# CYCLICITY IN BESOV-DIRICHLET SPACES FROM THE CORONA THEOREM

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ABSTRACT. Tolokonnikov's Corona Theorem is used to obtain two results on cyclicity in Besov-Dirichlet spaces.

À la mémoire de Mohamed Zarrabi

#### 1 INTRODUCTION

Let X be a Banach space of analytic functions in the unit disc  $\mathbb{D}$  such that the shift operator  $S: f(z) \to zf(z)$  is a continuous map of X into itself. The cyclic vectors in X are those functions f such that the polynomial multiples of f are dense in X. Beurling in [9] provided a complete characterization of cyclic vectors in the Hardy space; the cyclic vectors are precisely the outer functions. Cyclic vectors in the Dirichlet space were initially examined by Carleson in [12] and later by Brown and Shields in [10]. In this paper, we focus on studying cyclic vectors in Besov-Dirichlet spaces. Specifically, motivated by the inquiries raised by Brown and Shields [10, Question 3] regarding cyclic vectors in a general Banach space X of analytic functions :

Question: If  $f, g \in X$ , if g is cyclic, and if  $|f(z)| \ge |g(z)|$  for all  $z \in \mathbb{D}$  then must f be cyclic?

We extend some of Brown and Shields' results on cyclicity to Besov-Dirichlet spaces. We now introduce some notations. For  $p \ge 1$  and  $\alpha > -1$ , the Besov space,  $\mathcal{D}^p_{\alpha}$  is the set of holomorphic functions on  $\mathbb{D}$  such that

$$\mathcal{D}_{\alpha,p}(f) = \int_{\mathbb{D}} |f'(z)|^p dA_{\alpha}(z) < \infty,$$

where  $dA_{\alpha}(z) = (1+\alpha)(1-|z|^2)^{\alpha} dA(z)$  and dA(z) is the normalised Lebesgue measure on the disc. The Besov-Dirichlet space is equipped with the norm

$$\|f\|_{\mathcal{D}^p_{\alpha}}^p = |f(0)|^p + \mathcal{D}_{\alpha,p}(f).$$

The Besov-Dirichlet space  $\mathcal{D}^p_{\alpha}$  is the set of holomorphic functions f on  $\mathbb{D}$  whose derivative f' is a function of the Bergman space  $\mathcal{A}^p_{\alpha} = L^p(\mathbb{D}, dA_{\alpha}) \cap \operatorname{Hol}(\mathbb{D})$ , where  $\operatorname{Hol}(\mathbb{D})$  is the space of holomorphic functions on  $\mathbb{D}$ . Note that if p = 2 and  $\alpha = 1$ ,  $\mathcal{D}^2_1$  is the Hardy space  $H^2$  and if p = 2 and  $\alpha = 0$ , then  $\mathcal{D}^2_0$  is the classical Dirichlet space  $\mathcal{D}$ .

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Denote by  $[f]_{\mathcal{D}^p_{\alpha}}$  the smallest S-invariant subspace containing f, the vector subspace generated by  $\{z^n f, n \in \mathbb{N}\}$ . We say that  $f \in \mathcal{D}^p_{\alpha}$  is cyclic in  $\mathcal{D}^p_{\alpha}$  if

$$[f]_{\mathcal{D}^p_\alpha} = \mathcal{D}^p_\alpha.$$

The function  $f \in H^1$  is called outer function if it is of the form

$$f(z) = \exp \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) |d\zeta|, \qquad |z| < 1,$$

where  $\varphi$  is nonnegative function in  $L^1(\mathbb{T})$  such that  $\log \varphi \in L^1(\mathbb{T})$ . Note that  $|f| = \varphi$  a.e. on une unit circle  $\mathbb{T} = \partial \mathbb{D}$ .

