well-defined $O(2)$ equivariant map

$$
\mathcal{T}^{O(2)}: \mathcal{H}\left(\gamma, \mathfrak{s}_{0}\right) \rightarrow S W F(Y)
$$

From the construction, if we forget $I$ action, it is nothing but the construction of the original $S^{1}$-equivariant map given in DSS23, which is $S^{1}$-homotopy equivalence.
Proof of Corollary 1.6. We recall the process of drawing a graded root (up to overall grading shift, for simplicity) $R$ from a (finite) path $\gamma$ carrying the lattice homology of $(\Gamma, \mathfrak{s})$. Write $\gamma=\left\{\mathfrak{s}_{1}, \ldots, \mathfrak{s}_{n}\right\}$, where every $\mathfrak{s}_{i}$ restricts to $\mathfrak{s}$ on $Y_{\Gamma}$, and choose characteristic vectors $k_{i}$ that represent $\mathfrak{s}_{i}$. Then we consider the sequence

$$
k_{1}^{2}, \ldots, k_{n}^{2}
$$

Let $i_{1}, \ldots, i_{m}$ be the indices where the sequence achieves a local maximum. Then, for each $s=1, \ldots, m-1$, we consider the subsequence

$$
k_{i_{s}}^{2}, k_{i_{s}+1}^{2}, \ldots, k_{i_{s+1}}^{2}
$$

this sequence admits a global minimum, at an index which we denote as $j_{s}$, such that $k_{t}^{2} \geq k_{j_{s}}^{2}$ for any $t=i_{s}, \ldots, i_{s+1}$. Then $R$ is a graded root which consists of leaves $v_{1}, \ldots, v_{m}$, with $\operatorname{gr}\left(v_{t}\right)=k_{i_{t}}^{2}$, and angles $w_{1}, \ldots, w_{m-1}$ between the leaves (where $w_{t}$ lies between $v_{t}$ and $v_{t+1}$ ), with $\operatorname{gr}\left(w_{t}\right)=k_{j_{t}}^{2}$.

Now we calculate the Euler charcteristic of the fixed point locus. Since Euler characteristic can be computed using $\mathbb{Z}_{2}$ coefficient homology, it follows from Lemma 3.6 that

$$
\chi\left(S W F\left(\Sigma_{2}(K), \mathfrak{s}\right)^{I}\right)=\chi\left(\mathcal{H}(\gamma, \mathfrak{s})^{I}\right)
$$

It follows from a simple Mayer-Vietoris argument that

$$
\chi\left(\mathcal{H}(\gamma, \mathfrak{s})^{I}\right)=\sum_{i=1}^{n}(-1)^{\frac{k_{i}^{2}}{2}}-\sum_{j=1}^{n-1}(-1)^{\frac{\min \left(k_{j}^{2}, k_{j+1}^{2}\right)}{2}} .
$$

Then it follows from the choice of indices $i_{1}, \ldots, i_{m}$ and $j_{1}, \ldots, j_{m-1}$ that

$$
\chi\left(\mathcal{H}(\gamma, \mathfrak{s})^{I}\right)=\sum_{t=1}^{m}(-1)^{\frac{k_{i t}^{2}}{2}}-\sum_{t=1}^{m-1}(-1)^{\frac{k_{j_{t}^{2}}^{2}}{2}}=\operatorname{deg}(K)=\left|\sum_{v \in L(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}-\sum_{v \in A(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}\right|
$$

as desired.
4.11. Examples of $\left|\chi\left(S W F_{R}(K)\right)\right|$. We give several concrete examples of the computation of $\left|\chi\left(S W F_{R}(K)\right)\right|$ from Corollary 1.6. It is observed in KMT23b, Proof of Lemma 3.28] that $S W F_{R}(K)$ and $S W F_{R}(-K)$ are $V$-dual, where $-K$ denotes the mirror of $K$ and $V$ is some vector space. Therefore, we have

$$
\left|\chi\left(S W F_{R}(K)\right)\right|=\left|\chi\left(S W F_{R}(-K)\right)\right|
$$