The problem of characterizing the cyclic vectors in the Dirichlet space  $\mathcal{D}_0^2$  is much more difficultIn [10], Brown and Shields conjectured that a function f in the Dirichlet space  $\mathcal{D}$  is cyclic for the shift operator if and only if f is outer and its boundary zero set is of logarithmic capacity. The characterization of cyclic vector of  $\mathcal{D}_{\alpha}^p$  depends on the values of p and  $\alpha$ . More precisely our investigation is limited to the case  $\alpha + 1 \leq p \leq \alpha + 2$ . Indeed, If  $1 , then <math>H^p$  is continuously embedded in  $\mathcal{D}_{\alpha}^p$  see [22], hence every outer function  $f \in H^p$  is cyclic for  $\mathcal{D}_{\alpha}^p = \mathcal{A}_{\alpha-p}^p$ . On the other hand, if  $p > \alpha + 2$ , then  $\mathcal{D}_{\alpha}^p \subset \mathcal{A}(\mathbb{D}) = \operatorname{Hol}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$  becomes Banach algebra see [22], consequently the only cyclic outer function are the invertible functions, and then any function that vanishes at least at one point is not cyclic in  $\mathcal{D}_{\alpha}^p$ . Let  $\mathcal{A}(\mathbb{D})$  be the disc algebra. For  $f \in \mathcal{A}(\mathbb{D})$ , denote

$$\mathcal{Z}(f) = \{ \zeta \in \mathbb{T} : f(\zeta) = 0 \}$$

the zero set of f. Recall that Brown and Shields conjectured that a function f in the Dirichlet space  $\mathcal{D}$  is cyclic for the shift operator if and only if f is outer and its boundary zero set is of logarithmic capacity. Here, we will prove the following theorem.

**Theorem 1.** Let p > 1 such that  $\alpha + 1 \leq p \leq \alpha + 2$ . Let  $f \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  be an outer function such that  $\mathcal{Z}(f) = \{1\}$ , then f is cyclic in  $\mathcal{D}^p_{\alpha}$ .

The case of the classical Dirichlet space  $\mathcal{D}_0^2$  was discovered by Hedenmalm-Shields [21] and generalized by Richter-Sundberg [24]. This result was shown [22] for  $\alpha + 1 ,$ the method used for the proof is inspired by that of Hedenmalm and Shields [21]. Note $that our result also includes the case where <math>p = \alpha + 1$ . Thanks to [21, Theorem 3], Theorem 1 remains true if  $\mathcal{Z}(f) = \{1\}$  to a point is replaced by  $\mathcal{Z}(f)$  is countable. Our second main result is

**Theorem 2.** Let p > 1 such that  $1 + \alpha \leq p \leq \alpha + 2$ . Let  $f, g \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$|g(z)| \le |f(z)|, \qquad z \in \mathbb{D}.$$
(1)

If g is cyclic in  $\mathcal{D}^p_{\alpha}$  then f is cyclic in  $\mathcal{D}^p_{\alpha}$ .

This result generalizes that of Brown and Shields [10, Theorem 1], then Aleman [1, Corollary 3.3] for  $\mathcal{D}^2_{\alpha}$  spaces.

The proof of the two theorems is based on the Tolokonnikov Corona Theorem [28]. The idea of using Corona's theorem in this context goes back to Roberts for the Bergman space

[26], see also [2, 8, 7, 17]. For some results related to cyclic vectors, see [5, 6, 16, 18, 19, 20, 25, 24] and the references therein.

### 2 Proof of Theorem 1 and Theorem 4

We recall two results we will need for the proofs. The first is the Corona Theorem of Tolokonnikov [28]

**Theorem 3.** Let  $1 and Let <math>f_1, f_2 \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$\sup_{z\in\mathbb{D}}\left(|f_1(z)|+|f_2(z)|\right)>\delta>0.$$

Then there exists  $h_1, h_2 \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$\begin{cases} f_1(z)h_1(z) + f_2(z)h_2(z) = 1, & z \in \mathbb{D} \\\\ \|h_1\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \le \delta^{-\mathbf{A}} & and & \|h_2\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \le \delta^{-\mathbf{A}} \end{cases}$$

for some positive constant  $\mathbf{A} \geq 4$  independent of p and  $\alpha$ .

**Remarks 1.** If  $p > \alpha + 2$ , then  $\mathcal{D}^p_{\alpha} \subset \mathcal{A}(\mathbb{D})$ . If p = 2 and  $\alpha = 0$ ,  $\mathcal{D}^2_0 = H^2$  and we therefore find the classical Carleson-Corona Theorem [13]. In this case, the constant  $\mathbf{A} > 2$  instead of  $\mathbf{A} \ge 4$ . If  $\alpha = p - 2$ , Tolokonnikov [28] showed that  $\mathbf{A} = 4$ . Nicolau in [23] showed the Corona Theorem but without giving the quantitative version, see also [3, 14].

Let T be a bounded linear operator actings on an infinite dimensional complex Banach space X. The spectrum of T is denoted by  $\sigma(T)$ . The following corollary is easily obtained by Atzmon's Theorem [4] and Cauchy's inequalities.

**Corollary 1.** Let T be an invertible operator on Banach X such that  $\sigma(T) = \{1\}$ . Suppose that there exists  $k \ge 0$  and c > 0 such that for  $\varepsilon > 0$ , there exists  $c_{\varepsilon} > 0$ 

$$\begin{cases} \|(T-zI)^{-1}\| \le c_{\varepsilon} \exp \frac{\varepsilon}{1-|z|} \quad |z| < 1, \\ \|(T-zI)^{-1}\| \le \frac{c}{(|z|-1)^{k}} \quad |z| > 1, \end{cases}$$

then  $(I - T)^k = 0$ .

## **2.1 Proof of Theorem 1** Let $\lambda \in \mathbb{C}$ and put

$$\delta_{\lambda} := \inf_{z \in \mathbb{D}} |\lambda - z| + |f(z)|.$$

Since f is an outer function, by [27]

$$\lim_{|z| \to 1^{-}} (1 - |z|) \log 1 / |f(z)| = 0.$$

For all  $\varepsilon > 0$ , there is therefore  $c_{\varepsilon} > 0$  such that

$$|f(z)| \ge c_{\varepsilon} \exp \frac{-\varepsilon}{1-|z|}, \qquad z \in \mathbb{D}.$$
 (2)

Considering  $|\lambda| \neq 1$ , we distinguish two cases :

- If  $|z \lambda| \ge |1 |\lambda||/2$ , then  $\delta_{\lambda} \ge |1 |\lambda||/2$ .
- If  $|z \lambda| \le |1 |\lambda||/2$ , then  $|1 - |\lambda||/2 \ge |z - \lambda| \ge |(1 - |\lambda|) - (1 - |z|)| \ge |1 - |\lambda|| - |1 - |z||.$

Thus, we get  $1 - |z| \ge |1 - |\lambda||/2$  and by (2). We then have

$$|f(z)| \ge c_{\varepsilon} \exp \frac{-\varepsilon}{|1-|\lambda||}$$

Therefore, we finally get

$$\delta_{\lambda} \ge c_{\varepsilon} \exp \frac{-\varepsilon}{|1-|\lambda||}.$$

According to the Theorem 3, There is  $g, h \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$\begin{cases} (\lambda - z)g + fh = 1\\ \|g\|_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})} \le \delta_{\lambda}^{-\mathbf{A}} \text{ and } \|h\|_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})} \le \delta_{\lambda}^{-\mathbf{A}} \end{cases}$$

for some constant  $A \geq 4$ .

Let  $[f]_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})}$  be the ideal generated by f and let

$$\pi: \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D}) \to \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})}$$

be the canonical surjection. We have

$$(\lambda \pi(1) - \pi(z))^{-1} = \pi(g).$$

For  $|\lambda| < 1$ , we have

$$\begin{aligned} \|(\lambda \pi(1) - \pi(z))^{-1}\| &= \|\pi(g)\| \\ &\leq \|g\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \\ &\leq c_\varepsilon \exp \frac{\varepsilon}{1 - |\lambda|}. \end{aligned}$$

For  $|\lambda| > 1$ , we have

$$\begin{aligned} \|(\lambda \pi(1) - \pi(z))^{-1}\| &\leq \|(\lambda - z)^{-1}\|_{\mathcal{D}^{p}_{\alpha} \cap \mathcal{A}(\mathbb{D})} \\ &= \frac{1}{|\lambda| - 1} + \frac{1}{|\lambda|} + \left(\int_{\mathbb{D}} \frac{dA_{\alpha}(z)}{|\lambda - z|^{2p}}\right)^{1/p} \\ &\leq \frac{2}{|\lambda| - 1} + \frac{1}{(|\lambda| - 1)^{p}}. \end{aligned}$$

The spectrum of  $\pi$ ,  $\sigma(\pi) = \{1\}$ , by Corollary 1,  $(\pi(1) - \pi(\alpha))^{[p]+1} = 0$ , and we get  $(1-z)^{[p]+1} \in [f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}$ . Since  $(1-z)^{[p]+1}$  is cyclic in  $\mathcal{D}^p_\alpha$ , f is also cyclic in  $\mathcal{D}^p_\alpha$  and the proof is complete.