Thus, we do not need to care about the convention of knots about the mirrors here.
Example 4.9. Consider the plumbing graph


Then $Y_{\Gamma}$ is the double-branched cover of the pretzel knot $K=P(2,-3,-7)$. We will present a path of spin ${ }^{c}$ structures on $W_{\Gamma}$, presented in terms of homology classes in $H_{2}\left(W_{\Gamma} ; \mathbb{Z}\right)$, which carries the lattice homology of $\left(Y_{\Gamma}, \mathfrak{s}\right)$, where $\mathfrak{s}$ denotes the unique $\operatorname{spin}^{c}$ structure on $Y_{\Gamma}$.

We will use the following notation: classes in $H_{2}\left(Y_{\Gamma} ; \mathbb{Z}\right)$ are represented as quadruples $x=(a, b, c, d)$. This would mean that $x$ is the sum

$$
x=a\left[S_{-1}\right]+b\left[S_{-2}\right]+c\left[S_{-3}\right]+d\left[S_{-7}\right]
$$

where $S_{-n}$ denotes the node of $\Gamma$ whose self-intersection is $-n$. This setting is a bit different from the one that we used in the proof of Corollary 1.6, and thus the weight functions are defined differently. In fact, in this setting, the weight function is defined as

$$
w(x)=x^{2}+k \cdot x
$$

where $k=(0,1,1,1)$ is the spherical Wu class.
Now we consider the path

$$
\gamma=\{(-1,-1,-1,-1),(0,-1,-1,-1),(0,0,-1,-1),(0,0,0,-1),(0,0,0,0),(1,0,0,0)\}
$$

The sequence of weights are then given by

$$
w(\gamma)=\{2,0,0,0,0,2\}
$$

it is then easy to see that $\gamma$ carries the lattice homology of $\left(Y_{\Gamma}, \mathfrak{s}\right)$. In fact, a careful reader can observe that $\gamma$ is actually an almost $J$-invariant path in the sense of [DSS23, Definition 6.2]. From this data, we see that the $S^{1}$-equivariant lattice Floer homotopy type $\mathcal{H}(\gamma)$ (which is the same as the $S^{1}$-equivariant Seiberg-Witten homotopy type of $\Sigma(2,3,7)$ ) is given by $S^{2} \cup_{S^{0}} S^{2}$. Note that, since $\Sigma(2,3,7)$ is a homology sphere, it has only one $\operatorname{spin}^{c}$ structure, and thus we are dropping $\operatorname{spin}^{c}$ structures from our notations.

To see the $O(2)$-action on this homotopy type, we observe that $S^{2}$ and $S^{0}$ are actually given in terms of compactifications of $S^{1}$-representations as follows:

$$
S^{2}=\left(\mathbb{C}^{1}\right)^{+} \quad \text { and } \quad S^{0}=\left(\mathbb{C}^{0}\right)^{+}
$$

The $I$-action on complex representations are given by the complex conjugation, so we see that

$$
S W F_{R}(P(-2,3,7))=\mathcal{H}(\gamma)^{I} \simeq S^{1} \cup_{S^{0}} S^{1} \simeq S^{1} \vee S^{1} \vee S^{1}
$$

and thus $\left|S W F_{R}(P(-2,3,7))\right|=3 \cdot\left|\chi\left(S^{1}\right)\right|=3$. Note here that we take $\chi\left(S^{1}\right)=1$, as we are considering $S^{1}$ as a graded spectrum $\Sigma^{\infty} S^{1}$ and thus we are computing the Euler characteristic of its reduced homology.

For a sanity check, we will also use Theorem 1.5 and check that we get the same result. From the sequence of weights of lattice points on the given path $\gamma$, we see that the associated graded root is given as follows.