**2.2 Proof of Theorem 2** The proof of the Theorem 2 is deduced from the following two results.

**Lemma 1.** Let p > 1 such that  $1 + \alpha \leq p \leq \alpha + 2$ . Let  $f, g \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$ , if fg is cyclic, then both f and g are cyclic.

*Proof.* It suffices to show that g is cyclic. Let  $\sigma_n(f)$  denote the Fejér means of the partial sums of the power series for f. Since the  $\sigma_n(f)$  converges to f in  $\mathcal{D}^p_{\alpha}$ ,  $\sigma_n(f)g$  converge pointwise to fg in  $\mathbb{D}$  and

$$\|(\sigma_n(f)g - fg)'\|_{\mathcal{A}^p_{\alpha}}^p \le \|(\sigma_n(f) - f)\|_{\infty} \|g'\|_{\mathcal{A}^p_{\alpha}}^p + \|\sigma_n(f) - f\|_{\mathcal{D}^p_{\alpha}}^p \|g\|_{\infty},$$

we obtain  $\sigma_n g$  converge to fg in  $\mathcal{D}^p_{\alpha}$ , which completes the proof.

The constant N in the following theorem is related to that of the Corona Theorem. If  $\alpha = p - 2$ , we have N(p - 2, p) = 5.

**Theorem 4.** Let p > 1 be such that  $\alpha + 1 \le p \le \alpha + 2$ , there exists  $N = N(\alpha, p)$  which depends only on  $\alpha$  and p such that if  $f, g \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  with

$$|g(z)| \le |f(z)|, \qquad z \in \mathbb{D},$$

then  $[g^N]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$ .

*Proof.* Let  $\lambda \in \mathbb{C}$  and set

$$\inf_{z \in \mathbb{D}} \left\{ |1 - \lambda g(z)| + |f(z)| \right\} = \delta_{\lambda}.$$

Considering  $\lambda \neq 0$ , we have

• If  $|g(z)| \le \frac{1}{2|\lambda|}$ , then  $|1 - \lambda g(z)| \ge 1 - |\lambda||g(z)| \ge \frac{1}{2}$ . • If  $|g(z)| \ge \frac{1}{2|\lambda|}$  then  $|f(z)| \ge \frac{1}{2|\lambda|}$ 

From this, follows

$$\delta_{\lambda} \ge \frac{1}{2|\lambda|}.$$

According to the Theorem 3, there are  $F_{\lambda}, G_{\lambda} \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$\begin{cases} (1 - \lambda g)G_{\lambda} + fF_{\lambda} = 1, \\ \|F_{\lambda}\|_{\mathcal{D}^{p}_{\alpha} \cap \mathcal{A}(\mathbb{D})} \leq \delta_{\lambda}^{-\mathbf{A}} \quad \text{and} \quad \|G_{\lambda}\|_{\mathcal{D}^{p}_{\alpha} \cap \mathcal{A}(\mathbb{D})} \leq \delta_{\lambda}^{-\mathbf{A}} \end{cases}$$

for some constant  $\mathbf{A} > 4$ .

As before, we consider the canonical surjection

$$\pi: \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D}) \to \left(\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})\right) / [f]_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})}.$$

We have

$$(\pi(1-\lambda g))^{-1} = \pi(G_{\lambda})$$

and

$$\begin{aligned} \|(\pi(1-\lambda g))^{-1}\|_{\mathcal{D}^{p}_{\alpha}\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^{p}_{\alpha}\cap\mathcal{A}(\mathbb{D})}} &= \|\pi(G_{\lambda})\|_{\mathcal{D}^{p}_{\alpha}\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^{p}_{\alpha}\cap\mathcal{A}(\mathbb{D})}} \\ &\leq \|G_{\lambda}\|_{\mathcal{D}^{p}_{\alpha}\cap\mathcal{A}(\mathbb{D})} \\ &\leq 2^{\mathbf{A}}|\lambda|^{\mathbf{A}}. \end{aligned}$$

By Liouville Theorem,  $\pi(1-\lambda g)^{-1}$  is polynomial of degree at most [**A**]. Since  $|\lambda g| < 1$ ,  $\pi(1-\lambda g)^{-1} = \sum_{n\geq 0} \lambda^n \pi^n(g)$ . We obtain  $\pi^{[A]+1}(g) = 0$  which means that  $g^{[\mathbf{A}]+1} \in [f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}$  and hence  $[g^{[\mathbf{A}]+1}]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$ .

## 3 Refinement of the Theorem 2

We can improve the estimate (1) in Theorem 4, and thus obtain a possible more same conclusion. The improved estimate we are looking for is given by the following result.

**Theorem 5.** Let p > 1 such that  $\alpha + 1 \le p \le \alpha + 2$ . Let  $f, g \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$ . Suppose that  $\operatorname{Re}(g) \ge 0$  and there exists  $\gamma > 1$  such that.

$$|g(z)| \le \left(\log \frac{\|f\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}}{|f(z)|}\right)^{-\gamma}, \qquad z \in \mathbb{D}$$
(3)

then  $[g]_{\mathcal{D}^p_{\alpha}} \subset [f]_{\mathcal{D}^p_{\alpha}}$ .

*Proof.* We assume that  $||f||_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})} = 1$ . Let  $\lambda \in \mathbb{C}$ , we set

$$\inf_{z\in\mathbb{D}}\left\{|1-\lambda g(z)|+|f(z)|\right\}=\delta_{\lambda}.$$

Considering  $\lambda \neq 0$ , we distinguish two cases:

• If 
$$|g(z)| \le \frac{1}{2|\lambda|}$$
 then  $|1 - \lambda g(z)| \ge 1 - |\lambda||g(z)| \ge \frac{1}{2}$   
• If  $|g(z)| \ge \frac{1}{2|\lambda|}$ , then by (3)

$$|f(z)| \ge e^{-(2|\lambda|)^{\frac{1}{\gamma}}}.$$

From this follows

$$\delta_{\lambda} \ge e^{-(2|\lambda|)^{\frac{1}{\gamma}}}.$$

By Theorem 3, there exists  $F_{\lambda}$ ,  $G_{\lambda} \in \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$  such that

$$\begin{cases} (1 - \lambda g)G_{\lambda} + fF_{\lambda} = 1, \\ \|F_{\lambda}\|_{\mathcal{D}^{p}_{\alpha} \cap \mathcal{A}(\mathbb{D})} \leq \delta^{-\mathbf{A}} \quad \text{and} \quad \|G_{\lambda}\|_{\mathcal{D}^{p}_{\alpha} \cap \mathcal{A}(\mathbb{D})} \leq \delta^{-\mathbf{A}} \end{cases}$$

for some constant  $\mathbf{A} \geq 4$ .

Let  $\pi$  be the canonical surjection

$$\pi: \mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D}) \to \left(\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})\right) / [f]_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})}.$$

We have

$$\pi(1-\lambda g)^{-1} = \pi(G_\lambda)$$

and

$$\begin{aligned} \|\pi(1-\lambda g)^{-1}\|_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})}} &= \|\pi(G_{\lambda})\|_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_{\alpha}}} \\ &\leq \|G_{\lambda}\|_{\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})} \\ &\leq \frac{1}{\delta_{\lambda}^{\mathbf{A}}} \leq e^{\mathbf{A}(2|\lambda|)^{\frac{1}{\gamma}}}. \end{aligned}$$

Let  $\ell \in (\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})})^*$  with norm  $\|\ell\| = 1$  and define  $\varphi$  by

$$\varphi(\lambda) = \langle (\pi(1 - \lambda g))^{-1}, \ell \rangle.$$

The function  $\varphi$  is analytic on  $\mathbb C$  and

$$|\varphi(\lambda)| \le e^{c|\lambda|^{\frac{1}{\gamma}}} \tag{4}$$

where  $c = 2^{\frac{1}{\gamma}} \mathbf{A}$ . Since  $\gamma > 1$ , there exists  $\theta_{\gamma}$  such that  $\frac{\pi}{2}(2-\gamma) < \theta_{\gamma} < \frac{\pi}{2}\gamma$ . We suppose that  $\theta_{\gamma} < \pi$ . Consider now te sector  $S_{\theta_{\gamma}} = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta_{\gamma}\}.$ 