This graded root has three vertices, among which two of them are leaves. The leaves lie in degree 2, while the non-leaf vertex, which has only one angle, lies in degree 0 . Hence we see that Theorem 1.5 also gives the same result:

$$
\left|\chi\left(S W F_{R}(P(-2,3,7))\right)\right|=|(-1)+(-1)-1|=3
$$

Example 4.10. Instead of the pretzel knot $P(-2,3,7)$, we now consider the Montesinos knots $K_{n}$ given by negative-definite AR plumbing graphs of $\Sigma(2,3, n)$, where $n \geq 7$ and $n$ is relatively prime to 6 . In this case, one can use the computation of the $S^{1}$-equivariant Seiberg-Witten Floer homology of their double-branched covers $\Sigma_{2}\left(K_{n}\right)=\Sigma(2,3, n)$, which was already done in Man07, Section 7.2] to determine the graded root, and then use it to compute the value of $\left|\chi\left(S W F_{R}\left(K_{n}\right)\right)\right|$.

For simplicity, we will only present two cases: $n=12 k-5$ and $n=12 k+1$ for $k>0$. In the case $n=12 k-5$, which also covers the case of $P(2,-3,-7)$, the graded root is given as follows.


It has $2 k$ leaves in some even degree, which we consider to be at degree 2 after a suitable degree shift, and $2 k-1$ angles in degree 0 . Hence we have

$$
\left|\chi\left(S W F_{R}\left(K_{n}\right)\right)\right|=4 k-1
$$

On the other hand, if $n=12 k+1$, then the graded root looks like the following.


It has $2 k+1$ leaves in degree 2 (after a degree shift) and $2 k$ angles in degree 0 . Hence we get

$$
\left|\chi\left(S W F_{R}\left(K_{n}\right)\right)\right|=4 k+1
$$

The remaining cases can also be dealt similarly, and so we omit those.
Example 4.11. Let $\Gamma$ be a negative-definite AR plumbing graph, $W_{\Gamma}$ be the associated smooth 4-manifold, and $K_{\Gamma}$ be the associated arborescent knot. Since we know from Theorem 1.5 that the computation of the Euler characteristic $\left|\chi\left(S W F\left(\Sigma_{2}\left(K_{\Gamma}\right), \mathfrak{s}\right)^{I}\right)\right|$ depends only on the graded root of $(\Gamma, \mathfrak{s})$ for any $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ on $Y_{\Gamma}=\Sigma_{2}\left(K_{\Gamma}\right)$, their computations are now easy even in much more complicated cases.

We will give model computations for four additional cases, $Y_{1}, Y_{2}, Z_{1}, Z_{2}$, defined as follows.

$$
Y_{1}=\Sigma(3,5,7), \quad Y_{2}=\Sigma(5,8,13), \quad Z_{1}=\Sigma(3,4,11), \quad Z_{2}=\Sigma(5,7,17)
$$

They are Seifert manifolds and thus admit canonical plumbing graphs which we denote as $\Gamma_{Y_{1}}, \Gamma_{Y_{2}}, \Gamma_{Z_{1}}, \Gamma_{Z_{2}}$. We will denote their associated arborescent knots as $K_{Y_{1}}, K_{Y_{2}}, K_{Z_{1}}, K_{Z_{2}}$. Also, $Y_{1}, Y_{2}, Z_{1}, Z_{2}$ are all homology spheres, so they only have one $\operatorname{Spin}^{c}$-structures; hence we will drop them from our notations. The computation of their graded roots are given in [KŞ22, Figures 8 and 9]. Applying Theorem 1.5 then tells us the following.

$$
\left|\chi\left(S W F_{R}\left(K_{Y_{1}}\right)\right)\right|=\left|\chi\left(S W F_{R}\left(K_{Y_{2}}\right)\right)\right|=\left|\chi\left(S W F_{R}\left(K_{Z_{1}}\right)\right)\right|=\left|\chi\left(S W F_{R}\left(K_{Z_{2}}\right)\right)\right|=1
$$