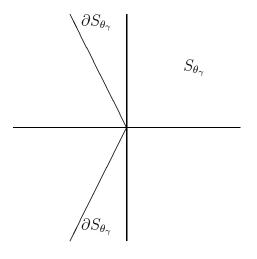


FIGURE 1. le secteur  $S_{\theta_{\gamma}}$ 

Let  $\lambda \in \partial S_{\theta_{\gamma}}$ , since  $\pi/2 < \theta_{\gamma} < \pi$ ,  $\operatorname{Re}(\lambda) \leq 0$  and  $\operatorname{Re}(\frac{1}{\lambda} - g(z)) \leq 0$ . We obtain  $|1 - \lambda g(z)| = |\lambda| |\frac{1}{\lambda} - g(z)|$   $\geq |\lambda| |\operatorname{Re}(\frac{1}{\lambda} - g(z))|$  $\geq |\lambda| \frac{|\operatorname{Re}(\lambda)|}{|\lambda|^2} = \frac{|\operatorname{Re}(\lambda)|}{|\lambda|}.$  Moreover,  $\lambda \in \partial S_{\theta_{\gamma}}$ , hence

$$\operatorname{Re}(\lambda) = |\lambda| \cos \theta_{\gamma}.$$

If we set  $C_{\gamma} = |\cos \theta_{\gamma}|^{-1} \neq 0$ , we get

$$\frac{1}{|1 - \lambda g(z)|} \le \frac{|\lambda|}{|\operatorname{Re}(\lambda)|} = C_{\gamma}$$

Then  $\varphi$  is analytic on  $\mathcal{S}_{\theta_{\gamma}}$ , continuous on  $\overline{\mathcal{S}_{\theta_{\gamma}}}$  and satisfies

$$\begin{cases} |\varphi(\lambda)| \leq e^{c|\lambda|^{\frac{1}{\gamma}}} & \text{for} \quad \lambda \in \mathcal{S}_{\theta_{\gamma}} \\ |\varphi(\lambda)| \leq C_{\gamma} & \text{for} \quad \lambda \in \partial \mathcal{S}_{\theta_{\gamma}} \end{cases}$$

Since  $\frac{1}{\gamma} < \frac{\pi}{2\theta_{\gamma}}$  with  $\theta_{\gamma} < \frac{\pi}{2}\gamma$ , by the Phragmén–Lindelöf principle for a sector  $S_{\theta_{\gamma}}$ , we

have

$$|\varphi(\lambda)| \leq C_{\gamma}, \qquad \lambda \in \mathcal{S}_{\theta_{\gamma}}.$$

The function  $\varphi$  is an entire function and satisfies (4) on  $\mathbb{C}$ . Again using the Phragmén–Lindelöf principle for a sector

$$\mathcal{S} = \mathbb{C} \setminus \mathcal{S}_{\theta_{\gamma}} = \{ \lambda \in \mathbb{C} : \theta_{\gamma} < \arg(\lambda) < 2\pi - \theta_{\gamma} \}.$$

Since  $2\pi - 2\theta_{\gamma}$  we get

$$\begin{cases} |\varphi(\lambda)| \leq e^{c|\lambda|^{\frac{1}{\gamma}}} & \text{on} \quad \lambda \in \mathcal{S} \\ |\varphi(\lambda)| \leq C_{\gamma} & \text{sur} \quad \lambda \in \partial \mathcal{S} \end{cases}$$
  
Since  $\theta_{\gamma} > \frac{\pi}{2}(2-\gamma), \frac{1}{\gamma} < \frac{\pi}{2\pi - 2\theta_{\gamma}}$  and  
 $|\varphi(\lambda)| \leq C_{\gamma} \quad \lambda \in \mathcal{S}. \end{cases}$ 