## 5. Concluding remarks

5.1. Calculations on Miyazawa's invariant. Miyazawa considered the mapping degree of the $\{ \pm 1\}$-framed real Bauer-Furuta invariants

$$
|\operatorname{deg}(S)| \in \mathbb{Z} /\{ \pm 1\}=\mathbb{Z}_{\geq 0} \quad \text { and } \quad|\operatorname{deg}(P)| \in \mathbb{Z} /\{ \pm 1\}=\mathbb{Z}_{\geq 0}
$$

for given a 2 -knot $S$ in $S^{4}$ and a given $\mathbb{R P}^{2}$-knot $P$ in $S^{4}$. We recall the following theorem, proven in Miy23:
Theorem 5.1. Let $K$ be a knot in $S^{3}$ with determinant one and $k, l$ be integers. We denote by $\tau_{k, \alpha}(K)$ the $k$-twisted $\alpha$-roll twisted spun knot in $S^{4}$. If $\frac{k}{2}+\alpha$ is odd, then we have

$$
\begin{equation*}
\left|\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)\right|=|\operatorname{deg}(K)| \tag{8}
\end{equation*}
$$

where $\operatorname{deg}(K)$ denotes the absolute value of the sign counting of the $( \pm 1)$-framed real Seiberg-Witten moduli space for $\Sigma_{2}(K)$ with the unique spin structure.

For the definition of twisted roll spun 2-knots, see Plo84, Section 1]. We shall rewrite the left-hand side of (8) in terms of real Seiberg-Witten Floer homotopy type of knots $S W F_{R}(K)$.

Proposition 5.2. For a knot $K$ in $S^{3}$, we have $|\operatorname{deg}(K)|=\left|\chi\left(S W F_{R}(K)\right)\right|$.
Proof. The proof needs a comparison between the critical point set of infinite-dimensional Morse functional and that of a finite-dimensional approximation of the functional. Basically, the analysis we need to do is similar to the arguments done in LM18, Section 7 and 9], although we only need to focus on critical point sets, not trajectories. Also, we do not need to consider the blow up of the configuration space since we forcus on the counting of framed moduli spaces. Such a comparison needs a careful analysis and the discussions rely on the compactness of the Seiberg-Witten equation. In our situation, we are just taking a fixed point part with respect to $I \in O(2)$, so such a compactness is still true. Thus, we will not repeat their argument here, instead, we write a sketch of the proof.

We first see the precise definition of the degree invariant $\operatorname{deg}(K)$. With respect to the unique spin structure on the double-branched cover $\Sigma_{2}(K)$ with a $\mathbb{Z}_{2}$-invariant Riemann metric on $\Sigma_{2}(K)$, we have the $O(2)$-invariant Chern-Simons Dirac functional on a global slice;

$$
C S D: \mathcal{C}_{K}:=\left(i \operatorname{Ker} d^{*} \subset i \Omega_{\Sigma_{2}(K)}^{1}\right) \oplus \Gamma(\mathbb{S}) \rightarrow \mathbb{R}
$$

Then, we consider the induced function on the fixed point set:

$$
C S D^{I}: \mathcal{C}_{K}^{I}:=\left(i \operatorname{Ker} d^{*}\right)^{I} \oplus \Gamma(\mathbb{S})^{I} \rightarrow \mathbb{R}
$$

We have an action of constant gauge transformations $\{ \pm 1\}$. Now, we take a perturbation that comes from cylinder functions

$$
f: \mathcal{C}_{K}^{I} \rightarrow \mathbb{R}
$$

such that all critical points of $C S D^{I}+f$ are non-degenerate, i.e. the Hessians on the critical point sets are invertible. The existence of such a perturbation is proven in Li23, 7.4. Proof of transversality]. After the perturbation, we can assume there are the unique reducible critical point $\left[\left(a_{0}, 0\right)\right]$ has stabilizer $\pm 1$, and the set of the other finite irreducible critical points have a free $\mathbb{Z}_{2}$ action comes from $[(a, \phi)] \rightarrow[(a,-\phi)]$. We also fix an orientation of a fiber of the determinant line bundle $\operatorname{det}\left(\operatorname{Ker} d\left(C S D^{I}+f\right)_{\left[\left(a_{0}, 0\right)\right]}\right)$ corresponding to the reducible $\left[\left(a_{0}, 0\right)\right]$. Induced from this orientation, we can define the absolute value of the signed counting of all critical points of $C S D^{I}+f$, which is denoted by $\operatorname{deg}(K)$. Since $\operatorname{deg}(K)$ is a counting of the $\{ \pm 1\}$-framed moduli space with respect to a fixed $\mathbb{Z}_{2}$-Riemann metric and a perturbation, $\operatorname{deg}(K)$ is independent of the choices of a $\mathbb{Z}_{2}$-invariant metric and a non-degenerate perturbation. Also, since $\operatorname{deg}(K)$ denotes the absolute value, $\operatorname{deg}(K)$ does not depend on the choices of an orientation of the determinant line bundle.