Then  $\varphi$  is bounded on  $\mathbb{C}$ , By Liouville Theorem,  $\varphi$  is a constant function

$$\varphi(\lambda) = \varphi(0) = \langle \pi^{-1}(1), \ell \rangle, \qquad \lambda \in \mathbb{C}.$$

Thus,  $\pi^{-1}(1 - \lambda g) = \pi^{-1}(1) = \pi(1)$ . For  $|\lambda g| < 1$ , we have  $\pi(1) = \pi^{-1}(1 - \lambda g) = \pi^{-1}(1 - \lambda g)$  $\sum_{n\geq 0} \lambda^n \pi^n(g)$ . Consequently  $\pi(g) = 0$  and  $g \in [f]_{\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})}$ , hence  $[g]_{\mathcal{D}^p_{\alpha}} \subset [f]_{\mathcal{D}^p_{\alpha}}$ . 

**Remarks 2.** We will construct two functions f and g that satisfy the condition of the Theorem 5, this answers a question of Sasha Borichev. A closed set E of the unit circle is said to be K-set (after Kotochigov), if there exists a positive constant  $c_E$  such that for any arc  $I \subset \mathbb{T}$ 

$$\sup_{\zeta \in I} dist(\zeta, E) \ge c_E |I|$$

where I denotes the length of I. K-sets arise as the interpolation sets for Hölder classes, as example the generalized cantor set [18, 19], we refer to [11, 15] for more details. Such a set fulfils the following condition

$$\frac{1}{|I|} \int_{I} \frac{|\zeta|}{dist(\zeta, E)^{\alpha}} \le |I|^{-\sigma}$$

for

$$\sigma < \left( \log\left(\frac{1}{1-c_E}\right) \right) / \left( \log\left(\frac{2}{1-c_E}\right) \right).$$

In particular, E has measure zero and  $\log dist(\zeta, E) \in L^1(\mathbb{T})$ . Let p > 1 such that  $\alpha + 1 \le p \le \alpha + 2$ . Let us now consider the outer function

$$|g(\zeta)| = dist(\zeta, E)^{\beta}, \qquad \zeta \in \mathbb{T}$$

Since E is K-set, by [11],  $\Re g(z) > 0$ ,

$$\Re g(z) \asymp |g(z)| \asymp dist(z, E)^{\beta}$$
 and  $|g'(z)| \asymp dist(z, E)^{\beta-1}$ ,  $z \in \mathbb{D}$ .

If  $1/(2+\alpha) < \beta < 1$ , then

$$\mathcal{D}_{\alpha,p}(g) \asymp \int_{\mathbb{D}} \frac{dA_{\alpha}(z)}{dist(z,E)^{p(1-\beta)}} \lesssim \int_{0}^{1} \frac{dr}{(1-r)^{(\alpha+2)(1-\beta)-\alpha}} < \infty.$$

so  $g \in \mathcal{A}(\mathbb{D}) \cap \mathcal{D}^p_{\alpha}$ . Now let  $1/\gamma = \kappa$ 

$$f(z) = \exp(-1/g^{\kappa}(z)), \qquad z \in \mathbb{D}.$$

We have  $f'(z) = \kappa \frac{g'(z)}{g(z)^{\kappa+1}} \exp(-1/g^{\kappa}(z))$ . Thus  $|f'(z)| \le |g'(z)|$  and  $f \in \mathcal{D}^p_{\alpha}$ .

Let us conclude this work with a final remark. Denote by  $c_0$  the logarithmic capacity and by  $c_{\alpha}$  the  $\alpha$ -capacity for  $0 < \alpha < 1$ . The case of Dirichlet spaces  $\mathcal{D}^2_{\alpha}$ ,  $0 \leq \alpha < 1$ , was studied in [18, 19]. In particular, it was shown in that if  $f \in \mathcal{D}^2_{\alpha} \cap \mathcal{A}(\mathbb{D})$ , is an outer function such that  $\mathcal{Z}(f)$  is a generalized cantor set, then f is cyclic in  $\mathcal{D}^2_{\alpha}$  if and only if  $c_{\alpha}(\mathcal{Z}(f)) = 0$ . We do not know if this result also holds for  $\mathcal{D}^p_{\alpha} \cap \mathcal{A}(\mathbb{D})$ .

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