Now, we relate $\operatorname{deg}(K)$ with $\left|\chi\left(S W F\left(\Sigma_{2}(K)\right)^{I}\right)\right|$. The spectrum $S W F\left(\Sigma_{2}(K)\right)^{I}$ was defined by taking the $\langle I\rangle$-fixed point part of the Seiberg-Witten Floer homotopy type $S W F\left(\Sigma_{2}(K), \mathfrak{s}_{0}\right)$, again $\mathfrak{s}_{0}$ denotes the unique spin structure on $\Sigma_{2}(K)$. Alternatively, we can describe $S W F\left(\Sigma_{2}(K)\right)^{I}$ as the Conley index of a finite-dimensional approximation of the flow with respect to the vector field $\operatorname{grad} C S D^{I}$. Let us say this construction briefly. Define $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K) \subset \mathfrak{C}_{K}^{I}$ to be the direct sums of the eigenspaces of the linear part of $\operatorname{grad}\left(C S D^{I}+f\right)$ whose eigenvalues are in $(-\lambda, \lambda]$, where $V_{-\lambda}^{\lambda}(K)$ is the eigenspace corresponding to the space of 1 -forms and $W_{-\lambda}^{\lambda}(K)$ is the eigenspace corresponding to spinors. Then we restrict the perturbed Chern-Simons Dirac functional $C S D^{I}+f$ to $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$. If we take $\lambda$ sufficiently large, the set of critical points of $C S D^{I}+f$ in $\mathfrak{C}_{K}^{I}$ is contained in $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$ and the Hessians of the restricted function

$$
\left.\left(C S D^{I}+f\right)\right|_{V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)}: V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K) \rightarrow \mathbb{R}
$$

on each critical point is invertible. Note that a comparison between the critical point sets of the infinitedimensional setting and a finite-dimensional Morse setting is given in LM18, Corollary 7.1.5, Corollary 7.2] in $S^{1}$-monopole Floer setting. ${ }^{10}$ A similar analysis enables us to see there is no other critical point of $\left.\left(C S D^{I}+f\right)\right|_{V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)}$ if we take $\lambda$ sufficiently large.

Now, we consider the gradient flow with respect to $\rho \operatorname{grad}\left(C S D^{I}+f\right)$ on $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$, where $\rho$ is a cut-off function appeared as in the case of the construction of the usual Seiberg-Witten Floer homotopy type. Then, one can prove this flow has an isolated invariant neighborhood, which is a big ball in $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$, again it is assumed to contain all critical points of $C S D^{I}+f$. Then, the Conley index of the vector field $\rho \operatorname{grad}\left(C S D^{I}+f\right)$ is described by a CW complex which has a handle decomposition coming from the Morse handle decomposition with respect to $C S D^{I}+f$. Therefore, it is not hard to see the Euler number of the Conley index is equal to the signed counting of the critical point set of $C S D^{I}+f$ restricted to $V_{-\lambda}^{\lambda}(K) \oplus W_{-\lambda}^{\lambda}(K)$. (See LM18, Theorem 2.4.3].) Thus, it is sufficient to see the sign coming from an orientation of the determinant line bundle and the sign comes from the Morse index with respect to $C S D^{I}+f$ are the same. This sign is equivalent to whether relative grading is odd or even with respect to the relative $\mathbb{Z}$-grading. Therefore, it is a comparison between the Morse index in the infinite-dimensional setting and the Morse index in a finitedimensional approximation. In [LM18, Corollary 9.1.3], such comparisons between the two degrees are given in the usual $S^{1}$-monopole Floer setting. A similar argument without essential change enables us to see the relative gradings in the infinite-dimensional setting and a finite-dimensional setting are the same. This completes the sketch of a proof.

Now, from Proposition 5.2, our result gives combinatorial computations of $\left|\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)\right|$, described in Corollary 1.4 and Corollary 1.6 .

On the other hand, for the standard $P_{0}=\mathbb{R} \mathbb{P}^{2}$ whose double cover is $\overline{\mathbb{C P}}^{2}$, we have

$$
\left|\operatorname{deg}\left(P_{0}\right)\right|=1
$$

and the connected sum formula

$$
|\operatorname{deg}(P \# S)|=|\operatorname{deg}(P)| \cdot|\operatorname{deg}(S)| \quad \text { and } \quad\left|\operatorname{deg}\left(S \# S^{\prime}\right)\right|=|\operatorname{deg}(S)| \cdot\left|\operatorname{deg}\left(S^{\prime}\right)\right|
$$

for general $\mathbb{R}^{2}$ knot whose double-branched cover has $b_{2}^{+}=0$ and 2-knots $S$ and $S^{\prime}$, which are again proven in Miy23.
Corollary 5.3. Let $\Gamma$ be a negative-definite AR-graph, $W_{\Gamma}$ be the associated plumbed 4-manifold with boundary $Y_{\Gamma}$, and consider the corresponding arborescent knot $K_{\Gamma}$. Let $\gamma$ be a path which carries the lattice homology of $(\Gamma, \mathfrak{s})$ for any $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ on $Y_{\Gamma}$. Suppose that the lattice homology of $(\Gamma, \mathfrak{s})$ is expressed as a graded root $R$, and the determinant of $K_{\Gamma}$ is one. Denote the sets of leaves and non-leaf vertices of $R$ by $L(R)$ and $N L(R)$, respectively, and shift the grading (if necessary) so that all vertices of $R$ lie on even degrees. Furthermore, we suppose

$$
\left|\sum_{v \in L(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}-\sum_{v \in N L(R)}(-1)^{\frac{\operatorname{gr}(v)}{2}}\right| \neq 1
$$

[^6]Then for integers $k, \alpha$ such that $\frac{k}{2}+\alpha$ is odd, the $k$-twisted $\alpha$-roll twisted spun $k n o t \tau_{k, \alpha}(K) \# P_{0}$ and $P_{0}$ are not smoothly isotopic.

Remark 5.4. As it is observed in Miy23, Theorem 4.47], $\tau_{k, \alpha}(K) \# P_{0}$ and $P_{0}$ have non-diffeomorphic complements for $k=0, \alpha=1$ and $K=P(-2,3,7)$. Note that the same proof works in general situations once we can ensure

$$
\operatorname{deg}\left(\tau_{k, \alpha}(K)\right)>1
$$

Under the same assumptions in Corollary 5.3, we see the complements of $\tau_{k, \alpha}(K) \# P_{0}$ and $P_{0}$ in $S^{4}$ are not diffeomorphic. Similarly, the corresponding statements of Miy23, Theorem 1.3] also hold for knots satisfying the assumptions of Corollary 5.3. It leads to giving a larger class of negative answers to Kir97, Problem 4.58]. See also Kam17, Epilogue].

### 5.2. Structual theorem of an $O(2)$-equivariant Bauer-Furuta invariant.

Proof of Theorem 1.7. For a given 2 -knot or $\mathbb{R}^{2}$ - $\operatorname{knot} S$ in $S^{4}$, we consider its double-branched covering space $\Sigma_{2}(S)$. We assume $b_{2}^{+}\left(\Sigma_{2}(S)\right)=0$ for the $\mathbb{R P}^{2}$-knot case. Then, we take the unique spin structure on $\Sigma_{2}(S)$ when $S$ is 2 -knot and the $\operatorname{spin}^{c}$ structure $\mathfrak{s}$ such that $c_{1}(\mathfrak{s})^{2}=-1$ when $S$ is $\mathbb{R}^{2} \mathbb{P}^{2}$-knot. Associated to it, we have an $O(2)$-equivariant map

$$
B F_{\Sigma_{2}(S), \mathfrak{s}}: W^{+} \rightarrow V^{+}
$$

with respect to the above spin or $\operatorname{spin}^{c}$ structure $\mathfrak{s}$. One can easily check that $W$ is isomorphic to $V$ as $O(2)$-representation spaces and the $O(2)$-equivariant stable homotopy class of $B F_{W, \mathfrak{s}}$ is an invariant of smooth isotopy classes of 2 -knots or such $\mathbb{R P}^{2}$ knots. If we take $\langle I\rangle \subset O(2)$-invariant part of $B F_{\Sigma_{2}(S), \mathfrak{s}}$, we recover the Miyazawa's invariant $\operatorname{deg}(S)$ as the mapping degree of $B F_{\Sigma_{2}(S), \mathfrak{s}}^{I}$. Such a homotopy class is determined by two quantities

$$
\operatorname{deg}\left(B F_{\Sigma_{2}(S), \mathfrak{s}}^{I}\right) \quad \text { and } \quad \operatorname{deg}\left(B F_{\Sigma_{2}(S), \mathfrak{s}}^{S^{1}}\right)
$$

by Theorem A.2. The latter one is +1 if we take a standard homology orientation. The first one is nothing but Miyazawa's invariant. The sign ambiguity corresponds to composing the permutation

$$
\left(z_{1}, z_{2}, z_{3} \ldots, z_{n}\right) \mapsto\left(z_{2}, z_{1}, z_{3}, \ldots, z_{n}\right): \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}
$$

This completes the proof.

## Appendix A. $O(2)$-Representations and $O(2)$-EQUivariant maps

A.1. $O(2)$ representations in our setting. We first see which representations of $O(2)$ appear in our situation.

Lemma A.1. Consider the Lie group $O(2)$, and identify its identity component with $U(1)$. Choose an order two element $I \in O(2)$ such that $O(2)$ is generated by $U(1)$ and $I$. Let $\rho: O(2) \rightarrow G L_{\mathbb{R}}(V)$ be a representation of $O(2)$, where $V=\mathbb{C}^{n}$ and $U(1)$ acts on $V$ via $\rho$ by complex multiplication. Then the action of $\rho(I)$ is the complex conjugation.

Proof. Since $O(2)$ is compact, its finite-dimensional representations over $\mathbb{R}$ decompose into a direct sum of irreducible representations. The list of all irreducible representations of $O(2)$ is described below (the proof is straightforward and thus omitted).

- 1-dimensional trivial representation $\mathbb{R}$;
- 1-dimensional flip reprsentation $\tilde{\mathbb{R}}$, where $O(2)$ acts through $\pi_{0}(O(2))$, which then acts on $\mathbb{R}$ by $\pm 1$;
- 2-dimensional reprsentations $\mathbb{C}_{q}$, indexed by positive integers $q$, where $I$ acts on $\mathbb{C} \cong \mathbb{R}^{2}$ by complex conjugation and $U(1)$ acts by the $q$-fold rotation. (When $q=1$, we denote $\mathbb{C}_{1}$ by $\mathbb{C}$, as $I$ acts on it by complex conjugation)
Hence $V$ decomposes into direct sums of several copies of $\mathbb{R}, \tilde{\mathbb{R}}$, and $\mathbb{C}_{q}$ for $q>0$. Observe that, among the irreducible representations of $O(2)$, the only one which induces a free action (outside the origin) of $U(1)$ is $\mathbb{C}$. Since $U(1)$ acts freely on $V \backslash\{0\}$ via $\rho$ by assumption, we deduce that $V=\mathbb{C}^{n}$ as $O(2)$-representations, and thus the action of $\rho(I)$ is the complex conjugation.
A.2. Equivariant version of Hopf's classification theorem. We review the equivariant version of Hopf's classification theorem written in tD87, Page 125], which was used to prove the existence of $O(2)$-equivariant map between the lattice homotopy type and the Seiberg-Witten Floer homotopy type.

Let $V$ and $W$ be an $O(2)$-representation. We denote by $V^{+}$and $W^{+}$the one-point compactifications of $V$ and $W$. Suppose the possible isotropy groups of $V^{+}$and $W^{+}$are

$$
\{e\}, S^{1},\langle I\rangle \subset O(2)
$$

For each isotopy group $G \subset O(2)$, we have the fixed point spheres $\left(V^{+}\right)^{G}$ and $\left(W^{+}\right)^{G}$, whose dimensions are written by $n_{V}(G)$ and $n_{W}(G)$. We further suppose

$$
n_{V}(G) \leq n_{W}(G)
$$

Let us define the set $\Phi(V, W, O(2))$ of the conjugacy classes of isotropy groups $G$ satisfying

$$
n_{V}(G)=n_{W}(G) \quad \text { and } \quad|W G|<\infty
$$

where $W G$ is the Weyl group given as $N G / G$. Here $N G$ denotes the normalizer of $G$ in $O(2)$. Thus, in our situation (assuming that $V$ and $W$ are in our universe $\mathbb{R}^{\infty} \oplus \widetilde{\mathbb{R}}^{\infty} \oplus \mathbb{C}^{\infty}$ ), we have

$$
\Phi(V, W, O(2)) \subset\left\{S^{1},\langle I\rangle\right\}
$$

In this situation, one can check that for any $G \in \Phi(V, W, O(2))$, the groups $\widetilde{H}^{n_{V}(G)}\left(\left(V^{G}\right)^{+}\right)$and $\widetilde{H}^{n_{W}(G)}\left(\left(W^{G}\right)^{+}\right)$ are isomorphic as $W G$-module. Under these assumptions, the following is proven in tD87, page 125, Theorem 4.11]:

Theorem A.2. Under the assumptions above, two $O(2)$-equivariant continuous maps

$$
f_{0}, f_{1}: V^{+} \rightarrow W^{+}
$$

are $O(2)$-equivariantly homotopic if $\operatorname{deg}\left(f_{0}^{G}\right)=\operatorname{deg}\left(f_{1}^{G}\right)$ for any $G \in \Phi(V, W, O(2))$.

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[^0]:    ${ }^{1}$ For the rest of the cables, it can be verified that they are not algebraically slice using Tristram-Levine Signatures Tri69 Lev69 and Kaw80b (see also CLR08, Theorem 6]).
    ${ }^{2}$ See KW18 Definition 2] for the precise definition of strongly rationally slice knots.

[^1]:    ${ }^{3}$ As it is pointed out in BS22 Remark 1.6], the invariants $\tau^{\#}$ and $\nu^{\#}$ vanish for rationally slice knots. In particular, from GLW19 Theorem 1.2], $\tau_{I}$ also vanishes.

[^2]:    ${ }^{4}$ For the definition of almost I-invariant path, see Subsection 4.3
    ${ }^{5}$ See AKS20 Section 4.4] for the definition of angles in a graded root.

[^3]:    ${ }^{6}$ For the construction of $O(2)$-equivariant Bauer-Furuta invariants, see Subsection 3.3
    ${ }^{7}$ For the definition of the coordinate changes, see Remark 4.7

[^4]:    ${ }^{8}$ The map $-I$ also induces another real involution on the configuration space. One can easily check the invariants $\delta_{R}, \underline{\delta}_{R}$ and $\bar{\delta}_{R}$ we will focus on in this paper do not depend on such choices.

[^5]:    ${ }^{9}$ It should be possible to write down this dependence on the choices of splittings explicitly. However, we do not need to do it in this paper, so we omit it.

[^6]:    ${ }^{10}$ Since they treat blown-up of a finite-dimensional approximation and comparison between Morse chain complexes. In our situation, we are just counting $\{ \pm 1\}$-framed critical points, we do not need to consider the blow-up configuration space.

